Lecture 11-1

- Pole Placement – Review
- Observer – controller
- Reference Input Tracking

Example Controllability and Control Canonical form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$\frac{Y(s)}{u(s)} = \frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$y = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_1 \end{bmatrix} \mathbf{x}$$

Controllable Canonical form

$$\dot{\mathbf{x}}_c = A_c \mathbf{x}_c + B_c u$$
$$y = C_c \mathbf{x} + D_c u$$

Control : $u = -K_c \mathbf{x}_c = -\begin{bmatrix} K_{c1} & K_{c2} & \cdots & K_{cn} \end{bmatrix} \mathbf{x}_c$

Example Controllability and Control Canonical form

Control : $u = -K_c \mathbf{x}_c = -\begin{bmatrix} K_{c1} & K_{c2} & \cdots & K_{cn} \end{bmatrix} \mathbf{x}_c$

$$\begin{aligned}
\dot{\mathbf{x}} &= A_c \mathbf{x}_c + B_c \left(-K_c \mathbf{x}_c \right) \\
&= \left(A_c - B_c \cdot K_c \right) \mathbf{x}_c \\
&= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(a_n + K_{c1}) & -(a_{n-1} + K_{c2}) & \cdots & -(a_1 + K_{c1}) \end{bmatrix} \mathbf{x}_c
\end{aligned}$$

Characteristic Eqn

$$s^n + (a_1 + K_n) s^{n-1} + \cdots + (a_{n-1} + K_2) s + (a_n + K_1) = 0$$

Characteristic Eqn for the desired pole locations :

$$s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n = 0$$

Then the necessary feedback gains

$$K_{c1} = -a_n + \alpha_n, \ K_{c2} = -a_{n-1} + \alpha_{n-1}, \ \cdots, \ K_{cn} = -a_1 + \alpha_1$$

General Form

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$
$$y = C\mathbf{x}$$

Transformation Matrix $\mathbf{x} = T\mathbf{x}_c$

$$T\dot{\mathbf{x}}_c = AT\mathbf{x}_c + Bu$$
$$\left.\begin{aligned}\dot{\mathbf{x}}_c &= T^{-1}AT\mathbf{x}_c + T^{-1}Bu = A_c\mathbf{x}_c + B_c u \\ y &= CT\mathbf{x}_c = C_c\mathbf{x}_c\end{aligned}\right\}$$

Controllable Canonical Form

State feedback control law

$$u = -K_c\mathbf{x}_c = -K_cT^{-1}\mathbf{x}$$

Transformation Matrix T $T = \underline{C}W$

$$\underline{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \qquad W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & \ddots & \cdots & 1 & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & 1 & & & \vdots \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Transformation Matrix T

$$T = \underline{C}W$$

Where $$\underline{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & \ddots & \cdots & 1 & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & 1 & & & \vdots \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T^{-1} = W^{-1}\underline{C}^{-1}$$

$$\mathbf{x} = T\mathbf{x}_c$$

Ackerman's Formula

Step 1. Converting (A,B) to (A_c, B_c), T

Step 2. Solving for the gain, K_c

Step 3. Converting back gain, $K = K_c T^{-1}$

$$
\begin{cases}
K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \underline{C}^{-1} \alpha_c(A) \\
\underline{C} = [\, B \;\; AB \;\; A^2 B \; \cdots \; A^{n-1} B \,] \\
\alpha_c(A) = A^n + \alpha_1 A^{n-1} + \; \cdots + \alpha_n I
\end{cases}
$$

α_i : *the coefficient of the "desired" characteristic polynomial*

Selection of Pole Location for Good Design

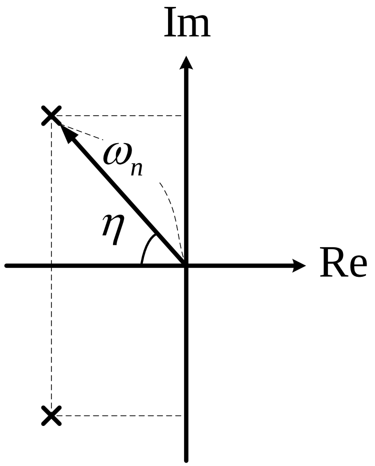
1. Dominant second-order poles
2. Symmetric Root Locus (SRL) – LQR Design

1. Dominant second-order poles

2nd order systems (p.139 Franklin)

Step response

$$H(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta)^2}=\frac{\omega_n^2}{(s+\sigma)^2+\omega_d^2}$$



$$\cos\eta=\zeta$$

Step input response

$$y(t)=1-e^{-\sigma t}\left(\cos(\omega_d t)+\frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

The Desired poles

$$p_c = [-\zeta\omega_n + \omega_d j\ ,\ -\zeta\omega_n - \omega_d j,\ -\alpha\ ,-\beta\ ,-\gamma]^T$$

Characteristic eqn : $(s^2 + 2\zeta\omega_n s + \omega_n^2)(s+\alpha)(s+\beta)(s+\gamma) = 0$

$$\alpha,\ \beta,\ \gamma\ >> \zeta\omega_n$$

<u>MATALB STATEMENT</u>

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$

A : (4x4) matrix

B : (4x1) matrix

pc : desired pole ex) pc=[-0.7+0.7j; -0.7-0.7j; -4;-4;-4]

K2 = acker(A,B,pc)=place(A,B,pc)

· Reference Input Tracking

Introducing the reference input with Full-State Feedback

The pole-placement design

$$u = -K\mathbf{x}$$

Introduce reference input

$$u = -K\mathbf{x} + r$$

; a non-zero steady-state error to step input

New control formula for zero s.s. error to any constant input

$$u = u_{ss} - K(\mathbf{x} - \mathbf{x}_{ss})$$

The system state equation

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$
$$y = C\mathbf{x} + Du$$

In the constant steady state

$$0 = A\mathbf{x}_{ss} + Bu_{ss}$$
$$y_{ss} = C\mathbf{x}_{ss} + Du_{ss}$$

We want $y_{ss} = r_{ss}$ For any value of r_{ss}

To do this, we make $\mathbf{x}_{ss} = N_x r_{ss}$

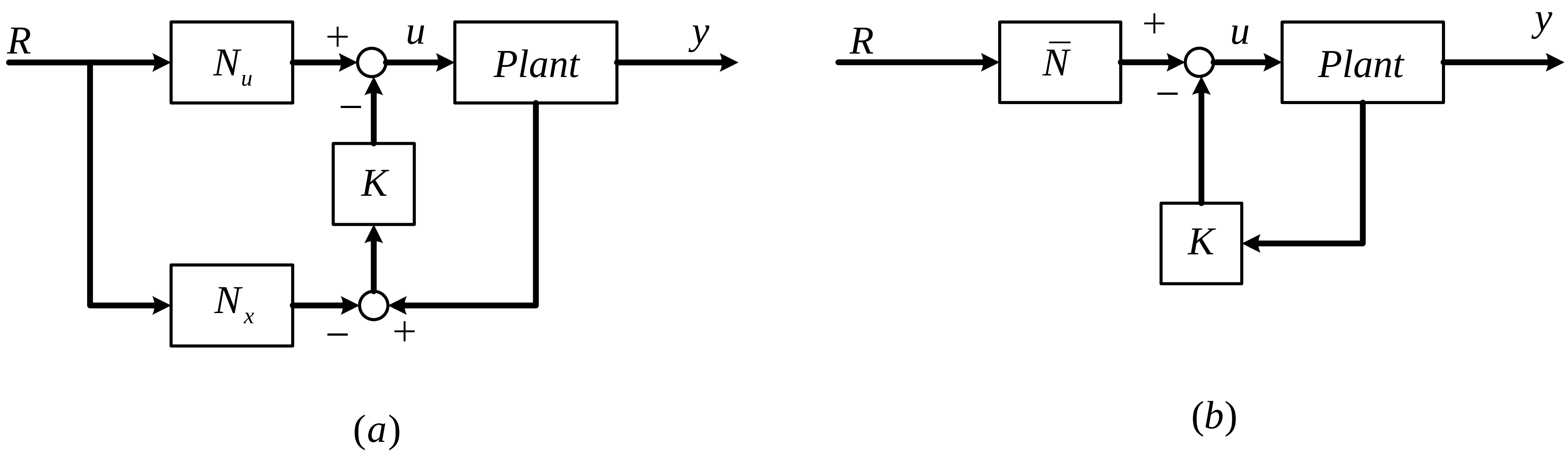
$$u_{ss} = N_u r_{ss}$$

Then, $$\begin{aligned} 0 &= AN_x r_{ss} + BN_u r_{ss} \\ r_{ss} &= CN_x r_{ss} + DN_u r_{ss} \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This eqn can be solved for N_x and N_u $$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

With this values

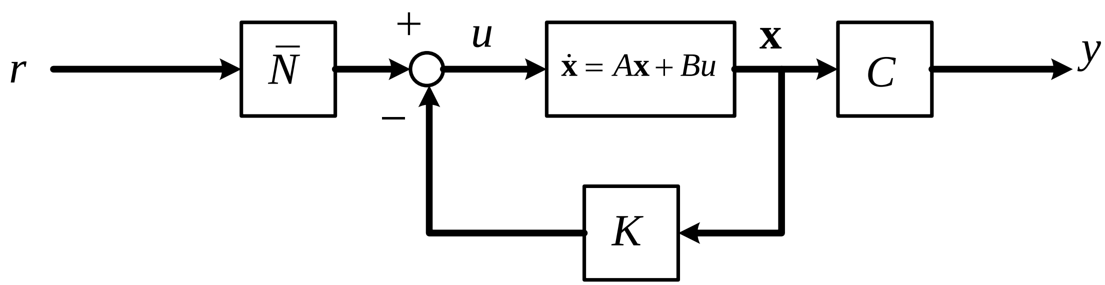
$$u = N_u r - K(\mathbf{x} - N_x r) = -K\mathbf{x} + (N_u + KN_x)r \qquad (a)$$
$$= -K\mathbf{x} + \bar{N}r \qquad (b)$$



(a) is better in practical implementation

(a) Zero s.s. error even if the gain K is slightly in error

(b) nonzero s.s. error if gains do not match exactly



$$\dot{\mathbf{x}} = (A - BK)\mathbf{x} + G\bar{N}r$$
$$y = C\mathbf{x}$$

The closed-loop zeros,

$$\det\begin{bmatrix} sI - (A - BK) & -B\bar{N} \\ C & 0 \end{bmatrix} = 0 \quad \Rightarrow \quad \det\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} = 0$$

(By elementary row and column operations)

<u>Conclusions</u>

When full state feedback is used as $u - -K\mathbf{x} + \bar{N}r$

The zeros remain unchanged by the feedback

(only closed loop poles have been changed by the state feedback)

Integral Control

Tracking $(r \neq 0)$

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$

$$y = C\mathbf{x}$$

(for simplicity, assume D=0)

$$u = u_{ss} - K(\mathbf{x} - \mathbf{x}_{ss}) = N_u r - K(\mathbf{x} - N_x r)$$

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B(N_u r - K\mathbf{x} + KN_x r) = (A - BK)\mathbf{x} + B(\underbrace{N_u + KN_x}_{\bar{N}})r$$

$$(sI - A + BK)X(s) = B\bar{N}r(s)$$

$$X(s) = (sI - A + BK)^{-1} B\bar{N}r(s)$$

$$y(s) = CX(s) = C(sI - A + BK)^{-1} B\bar{N}r(s)$$

$$\frac{Y(s)}{R(s)} = C(sI - A + BK)^{-1} B\bar{N} \ : \ \text{(closed loop DC gain)}$$

$$\bar{N} = N_u + KN_x \ : \ \text{Parameter dependent}$$

Parameter error → Steady state error

The closed loop DC gain

: the ratio of the output of a system to its input

(presumed constant) after all transients have decayed

State Feedback $\dot{\mathbf{x}} = A\mathbf{x} + Bu$ $\quad u = u_{ss} - K(\mathbf{x} - \mathbf{x}_{ss})$

$y = C\mathbf{x}$

$$\dot{\mathbf{x}} = (A - BK)\mathbf{x} + B\bar{N}r$$
$$y = C\mathbf{x}$$

$$Y(s) = C(sI - A + BK)^{-1} B\bar{N}R(s)$$

DC gain ; $\lim_{t\to\infty} \dfrac{y}{r} = \lim_{s\to 0} sG(s)\dfrac{1}{s} = C(-A + BK)^{-1} B\bar{N}$

DC gain is not 1 if there exist parametr errors or parameter variations

$$\bar{N} = N_u + KN_x$$

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} A^{-1} & \alpha \\ \beta & \gamma \end{bmatrix}$$

State space design :

Plant parameter variations → s.s. error

⇒ Integral Control and Robust Tracking

Integral control

The system $\dot{\mathbf{x}} = A\mathbf{x} + Bu + B_1 u$

$y = C\mathbf{x}$

Augment the plant state with the extra state x_I which obeys the differential equations

$$\dot{x}_I = C\mathbf{x} - r = e$$

thus $$x_I = \int_0^t e\,dt$$

The augmented state equations become

$$\begin{bmatrix} \dot{x}_I \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix} \begin{bmatrix} x_I \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u - \begin{bmatrix} 1 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} w$$

The feedback law

$$u = -\begin{bmatrix} K_I & K_0 \end{bmatrix} \begin{bmatrix} x_I \\ \mathbf{x} \end{bmatrix}$$

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\left[-K_I x_I - K_0\mathbf{x}\right] + B_1 w \\ \dot{x}_I = C\mathbf{x} - r \\ y = C\mathbf{x} = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} x_I \\ \mathbf{x} \end{bmatrix} \end{cases}$$

$$(sI - A + BK_0)X(s) = -BK_I X_I(s) + B_1 w(s)$$

$$sX_I(s) = CX(s) - R(s)$$

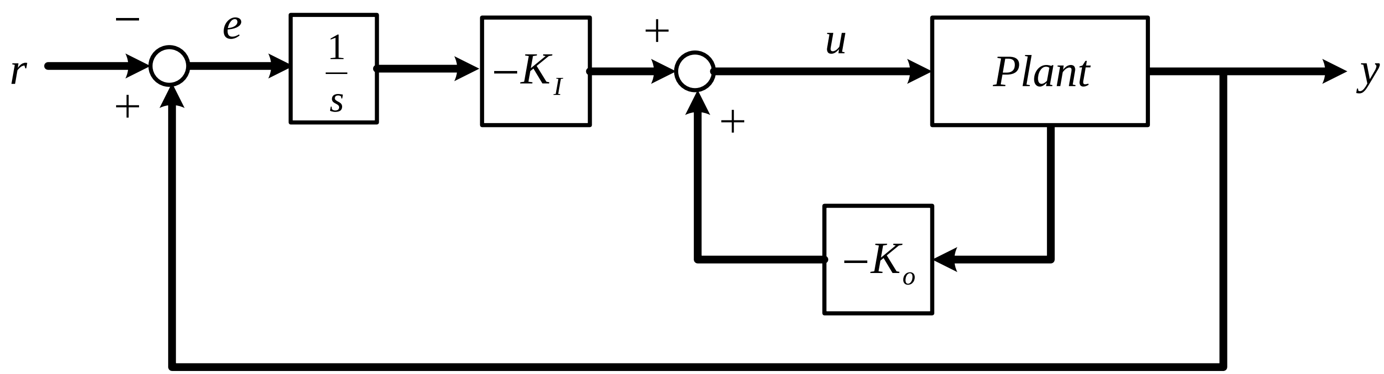
Plugging $X_I(s)$ into the eq

$$(sI - A + BK_0)X(s) = -BK_I \frac{1}{s}\left(CX(s) - R(s)\right) + B_1 W(S)$$

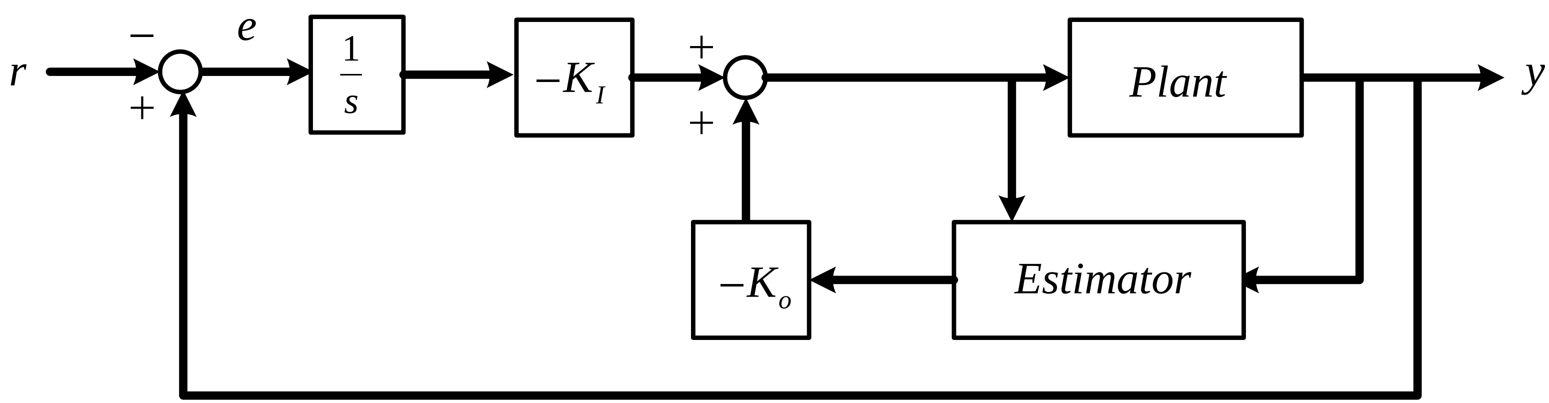
$$\begin{cases} \left[s\left(sI - A + BK_0\right) + BK_I C\right] X(s) = BK_I R(s) + sB_1 W(s) \\ X(s) = \left[s\left(sI - A + BK_0\right) + BK_I C\right]^{-1} \left[BK_I R(s) + sB_1 W(s)\right] \\ \dfrac{Y(s)}{R(s)} = C\left[s\left(sI - A + BK_0\right) + BK_I C\right]^{-1} BK_I \end{cases}$$

$$X_I(s) = \frac{1}{s}\left[CX(s) - R(s)\right] = \frac{1}{s}\left[C\left\{s\left(sI - A + BK_0\right) + BK_I C\right\}^{-1} BK_I - 1\right] R(s)$$

D.C. gain $= 1$ for any K_I, K_0



Integral Control



Integral Control with Estimator

End of Lecture Note 11