10-2. Controllability

State Transfer

Consider

$$\dot{x} = Ax + Bu \qquad or \qquad x(k+1) = Fx(k) + Gu(k)$$

Over time interval $[t_i, t_f]$

(We say) input $u : [t_i, t_f] \to \boldsymbol{R}^m$ steer, or transfers state from $x(t_i)$ to $x(t_f)$

Questions :

- where can $x(t_i)$ be transferred to at $t = t_f$
- how quickly can $x(t_i)$ be transferred to some x_{target}
- how do we find a u that transfers $x(t_i)$ to $x(t_f)$?
- how do we find a ‘small’ or ‘efficient’ u that transfers $x(t_i)$ to $x(t_f)$?

Controllability and Reachability

Definition :Controllability (Cont. Time Case)

A System described by

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

is said to be controllable if any initial state

$\boldsymbol{x}(\boldsymbol{t}_0)$ can be transferred to any final state

$\boldsymbol{x}(\boldsymbol{t}_f)$ in a finite time by some control $\{\boldsymbol{u}(\tau);\boldsymbol{t}_0 \le \boldsymbol{t} \le \boldsymbol{t}_f\}$

Definition :Controllability (Disc. Time Case)

A System described by

$$x(k+1) = Fx(k) + Gu(k)$$

is said to be controllable if any initial state can be transferred to any

final state $\boldsymbol{x}(0)$ in a finite time N by some control sequence $\boldsymbol{x}(\boldsymbol{N})$

$$\{\boldsymbol{u}(\boldsymbol{k});\boldsymbol{k} = 0,\cdots,\boldsymbol{N}\}$$

Another definitions : Controllability and Reachability

Controllability : A control system is defined to be state controllable if, given an arbitrary initial state $\boldsymbol{x}(0)$, it is possible to bring the state to the origin of the state space in a finite time interval, provided the control vector is unconstrained (unbounded)

Reachability : A system is defined to be state reachable if, starting from the origin of the state space, the state can be brought to an arbitrary point in the state space in a finite time period, provided the control vector is unconstrained.

Reachable set

Define $\boldsymbol{R}_t \leq \boldsymbol{R}^n$ as the set of points reachable in t seconds for $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$

$$R_t = \left\{ \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \,\middle|\; u : [0, t] \to R^m \right\}$$

and in $\boldsymbol{k}$ steps for DT system $\boldsymbol{x}(\boldsymbol{k}+1) = \boldsymbol{G}\boldsymbol{x}(\boldsymbol{k}) + \boldsymbol{H}\boldsymbol{u}(\boldsymbol{k})$

$$R_k = \left\{ \sum_{i=0}^{k-1} \underset{\boldsymbol{G}}{\cancel{A}}^{k-1-i} \underset{\boldsymbol{H}}{\cancel{B}} u(i) \,\middle|\; u(i) \in R^m \right\}$$

Differential Equations

$d.e. \quad \dot{x} = P(x,t)$

$$x(t) \in R^n,\ t > 0,\ P : R^n \times R \rightarrow R^n$$

$i.c. \quad x(t_0) = x_0$

Two Conditions.

(a) $P(x,t)$: finite number of discontinuities.

$t \rightarrow P(x,t)$: continuous and, at $t_i \in D$ (set of possible discontinuity points) has finite left- and right- hand limits at t_i

(b) Lipschitz conditions

$$\| P(\xi,t) - P(\xi^{'},t) \| \le \| \xi - \xi^{'} \|$$

$K(t)$: piecewise continuous function

<u>Fundamental Theorem</u>

If the function $P(x,t)$ satisfies assumption (a) and (b), then

(1) For each $x_0 \in R^n$ and each $t_0 \in R_+$ there is a continuous function such that

(2) Is unique and is called the solution of the d.e.

Differential Equations

<u>Fundamental Theorem</u>

If the function $P(x,t)$ satisfies assumption (a) and (b), then

(1) For each $x_0 \in R^n$ and each $t_0 \in R_+$ there is a continuous function such that

$$\Phi(t_0)|= x_0 \ \ and$$

$$\dot{\Phi}(t) = P(\Phi(t),\ t) \qquad \forall t \in R_+ \ \ and \ \ t \notin D$$

(2) Φ is unique and is called <u>the solution</u> of the d.e.

Differential Equations

2. Calculation of e^{At} by Laplace Transforms and Caley Hamilton Theorem.

$$Let \quad \hat{\Phi}(s) = \mathcal{L}[\Phi(t_0)]$$

$$\text{since, } \frac{d\Phi(t_0)}{dt} = A\Phi(t,0)$$

$$s\hat{\Phi}(s) - \Phi(0,0) = A\hat{\Phi}(s)$$

$$(sI - A)\hat{\Phi}(s) = I$$

$$hence \quad \hat{\Phi}(s) = (sI - A)^{-1}$$

$$crammer's\ rule\ applies,\ A^{-1} = \frac{1}{\det(A)} Adj(A)$$

$$\rightarrow \hat{\Phi}(s) = \frac{B(s)}{d(s)} = \frac{s^{n-1}B_0 + s^{n-2}B_1 + \cdots + sB_{n-2} + B_{n-1}}{s^n + d_1 s^{n-1} + d_2 s^{n-2} + \cdots + d_n}$$

$$where\ d(s) = \det(sI - A),$$

$$B_i = n \times n \quad real\ matrices$$

Differential Equations

Theorem*

Assuming that d(s) is known, B_k can be successively calculated by the formulas

$$
\begin{aligned}
&B_0 = I \\
&B_1 = B_0 A + d_1 I \\
&B_0 = B_1 A + d_2 I \\
&\qquad\vdots \\
&B_k = B_{k-1} A + d_k I \\
&\qquad\vdots \\
&B_{n-1} = B_{n-2} A + d_{n-1} I \\
&0 = B_{n-1} A + d_n I
\end{aligned}
$$

Differential Equations

Proof: $$\Phi(s) = \frac{B(s)}{d(s)} = \frac{s^{n-1}B_0 + s^{n-2}B_1 + \cdots + sB_{n-2} + B_{n-1}}{s^n + d_1 s^{n-1} + d_2 s^{n-2} + \cdots + d_n}$$

$$= (sI - A)^{-1}$$

$$post\, multiply\ (sI - A)d(s)$$

$$\to d(s)I = (s^{n-1}B_0 + s^{n-2}B_1 + \cdots + sB_{n-2} + B_{n-1})(sI - A)$$

$$= s^n B_0 + (B_1 - B_0 A)s^{n-1} + (B_2 - B_1 A)s^{n-2} + \cdots$$

$$+ (B_{n-1} - B_{n-2}A)s + (-B_{n-1}A)$$

Compare both side (End of Proof)

Differential Equations

$$premultiply\ (sI - A)d(s)$$

$$\rightarrow Id(s) = (sI - A)(s^{n-1}B_0 + s^{n-2}B_1 + \cdots + sB_{n-2} + B_{n-1})$$

$$= s^n B_0 + (B_1 - B_0 A)s^{n-1} + (B_2 - B_1 A)s^{n-2} + \cdots$$

$$\vdots$$

$$B_0 = I$$

$$B_1 = AB_0 + d_1 I$$

$$B_0 = AB_1 + d_2 I$$

$$\vdots$$

$$B_k = AB_{k-1} + d_k I$$

$$\vdots$$

$$B_{n-1} = AB_{n-2} + d_{n-1} I$$

$$0 = AB_{n-1} + d_n I$$

Cayley Hamilton Theorem

$$\begin{cases} \Delta(s) = \det(A - sI) \\ \qquad = (-1)^n d(s) \\ \qquad = (-1)^n \det(A - sI) \\ \qquad\quad \textit{characteristic eq. of } A \\ \Delta(\lambda) = 0;\ \lambda : \textit{zero of } \Delta(s) \\ \qquad\quad \textit{eigenvalues of } A \end{cases}$$

For any square matrix A,

$$\Delta(A) = 0$$

Proof: It is equivalent to show d(A)=0 use Theorem*

$$\begin{aligned} 0 &= B_{n-1}A + d_n I \\ &= \left(B_{n-2}A + d_{n-1}I\right)A + d_n I = B_{n-2}A^2 + d_{n-1}A + d_n I \\ &= \left(B_{n-3}A + d_{n-2}I\right)A^2 + d_{n-1}A + d_n I = B_{n-3}A^3 + d_{n-1}A^2 + d_{n-1}A + d_n I \\ &\ \ \vdots \\ &= d(A) \\ &= A^n + d_1 A^{n-1} + d_2 A^{n-2} + \cdots + d_{n-1}A + d_n I \end{aligned}$$

<u>Remark</u>: Cayley-Hamilton Theorem implies that for any nxn matrix with elements in a <u>field</u> F, A^n is a linear combination of I, A, A^2, …, A^{n-1}

Cayley Hamilton Theorem

The concept of field

Let F be a set of elements α, β, γ, … .
The set of F will be called a field iff

$$(A) \quad \alpha, \beta \in F, \qquad (\alpha+\beta) \in F$$

$$(A1) \qquad \alpha+\beta=\beta+\alpha \qquad \forall \alpha, \beta \in F \qquad (commutivity)$$

$$(A2) \qquad \alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma \qquad \forall \alpha, \beta, \gamma \in F \qquad (associativity)$$

$$(A3) \quad \exists 0, \quad \alpha+0=\alpha \qquad \forall \alpha \in F \qquad (additive\ identity)$$

$$(A4) \quad \exists(-\alpha), \alpha+(-\alpha)=0 \qquad \forall \alpha \in F \qquad (additive\ inverse)$$

$$(M) \quad \alpha, \beta \in F, \qquad \alpha \cdot \beta \in F$$

$$\alpha \cdot \beta = \beta \cdot \alpha$$

$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$$

$$\alpha \cdot 1 = \alpha$$

$$\alpha \cdot \alpha^{-1} = 1$$

$$(D) \quad \alpha \cdot (\beta+\gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

Ex. Field of R, field of C, field of rational functions field of binary numbers.

Controllability

Now consider a continuous system

$$\dot{x} = Ax + Bu$$

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

Let $t_0=0$, time-invariant,

$$\Phi(t, t_0) = \Phi(t, 0) = \Phi(t) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

$$= I + At + \frac{1}{2!} A^2 t^2 + \cdots + \frac{1}{n!} A^n t^n + \cdots$$

A

Cayley Hamilton Theorem

Therefore,

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t}\left(I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \cdots \right)B(\tau)u(\tau)d\tau$$

$$= \Phi(t,t_0)x_0 + \int_{t_0}^{t} B(\tau)u(\tau)d\tau + \int_{t_0}^{t} A(t-\tau)B(\tau)u(\tau)d\tau$$

$$+ \int_{t_0}^{t} \frac{1}{2!}A^2Bu(\tau)(t-\tau)^2 d\tau + \cdots + \int_{t_0}^{t} \frac{1}{n!}A^nBu(\tau)(t-\tau)^n d\tau + \cdots$$

(By the C-H Theorem),

Since A^k for k>n can be represented as a linear combination of

$$A^{n-1}, A^{n-2}, \cdots, A, I\,(A^0)$$

$$i.e.\;\; A^n = -d_1A^{n-1} - d_2A^{n-2} - \cdots - d_{-11}A - d_nI$$

$$d(A) = 0$$

By C-H theorem

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A + \cdots + \alpha_{n-1}(t)A^{n-1}$$

therefore

$$\begin{aligned} x(t) &= \Phi(t)x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ &= \Phi(t)x_0 + \int_0^t \left(\sum_{i=0}^{n-1} \alpha_i(\tau)A^i \right) Bu(t-\tau)d\tau \\ &= \Phi(t)x_0 + \sum_{i=0}^{n-1} A^i B \int_0^t \alpha_i(\tau)u(t-\tau)d\tau \\ &= \Phi(t)x_0 + C \cdot \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} \end{aligned}$$

$$z_i(t) = \int_0^t \alpha_i(\tau)u(t-\tau)d\tau$$

x(t) is in range(C)

thus,

$$x(t) = \Phi(t)x_0 + \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \cdot \begin{bmatrix} \xi_1(u) \\ \xi_2(u) \\ \vdots \\ \xi_n(u) \end{bmatrix}$$

If $R(\underline{C})=range(\underline{C})=R^n$

i.e, rank $\underline{C} = n$

then

x(t) can be transferred from x_0 to any state in R^n by some control inputs

range : a subspace

range(C) : a subspace of R^n

$$y = Cx \quad , \quad \forall x \in R^n$$

defined by a function or transformation that maps

$$x \in R^n \quad \text{into} \quad y \in R^m$$

$$m \leq n$$

$$range(C) = \left\{ y \,\middle|\, y = Cx, \forall x \in R^n \right\}$$

Theorem (Controllability)

The continuous system

is controllable if and only if $\dot{x} = Ax + Bu$

$$rank\,\mathrm{W_c} = n \qquad (rank\ \underline{C} = n)$$

Where n : the order of the system

$\underline{\mathrm{C}} = \mathrm{W_c}$: the controllability matrix

$$\underline{C} = \mathrm{W_c} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

End of 10-2