# **Aircraft Structural Analysis**

#### Chapter 2 Statically Determinate Structures



## 2.1 Introduction

The purpose of this chapter is to review and Reinforce the principle of static equilibrium within the context of some basic types of aircraft structures. A structure may be defined as an assemblage of materials that is intended to sustain loads. How well the intention is realized depends on the design, and that depends (among other things) on how well the shape of the structure and the properties of the selected materials accommodate the predicted internal loads.

Therefore, it is important for a structural designer, in spite of—and aided by—digital computers, to develop a keen insight for predicting and visualizing load paths throughout a structure. The ability to do so largely depends on how well one has mastered the skills of sketching accurate free-body diagrams and properly applying the equilibrium equations to them, which will be one of our primary concerns here.

## 2.1 Introduction

- Statically determinate structures
  - Pinned and rigid-jointed frames
  - Stiffened shear webs
  - Thin-Walled beams and torque tubes

But, most real structures are statically indeterminate with redundant elements.

A truss, also called a pin-jointed bars, is an idealized skeletal or "sticklike" structure composed of slender rods joined together by smooth pins at the joints, also called nodes.

None of the smooth pins can apply a "couple" to the rods connected to it. Each member withstands tensile and compressive forces can not transfer couple moment.



**Figure 2.2.1** (a) Plane frame members held together by a single pin. (b) A riveted or bolted connection. (c) The idealized pin joint.

Statically determinated Truss :

Number of Triangles	1
Number of Joints	3
Number of members	3



. 2j = m + 3





- Externally statically determinate
- Minimally stable
- Internally statically determinate









Figure 2.2.5

Examples of supports inadequate to restrain rigid-body translation (a) and rotation (b).

(a) (b)

**Figure 2.2.6** Unstable trusses made rigid by properly located additional supports.

2j = m + r, and the supports restrain rigid-body motion

- (r: The total number of reactions)
- -> Statically determinate

After obtaining of internal forces in the bar elements, we can compute stresses by

$$\sigma = \frac{N}{A}$$

With compressive load, check buckling by the *Euler Column* formular

$$N_{CY} = \frac{\pi^2 EI}{L^2}$$



Table 2.2.1Moments of inertia for some simple sections.

Cross section	+d +	$\begin{array}{c} \leftarrow d \rightarrow \\ \downarrow \\ d \\ \downarrow \end{array}$	$ \begin{array}{c}                                     $	$\frac{d}{d} = \frac{1}{t^2}$
Area	$\frac{\pi d^2}{4}$	$d^2$	πtd	4 <i>td</i>
Moment of inertia	$\frac{\pi d^4}{64}$	$\frac{d^4}{12}$	$\frac{\pi \iota d^3}{8}$	$\frac{2td^3}{3}$

Solution of SD Truss Structures

1. Make equilibrium at nodes with unknown member forces and solve the obtained simultaneous equations.

2. Find the reaction forces at the supports by equilibrium of whole structures( Utilize FBD ) and get the values of member loads through the equilibrium at node by using the known support reaction forces.

#### Example 2.2.1

All members of the truss in Figure 2.2.8 are to be fabricated from the same stock of thin-walled, round, steel tubing, the section properties of which are listed in Table 2.2.2. Select the lightest weight tubing for which the axial stress in any rod of the truss does not exceed 25,000 psi in tension or compression and the critical buckling load is not exceeded. For steel,  $E = 30 \times 10^6 psi$ .



Figure 2.2.8 (a) Truss with loads and dimensions. (b) Truss as a free body, showing support reactions, and the chosen joint and member numbering scheme.







Member loads are given in pounds (+ = tension, - = compression).

#### 3. Method of Section

Section the truss into two bodies so as to expose the force in that member.

Then write the equilibrium equations for the free body on either side of the section and solve them for the unknown force.



Figure 2.2.10

(a) Cantilevered truss with a transverse section through the center bay. (b) Free-body diagram to the left of the cut, revealing the member forces in that bay.

$$\begin{split} \sum F_y &= 0 & \Rightarrow & N_{5.8} = \sqrt{2} \ P \\ \sum M_5 &= 0 & \Rightarrow & N_{6.8} = 2P \\ \sum F_x &= 0 & \Rightarrow & N_{5.7} = -3P \end{split}$$

## 2.3. Space Trusses

To avoid the rigid body motion, the structure should be constrained in three orthogonal translation and three rotational directions.

I.	1	2	3	4 …	1.7
j	3	4	5	6 …	i+3=j
m	3	5	7	9	3i+3=m
				- 8	

... 3j = m + 6

In three dimension, the structure is in SD if it satisfies 3j = m + r, r: support reaction

Since each node create three equilibrium equations.

## 2.3. Space Trusses

#### Example 2.3.1

Using the method of joints, calculate all of the member loads the truss in Figure 2.3.1 in terms of the loads P and Q applied as shown.



Figure 2.3.1 A space truss that is both internally and externally statically determinate. Figure 2.3.2 Free-body of

Figure 2.3.2 Free-body diagrams of nodes 1 through 5 of the truss in Figure 2.3.1.

The nodal coordinates are given in the table.

## 2.3. Space Trusses

Node 1 The equilibrium of node 1,

 $N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3 + P\mathbf{k} + Q\mathbf{j} = 0$ 

Substituting unit vectors into equilibrium equation,

 $\mathbf{e}_1 = \mathbf{j}$   $\mathbf{e}_2 = -0.8208\mathbf{i} + 0.5472\mathbf{j} + 0.1642\mathbf{k}$   $\mathbf{e}_3 = -0.8208\mathbf{i} + 0.5472\mathbf{j} - 0.1642\mathbf{k}$ 

 $(-0.8208N_2 - 0.8208N_3)\mathbf{i} + (N_1 + 0.5472N_2 + 0.5472N_3 + Q)\mathbf{j} + (0.1642N_2 - 0.1642N_3 + P)\mathbf{k} = 0$ 

Setting the x, y, and z components of this vector equation equal zero,

 $-0.8208N_2 - 0.8208N_3 = 0$  $N_1 + 0.5472N_2 + 0.5472N_3 = -Q$  $0.1642N_2 - 0.1642N_3 = -P$ 

 $N_1 = -Q$   $N_2 = -3.046P$   $N_3 = 3.046P$ 

### 2.4. Simple Beams

A simple beam is a slender, homogeneous bar that bends without twisting.









### 2.4. Simple Beams









2 (a) Simply supported beam. (b) Internal shear and bending moment as a function of distance from the left end.





Free-body diagram of a differential beam segment, showing the internal shears and bending moments and a differential portion of the externally applied distributed load.







- Shear Panel : The Structure with a thin sheet of materials to which a rod is bonded along each edge.
- If we assume the panel only carries shear forces, then the structure becomes SD.
- \* Relaxing this assumption will be treated in Ch.4.
- Flange : top and bottom rods
- Web : the panel
- Stiffner : vertical rod



Figure 2.5.1 Stiffened shear web acting as an idealized beam.







S : pure shear load

q: shear flow (shear force per unit length)

Summing moments about point 1,

 $(S_2 \sin \theta)a = S_3(b \sin \theta) \implies (\bar{q}_2 b)a = (\bar{q}_3 a)b \implies \bar{q}_2 = \bar{q}_3$ Summing moments about any two of the remaining three corners,  $\bar{q}_1 = \bar{q}_2 = \bar{q}_3 = \bar{q}_4 = \bar{q}$ 

The average shear flow is constant around the panel.

If we extend the differential parallelogram in any direction and parallel to the sides of the panel, we see that the shear flow throughout and around the sides of a parallelogram panel is constant.





#### Example 2.5.1

Find the shear flow in the structure shown in Figure 2.5.4 and the flange loads at a section 75cm from the left end.



**Figure 2.5.4** (a) Cantilevered parallelogram stiffened shear panel. (b) Shear and axial loads 0.75 m from free end. (c) Constant shear flows around the panel.

#### Example 2.5.1



Figure 2.5.5

(a) Trapezoidal shear panel. (b) Internal shear flow q related to the base shear flow  $q_{
m 0}.$ 





$$p'(q'\overline{a'b'}) = p(q\overline{ab}), \quad \frac{\overline{a'b'}}{p'} = \frac{\overline{ab}}{p}$$
  
 $q = \frac{\beta}{p^2}$ 

$$S = \int_{0}^{l} q \, ds = \int_{p_0}^{p_L} q \frac{dp}{\sin \alpha} = \int_{p_0}^{p_L} \frac{\beta}{p^2} \frac{dp}{\sin \alpha}$$
$$= \frac{p_L - p_0}{\sin \alpha} \frac{\beta}{p_L p_0}$$
$$S / \frac{p_L - p_0}{\sin \alpha} = S / l$$
$$= \bar{q}$$
$$\bar{q} = \frac{\beta}{p_0 p_L}$$
$$q = \bar{q} \frac{h_0 h_L}{h^2}$$
$$\bar{q}(h) = \frac{\beta}{p_0 p} = \frac{\bar{q} p_0 p_L}{p_0 p} = \bar{q} \frac{p_L}{p}$$
$$= \bar{q} \frac{h_L}{h}$$

#### Example 2.5.2

Find the shear flow in the web of the tapered beam shown in Figure 2.5.7. Also, calculate the average shear flow on each of the panel.













If the number of equilibrium equations equal the number of unknowns, the stiffened web structure is statically determinate.

$$n_{rods} + n_{panels} + n_{reactions} = 2n_{nodes}$$
 [2.5.7]

#### Example 2.5.3

Find the structure in Figure 2.5.11, calculate the shear flows in each of the three panels and the maximum load in the stiffeners.



**Figure 2.5.11** (a) Stiffened web structure, with the chosen node, rod, and panel numbering. (b) Free-body diagram, showing the applied loads and the reactions.

#### Example 2.5.3







Figure 2.5.13

Free-body diagram resulting from a vertical section through panel 3.



Figure 2.5.14Free-body diagram of topmost stiffener in Figure2.5.11.

Example 2.5.3





Cylindrical sheet and Conical surface in pure shear

Element equilibrium in the axial direction,

 $\sum F_x = 0: \quad \Rightarrow \quad -qdx + q'dx = 0 \quad \Rightarrow \quad q' = q \quad \text{Cylindrical sheet}$  $\sum F_x = 0: \quad \Rightarrow \quad (q'd\mathbf{l}') \cdot \mathbf{i} - (qd\mathbf{l}) \cdot \mathbf{i} = 0 \quad \Rightarrow \quad q' = q \quad \text{Conical surface}$ 

-> Shear flow is constant around the cross section.







Consider the curve with constant shear flow q joining points B and C

The y and z components of the resultant force R are

 $R_{y} = q \Delta y \qquad R_{z} = q \Delta z \qquad [2.5.9]$  $R = q \sqrt{\Delta y^{2} + \Delta z^{2}} = qL \qquad [2.5.10]$ 

The moment dT of the shear flow q acting on element ds at point P is

 $d\mathbf{T} = \mathbf{r} \times qd\mathbf{s}$ =  $qds(r\sin\phi)\mathbf{i}$ =  $q(hds)\mathbf{i}$ 

The total moment about O of the shear flow is

T = 2Aq [2.5.11]

From equation [2.5.10] and [2.5.11],

$$e = \frac{2A}{L}$$
 [2.5.12]

(e: perpendicular distance from point O to R)





Figure 2.5.20

(a) Uniform shear flow q on a curved web. (b) Area enclosed by the web and the lines joining each end of the web to point O.



Figure 2.5.21

The resultant  ${\bf R}$  of the constant shear flow in (a) has a line of action located as shown in (b).



 $\frac{2A_1}{l}$   $\frac{a}{ql}$   $\frac{2A_2}{a'}$   $\frac{2A_2}{l}$ 

(c)

Figure 2.6.1

(a)

- 3.

(a) Constant shear flow on a thinwalled closed section. (b) Closed section viewed as two open sections. (c) Shear flow resultants on each section. (d) Pure couple resultant for the closed section.

(b)

(d)

Constant shear flow q on a closed section is equivalent to a pure couple of magnitude 2Aq.

(e)



(a)

Figure 2.6.2

The same torque **T** is applied to a closed thin-walled section whose enclosed area approaches zero, moving from (a) to (d).

ÝΤ

(d)

## b T T $\frac{3T}{b^2}$

 $bt^2$ 

Figure 2.6.3 Shear stress due to torsion in a thin-walled open section.

## An exact approach using the theory of elasticity

#### Example 2.6.1

Figure 2.6.4 shows an idealized beam comprised of two flanges and a curved, thin web that has a semi-elliptical shape. A 3 kN vertical shear load is applied to the free end. Calculate the shear flow and find the horizontal location where the shear force bust be applied to produce no torsion.



Figure 2.6.4

Idealized cantilever beam with a semielliptical web.

A 500 lb shear load is applied at the free end.

#### Example 2.6.2

Calculate the shear flow in the walls of the closed section subjected to pure torsion, shown in Figure 2.6.6.





#### Example 2.6.3

Figure 2.6.7 shows a 50-inch span of a tapered box beam. At the left end, where the indicated loads are applied, there is a rigid rib at which the flange loads are zero. Other ribs (not shown) of varying size are spaced along the beam to maintain the form of the cross section. Calculate the shear flows and flange loads at the 50-inch station, which lies between two ribs.



**Figure 2.6.7** Free-body diagram of a tapered box beam, showing the three flange loads and three shear flows at the 50-inch station.



**Figure 2.6.9** Torque box in which all four corners intersect in a common point *P*.



[2.6.1]



Figure 2.6.10

(a) Idealized torque box with pure torsion applied to each end. Webs are referenced by numbers enclosed by squares. Flange members are referenced by numbers enclosed by circles. (b) The corresponding shear flows at each end and on an intermediate section.

$$q^{(1)}(x) = q^{\overline{(i)}} \frac{h^{(i)}(0)h^{(i)}(L)}{h^{(i)}(x)^2}$$
[2.6.2]  
$$\bar{q} = \frac{T}{2\bar{A}}$$
[2.6.3]  
$$\bar{A} = \left[\frac{h^{(1)}(L)h^{(2)}(0) + h^{(2)}(L)h^{(3)}(0)}{2}\right] \frac{h^{(4)}(0)}{h^{(2)}(0)}$$
[2.6.4]

#### Example 2.6.4

The idealized, stiffened web torque box structure in Figure 2.6.11(a) is span, depth, and chord. Given that it transmits a pure torque of 42,000 in-lb, calculate the shear flows and flange loads at 20-in intervals.



Figure 2.6.11 (a) Torque box with pure torsion applied to each end. (b) The shear flows on intermediate sections.

Numbers in parentheses are the lengths, in inches, of the inclined edges of the box.

#### Example 2.6.4



Figure 2.6.12

Relationship between flange load and average shear flows in adjacent webs (stringer 1 illustrated).

Figure 2.6.13

(a) Shear flow and (b) flange load variations with span for the torque box in Figure 2.6.11.



Frames, like trusses, are skeletal structures composed of slender member.

However, unlike trusses, the members of a frame transmit shear and bending, as well as axial loads.









24 equilibrium equations, 27 unknowns

-> Statically indeterminate

uuu

(d)

#### Example 2.7.1

Find the location and value of the maximum bending moment in the semicircular frame shown in Figure 2.7.3



**Figure 2.7.3** (a) Semicircular frame with an intermediate transverse point load. (b) Free-body diagram of the complete frame. (c) Free-body diagram of a section at  $\theta < 30^{\circ}$ . (d) Free-body diagram for a section at  $\theta > 30^{\circ}$ 

#### Example 2.7.2

Find the location and value of the maximum bending moment in the frame in Figure 2.7.5. The semicircular portion of the structure is acted on by a uniform shear flow, while point loads are applied at the endpoints A and A' of the horizontal elements. The given shear flow and point loads form a self-equilibrating set.



**Figure 2.7.5** (a) Semicircular frame with self-equilibrating applied load system. (b), (c), & (d) Free-body diagrams of each element of the frame, revealing the internal forces on transverse sections.

#### Example 2.7.2



- Figure 2.7.6
- Area of the segment of a circle spanned by the angle  $\theta$ .





Compression occurs on the side of the frame on which the bending moment diagram is plotted.

#### Example 2.7.3

Figure 2.7.8(a) shows a frame built into a wall at point W, with a 20 kN load P applied at point D. Calculate the magnitudes of the shear force, bending moment, and torsional moment acting on a transverse section through the frame at point O, located at some distance from the built-in support.





Use vector notation.

#### Example 2.7.4

Find the location and magnitude of the maximum bending moment in the frame of Figure 2.7.9. The support at 1 is capable of exerting reactive forces in y- and z-directions and couples about those axes. The support at 6 can exert forces only in the xand y-directions.



<u>Vector notation</u> <u>unnecessary.</u>

Figure 2.7.9 Statically determinate frame and its free-body diagram.

#### Example 2.7.4

Start at point 6 and move from member to member, calculating the forces and moments at each section using the free-body diagrams.





**Figure 2.7.11** Bending moment distribution for the frame in Figure 2.7.9.

**Figure 2.7.10** Free-body diagrams showing the internal forces acting on each transverse section of the frame in Figure 2.7.9.