

System Control

9. Control systems analysis and design in state space: State Feedback and pole placement

Professor Kyongsu Yi

©2014 VDCL

**Vehicle Dynamics and Control Laboratory
Seoul National University**

1. Solution of the state equation
2. Controllability
3. Observability
4. Pole placement
5. Design of servo systems
6. State observer
7. Observer-controller
8. Quadratic Optimal Regulator(tentative)
9. Robust control systems (tentative)

Lecture note 3

State, State equation, transformation,
Transfer function and state equation

State Feedback

- **State Feedback**

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

$$\dot{x} = (A - BK)x$$

(A, B) Controllable : arbitrary pole assignment

In general, full states are not available → Observer.

$$y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$$e = x - \hat{x}$$

$$\dot{e} = (A - LC)e$$

(A, C) Observable : arbitrary pole assignment

$$\hat{x} \rightarrow x$$

$$u = -K\hat{x} \rightarrow 2n \text{ system}$$

$$\dot{x} = Ax + Bu$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

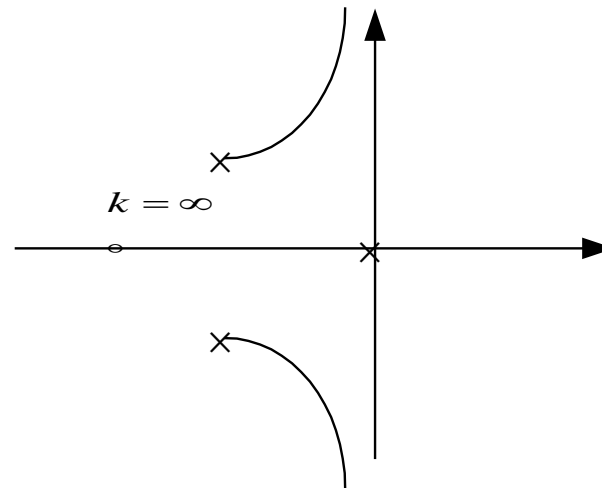
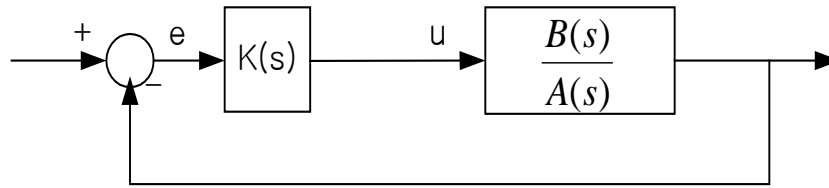
$$y = Cx$$

$$u = -K\hat{x}$$

$2n$ eigenvalues.

$(A - BK), (A - LC)$: separation property

State Feedback Control and Pole Placement



Location of closed loop poles are restricted in root locus.

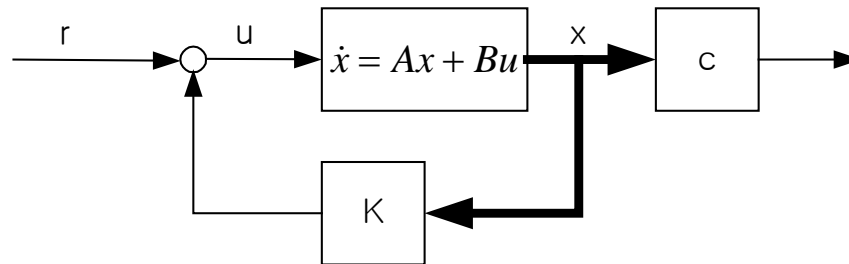
Q: Is it possible to assign arbitrary closed loop poles?

A: Yes, If the system is controllable & all state variables are feed back single input system in controllable canonical form.

State Feedback Control and Pole Placement

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$\det(sI - A) = s^n + a_1s^{n-1} + \cdots + a_n = 0$$



$$u = -Kx + r$$

$$= -[k_1 \quad k_2 \quad k_3 \quad \cdots \quad k_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + r$$

State Feedback Control and Pole Placement

Closed loop System

$$\dot{x} = Ax + B[-Kx + r]$$

$$= [A - BK]x + Br$$

$$= \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ k_1 & \cdots & \cdots & k_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ -(a_n + k_1) & \cdots & \cdots & \cdots & -(a_1 + k_n) \end{bmatrix} x + Br$$

State Feedback Control and Pole Placement

Closed loop Ch. Eq

$$s^n + (a_1 + k_n)s^{n-1} + \dots + (a_{n-1} + k_2)s + (a_n + k_1) = 0$$

$\{\lambda_1, \lambda_2, \dots, \lambda_n\}$: Desired closed loop poles

λ : symmetric with respect to the real axis of s-plane

$$(s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_n) = 0 \Rightarrow$$

$$s^n + [-(\lambda_1 + \lambda_2 + \dots + \lambda_n)]s^{n-1} + [\lambda_1\lambda_2 + \dots]s^{n-2} + [(-\lambda_1)(-\lambda_2) \dots (-\lambda_n)] = 0$$

$$\Rightarrow s^n + a_{c1}s^{n-1} + a_{c2}s^{n-2} + \dots + a_{cn} = 0$$

$$a_{n+1-i} + k_i = a_{c(n+1-i)}$$

k_i : State feed back control gain

State Feedback Control and Pole Placement

Single-Input Controllable System not in controllable canonical form

$$\dot{x} = Ax + Bu, \quad \text{rank}[B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] = n$$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & 1 \\ \vdots & & & a_1 & 1 & 0 \\ a_2 & \ddots & \ddots & 0 & 0 \\ a_1 & \ddots & & \vdots & \vdots \\ 1 & & & 0 & 0 \end{bmatrix} \begin{matrix} \Rightarrow M \\ , M^{-1} \text{ exist} \end{matrix}$$

$$T = MW \quad T^{-1} = W^{-1}M^{-1} \quad x = Tx_c$$

$$T\dot{x} = ATx_c + Bu$$

$$T\dot{x}_c = T^{-1}ATx_c + T^{-1}Bu \quad : \text{Controllable Canonical form}$$

$$= A_c x_c + B_c u$$

$$u = -K_c x_c + V$$

$$= -\underset{K}{K_c} T^{-1} x + V$$

Controllability \Rightarrow arbitrary

\Leftarrow pole assignment.

Necessary and sufficient condition

State Feedback Control and Pole Placement

Discrete Time Case

$$x(k+1) = Gx(k) + Hu(k) \quad , \quad u(k) = -Kx(k)$$
$$= (G - HK)x(k)$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ & & \ddots & & 0 \\ & & & & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} x(k)$$

Nilpotent matrix

- Characteristic Equation

$$z^n = (z - 0)(z - 0) \cdots (z - 0) = 0$$

$$(G - HK)^n = 0$$

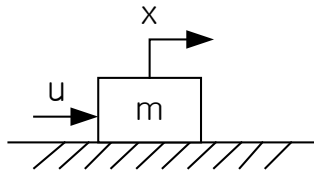
- Dead beat response (Finite Settling Time control)

$$x(n) = (G - HK)^n x(0) = 0 \quad \text{for any } x(0)$$

In general, Dead beat response should not be attempted with a small sampling time.

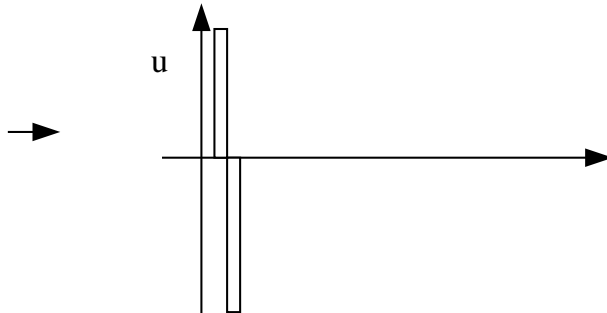
State Feedback Control and Pole Placement

Ex



$$\frac{x}{u} = \frac{1}{ms^2} \rightarrow \frac{T^2(z+1)}{2(z-1)^2}$$

If $T=0.0000001$ sec



Q: Is X directly accessible for state feed back?

A: No, in general.

State Feedback Control and Pole Placement

Observer / State estimator

\hat{x} ; *state estimator*

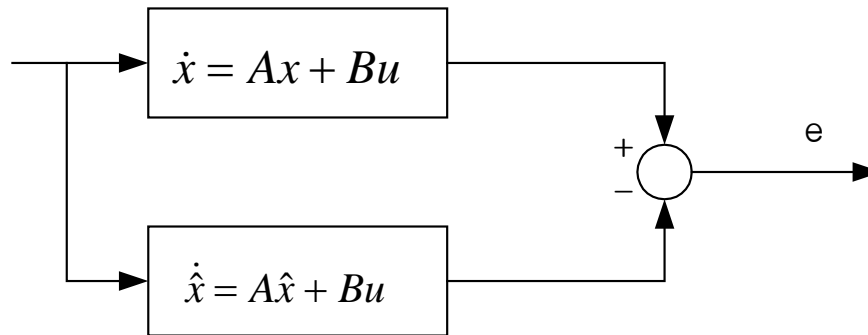
$\hat{x} \rightarrow x$ *in some sense*

Looks reasonable to replace

$u = -Kx$ by $u = -K\hat{x}$

$\dot{x} = Ax + Bu$, $y = Cx$

$\dot{\hat{x}} = A\hat{x} + Bu$



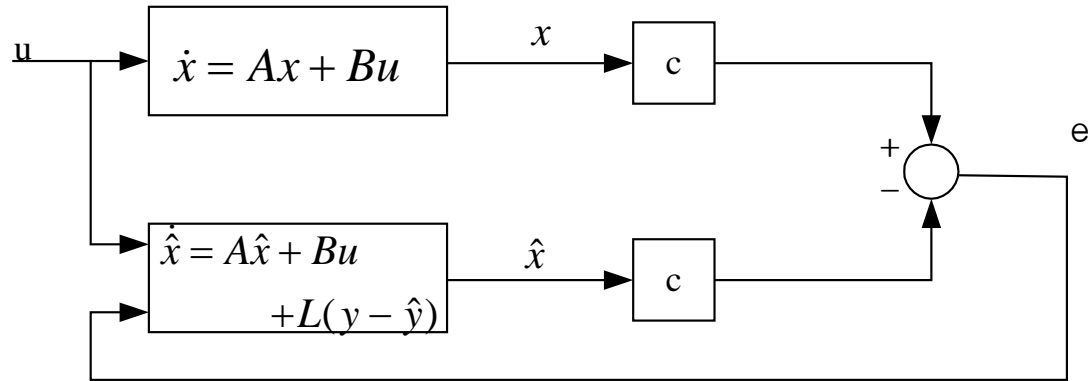
$$e = x - \hat{x}$$

$$\dot{e} = (\dot{x} - \dot{\hat{x}}) = A(x - \hat{x}) = Ae$$

State Feedback Control and Pole Placement

If A is asymptotically stable, then $\lim_{t \rightarrow \infty} e(t) = 0$

If A is not stable, then $e \rightarrow \infty$ as $t \rightarrow \infty$



$$\dot{\hat{x}} = A\hat{x} + Bu + L(cx - c\hat{x}) = (A - LC)\hat{x} + Bu + LCx$$

$$\dot{e} = (A - LC)e$$

State Feedback Control and Pole Placement

System in obs. Canonical form

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & \\ -a_3 & & & \ddots & \\ \vdots & & & & 1 \\ -a_n & 0 & & & 0 \end{bmatrix} \quad C = [1 \quad 0 \quad \cdots \quad 0] \quad L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}$$

$$A - LC = \begin{bmatrix} -(a_1 + l_1) & 1 & 0 & \cdots & 0 \\ -(a_2 + l_2) & 0 & 1 & & \\ \vdots & & & \ddots & \\ \vdots & & & & 1 \\ -a_n & 0 & & & 0 \end{bmatrix}$$

(A,C) observable \Leftrightarrow Arbitrary pole assignment for (A-LC)

State Feedback Control / Observer

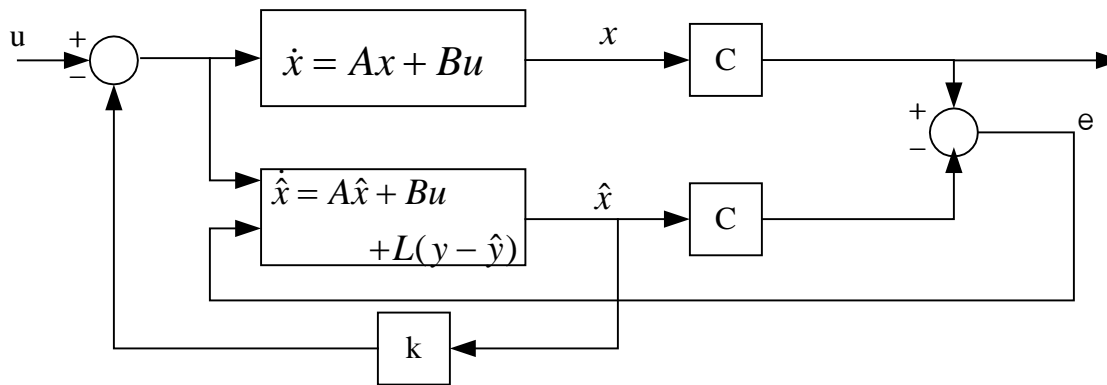
$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

$$\hat{y} = C\hat{x}$$

$$u = -K\hat{x} + r$$



$$\dot{x} = Ax + B(-K\hat{x} + r)$$

$$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x} + r) + L(Cx - C\hat{x})$$

State Feedback Control / Observer

The order of the overall system is $2n \Rightarrow 2n$ eigenvalues

$$\frac{d}{dt} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r$$

$$e = x - \hat{x}$$

$$\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= Ax + B(-K\hat{x}) + BKx \\ &= (A - BK)x + BK(e) \end{aligned}$$

$$\dot{e} = (A - LC)e$$

$$\frac{d}{dt} \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

State Feedback Control / Observer

- Eigenvalues are form

$$\begin{aligned}\det(sI - A) &= \det \begin{bmatrix} sI - (A - BK) & -BK \\ 0 & sI - (A - LC) \end{bmatrix} \\ &= \det(sI - (A - BK)) \det(sI - (A - LC)) \\ &= 0 \qquad \qquad \qquad \text{observerpoles}\end{aligned}$$

- n eigenvalues = regulator poles

$$\begin{aligned}x &= Ax + Bu \\ u &= -Kx \quad \Rightarrow \text{separation property}\end{aligned}$$

- Selection of Best L

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \dot{\hat{x}} &= (A - LC)\hat{x} \\ \dot{e} &= (A - LC) - Ln\end{aligned}$$

State Feedback Control / Observer

- In general

- i) For fast error convergence \Rightarrow Large L
- ii) If y is not reliable, contaminated by disturbance, and measurement \Rightarrow small L

-Thus,

i) Select several, L is choose several pole locations of the observer

ii) Simulation

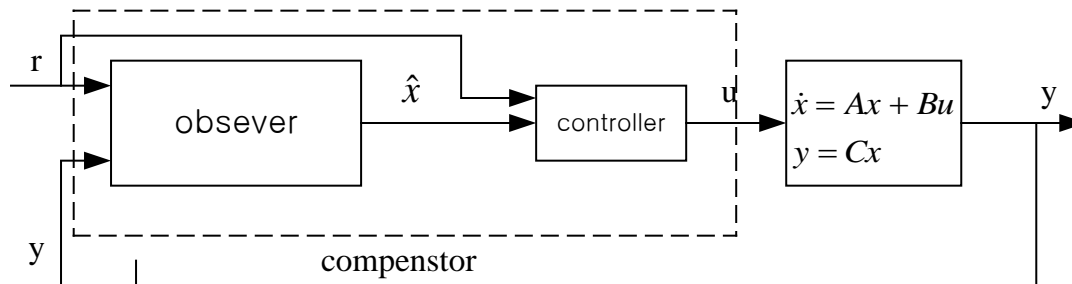
iii) Choose the best L , from the view point of overall system performance.

\rightarrow Compromise between speedy response and sensitivity to disturbances and noise

\rightarrow Kalman Filter

State Feedback Control / Observer

$$\begin{aligned}\dot{x} &= Ax + Bu \\ u &= -K\hat{x} + r \\ \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y})\end{aligned}$$



$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + B(-K\hat{x} + r) + LC(y - C\hat{x}) \\ (sI - A + BK + LC)\hat{x}(s) &= -L(-Y(S))\end{aligned}$$

$$\begin{aligned}U &= -K\hat{x}(s) + r \\ &= K(sI - A + BK + LC)^{-1}(-Y) + \dots\end{aligned}$$

End of state space design

Discrete Time Observer

$$x(k+1) = G \cdot x(k) + H \cdot u(k)$$

$$y(k) = Cx(k)$$

Observer

$$\hat{x}(k+1) = G \cdot \hat{x}(k) + H \cdot u(k) + K_e [y(k) - C\hat{x}(k)]$$

$$e(k) = x(k) - \hat{x}(k)$$

$$e(k+1) = (G - K_e C)e(k)$$

$\hat{x}(k+1)$ is given in terms of $\underbrace{\{y(0), y(1), \dots, y(k)\}}_{y_k}$

$$\ddots \left(\begin{array}{l} \hat{x}(k+1|k) : \text{estimate of } x(k+1) \text{ based on } y_k \\ \hat{x}(k+1|k+1) : \text{estimate of } x(k+1) \text{ based on } y_{k+1} \end{array} \right)$$

$-\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K_e [y(k+1) - C\hat{x}(k+1|k)]$: Current Observer.

Error Dynamics

$$\begin{aligned}
 e(k+1|k) &= x(k+1) - \hat{x}(k+1|k) \\
 &= Gx(k) + Hu(k) - [Gx(k|k) + Hu(k)] \\
 &= Gx(k) - G[\hat{x}(k|k-1) + K_e[Cx(k) - C\hat{x}(k|k-1)]] \\
 &= G(I - K_e C) \cdot e(k|k-1)
 \end{aligned}$$

$$\begin{aligned}
 e(k+1|k+1) &= x(k+1) - \hat{x}(k+1|k+1) \\
 &= \underbrace{(I - K_e C)G}_{\substack{G - K_e C \cdot G \\ C^*}} \cdot e(k|k) \quad : (G, C^*) \text{Observable} \\
 &\quad \rightarrow \text{arbitrary pole assignment}
 \end{aligned}$$

$$y = Cx + n \quad (n : \text{measurement noise})$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + n$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L \cdot (y - C\hat{x})$$

$$\dot{e} = (A - LC)e - L \cdot n$$

→ : Optimal compromise between convergence speed

and noise effects → Kalman Filter. → $\min E[(x - \hat{x})^T (x - \hat{x})]$

$$u = -Kx$$

Vector Differential

$$(1) \underbrace{f(x)}_{n \times 1} : \underbrace{x}_{n \times 1} \rightarrow \underbrace{y}_{n \times 1}$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_1} \\ \vdots & & & \\ \vdots & & & \\ \frac{\partial f_1}{\partial x_1} & \dots & \dots & \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_1}{\partial x_1} & & & \frac{\partial f_1}{\partial x_1} \end{bmatrix}$$

$$(2) R \cdot u \quad R: n \times n, \quad u: n \times 1$$

$$\frac{\partial Ru}{\partial u} = R$$

$$\frac{\partial u^T R}{\partial u} = \left[\frac{\partial R^T u}{\partial u} \right] = [R^T]^T = R$$

Vector Differential

$$(3) \underbrace{u^T}_{1 \times n} \cdot \underbrace{R}_{n \times n} \cdot \underbrace{u}_{n \times 1}$$

$$\frac{\partial u^T R u}{\partial u} = 2u^T R$$

$$\underbrace{x^T}_{1 \times n} \underbrace{u}_{n \times 1} \frac{\partial x^T u}{\partial u} = x^T$$

$$u^T x \quad \frac{\partial u^T x}{\partial u} = x^T$$

Review

1. Key Concepts

- Modeling
- Control System Design
- Controller Design
- Linear, Nonlinear Systems
- Design Approach

2. System Representations

- Laplace Transform, Z-Transform
- Graphical representation, block diagram, signal flow graph
- Transfer Function
- State Equation (cont. discrete.)
- State Eq. T.F.
- Canonical forms
- Solution of Linear state Eq. : cont. discrete.
- Discrete Representation of cont. time systems
- Relationships between cont. and discrete time eigenvalues
- Sampling of cont. time systems

3. Dynamic Properties of the L.T.I Systems

- Stability. Definitions
- Routh stability Criterion
- Bilinear Transformation
- Lyapunov function
- Stability / In-stability theorems
- Controllability / Observability definitions : cont. discrete Sufficient Conditions for controllability and observability
- Pole – Zero cancellations

4. Analysis and Design

- Root locus. Design concepts. Pole-Zero additions
- Frequency response
- Nyquist stability Criterion
- Lead – Lag compensators
- Robustness to modeling errors – Nyquist stability Criterion
- MIMO Nyquist stability Criterion
- Singular values
- Pole placement – state feedback
- Observer / State estimation
- Discrete time current observer
- Separation property
- LQ optimal control