Lecture 10-3. Observability and state Estimation

- State variable feedback plays a key role in the state space approach to the linear feedback control system

- However, often we do not have direct access to all the state variables

- They must be computed from inputs and measurable outputs

Consider the discrete-time system

$$x(k+1) = Gx(k) + Hu(k), \qquad x(0) = x_0 \qquad (*1)$$

$$y(k) = Cx(k) \qquad\qquad x(k) \in R^n, u(k) \in R^m, \; y(k) = R^p$$

The solution

$$x(k) = \underbrace{G^k x_0}_{\text{free response}} + \underbrace{\sum_{j=0}^{k-1} G^{k-1-j} Hu(j)}_{\text{forced-response}}$$

$$y(k) = Cx(k) = \overbrace{CG^k x_0} + C\overbrace{\sum_{j=0}^{k-1} G^{k-1-j}} Hu(j)$$

$$u(j) \quad j = 0, \ldots, k-1 \quad [0, k-1] : \quad \textit{known} \quad \textit{inputs}$$

$$y(j) \qquad\qquad\qquad\qquad : \quad \textit{measured outputs}$$

The question is how and when the state can be determined from the output measurements.

If x_0 is found, we can determine x(k) based on the sol of the state equation

Definition (observability) : A system described by

$$x(k+1) = Gx(k) + \ (Hu(k)\ ), \qquad x(0) = x_0$$
$$y(k) \ = Cx(k)$$

is said to be observable if every initial state x_0 can be exactly determined from the output measurements, y(j), in finite time steps, $0 \le j \le k$

Once x_0 is determined, x(k) can be computed

$$\begin{bmatrix} y(0) \\ \vdots \\ y(k-1) \end{bmatrix} = \underline{O}_k x(0) + T_k \begin{bmatrix} u(0) \\ \vdots \\ u(k-2) \end{bmatrix}$$

$$y(0) = Cx(0)$$
$$y(1) = Cx(1) = CG^1 x_0 + CHu(0)$$
$$y(2) = Cx(2) = CG^2 x_0 + CGHu(0) + CHu(1)$$
$$\vdots$$
$$y(k-1) = CG^{k-1}x(0) + C\sum_{j=0}^{k-2} G^{k-2-j} Hu(j)$$

$$\underline{O}_k = \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{k-1} \end{bmatrix} \qquad T_k = \begin{bmatrix} 0 & 0 & \cdots & \cdots \\ CH & 0 & 0 & \cdots \\ \vdots & & \ddots & \\ CG^{k-2}H & CG^{k-3}H & \cdots & CH \end{bmatrix}$$

$$\underline{O}_k x(0) = \begin{bmatrix} y(0) \\ \vdots \\ y(k-1) \end{bmatrix} - T_k \begin{bmatrix} u(0) \\ \vdots \\ u(k-1) \end{bmatrix}$$

RHS is known, x(0) is to be determined

x(0) can be uniquely determined

if and only if $rank(\underline{O}_k) = n$ $\quad \begin{pmatrix} N(\underline{O}_k) = \{0\} \\ R(\underline{O}_k) = R^n \end{pmatrix}$

if $x(0) = N(\underline{O}_k)$ and $u = 0,\ y_{(j)} = 0,\ k = 0,1,\cdots,k-1$

by Cayley-Hamilton theorem

G^k is linear combination of $G^0,\cdots,G^{n-1}$ (for any k)

Hence, k ≥ n , $\begin{matrix} rank(\underline{O}_k) = rank(\underline{O}) \\ N(\underline{O}_k) = N(\underline{O}) \end{matrix}$ where $\underline{O} = \underline{O}_n = \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix}$ observability matrix

$N(\underline{O})$: unobservable subspace

System is called observable if $N(\underline{O}) = \{0\}$

i.e. $Rank(\underline{O}) = n$

Theorem (observability) :

The discrete time system is observable if and only if

$$rank(\underline{O}) = n$$

Where, $\underline{O}$ is the observability matrix defined by

$$\underline{O} = \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix}$$

Proof

For a given arbitrary initial state, x_0 and $u(i)=0$

$$y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix} x_0 = \underline{O} x_0$$

If rank $\underline{O}$ = n, then there exist n independent $\underline{O}_r$ row vectors $m_{r1}, m_{r2}, \ldots, m_{rn}$ of $\underline{O}$ and from the above equation we can have

$$\begin{bmatrix} y_{r1} \\ \vdots \\ y_{m} \end{bmatrix} = \begin{bmatrix} m_{r1} \\ \vdots \\ m_{rm} \end{bmatrix} x_0 = \underline{O}_r x_0$$

$\underline{O}_r$ is nonsigular and $\underline{O}_r^{-1}$ exists. Thus x_0 can be uniquely determined from $y_{r1}, \ldots, y_{rm}$

Continuous-time Observability

Continuous-time system(without sensor noise and disturbance)

$$\dot{x} = Ax + Bu, \quad y = Cx + Bu$$

The question : How and when the state can be determine from the output measurements?

Let's look at derivatives of y :

$$y = Cx + Du$$

$$\dot{y} = C\dot{x} + D\dot{u} = CAx + CBu + D\dot{u}$$

$$\ddot{y} = CA^2 x + CABu + CB\dot{u} + D\ddot{u}$$

$$\vdots$$

$$y^{(n-1)} = CA^{n-1}x + CA^{n-2}Bu + CA^{n-3}B\dot{u} + \ldots + CBu^{(n-2)} + Du^{(n-1)}$$

$$hence, \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x + \begin{bmatrix} D & 0 & \ldots & \\ CB & D & 0 & \\ \vdots & \vdots & \vdots & \\ CA^{n-2}B & CA^{n-3}B & \ldots & CBD \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}$$

$$y^{(n-1)} = \underline{O}x + Tu^{(n-1)}$$

Rewriting $\underline{O}x = \underbrace{y^{(n-1)} - Tu^{(n-1)}}_{\text{known}}$

Hence, if $N(\underline{O}_k) = \{0\}$, we can determine x(t)

from derivatives of u(t) and y(t) up to order n−1 .

i.e., if $N(\underline{O}_k) = \{0\}$, i.e., $rank(\underline{O}) = n$

then the system is observable

$$x = F\left(y^{(n-1)} - Tu^{(n-1)}\right)$$

$$F : left\ inverse\ of\ \underline{O}$$

Definition

(Observability) C-T systems :

If every x0 can be determined by measuring output y in a “finite” time, the system is “completely observable” or “observable”

Theorem : Continuous system is observable if and only if

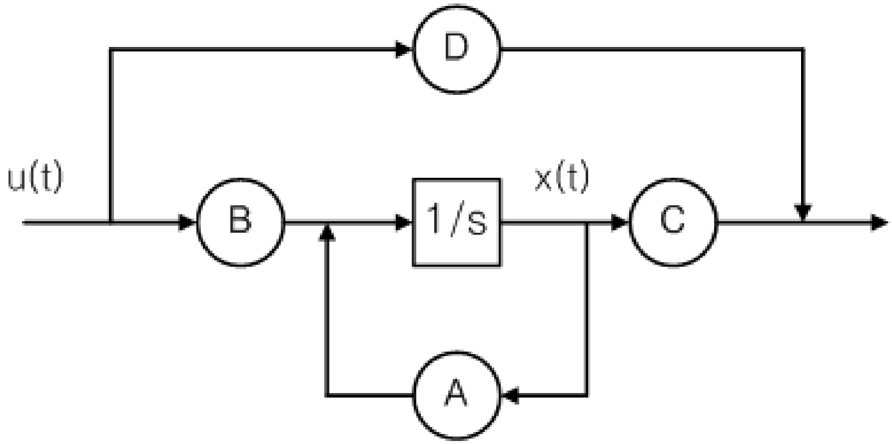
$$rank(\underline{O}) = n$$

Observability-Controllability duality

Let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be dual of system (A, B, C, D)

i.e.,

Original system $\dot{x} = Ax + Bu, \; y = Cx + Du$

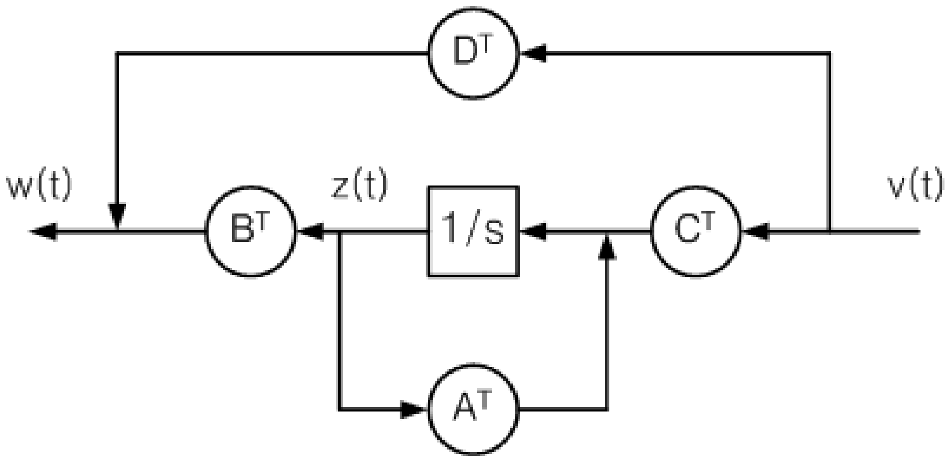


Dual system : transpose all matrices

Swap inputs and outputs

Reverse directions of signal flow arrows

Observability-Controllability duality



$$\dot{z}(t) = A^T z(t) + C^T v(t) = \tilde{A} z(t) + \tilde{B} u(t)$$
$$w(t) = B^T z(t) + D^T v(t) = \tilde{C} z(t) + \tilde{D} v(t)$$

$$\tilde{A} = A^T, \;\; \tilde{B} = C^T, \;\; \tilde{C} = B^T, \;\; \tilde{D} = D^T$$

Observability-Controllability duality

Controllability matrix $\tilde{C}$ of dual system

$$\begin{aligned}\tilde{C} &= [\tilde{B} \quad \tilde{A}\tilde{B} \quad \cdots \quad \tilde{A}^{n-1}\tilde{B}] \\ &= [C^T \quad A^T C^T \quad \cdots \quad (A^T)^{n-1} C^T] \\ &= \underline{O}^T\end{aligned}$$

Similary, $\tilde{O} = C^T$

Thus, system is observable (controllable) if and only if dual system is controllable (observable)

In fact, $N(\underline{O}) = range(\underline{O}^T)^{\perp} = range(\tilde{C})^{\perp}$

i.e., unobservable subspace is orthogonal complement of controllable subspace of dual system

End of 10-3