System Control

10-1. Solution of State Equation

Professor Kyongsu Yi

Vehicle Dynamics and Control Laboratory
Seoul National University

Solution of State Equation

State Transition matrix

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t) \qquad , \quad A, B, C, D$$ Constant real (or Complex)

$u(t)$: a piecewise continuous function with values in $\boldsymbol{R}^r$

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t-\tau)Bu(\tau)d\tau$$

Let $t_0 = 0$

$$x(t) = \Phi(t, \overset{0}{\cancel{t_0}})x(t_0)$$ is solution of $\dot{x} = Ax(t)$

if $$\frac{d}{dt}x(t) = \frac{d}{dt}\Phi(t, \overset{0}{\cancel{t_0}})x(0) = A\Phi(t,0)x(0) = Ax(t)$$

the conditions $\Phi(t,0)$ should satisfy are :

1) $$\frac{d}{dt}\Phi(t,0) = A\Phi(t,0), \quad \forall t \geq 0$$

2) $$\Phi(0,0) = \mathrm{I}$$

Let us solve the matrix differential eq. by iteration(existence of the solution and uniqueness)

$$\Phi_0(t) = \mathrm{I}, \quad \frac{d}{dt}\Phi(t,0) = A\Phi(t,0)$$

$$\Phi_1(t) = \mathrm{I} + \int_0^t A \cdot \mathrm{I} d\tau = \mathrm{I} + At$$

$$\Phi_2(t) = \mathrm{I} + \int_0^t A \cdot \Phi_1(t) dt = \mathrm{I} + At + A^2 \frac{t^2}{2}$$

$$\vdots$$

$$\Phi_k(t) = \mathrm{I} + \int_0^t A \cdot \Phi_{k-1}(\tau) d\tau = \mathrm{I} + At + A^2 \frac{t^2}{2!} + \cdots + A^k \frac{t^k}{k!}$$

$$\Phi(t,0) = \mathrm{I} + At + A^2 \frac{t^2}{2!} + \cdots + A^k \frac{t^k}{k!} + \cdots$$

Let's define $\Phi(t,0) = e^{At}$, $e^{at} = 1 + at + \frac{1}{2}a^2t^2 + \ldots + a^k \frac{t^k}{k}$

Properties of STM (State, Transition Matrix)

1. $\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0)$ for any $t_0,\ t_1,\ t_2$

2. $\Phi(0) = \mathrm{I}$

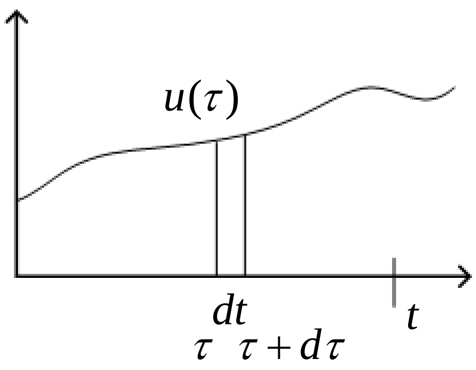
3. $\Phi(t)\Phi(t) = \Phi^2(t) = \Phi(2t)$ $\quad (e^{At})^2 = e^{A\cdot 2t} = \Phi(2t)$ $\quad \Phi^q(t) = \Phi(qt)$

4. $\Phi^{-1}(t) = \Phi(-t)$

5. $\Phi(t)$ is non singular for all finite values of t . (inverse exists)

Solutions

$$\dot{x}(t) = Ax(t) + Bu(t)$$



1) $\quad x(\tau) = 0$

$$x(\tau + d\tau) = B \cdot u(\tau)d\tau$$

$$x(t) = \Phi(t - (\tau + d\tau)) \cdot x(\tau + d\tau)$$

$$= \Phi(t - (\tau + d\tau)) \cdot Bu(\tau)d\tau$$

$$x(t) = \int_0^t \Phi(t - \tau)Bu(\tau)d\tau$$

2) $\quad u(\tau) = 0$

$$x(t) = \Phi(t - 0)x(0)$$

$\Rightarrow$ Complete solution

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t - \tau)Bu(\tau)d\tau$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$sX(s) - x(0) = Ax(s) + Bu(s)$$

$$(s\mathrm{I} - A)X(s) = x(0) + Bu(s)$$

$$\underline{X(s) = (s\mathrm{I} - A)^{-1}x(0) + (s\mathrm{I} - A)^{-1}Bu(s)} \qquad (1)$$

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \dot{x} = Ax\,, \quad x(t) = \Phi(t,0)x(0) = e^{At}x(0) \quad , \frac{d}{dt}(\Phi(t,0)) = A\Phi(t,0)$$

$$e^{-At}\dot{x} - e^{-At} \cdot A \cdot x = e^{-At}Bu$$

$$\frac{d}{dt}[e^{-At} \cdot x] = e^{-At}Bu$$

$$e^{-At}x(t) = \int_0^t e^{-At}Bu(\tau)d\tau$$

$$-e^{-A0}x(0)$$

$$x(t) = e^{At}x(0) + \int_0^t e^{-A(t-\tau)}Bu(\tau)d\tau \qquad (2)$$

from (1)

$$x(t) = L^{-1}[X(s)]$$

$$= L^{-1}[(s\mathrm{I} - A)^{-1}]x(0) + L^{-1}[(s\mathrm{I} - A)^{-1}Bu(s)]$$

$$\Rightarrow \quad L^{-1}[(s\mathrm{I} - A)^{-1}] = e^{At} = \Phi(t)$$

$$L^{-1}[(s\mathrm{I} - A)^{-1}Bu(s)] = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$(sI - A)^{-1} = L[e^{At}]$$
$$= L[I + At + \frac{1}{2!}A^2t^2 + \cdots]$$
$$= \frac{1}{s}I + A\frac{1}{2}\cdot\frac{1}{s^2} + A^2\frac{1}{s^3} + \cdots$$

cf) $$\left[\begin{aligned} \frac{1}{s}(I - \frac{1}{s}A)^{-1} &= \frac{1}{s}(I + \frac{1}{s}A + \frac{1}{s^2}A^2 + \frac{1}{s^3}A^3 + \cdots) \\ &= \frac{1}{s} + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \cdots \end{aligned}\right.$$

$$(I - C)^{-1} = I + C + C^2 + C^3 + \cdots$$ if series converges

$(sI - A)^{-1}$: the resolvent of A

- series expansion of resolvent:

$$(sI - A)^{-1} = \frac{1}{s}(I - \frac{A}{s})^{-1} = \frac{1}{s}(I + \frac{A}{s} + \frac{A^2}{s^2} + \cdots)$$
$$= \frac{1}{s}\cdot I + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$$ valid for $|s|$ large enough

so $\Phi(t) = L^{-1}((sI - A)^{-1}) = I + tA + \frac{(tA)^2}{2!} + \cdots$
$$= e^{At}$$

define matrix exponential as $e^M = I + M + \frac{M^2}{2!} + \cdots$

Picard's Iteration Method

(1) $y' = f(x,y), \quad y(x_0) = y_0$ $\qquad\qquad y' = f(y,t), \quad y(t_0) = y_0$

$$y(t) = y_0 + \int_{t_0}^{t} f[t, y(\tau)]d\tau$$

$$y_1(t) = y_0 + \int_{t_0}^{t} f[t, y_0(\tau)]d\tau$$

$$y_2(t) = y_0 + \int_{t_0}^{t} f[t, y_1(\tau)]d\tau$$

$$\vdots$$

$$y_n(t) = y_0 + \int_{t_0}^{t} f[t, y_{n-1}(\tau)]d\tau$$

$$n \to \infty, \quad y_n(t) \to y(t)$$

Thm.1 Existence

If $f(t,y)$ is continuous at all points (t,y), in some rectangle and bounded in R

$$|f(t,y)| \le K$$

for all (t,y) in R

Then the initial value problem (1) has at least one solution y(t)

Picard's Iteration Method

Thm.2 Uniqueness

If $f(t,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous at all points (t, y) in that rectangle R and bounded ,say

$$|f| \le K,\ \left|\frac{\partial f}{\partial y}\right| \le M$$

(1) has at most one solution y(t), and it can be obtained by Picard's method,

$$y_n(t) = y_0 + \int_{t_0}^{t} f[\tau, y_{n-1}(\tau)]d\tau \quad n = 1, 2\ldots$$

Converses to that solution y(t)

1. Diagonal

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$= \begin{bmatrix} 1+\lambda_1 t + \frac{1}{2}\lambda_1{}^2 t^2 + \cdots & 0 & 0 \\ 0 & 1+\lambda_2 t + \frac{1}{2}\lambda_2{}^2 t^2 + \cdots & 0 \\ 0 & 0 & 1+\lambda_3 t + \frac{1}{2}\lambda_3{}^2 t^2 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$$

2. Jordan Form

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2 e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

3. General A : Diagonalize

$$\dot{x} = Ax + Bu$$

$$x = p\hat{x}$$

$$AP_i = \lambda_i P_i$$

$$AP = P\Lambda, \quad P^{-1}AP = \Lambda$$

$$\dot{\hat{x}} = P^{-1}AP\hat{x} + P^{-1}Bu = \Lambda\hat{x} + P^{-1}Bu$$

$$\hat{x}(t) = e^{\Lambda t}\hat{x}(0) + \int e^{\Lambda(t-\tau)}P^{-1}Bu(\tau)d\tau$$

$$x(t) = \underbrace{Pe^{\Lambda t}P^{-1}}_{e^{At}}x(0) + \int Pe^{\Lambda(t-\tau)}P^{-1}Bu(\tau)d\tau$$

End of 10-1