Inverted Pendulum System
Fall 2014

Output Feedback vs. State Space Approach for Tracking

Inverted Pendulum System

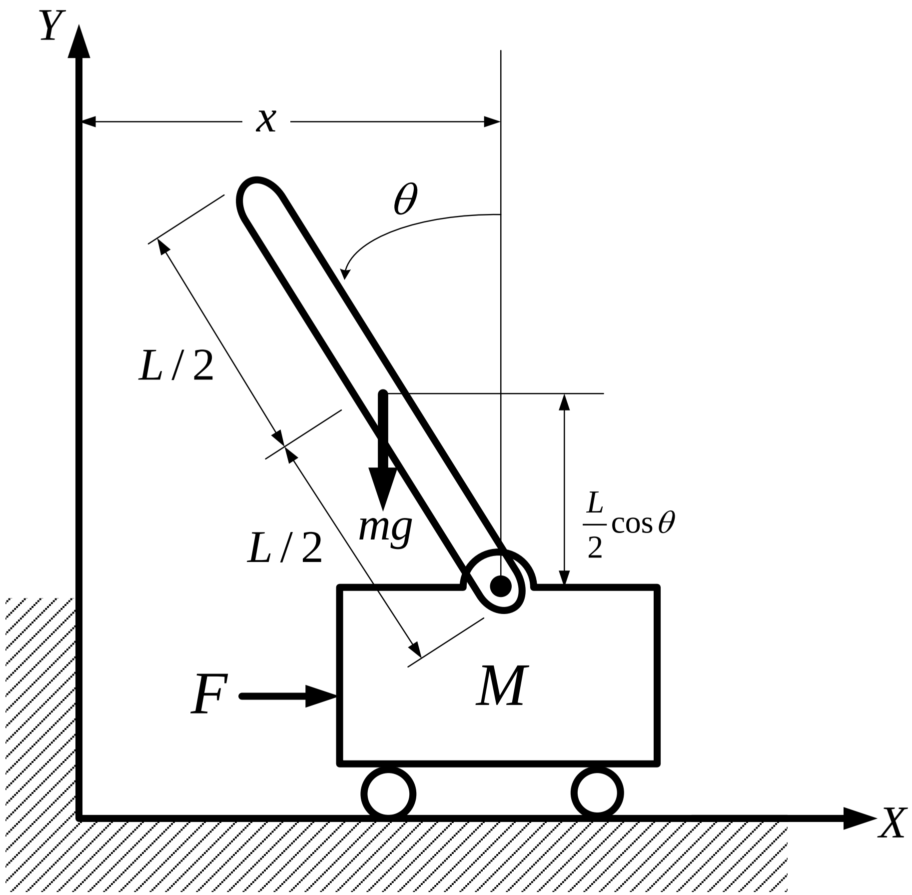
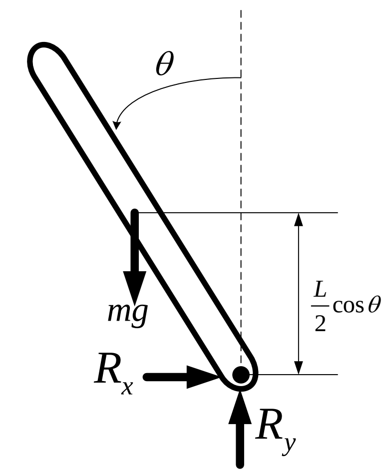


Fig 1. Inverted Pendulum System

Free Body Diagram : Rod & Cart

(a)



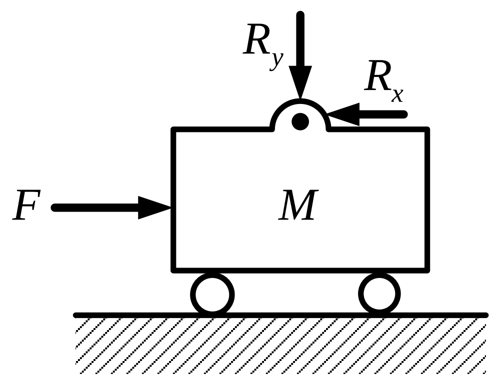
About bar having mass 'm'

$$\sum F_x = R_x = m\ddot{x}_m \quad \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

$$\sum F_y = R_y - mg = m\ddot{y}_m \quad \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

$$\sum M = R_x \frac{L}{2}\cos\theta + R_y \frac{L}{2}\sin\theta = I\ddot{\theta} \quad \cdots\cdots(3)$$

(b)



About cart having mass 'M'

$$\sum F_x = F - R_x = M\ddot{x} \quad \cdots\cdots\cdots(4)$$

Fig 2.
(a) free-body diagram of rod
(b) free-body diagram of cart

Variable Relation

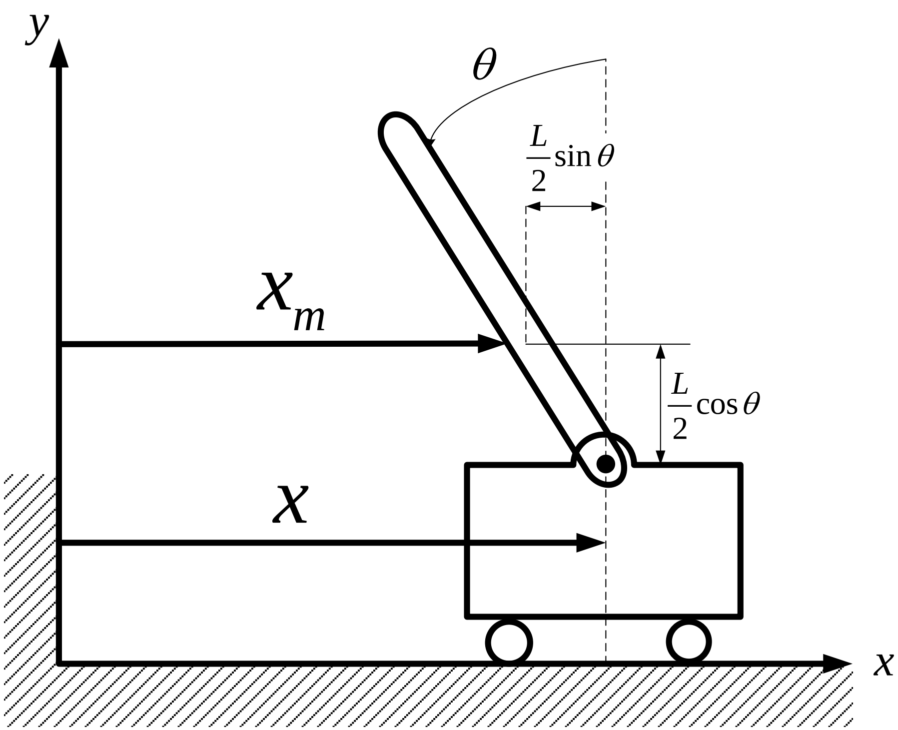


Fig 3. Relation between two center of gravities

$$x_m = x - \frac{L}{2}\sin\theta$$

$$y_m = \frac{L}{2}\cos\theta$$

$$\dot{x}_m = \dot{x} - \frac{L}{2}\dot{\theta}\cos\theta$$

$$\dot{y}_m = -\frac{L}{2}\dot{\theta}\sin\theta$$

$$\ddot{x}_m = \ddot{x} + \frac{L}{2}\dot{\theta}^2\sin\theta - \frac{L}{2}\ddot{\theta}\cos\theta$$

$$\ddot{y}_m = -\frac{L}{2}\dot{\theta}^2\cos\theta - \frac{L}{2}\ddot{\theta}\sin\theta$$

Equations & Desired State

From above relations, equations (1) ~ (4) *can be rewritten as follow*

(1) $$R_x = m\left(\ddot{x} + \frac{L}{2}\dot{\theta}^2 \sin\theta - \frac{L}{2}\ddot{\theta}\cos\theta\right)$$

(2) $$R_y = mg + m\left(-\frac{L}{2}\dot{\theta}^2 \cos\theta - \frac{L}{2}\ddot{\theta}\sin\theta\right)$$

(3) $$I\ddot{\theta} = m\ddot{x}\frac{L}{2}\cos\theta - m\left(\frac{L}{2}\right)^2\ddot{\theta} + mg\frac{L}{2}\sin\theta \quad \cdots\cdots (5)$$

(4) $$(M+m)\ddot{x} + m\frac{L}{2}\left(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta\right) = F \quad \cdots\cdots (6)$$

Initial condition

$$\theta(0) = \theta_0$$

$$x(0) = 0$$

Equations & Desired State

From equation (5)

$$\ddot{x} = \frac{1}{m\frac{L}{2}\cos\theta}\left\{\left[I + m\left(\frac{L}{2}\right)^2\right]\ddot{\theta} - mg\frac{L}{2}\sin\theta\right\}$$

put *this* into (6) and arrange

$$\ddot{\theta} = \frac{1}{(M+m)\left(I + m(L/2)^2\right) - m^2(L/2)^2\cos^2(\theta)}\left[-m^2(L/2)^2\sin(\theta)\cos(\theta)\dot{\theta}^2 + (M+m)mg\frac{L}{2}\sin(\theta) + m(L/2)\cos(\theta)F\right]$$

and from this, $\ddot{x}$ *also can be*

$$\ddot{x} = \frac{1}{(M+m)\left(I + m(L/2)^2\right) - m^2(L/2)^2\cos^2(\theta)}\left[m^2(L/2)^2\sin(\theta)\cos(\theta)g - m\left(I + m(L/2)^2\right)(L/2)\sin(\theta)\dot{\theta}^2 + \left(I + m(L/2)^2\right)F\right]$$

1. Nonlinear System Control

Nonlinear System + **Angle(θ) and Angle Rate($\dot{\theta}$) Controller**

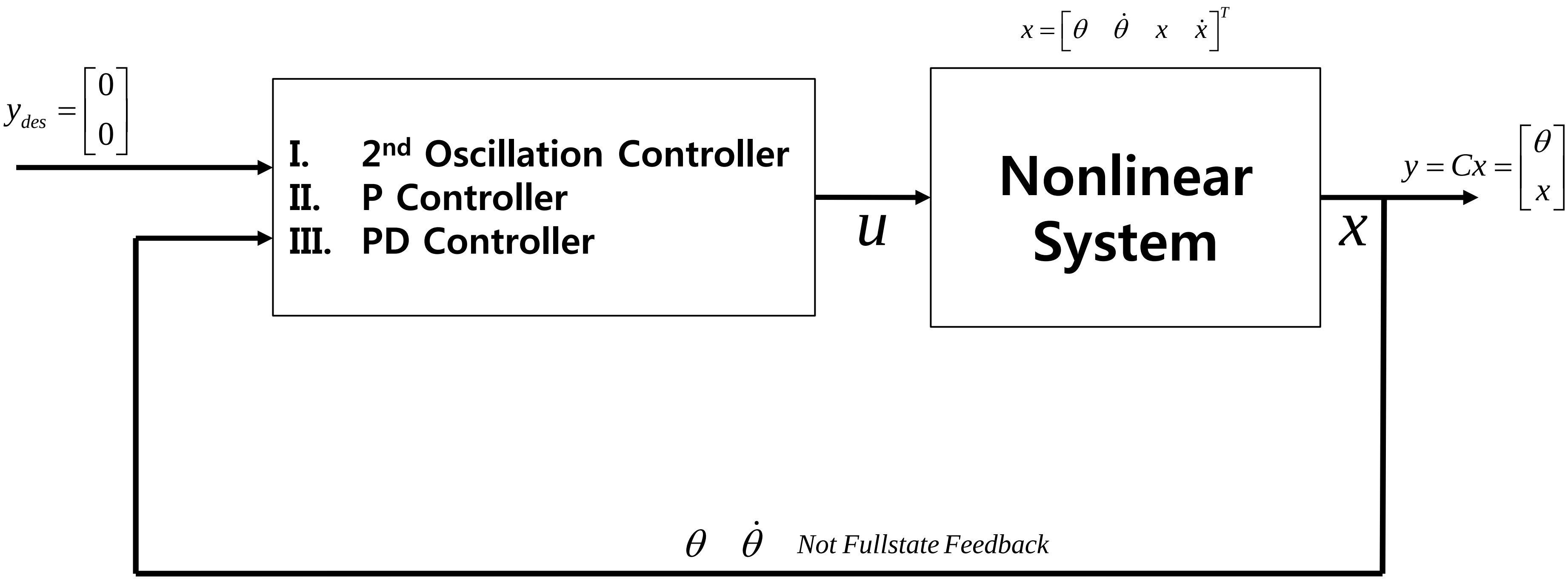


Fig 4. Block diagram for nonlinear system control

θ Control : by Force control

Desired state is that θ *goes to zero at* $t \to \infty$

(*We do not care about* $x(t)$)

$$\lim_{t\to\infty} \theta(t) = 0$$

$$\ddot{\theta} = \frac{1}{(M+m)\left(I + m(L/2)^2\right) - m^2 (L/2)^2 \cos^2(\theta)} \left[-m^2 (L/2)^2 \sin(\theta)\cos(\theta)\dot{\theta}^2 + (M+m)mg\frac{L}{2}\sin(\theta) + m(L/2)\cos(\theta)F \right]$$

From above equation, we want to make $\theta \to 0$ *by controlling appropriate* $F(t)$

F *is a control input* $u(t)$, $F(t) = u(t)$

Three ways of control using force are introduced

I. Make above *equation about* $\ddot{\theta}$ *like* $\ddot{\theta} = -2\zeta\omega_n\dot{\theta} - \omega_n^2\theta$

II. $u(t) = F = -K\theta(t)$

III. $u(t) = F = -K\theta(t) - C\dot{\theta}$

θ Control : by Force control

I

$$\ddot{\theta} = \frac{1}{(M+m)\left(I+m(L/2)^2\right) - m^2 (L/2)^2 \cos^2(\theta)} \left[-m^2 (L/2)^2 \sin(\theta)\cos(\theta)\dot{\theta}^2 + (M+m)mg\frac{L}{2}\sin(\theta) + m(L/2)\cos(\theta)F \right]$$

Let $A = \dfrac{(M+m)}{m\frac{L}{2}\cos\theta}\left[I + m\left(\frac{L}{2}\right)^2 \right] - m\frac{L}{2}\cos\theta$ and $B = \dfrac{(M+m)}{m\frac{L}{2}\cos\theta}\dfrac{L}{2}\sin\theta$

Then above equation can be expressed as follows

$$\ddot{\theta} = -\frac{1}{A} m \frac{L}{2} \dot{\theta}^2 \sin\theta + \frac{B}{A} \cdot mg + \frac{1}{A} F$$

If θ satisfies the following equation,

$$\ddot{\theta} = -2\zeta\omega_n\dot{\theta} - \omega_n^2\theta$$

$$\therefore \; u(t) = F = -2A \cdot \zeta\omega_n \cdot \dot{\theta} - A \cdot \omega_n^2 \cdot \theta - B \cdot mg + m\frac{L}{2}\sin\theta \cdot \dot{\theta}^2$$

With appropriate values of ζ and ω_n, θ can be zero at $t \to \infty$

θ Control : by Force control

From the equation $\ddot{\theta} = -\frac{1}{A} m \frac{L}{2} \dot{\theta}^2 \sin\theta + \frac{B}{A} \cdot mg + \frac{1}{A} F$

We want to control by

II $u(t) = F = -K\theta(t) \quad (K > 0)$

$$\ddot{\theta} = -\frac{1}{A} m \frac{L}{2} \dot{\theta}^2 \sin\theta + \frac{B}{A} \cdot mg - \frac{K}{A}\theta$$

In this case, θ *goes to zero when* $t \to \infty$?

III $u(t) = F = -K\theta(t) - C\dot{\theta} \quad (K > 0, C > 0)$

$$\ddot{\theta} = -\frac{1}{A} m \frac{L}{2} \dot{\theta}^2 \sin\theta + \frac{B}{A} \cdot mg - \frac{1}{A}\left(K\theta + C\dot{\theta}\right)$$

In this case, θ *goes to zero when* $t \to \infty$?

Simulation : Force Control

Simulation Condition

$\theta(0) = 0.1 rad$ $\quad x(0) = 0\ m$ $\quad \zeta = 0.5$ $\quad K = 30$

$\dot{\theta}(0) = 0\, rad/s$ $\quad \dot{x}(0) = 0\, m/s^2$ $\quad \omega_n = 0.7\, rad/s$ $\quad C = 15$

I Simulation Results Plots

— : ζ, ω_n control --- : P control - - - : PD control

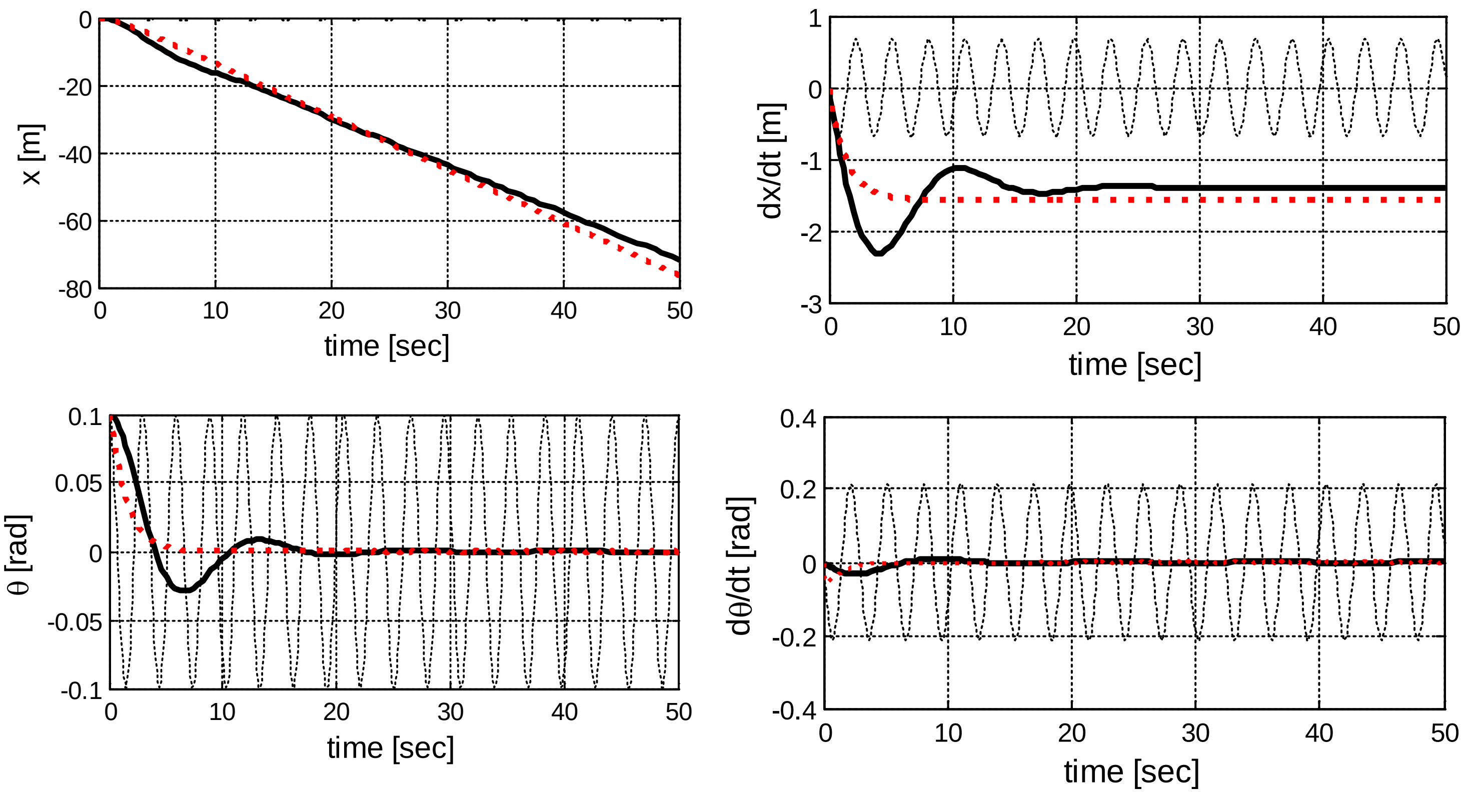


Fig 5. Simulation results of nonlinear system for three kinds of control

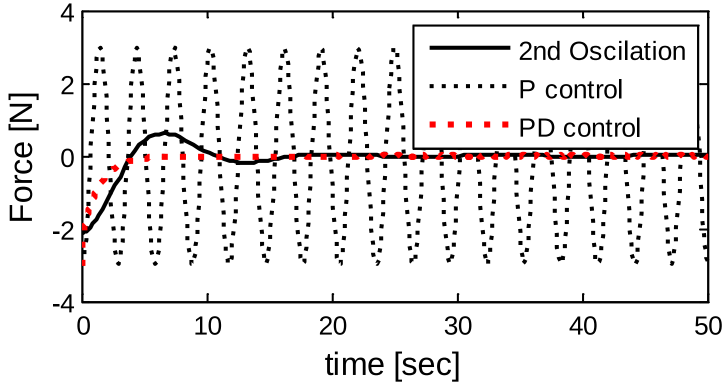
Simulation : Animation

Simulation Condition

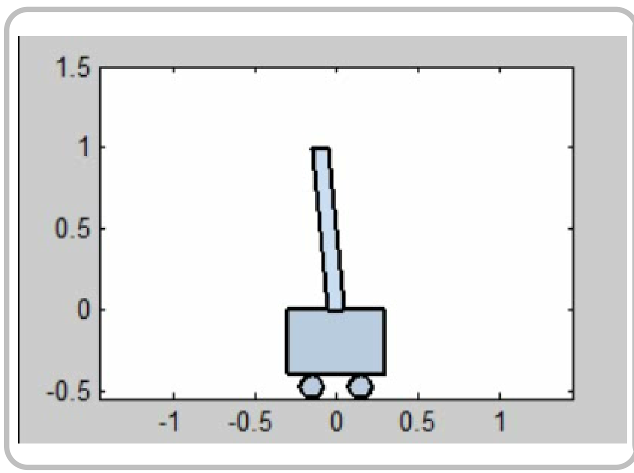
$\theta(0) = 0.1 rad$ $\quad x(0) = 0\ m$ $\quad \zeta = 0.5$ $\quad K = 30$

$\dot{\theta}(0) = 0\, rad/s$ $\quad \dot{x}(0) = 0\, m/s^2$ $\quad \omega_n = 0.7\, rad/s$ $\quad C = 15$

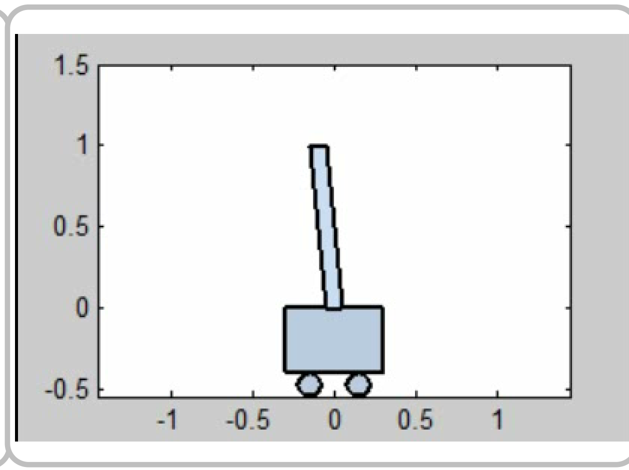
II Force Plot and Simulation Results Animation



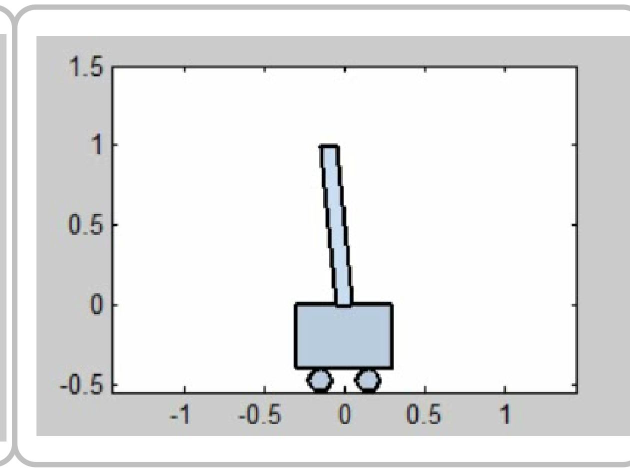
— : ζ, ω_n control
--- : P control
- - - : PD control



(a) 2nd Oscillation Control



(b) P Control



(c) PD Control

Fig 6. Simulation results Animation

After run simulation model

Open '$RUN\_file.m$' and execute

$\Rightarrow$ *Four graphs* (θ vs t, $\dot{\theta}$ vs t, x vs t, $\dot{x}$ vs t)

&

Animation

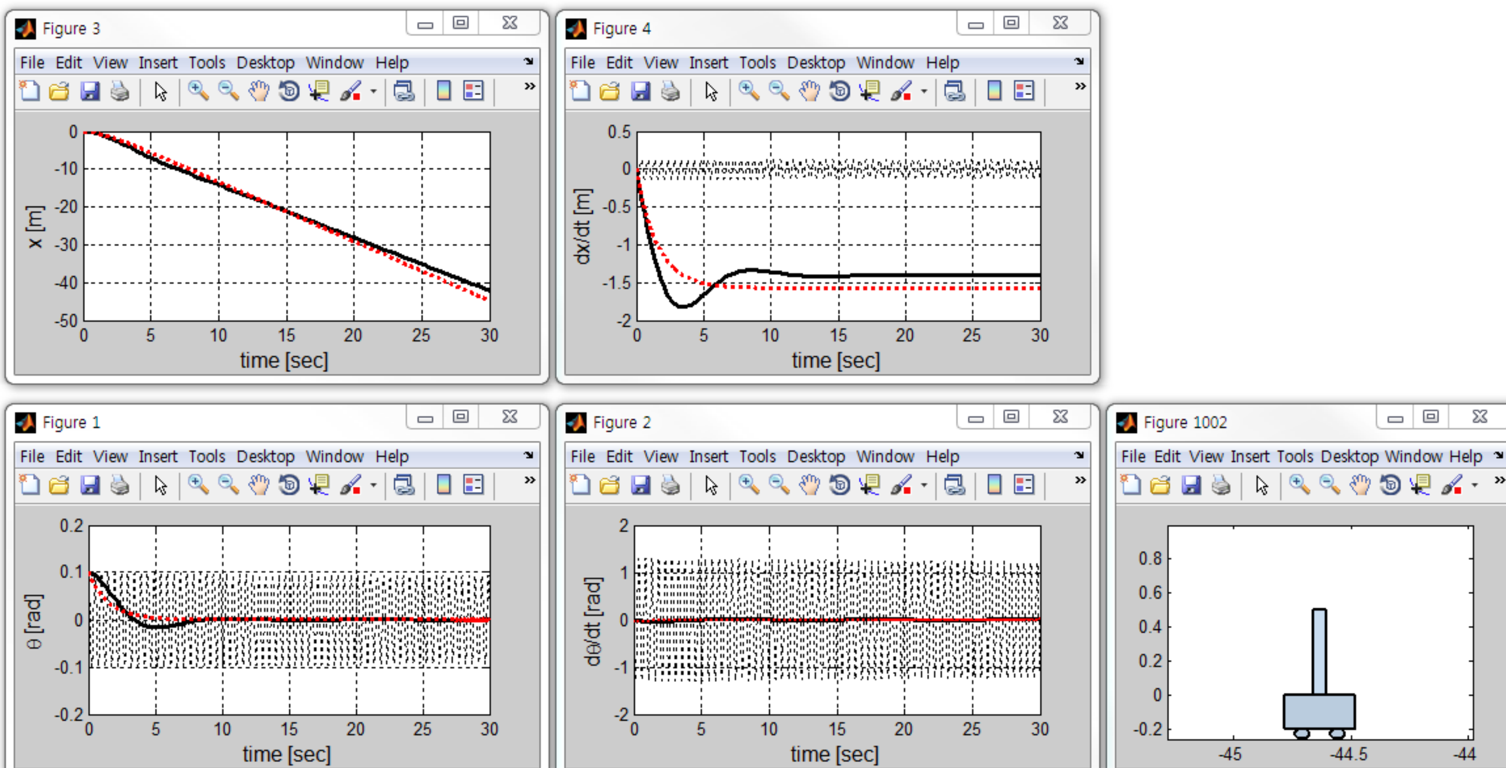
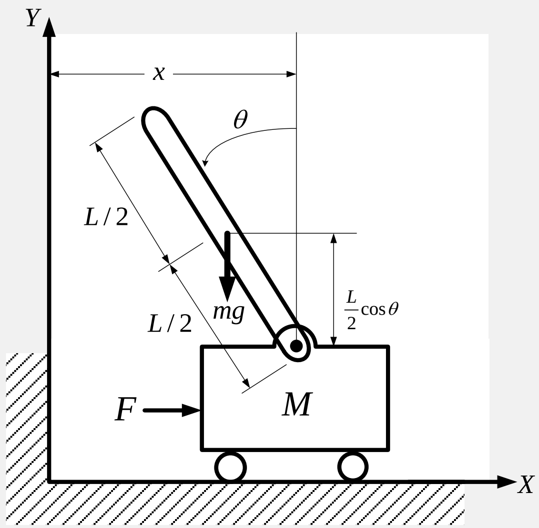


Fig 7. Screenshot after run simulation

Nonlinear System



Equation of motions

$$I\ddot{\theta} = m\ddot{x}\frac{L}{2}\cos\theta - m\left(\frac{L}{2}\right)^2\ddot{\theta} + mg\frac{L}{2}\sin\theta$$

$$(M+m)\ddot{x} + m\frac{L}{2}\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right) = F$$

Control input : $u(t) = F$

Initial Condition : $\theta(0) = \theta_0,\ \dot{\theta}(0) = 0,\ x(0) = 0,\ \dot{x}(0) = 0$

From combining above two equations

$$\ddot{\theta} = \frac{B}{A}mg - \frac{1}{A}m\frac{L}{2}\dot{\theta}^2\sin\theta + \frac{1}{A}F$$

$$\ddot{x} = C\left[-\frac{1}{A}m\frac{L}{2}\sin\theta\dot{\theta}^2 + \frac{B}{A}mg + \frac{1}{A}F\right] - g\tan\theta$$

$$\dot{\mathbf{x}} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{B}{A}mg - \frac{1}{A}m\frac{L}{2}\dot{\theta}^2\sin\theta \\ x_4 \\ C\left[-\frac{1}{A}m\frac{L}{2}\sin\theta\dot{\theta}^2 + \frac{B}{A}mg\right] - g\tan\theta \end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{A}\\ 0\\ \frac{C}{A}\end{bmatrix}F(t)$$

Where $A = \dfrac{(M+m)}{m\frac{L}{2}\cos\theta}\left[I + m\left(\frac{L}{2}\right)^2\right] - m\frac{L}{2}\cos\theta$, $B = \dfrac{(M+m)}{m\frac{L}{2}\cos\theta}\dfrac{L}{2}\sin\theta$ $\quad C = \dfrac{I + m(L/2)^2}{m(L/2)\cos\theta}$

Angle and Angle Rate Controller

Control Strategy

$-How\ to\ control\ u(t) = F(t)?$

$-Desired\ state\ when\ t \to \infty\ is\ \theta \to 0$

i) Applying a assumption that θ satisfies 2nd oscillation, bellow equation comes

$$\ddot{\theta} = \frac{B}{A}mg - \frac{1}{A}m\frac{L}{2}\dot{\theta}^2 \sin\theta + \frac{1}{A}F = -2\zeta\omega_n\dot{\theta} - \omega_n^2\theta$$

In this case, control input becomes

$$u(t) = F(t) = A\left(-2\zeta\omega_n\dot{\theta} - \omega_n^2\theta\right) - Bmg + m\frac{L}{2}\sin\theta \cdot \dot{\theta}^2$$

ii) P control

$$u(t) = F(t) = -K\theta$$

iii) PD control

$$u(t) = F(t) = -K\theta - C\dot{\theta}$$

2. Linear System Control

Linear System + **Full State Feedback Controller** / **Reference Tracking Controller**

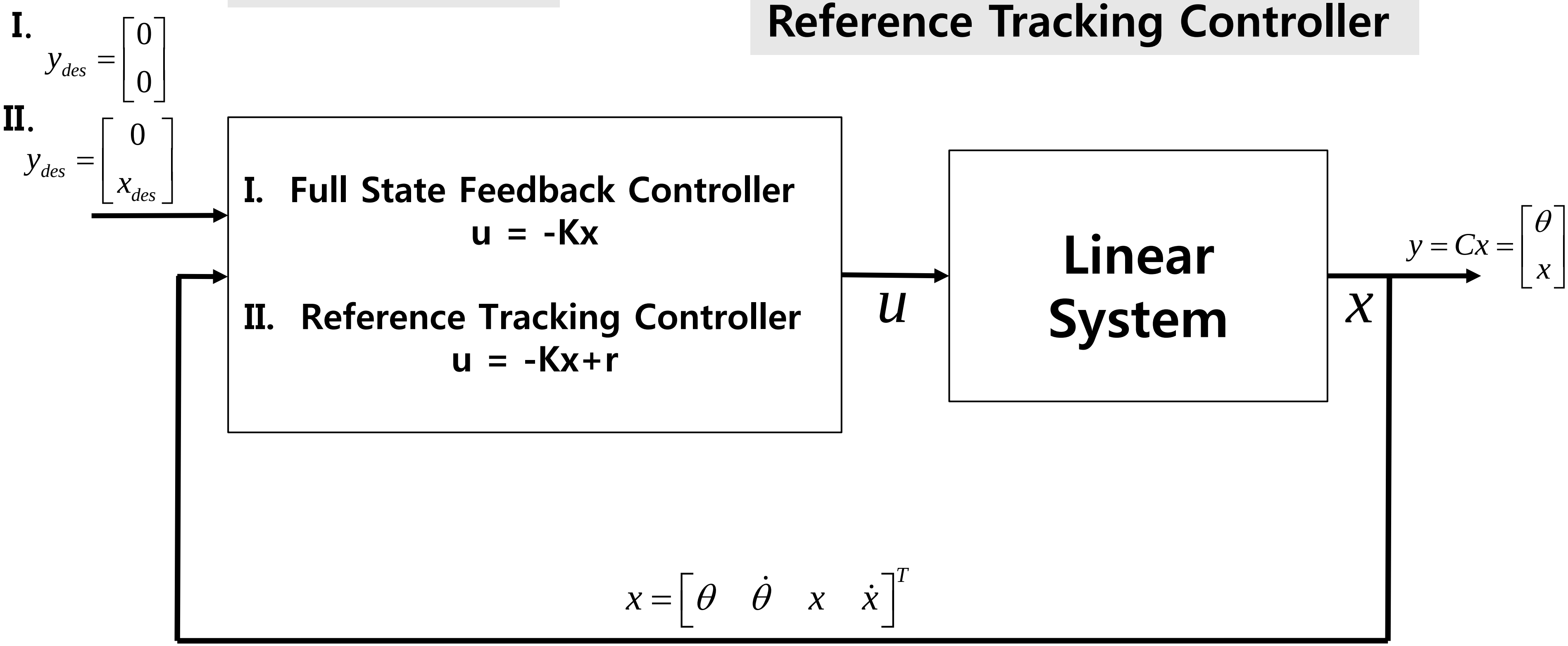


Fig 8. Block diagram for linear system control

Linear Model

$$By\ \ small\ \ angle\ \ assumption,\ \sin(\theta)\approx\theta,\ \cos(\theta)\approx 1,\ \dot{\theta}^2\approx 0$$

then, $\dot{\mathbf{x}}=\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\end{bmatrix}=\begin{bmatrix}x_2\\\frac{B}{A}mg-\frac{1}{A}m\frac{L}{2}\dot{\theta}^2\sin\theta\\x_4\\C\left[-\frac{1}{A}m\frac{L}{2}\sin\theta\dot{\theta}^2+\frac{B}{A}mg\right]-g\tan\theta\end{bmatrix}+\begin{bmatrix}0\\\frac{1}{A}\\0\\\frac{C}{A}\end{bmatrix}F(t)$ becomes as follows

$$\dot{\mathbf{x}}=\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ \dfrac{mg(L/2)}{I+m(L/2)^2-m^2(L/2)^2/(M+m)} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ \left(\dfrac{\left(I+m(L/2)^2\right)}{I+m(L/2)^2-m^2(L/2)^2/(M+m)}-1\right)g & 0 & 0 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}0\\ \dfrac{m(L/2)/(M+m)}{I+m(L/2)^2-m^2(L/2)^2/(M+m)}\\ 0\\ \dfrac{\left(I+m(L/2)^2\right)/(M+m)}{I+m(L/2)^2-m^2(L/2)^2/(M+m)}\end{bmatrix}F(t)$$

where $\mathbf{x}=[x_1\ \ x_2\ \ x_3\ \ x_4]^T=\left[\theta\ \dot{\theta}\ x\ \dot{x}\right]^T$

Where $A=\dfrac{(M+m)}{m\frac{L}{2}}\left[I+m\left(\frac{L}{2}\right)^2\right]-m\frac{L}{2}$, $B=\dfrac{(M+m)}{m\frac{L}{2}}\dfrac{L}{2}\theta$ $C=\dfrac{I+m(L/2)^2}{m(L/2)}$

Linear Model : Feedback Control

$When\ a\ system\ is\ defined\ as\ \dot{\mathbf{x}} = A\mathbf{x} + Bu,\ we\ want\ to\ control\ \mathbf{x}\ to\ make\ \mathbf{x}(\infty) = 0$

$Let\ \ u = -K\mathbf{x}$

$Then,\ \dot{\mathbf{x}} = A\mathbf{x} - Bu = (A - BK)\mathbf{x} \quad \Rightarrow \quad \mathbf{x} = e^{(A-BK)t}$

$(\mathbf{x} : n \times 1\ matrix,\ \ A : n \times n\ matrix,\ \ B : n \times 1\ matrix,\ \ K : 1 \times n\ matrix)$

$Appropriate\ gain\ K\ makes\ the\ system\ stable$

$where\ \ \mathbf{x} = [x_1\ x_2\ x_3\ x_4]^T = \left[\theta\ \dot{\theta}\ x\ \dot{x}\right]^T,\ K = \left[k_1\ \ k_2\ \ k_3\ \ k_4\right]$

$If\ eigenvalues(poles)\ of\ matrix\ A - BK\ have\ negative-real\ part,$
$then\ \ \mathbf{x}\ goes\ to\ zero\ at\ \ t \rightarrow \infty$

The control methods, which were introduced at the 'Nonlinear system' part,
were failed to control state 'x' to be zero at $t \rightarrow \infty$
On the otherhand, linear feedback control make the system to be stable about all state

$Command$ 'acker' $can\ be\ used\ to\ get\ desired\ gain\ K$

Simulation : Linear Feedback Control

Simulation Condition

$\theta(0) = 0.1 rad$ $\quad x(0) = 0\ m$

$\dot{\theta}(0) = 0\, rad/s$ $\quad \dot{x}(0) = 0\, m/s^2$

$$K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$$

Simulation Results Plots

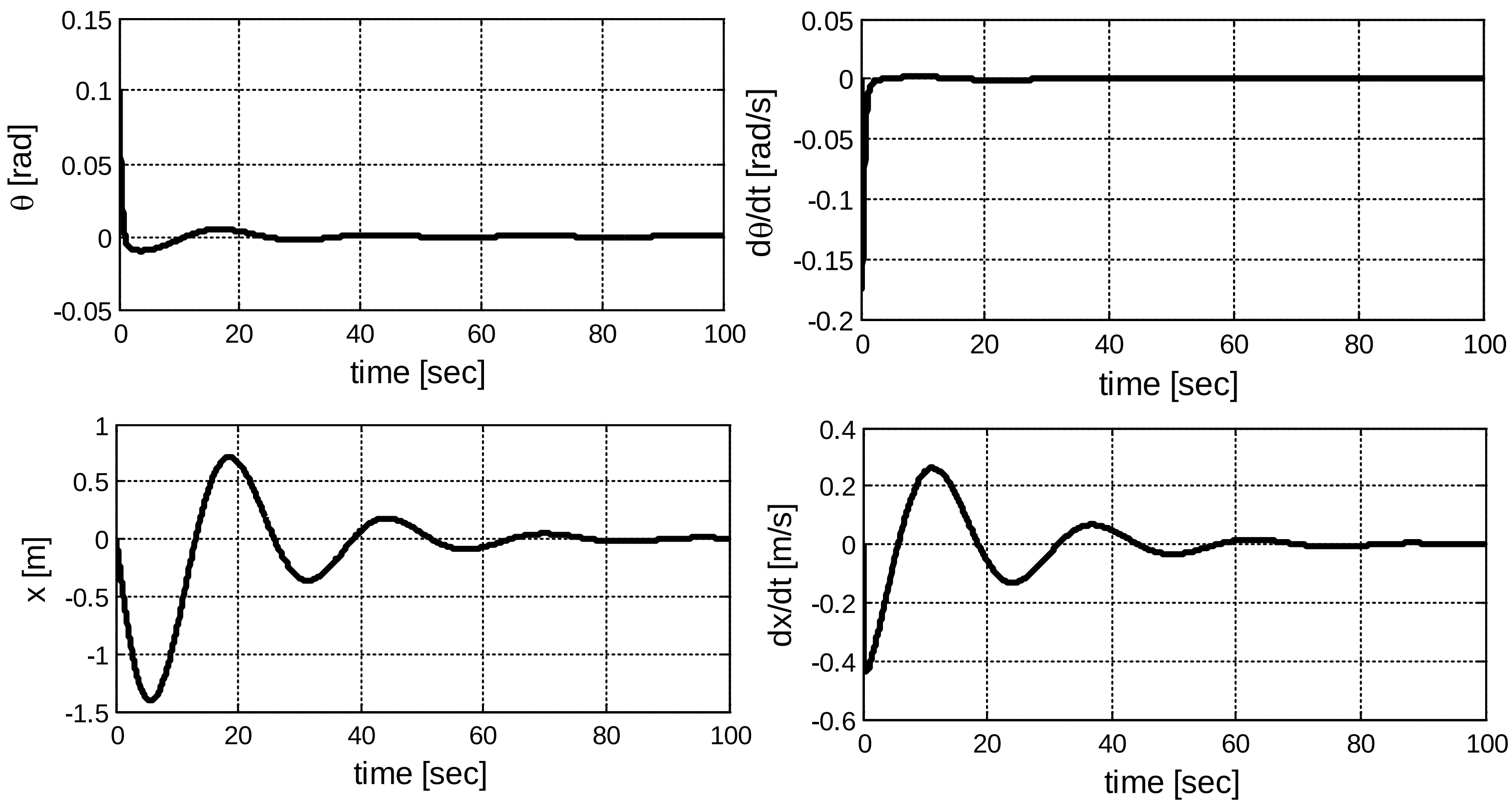


Fig 9. Simulation results of linear system for tracking control

Simulation : Linear Feedback Control - Animation

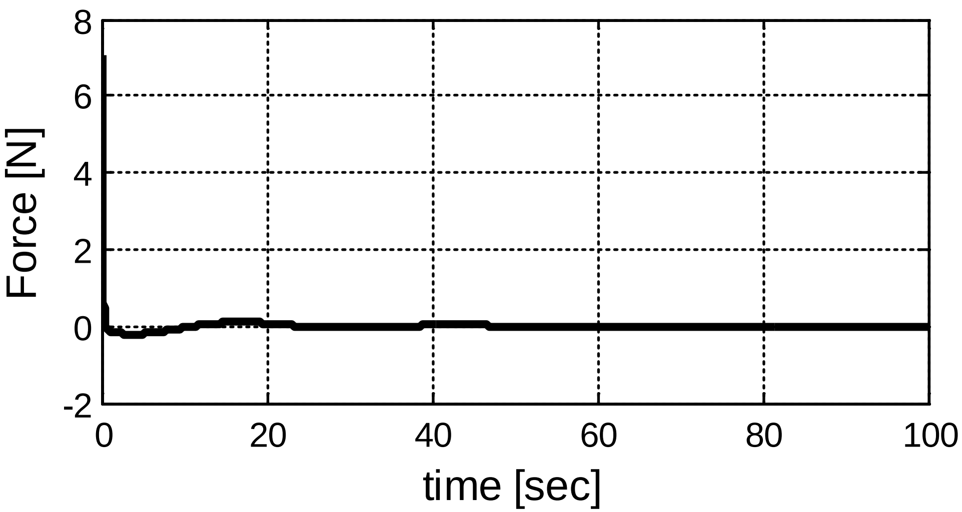
Simulation Condition

$\theta(0) = 0.1 rad$ $x(0) = 0\ m$

$\dot{\theta}(0) = 0 rad/s$ $\dot{x}(0) = 0 m/s^2$

$$K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$$

Force Plot and Simulation Results Animation



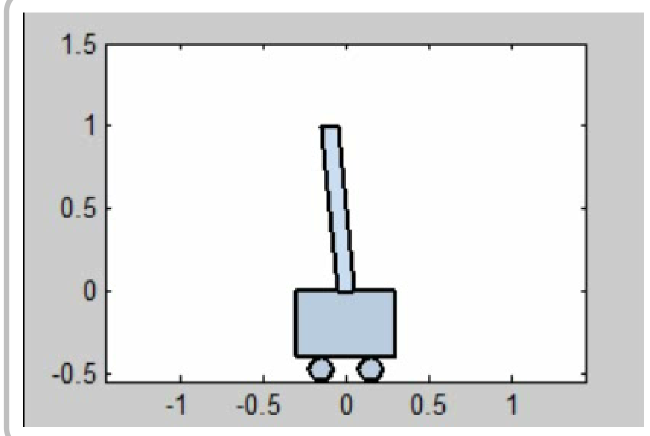


Fig 10. Feedback Control Input and Simulation results Animation

Linear Model : Tracking Problem

We discussed linear feedback control.
This make it possible that all the states go to zero at $t \to \infty$.
But what if we want to make the states converge to specific values?

⇨ *Q) How to make the states converge to nonzero values*

Reference Input Tracking

Introduce reference input $u = -K\mathbf{x} + r$

Let x_{ss} *and* u_{ss} *as state* x *and input* u *repectively at steady state*

Then, $u = u_{ss} - K(x - x_{ss})$

if the system is like $\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + Bu \\ y &= C\mathbf{x} + Du \end{aligned}$ *where* $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^T = \left[\theta \ \dot{\theta} \ x \ \dot{x}\right]^T$,

At the steady state, this system becomes $\begin{aligned} 0 &= A\mathbf{x}_{ss} + Bu_{ss} \\ y_{ss} &= C\mathbf{x}_{ss} + Du_{ss} \end{aligned}$ $\cdots\cdots(*)$ $\left(\because \text{At steady state, } \dot{\mathbf{x}} = 0\right)$

Linear Model : Tracking Problem

We want to make $y_{ss} = r_{ss}$ *for any value of* r_{ss}

To do this, asssume that $\begin{aligned} \mathbf{x}_{ss} &= N_x r_{ss} \\ u_{ss} &= N_u r_{ss} \end{aligned}$ *and put these equations to* (*)

Then

$$
\begin{aligned}
0 &= AN_x r_{ss} + BN_u r_{ss} \\
r_{ss} &= CN_x r_{ss} + DN_u r_{ss}
\end{aligned}
$$

It can be also written as a matrix form like $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix}$

Assume that the inverse of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *is exists, then this equation can be solved for* N_x *and* N_u

$$
\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix}
$$

From above relation

$$
\begin{aligned}
u &= N_u r - K\left(x - N_x r\right) = -K\mathbf{x} + (N_u + K\,N_x)r \\
&= -K\mathbf{x} + \bar{N} r
\end{aligned}
$$

where $\bar{N} = N_u + K\,N_x$

Simulation : Reference Tracking Control

Simulation Condition

$\theta(0) = 0.1 rad$ $\quad x(0) = 0\ m$ $\quad K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

$\dot{\theta}(0) = 0\, rad/s$ $\quad \dot{x}(0) = 0\, m/s^2$ $\quad x_{ss} = r_{ss} = 5$

Simulation Results Plots

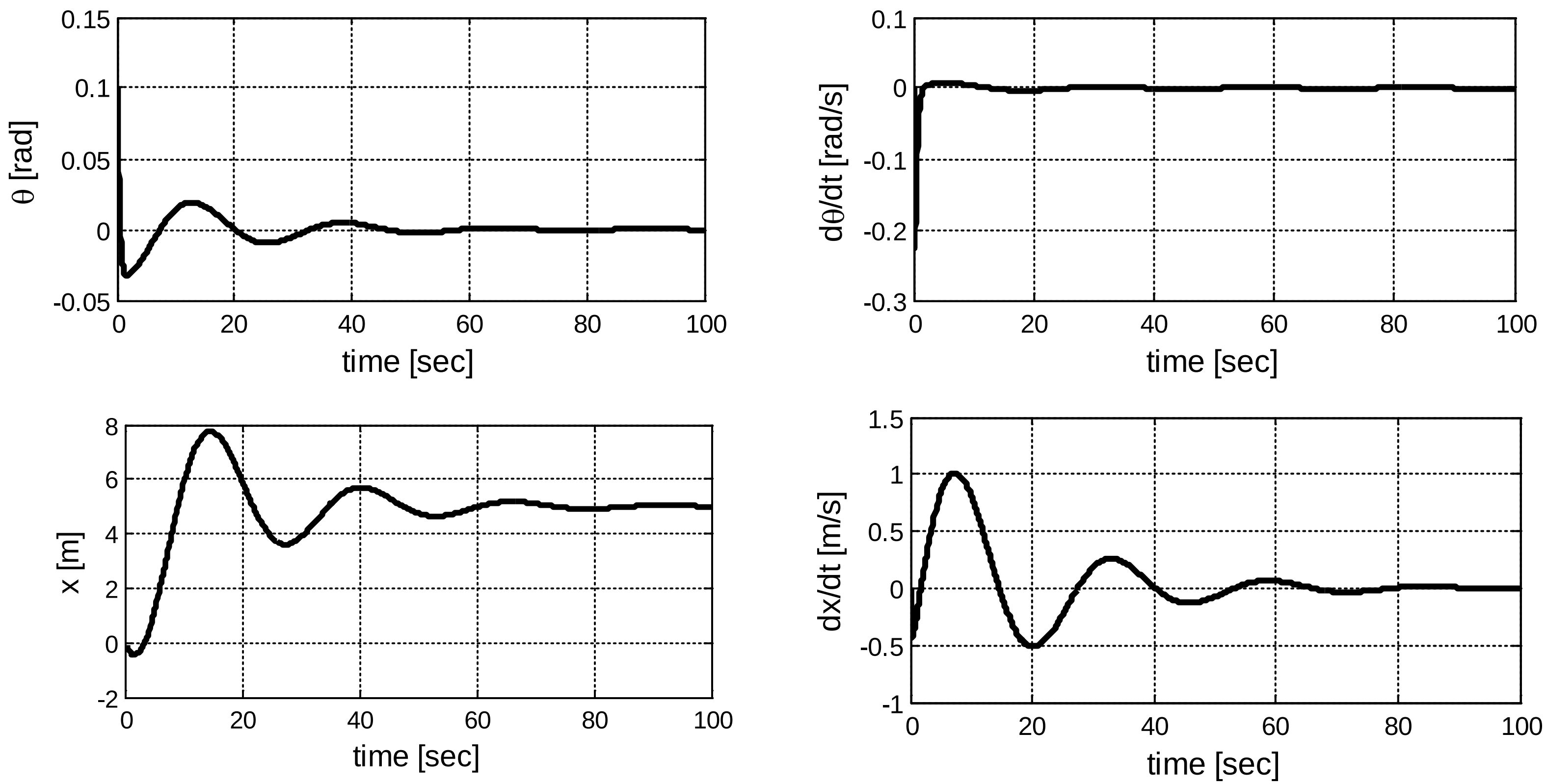
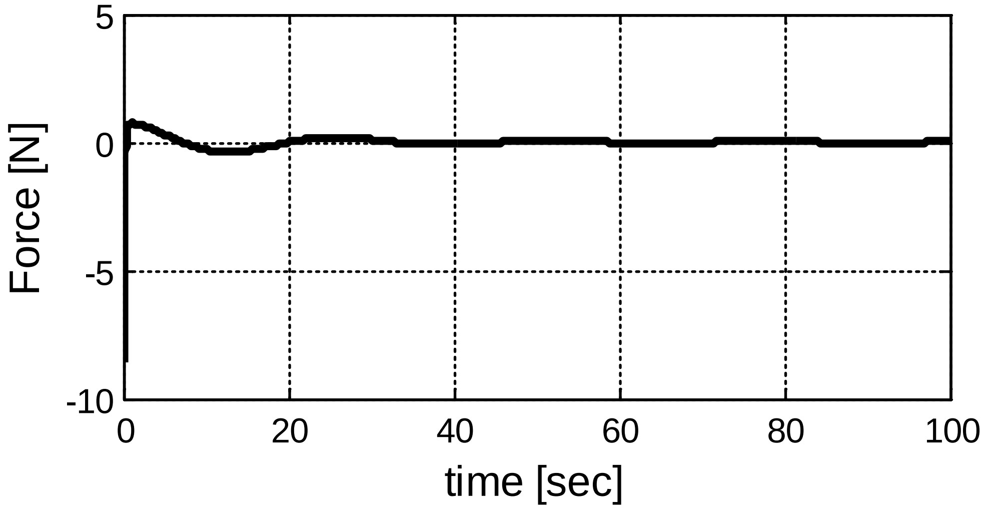


Fig 11. Simulation results of linear system for tracking control

Simulation : Reference Tracking Control - Animation

Simulation Condition			
	$\theta(0) = 0.1rad$	$x(0) = 0\ m$	$K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$
	$\dot{\theta}(0) = 0\,rad/s$	$\dot{x}(0) = 0\,m/s^2$	$x_{ss} = r_{ss} = 5$

Force Plot and Simulation Results Animation



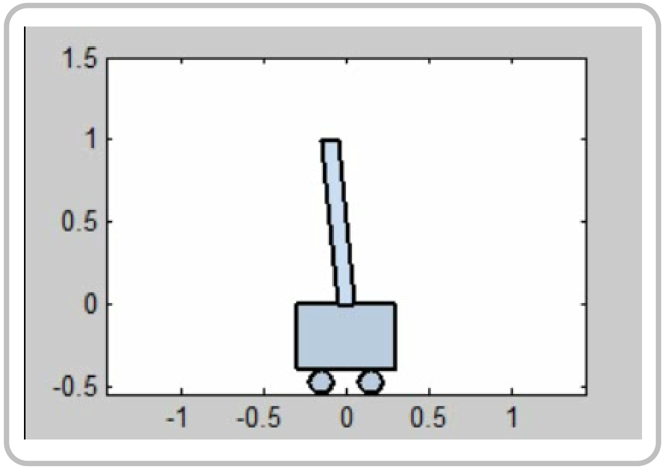


Fig 12. Reference Tracking Input and Simulation results Animation

3. Nonlinear System with Linear Controller

Nonlinear System + Full State Feedback Controller / Reference Tracking Controller

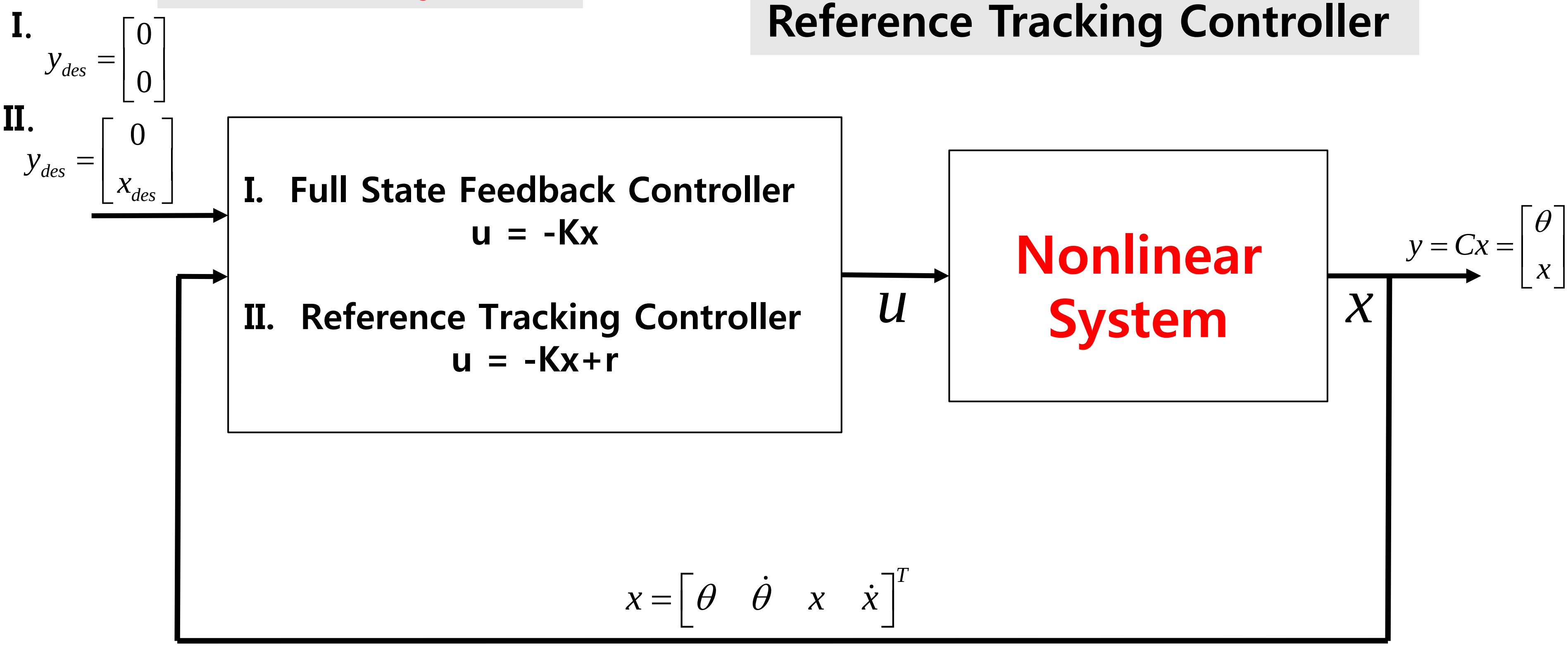
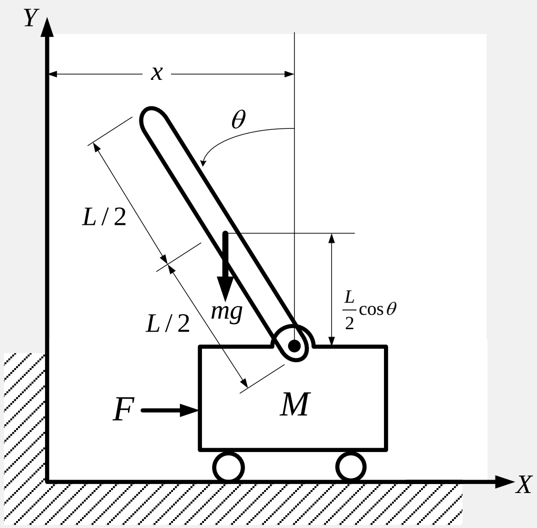


Fig 8. Block diagram for nonlinear system with linear controller

Nonlinear System



Equation of motions

$$I\ddot{\theta} = m\ddot{x}\frac{L}{2}\cos\theta - m\left(\frac{L}{2}\right)^2\ddot{\theta} + mg\frac{L}{2}\sin\theta$$

$$(M+m)\ddot{x} + m\frac{L}{2}\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right) = F$$

Control input : $u(t) = F$

Initial Condition : $\theta(0)=\theta_0,\ \dot{\theta}(0)=0,\ x(0)=0,\ \dot{x}(0)=0$

From combining above two equations

$$\ddot{\theta} = \frac{B}{A}mg - \frac{1}{A}m\frac{L}{2}\dot{\theta}^2\sin\theta + \frac{1}{A}F$$

$$\ddot{x} = C\left[-\frac{1}{A}m\frac{L}{2}\sin\theta\dot{\theta}^2 + \frac{B}{A}mg + \frac{1}{A}F\right] - g\tan\theta$$

$$\dot{\mathbf{x}} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{B}{A}mg - \frac{1}{A}m\frac{L}{2}\dot{\theta}^2\sin\theta \\ x_4 \\ C\left[-\frac{1}{A}m\frac{L}{2}\sin\theta\dot{\theta}^2 + \frac{B}{A}mg\right] - g\tan\theta \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{A} \\ 0 \\ \frac{C}{A}\end{bmatrix} F(t)$$

Where $A = \dfrac{(M+m)}{m\frac{L}{2}\cos\theta}\left[I + m\left(\frac{L}{2}\right)^2\right] - m\frac{L}{2}\cos\theta$, $B = \dfrac{(M+m)}{m\frac{L}{2}\cos\theta}\frac{L}{2}\sin\theta$ $C = \dfrac{I + m(L/2)^2}{m(L/2)\cos\theta}$

Linear Feedback Control

When a system is defined as $\dot{\mathbf{x}} = A\mathbf{x} + Bu$, *we want to control* $\mathbf{x}$ *to make* $\mathbf{x}(\infty) = 0$

Let $u = -K\mathbf{x}$

Then, $\dot{\mathbf{x}} = A\mathbf{x} - Bu = (A - BK)\mathbf{x} \quad \Rightarrow \quad \mathbf{x} = e^{(A-BK)t}$

$(\mathbf{x} : n \times 1 \; matrix, \; A : n \times n \; matrix, \; B : n \times 1 \; matrix, \; K : 1 \times n \; matrix)$

Appropriate gain K makes the system stable

where $\mathbf{x} = [x_1 \; x_2 \; x_3 \; x_4]^T = \left[\theta \; \dot{\theta} \; x \; \dot{x}\right]^T, \; K = \left[k_1 \; k_2 \; k_3 \; k_4\right]$

If eigenvalues(poles) of matrix $A - BK$ *have negative-real part,*
then $\mathbf{x}$ *goes to zero at* $t \to \infty$

The control methods, which were introduced at the 'Nonlinear system' part,
were failed to control state 'x' to be zero at $t \to \infty$
On the otherhand, linear feedback control make the system to be stable about all state

Command 'acker' *can be used to get desired gain K*

Nonlinear System vs Linear System with Linear Feedback Control

Simulation Condition

$\theta(0) = 0.1 rad$ $\quad x(0) = 0\ m$

$\dot{\theta}(0) = 0\, rad/s$ $\quad \dot{x}(0) = 0\, m/s^2$

$K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

Simulation Results Plots

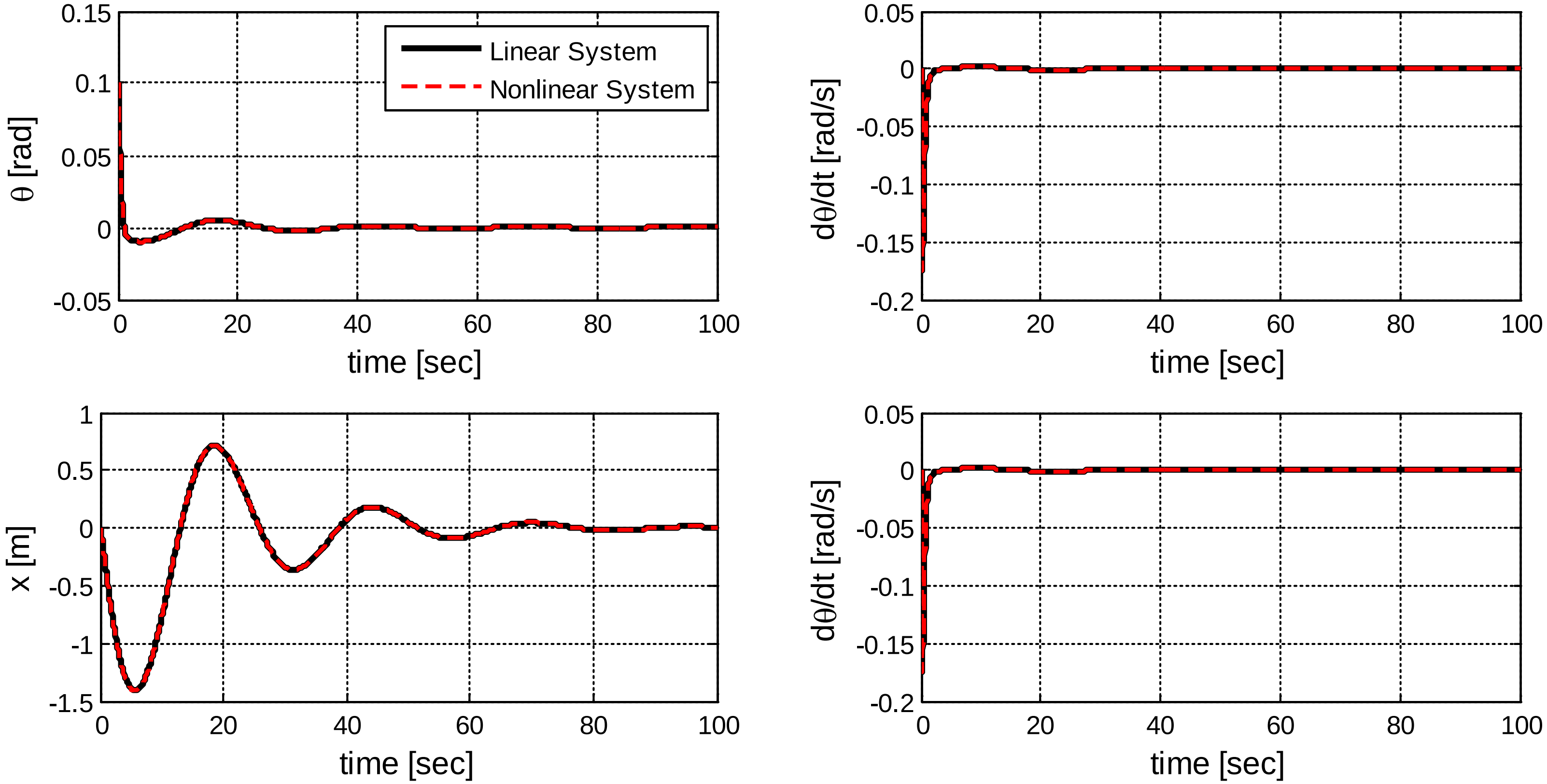


Fig 9. Simulation results for linear Feedback control

Nonlinear System vs Linear System with Linear Feedback Control

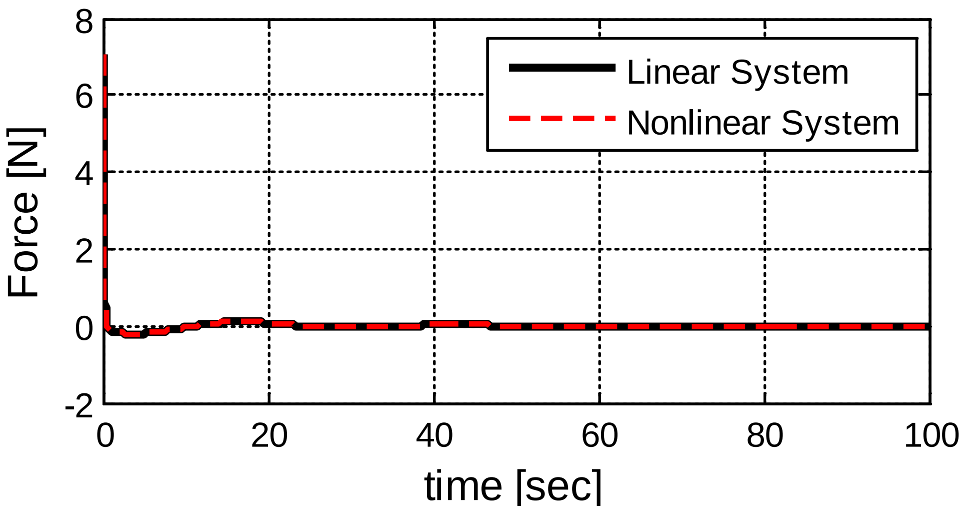
Simulation Condition

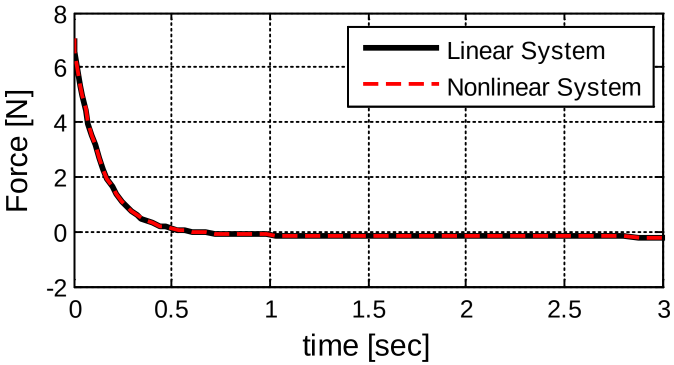
$\theta(0) = 0.1 rad$ $\quad x(0) = 0\ m$

$\dot{\theta}(0) = 0\, rad/s$ $\quad \dot{x}(0) = 0\, m/s^2$

$K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

Force Plot and Simulation Results Animation





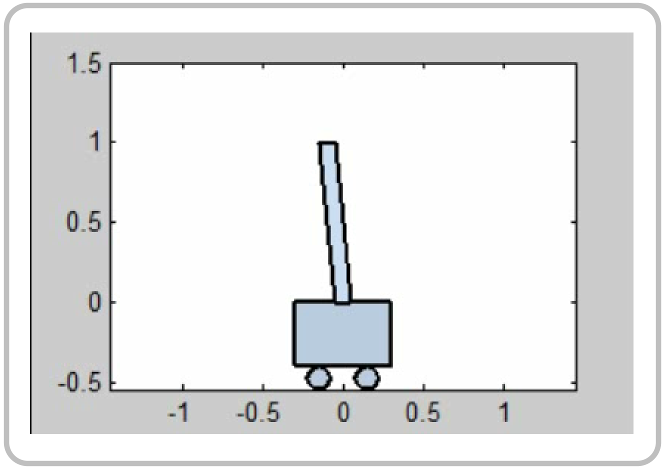


Fig 10. Feedback Control Input (Nonlinear system) and Simulation results Animation

Nonlinear System vs Linear System with Linear Feedback Control

Simulation Condition

$\theta(0) = 1.57\, rad$ $\quad x(0) = 0\ m$

$\dot{\theta}(0) = 0\, rad/s$ $\quad \dot{x}(0) = 0\, m/s^2$

$K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

Simulation Results Plots

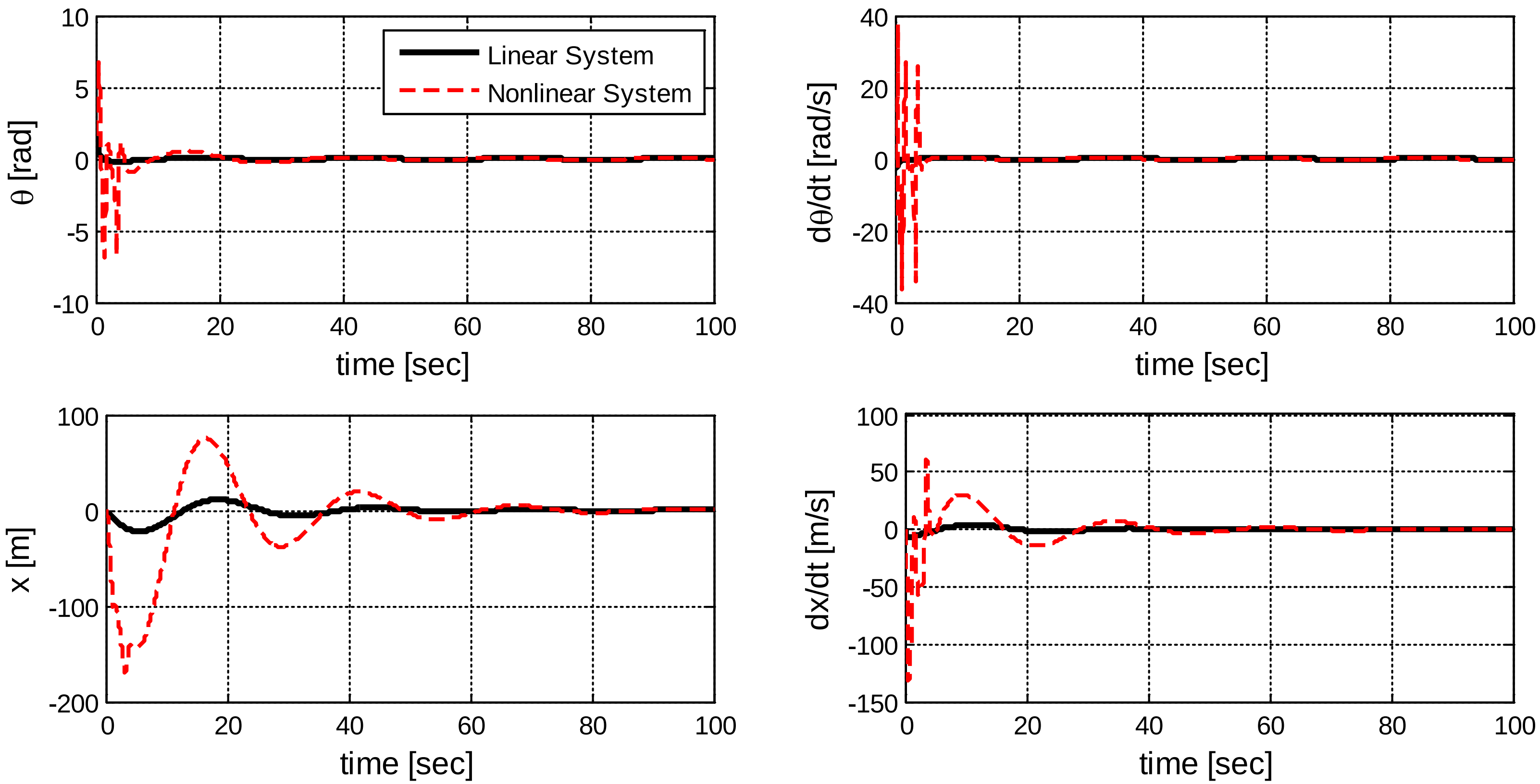


Fig 11. Simulation results for linear feedback control

Nonlinear System vs Linear System with Linear Feedback Control

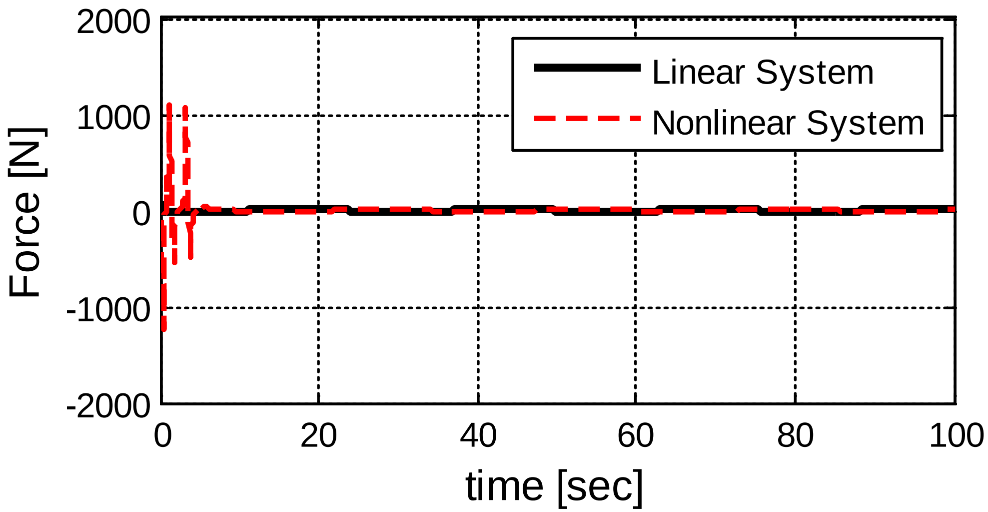
Simulation Condition

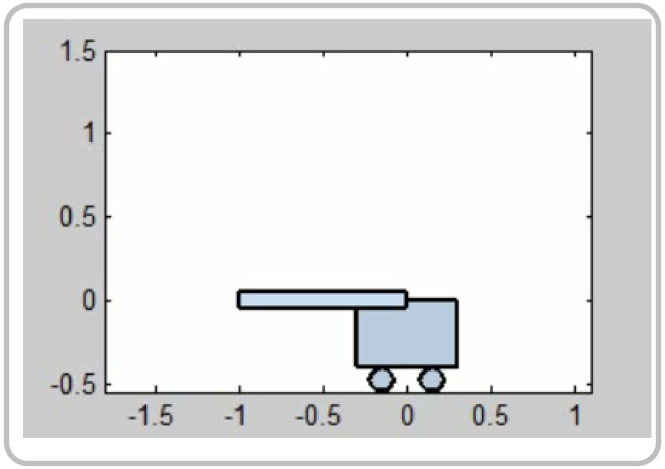
$\theta(0) = 1.57\, rad$ $\quad x(0) = 0\ m$

$\dot{\theta}(0) = 0\, rad/s$ $\quad \dot{x}(0) = 0\, m/s^2$

$K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

Force Plot and Simulation Results Animation





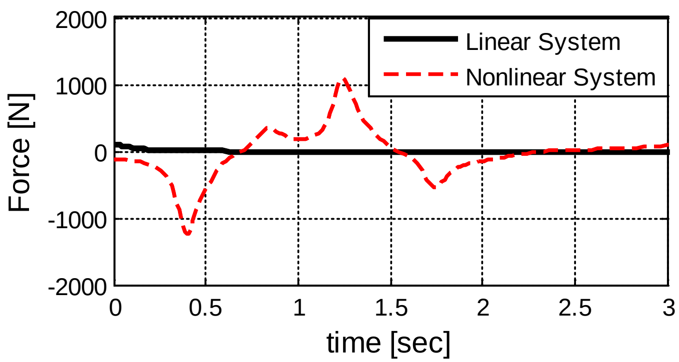


Fig 12. Feedback Control Input (Nonlinear system) and Simulation results Animation

Nonlinear System with Linear Feedback Control

Simulation Condition

$\theta(0) = 2.44\,rad$ $\quad x(0) = 0\ m$

$\dot{\theta}(0) = 0\,rad/s$ $\quad \dot{x}(0) = 0\,m/s^2$

$K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

Simulation Results Plots

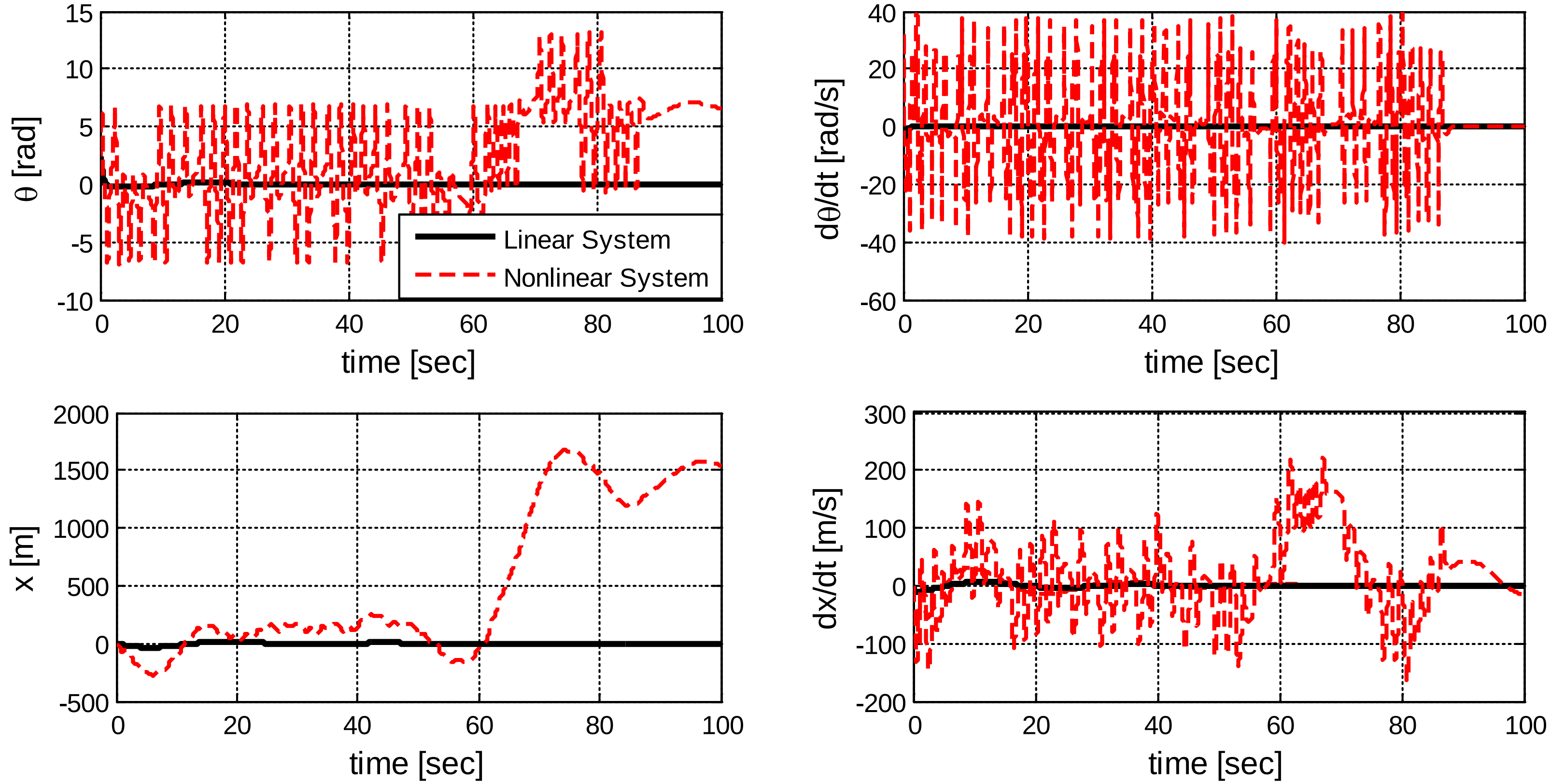


Fig 13. Simulation results for linear feedback control

Nonlinear System with Linear Feedback Control - Animation

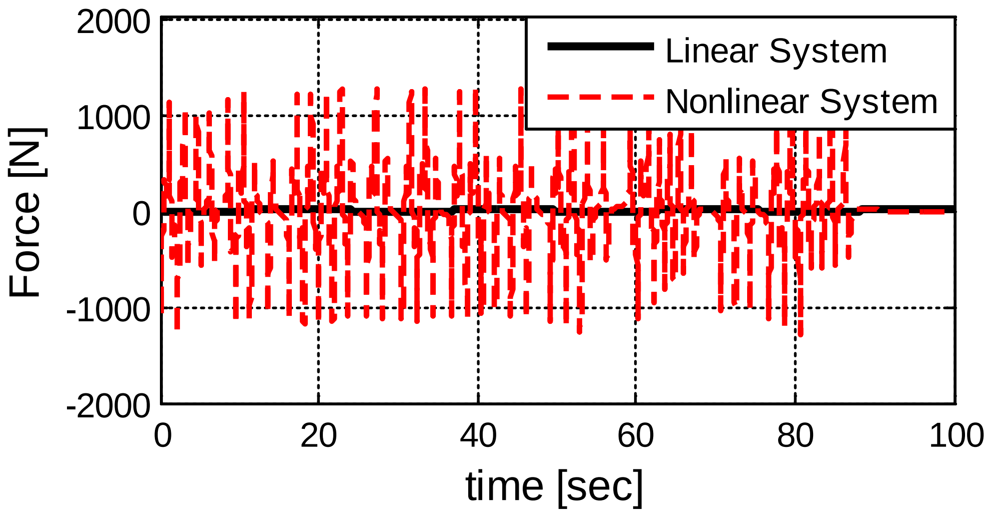
Simulation Condition

$\theta(0) = 2.44\,rad$ $\quad x(0) = 0\ m$

$\dot{\theta}(0) = 0\,rad/s$ $\quad \dot{x}(0) = 0\,m/s^2$

$K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

Force Plot and Simulation Results Animation



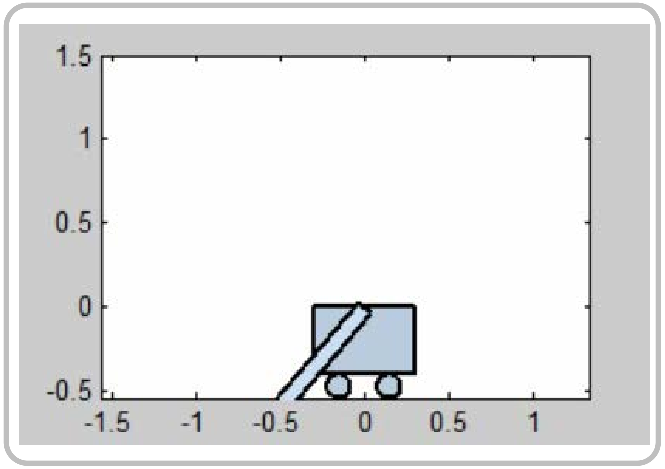


Fig 14. Feedback Control Input (Nonlinear system) and Simulation results Animation

Tracking Problem

We discussed linear feedback control.

This make it possible that all the states go to zero at $t \to \infty$.

But what if we want to make the states converge to specific values?

⇨ *Q) How to make the states converge to nonzero values*

Reference Input Tracking

Introduce reference input $u = -K\mathbf{x} + r$

Let x_{ss} *and* u_{ss} *as state x and input u repectively at steady state*

Then, $u = u_{ss} - K(x - x_{ss})$

if the system is like
$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$
$$y = C\mathbf{x} + Du$$
where $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^T = \left[\theta \ \dot{\theta} \ x \ \dot{x}\right]^T$,

At the steady state, this system becomes
$$0 = A\mathbf{x}_{ss} + Bu_{ss}$$
$$y_{ss} = C\mathbf{x}_{ss} + Du_{ss}$$
$\cdots\cdots(*)$ $\left(\because \text{At steady state, } \dot{\mathbf{x}} = 0\right)$

Tracking Problem

$We\ want\ to\ make\ y_{ss} = r_{ss}\ \ for\ any\ value\ of\ \ r_{ss}$

$To\ do\ this,\ asssume\ that\quad \begin{aligned}\mathbf{x}_{ss} &= N_x r_{ss}\\ u_{ss} &= N_u r_{ss}\end{aligned}\quad and\ \ put\ these\ equations\ to\ (*)$

$$Then\quad \begin{aligned}0 &= AN_x r_{ss} + BN_u r_{ss}\\ r_{ss} &= CN_x r_{ss} + DN_u r_{ss}\end{aligned}$$

$It\ can\ be\ also\ written\ as\ a\ matrix\ \ form\ like\quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix}$

$Assume\ that\ the\ inverse\ of\ \begin{bmatrix} A & B \\ C & D \end{bmatrix}\ is\ exists,\ then\ this\ equation\ can\ be\ solved\ \ for\ N_x\ and\ N_u$

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} \underline{0} \\ 1 \end{bmatrix}$$

$From\ above\ relation$

$$\begin{aligned}u &= N_u r - K\left(x - N_x r\right) = -K\mathbf{x} + (N_u + K\,N_x)r\\ &= -K\mathbf{x} + \bar{N}r\end{aligned}\qquad where\quad \bar{N} = N_u + K\,N_x$$

Nonlinear System vs Linear System with Reference Tracking Control

Simulation Condition

$\theta(0) = 0.1 rad$ $x(0) = 0\ m$ $K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

$\dot{\theta}(0) = 0\, rad/s$ $\dot{x}(0) = 0\, m/s^2$ $x_{ss} = r_{ss} = 5$

Simulation Results Plots

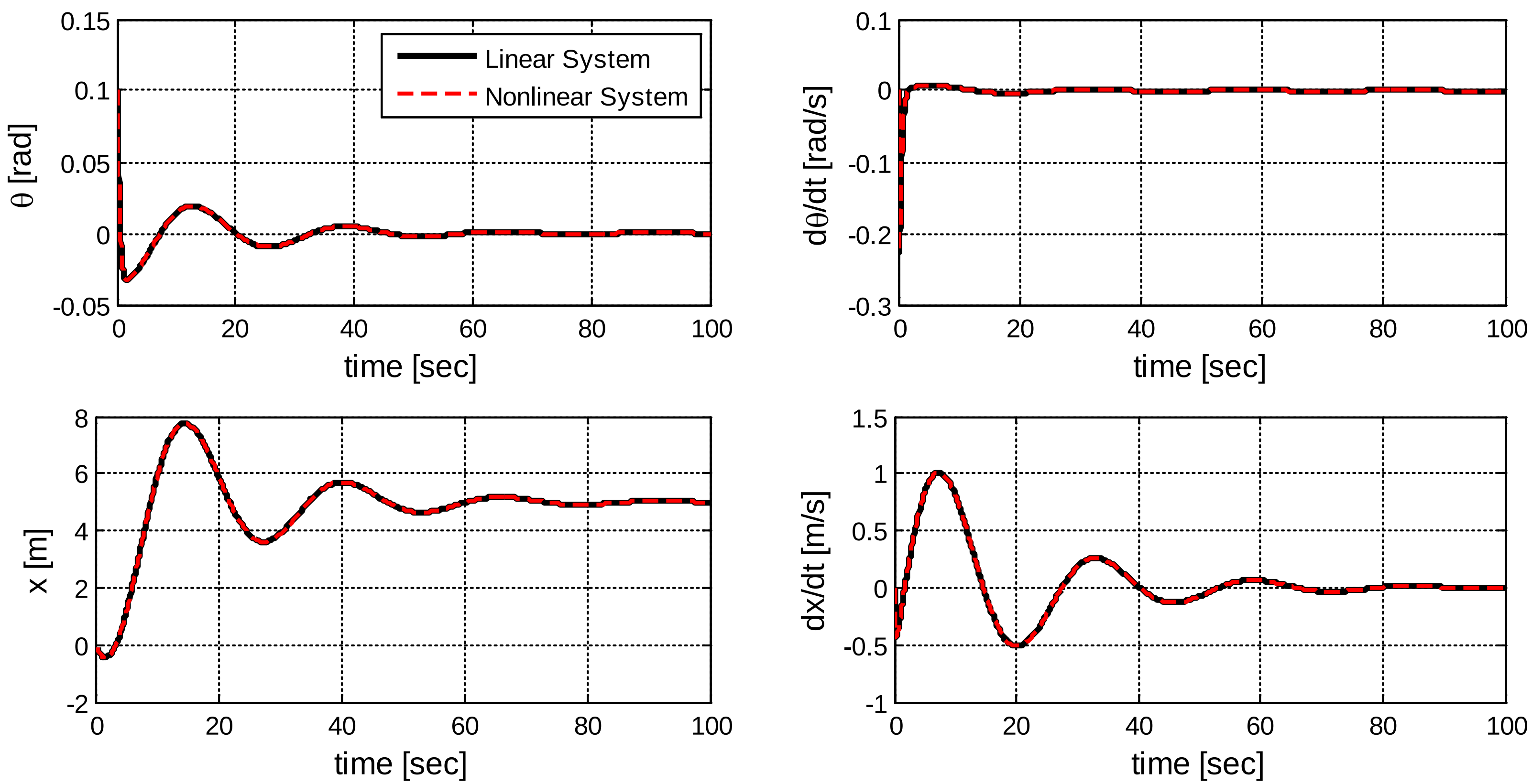


Fig 15. Simulation results for tracking control

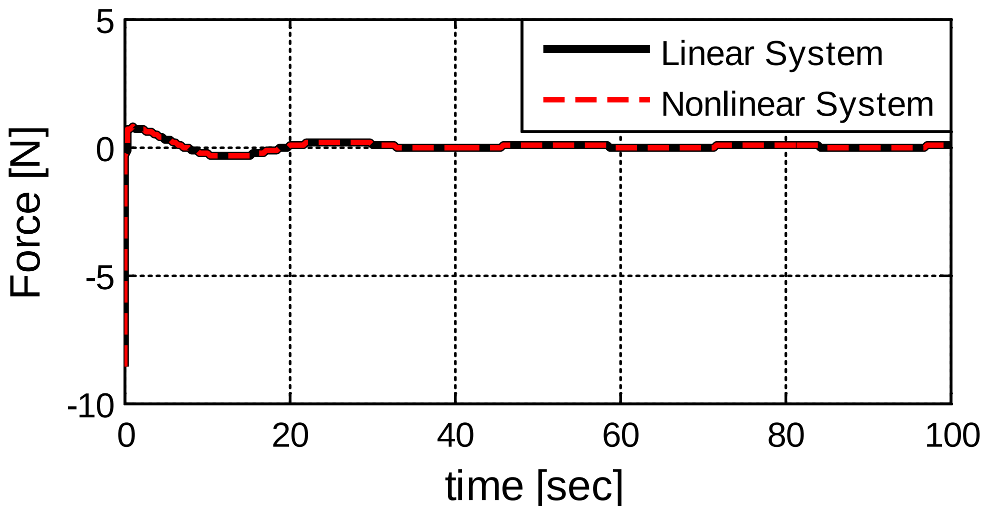
Nonlinear System vs Linear System with Reference Tracking Control

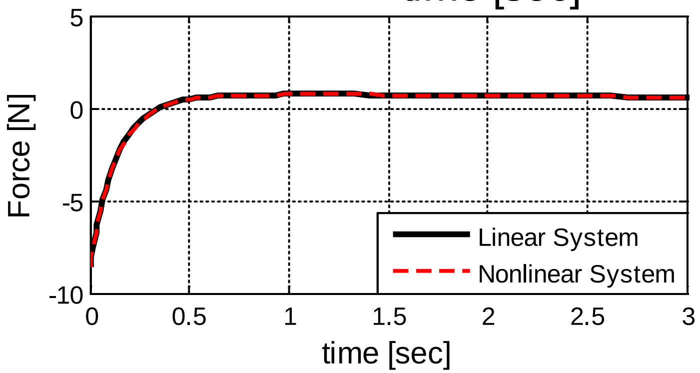
Simulation Condition

$\theta(0) = 0.1 rad$ $\quad x(0) = 0\ m$ $\quad K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

$\dot{\theta}(0) = 0\, rad/s$ $\quad \dot{x}(0) = 0\, m/s^2$ $\quad x_{ss} = r_{ss} = 5$

Force Plot and Simulation Results Animation





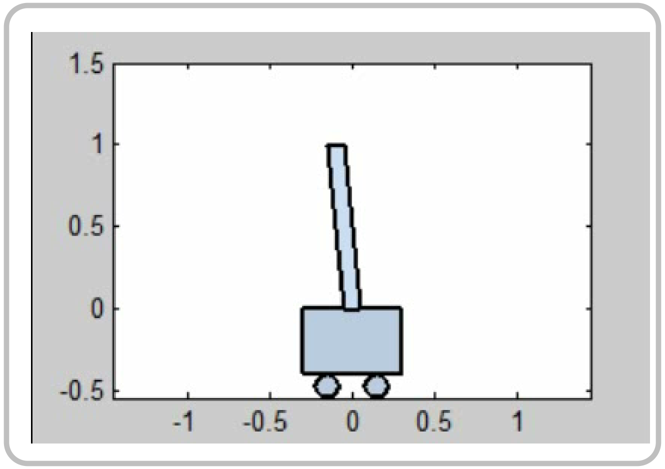


Fig 16. Reference Tracking Input (Nonlinear system) and Simulation results Animation

Nonlinear System vs Linear System with Reference Tracking Control

Simulation Condition

$\theta(0) = 1.57\,rad$ $\quad x(0) = 0\ m$ $\quad K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

$\dot{\theta}(0) = 0\,rad/s$ $\quad \dot{x}(0) = 0\,m/s^2$ $\quad x_{ss} = r_{ss} = 5$

Simulation Results Plots

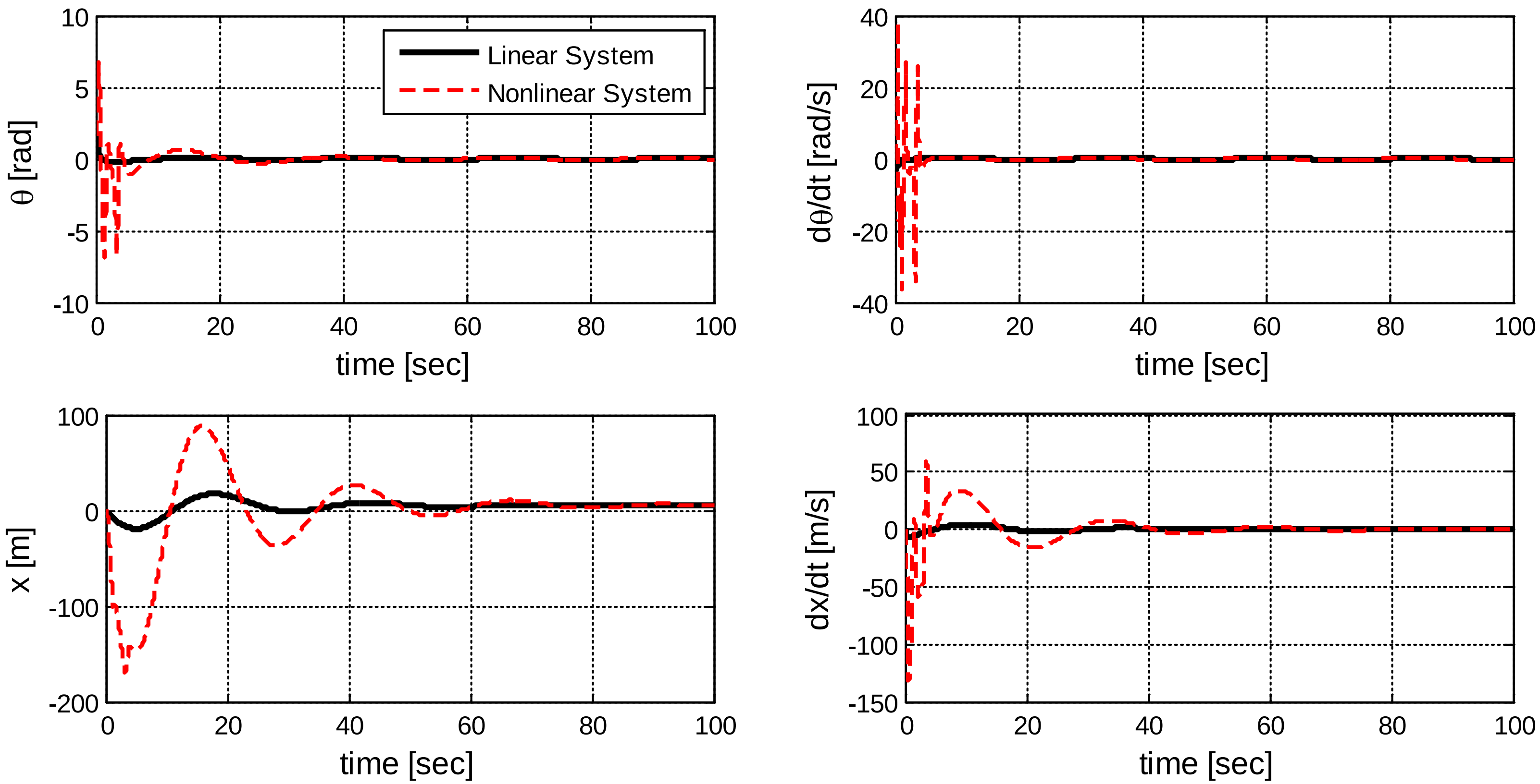


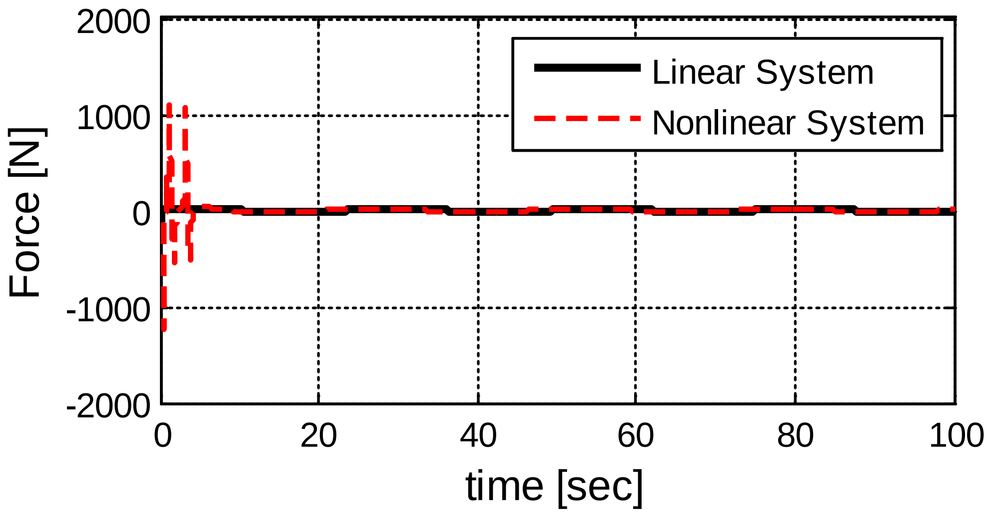
Fig 17. Simulation results for tracking control

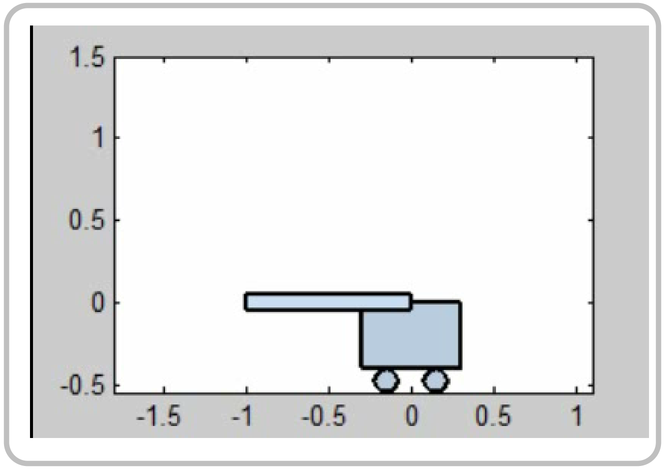
Nonlinear System vs Linear System with Reference Tracking Control

Simulation Condition

$\theta(0) = 1.57\,rad$	$x(0) = 0\ m$	$K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$
$\dot{\theta}(0) = 0\,rad/s$	$\dot{x}(0) = 0\,m/s^2$	$x_{ss} = r_{ss} = 5$

Force Plot and Simulation Results Animation





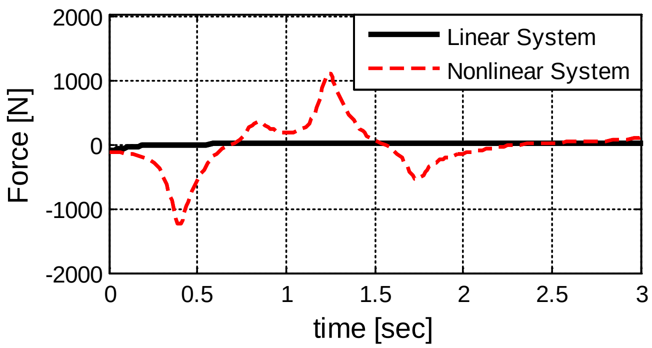


Fig 18. Reference Tracking Input (Nonlinear system) and Simulation results Animation

Nonlinear System vs Linear System with Reference Tracking Control

Simulation Condition

$\theta(0) = 3.05\,rad$ $\quad x(0) = 0\ m$ $\quad K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

$\dot{\theta}(0) = 0\,rad/s$ $\quad \dot{x}(0) = 0\,m/s^2$ $\quad x_{ss} = r_{ss} = 5$

Simulation Results Plots

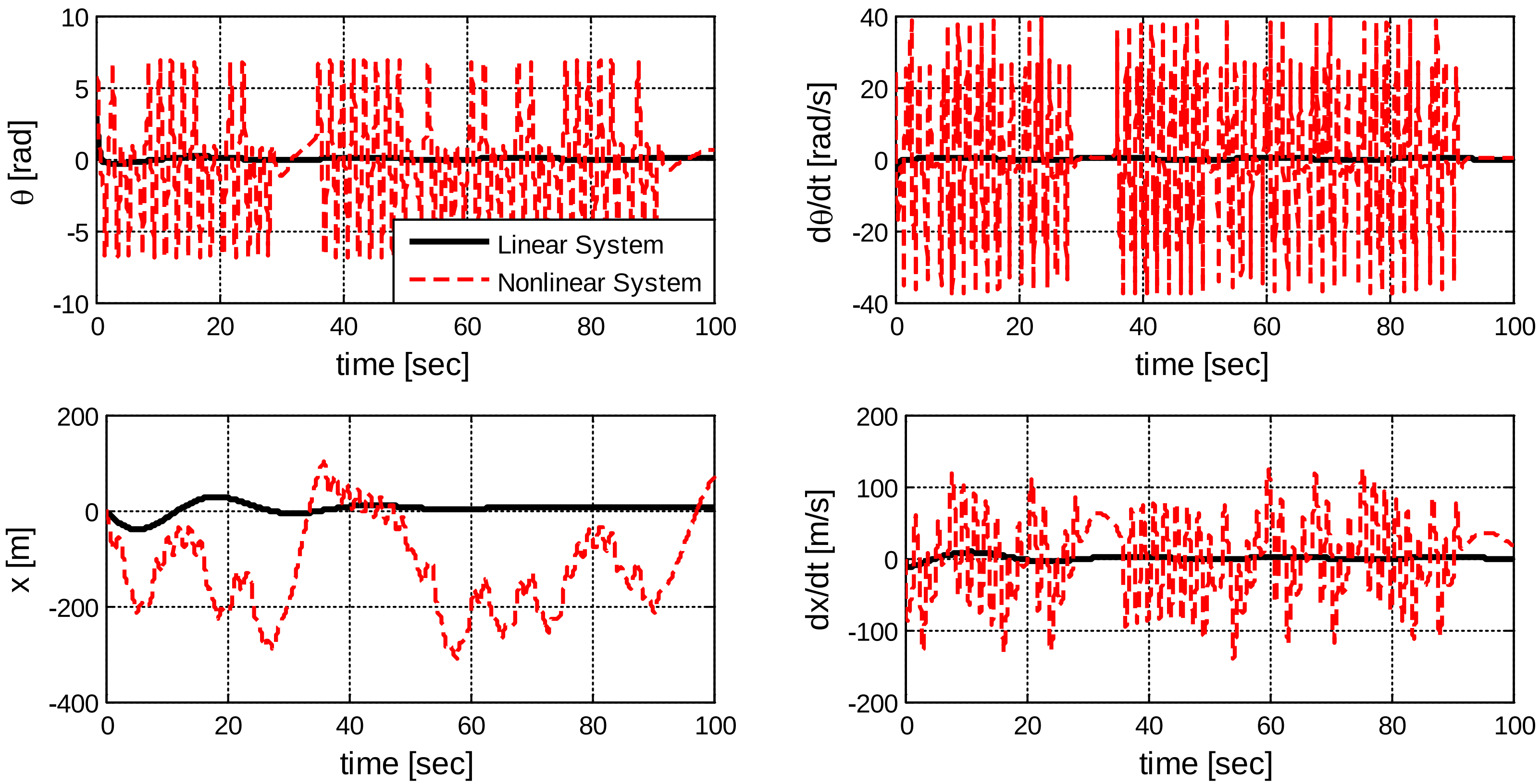


Fig 19. Simulation results for tracking control

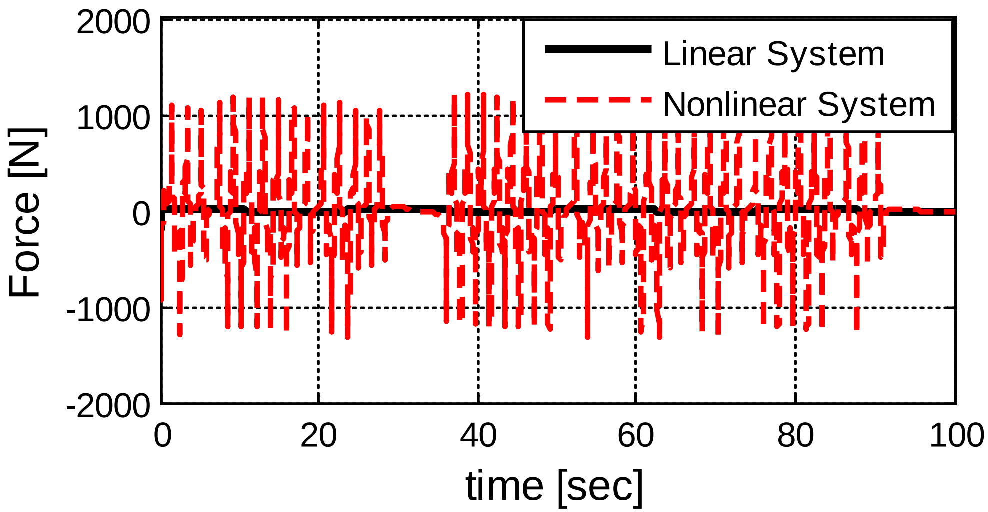
Nonlinear System vs Linear System with Reference Tracking Control

Simulation Condition

$\theta(0) = 3.05\,rad$ $\quad x(0) = 0\ m$ $\quad K = [70.5,\ 22.7,\ -0.3,\ -0.6483]$

$\dot{\theta}(0) = 0\,rad/s$ $\quad \dot{x}(0) = 0\,m/s^2$ $\quad x_{ss} = r_{ss} = 5$

Force Plot and Simulation Results Animation



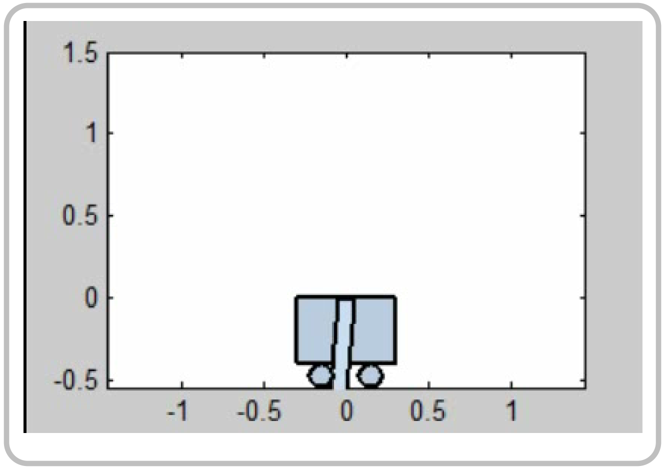


Fig 20. Reference Tracking Input (Nonlinear system) and Simulation results Animation

Appendix. A

How to use

Open ‘RUN_file.m’

1. Simulation(시간, 초기조건 등) 환경 설정
2. Parameter(질량, 길이 등) 값 설정
3. Gain 값 설정
4. 실행
5. ‘Simulaton_file.m’이 실행된다.

‘Simulation_file.m’

5. Nonlinear Equation을 실시간으로 계산(By function ‘*ODE4*’)
6. State의 실시간 값들이 얻어짐(x,theta…)
7. State의 시간에 대한 그래프 출력

How to use – RUN_file.m

Open 'RUN_file.m' → Simulation Setting

RUN file

```
global m M L I g Kp Kd zeta wn mode Kf A B u_ss x_ss;
%% Simulation Configuration

t0=0;                       % Initial Time [sec]
tf=100;                      % Final Time that simulation ends [sec]
del_t = 0.05;               % Sampling time [sec]

%% Initial Condition

x0 = 0;                     % Initial Position of Cart [m]
d_x0 = 0;                   % Initial velocity of Cart [m/s]
theta_0 = 0.1;            % initial Condition of Angle 1 [rad]
d_theta_0 = 0;              % initial Condition of Angular Velocity [rad/s]

%% Tracing Control
rss = 0;          % Desried 'x'

%% System Type Selection

mode = 3;                   %  < Choose the mode >
                            %    mode 1 : Nonlinear System +
                            %             Angle and Angle rate Controller
                            %    mode 2 : Linear System
                            %    mode 3 : Nonlinear System +
                            %             Linear Feedback Controller

%% Dynamic Parameters

m=0.1;                       % Mass of rod [kg]
M=2;                         % Mass of cart [kg]
L=1;                         % Length of rod [m]
I = m*(L/2)^2;               % Moment of inertia of rod [kg*m^2]
g = 9.81;                    % Gravity [m/s^2]
```

Global 변수 선언

시뮬레이션에 필요한 함수인 ode45에 쓰일 변수를 선언.
부록B에 ode45에 대해서 설명해두었다.

Simulation time 설정

t0 : 초기 시간(=0)
tf : simulation 종료 시간
del_t : sampling time 설정

Initial Condition 설정

x0 : Cart의 초기 위치
d_x0 : Cart의 초기 속도
theta_0 : Rod의 초기 각도
d_theta_0 : Rod의 초기 각속도

Desired X 좌표 설정

rss : 목표 도달 X 좌표

Dynamic Parameter 설정

m : Rod의 질량
M : Cart의 질량
L : Rod의 길이
I : Rod의 moment of inertia (자동으로 계산)
g : 중력가속도 (고정값)

Control Gain 설정

K : P gain 값 설정 (양수)
C : D gain 값 설정 (양수, 0일 시에는 P Control)
Zeta : 댐핑 값 설정
wn : 공진주파수 설정

How to use – Simulation_file.m

▶ ***'RUN_file'***을 실행시키면 자동으로 ***'Simulation_file.m'***이 실행된다.

```
%% Simulation File

% - Run ode function to calculate 'theta' in real time
% - If mode selection is invalid, display the messege
% - Display result plot
% - Display animation

%% Run Simulation
  [t,X]=ode45(@OSC_control,[0, tf],[theta_0, d_theta_0, x0, d_x0]);
  figure(1);
  set(figure(1),'Position',[200,400,400,200]);
  plot(t,X(:,1),'k','linewidth',2.3);
  grid on;
  xlabel('time [sec]','Fontsize',12);
  ylabel('₩theta [rad]','Fontsize',12);
  figure(2);
  set(figure(2),'Position',[620,400,400,200]);
  plot(t,X(:,2),'k','linewidth',2.3);
  grid on;
  xlabel('time [sec]','Fontsize',12);
  ylabel('d₩theta/dt [rad]','Fontsize',12);
  figure(3)
  set(figure(3),'Position',[200,700,400,200]);
```

ode45

Ode45라는 함수를 통해 nonlinear equation을 실시간으로 풀어낼 수 있다.(부록B에 ode45에 대해 설명)

State의 실시간 값들이 얻어짐

앞서 설정한 simulation time동안의 매 순간의 data 값(카트 위치, 바의 각도 등)들이 ode45 함수의 결과로 얻어진다.

결과의 시간에 대한 그래프 출력

이렇게 구해진 실시간 데이터 결과들을 시간에 대한 그래프로 출력한다.

Appendix. B

부록 : ode45

▶ ODE45 ?

MATALB에 내장돼있는 함수 & 미분방정식을 실시간으로 풀어주는 함수
이 때 exact solution을 구하는 것이 아닌, 수치해석 방법으로 실시간 해를 구한다.
Runge-Kutta method를 사용하는 numerical solver
내가 풀고자 하는 미분방정식을 풀어주는 함수를 만든다.

▶ Why ode45?

선형 뿐 아니라 비선형 미분방정식도 풀 수 있으며

사용방법 및 수식작성이 직관적이다.

▶ How to use?

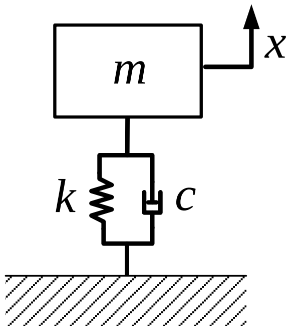
함수 이름 작성, 변수 선언
(이 때 함수 이름은 저장 시에도 동일한 이름으로 저장해야 한다.)

$\dot{\mathbf{x}} = f\left(\mathbf{x}\right)$ 꼴 그대로 수식으로 옮긴다.

부록 : ode45

Example

$$m\ddot{x}+c\dot{x}+kx=0,\quad x(0)=x_0, \dot{x}(0)=\dot{x}_0$$



1. 함수의 이름을 정하고 변수를 차례로 적어준다.(이때는 시간과 state x)

```
function dx=function_name(t,x)
```

미분 term을 나타내는 변수, state를 한번 미분한 것을 의미한다. 즉, $\dot{x}$를 의미 (dx)

함수이름(저장 시 동일한 이름으로 저장해야한다.) (function_name)

2. 미분방정식에 쓰이는 global 변수(state 제외)를 선언한다.

```
global m c k
```

이 함수에서 이 변수들이 쓰이기 때문에 이 함수가 돌아가기 전,
이 변수들이 workspace 상에 선언되어 있어야 한다.

3. $\dot{\mathbf{x}}=f(\mathbf{x})$ 꼴로 식을 만들어주고 code에 그대로 옮겨 적는다.

$$\mathbf{x}=\begin{bmatrix} x \\ \dot{x} \end{bmatrix}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{c}{m}x_2-\frac{k}{m}x_1 \end{aligned}$$

```
dx = zeros(2,1);
dx(1) = x(2);
dx(2) = -(c/m)*x(2)-(k/m)*x(1);
```

dx가 (2x1)행렬임을 먼저 선언해야 한다.

부록 : ode45

▶ 문제의 미분방정식을 푸는 ode45를 MATLAB에 작성하면 다음과 같다.

Example

$$m\ddot{x}+c\dot{x}+kx=0,\quad x(0)=x_0, \dot{x}(0)=\dot{x}_0$$

```
function dx=function_name(t,x)

global m c k

dx = zeros(2,1);
dx(1) = x(2);
dx(2) = -(c/m)*x(2)-(k/m)*x(1);
```

▶ 그럼 이 때 다음과 같이, 앞에서 만든 ode45를 이용한 m-file을 작성하여 실행하면 다음과 같다.

```
global m c k

simtime = 10;
samtime = 0.1;

m = 10;
c = 20;
k = 100;

x0 = 2;
d_x0 = 5;

[t,X] = ode45(@function_name,[0, simtime],[x0,d_x0]);
```

사용할 함수 이름 (@function_name)

t의 초기, 말기 값 ([0, simtime])

State의 초기 값(x1, x2 차례로) ([x0,d_x0])