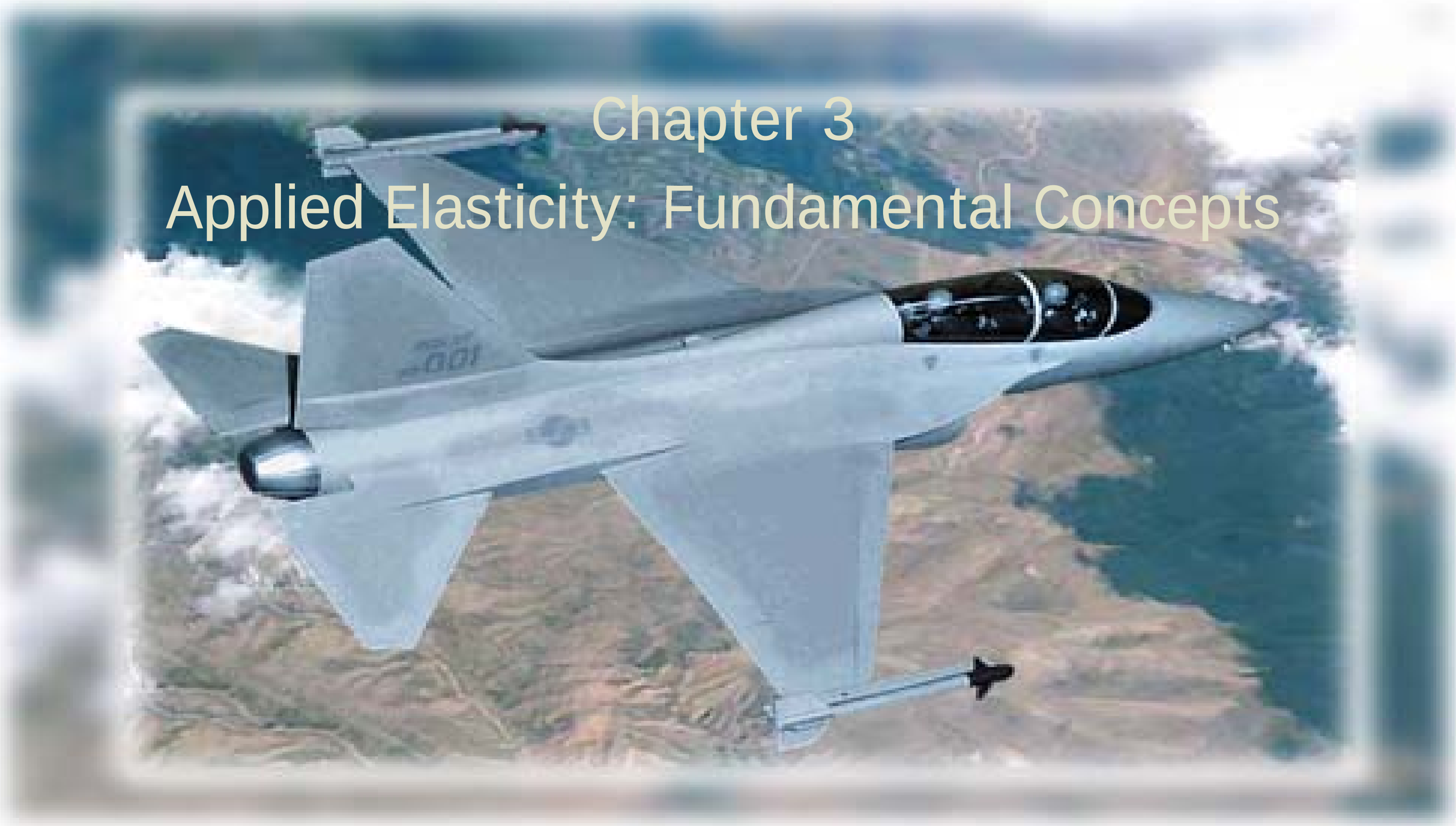
Aircraft Structural Analysis



Chapter 3

Applied Elasticity: Fundamental Concepts

3.1 Introduction

- The Airplane, that most fascinating and complex of man-made systems, epitomizes engineering at its finest. We focus in this text on the airframe, the “skeleton” of ribs and spars and other assorted structural members hidden the shiny, fragile skin of the modern airplane. The airframe must be strong, rigid, and durable, yet as light in weight as safety will allow. It must also fit within the streamlined shape defined by aerodynamics and the mission the airplane was built to serve.

To understand the structures of today’s aircraft, it is helpful to look back over the evolution of the airframe since the beginning of the twentieth century. It is sufficient to restrict our attention to the development of fixed-wing aircraft. We nevertheless acknowledge the significant structural dynamics problems that have attended the evolution of rotary-wing aircraft, which began in 1907.

3.1 Introduction

From the Newton's 2nd Law , we have *F = ma.*

The vector sum of forces becomes zero in the static equilibrium state.

$$\sum \underset{\sim}{F} = 0 \quad \text{And} \quad \sum \underset{\sim}{M}_p = 0$$

And the 2nd law can be extended in terms of momenta as,

$$\sum \underset{\sim}{F} = \dot{\underset{\sim}{L}}_{cm} \qquad \text{And} \qquad \sum \underset{\sim}{M}_p = \dot{\underset{\sim}{H}}_{cm}$$

cm : Center of mass

L_{cm} : linear momentum

H_{cm}: angular momentum

From D'Alembert's Principle, a state of Dynamic Equilibrium could be written by

$$\sum \underset{\sim}{F} + (-\dot{\underset{\sim}{L}}_{cm}) = 0 \quad \text{And} \quad \sum \underset{\sim}{M}_p + (-\dot{\underset{\sim}{H}}_{cm}) = 0$$

3.1 Introduction

- Continuum assumption : Continuum Mechanics
- Continuum : Solid, Fluid
- Solid : Elastic, Plastic….

- Elasticity : ability to bounce back
  - Linear elasticity : Displacements are proportional to the applied load
  - Nonlinear elasticity : Displacements are not proportional to the applied load.
    - Rubber band, Inflating a balloon, buckling a meter stick

3.1 Introduction

- Structural analysis contains many time-tested <u>formulas based on assumptions</u>.
- But, we need to know the theory of elasticity
  - to describe the behavior of elastic solid in precise detail
  - to assess the consequences of simplifying assumptions
- Topics in this chapter
  - Stress and strain.
  - Stress equilibrium and strain compatibility.
  - Stress and strain transformations.
  - Principal stress and strain.
  - Generalized Hooke's Law.
  - Saint-Venant's principle.
  - Strain energy.
  - Anisotropy.
  - Failure theories for steady and fluctuating loads.
  - Margins of safety.

3.2 Stress

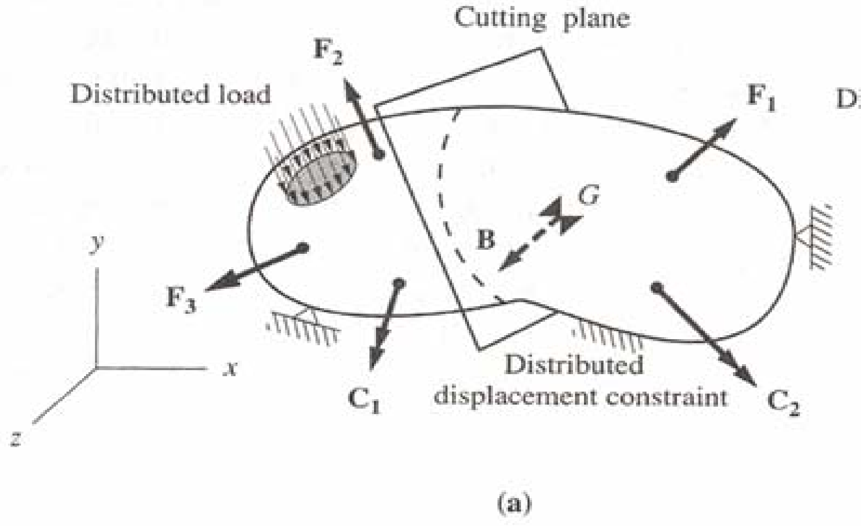


Figure 3.2.1: (a) A solid body, loaded and constrained in an arbitrary fashion. (b) Force acting on a small area of a cutting plane through point P in the solid. G and G′ are the centers of mass.

F: point load

C: point couple

B: body force

$\Delta \mathbf{F}^{(n)}$: Net force acting at point P

depends on orientation of the cutting plane, easily depend on n

ΔA : small area surrounding point P

3.2 Stress

Traction $$\mathbf{T}^{(n)} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}^{(n)}}{\Delta A}$$

$$\mathbf{T}^{(n)} = \boldsymbol{\sigma}_n + \boldsymbol{\tau}_n$$

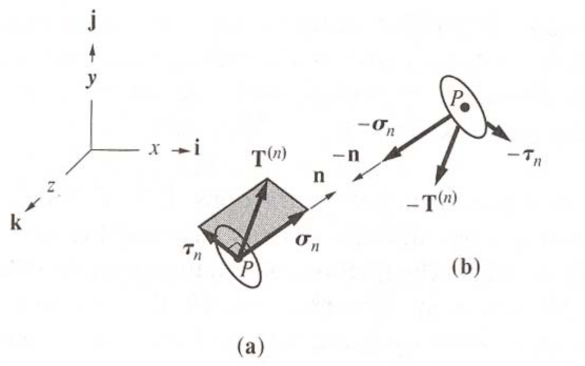


Figure 3.2.2 (a) Normal and shear components of the traction on an arbitrary plane through *P*. (b) Traction components on the opposite side of the cutting plane at *P*.

By defining the *xyz* axes,

Traction force vector can be expressed

in the planes normal to the three ax

$$\mathbf{T}^{(x)} = \sigma_x \mathbf{i} + \tau_{xy} \mathbf{j} + \tau_{xz} \mathbf{k}$$
$$\mathbf{T}^{(y)} = \tau_{yx} \mathbf{i} + \sigma_y \mathbf{j} + \tau_{yz} \mathbf{k}$$
$$\mathbf{T}^{(z)} = \tau_{zx} \mathbf{i} + \tau_{zy} \mathbf{j} + \sigma_z \mathbf{k}$$

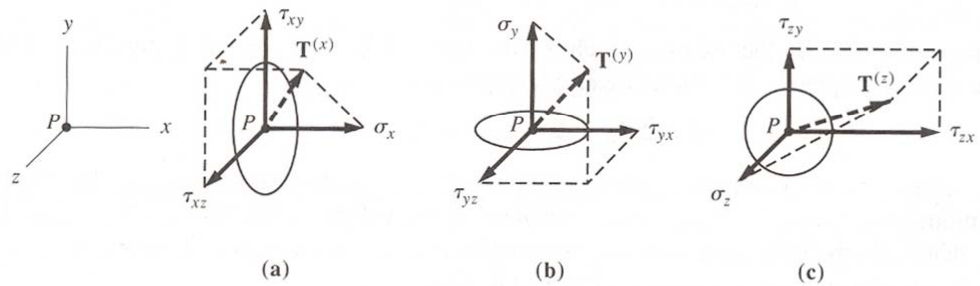


Figure 3.2.3 Positive stress components on positively-oriented cutting planes normal to: (a) the *x* axis, (b) the *y* axis, and (c) the *z* axis.

3.2 Stress

From the force equilibrium in the prism,

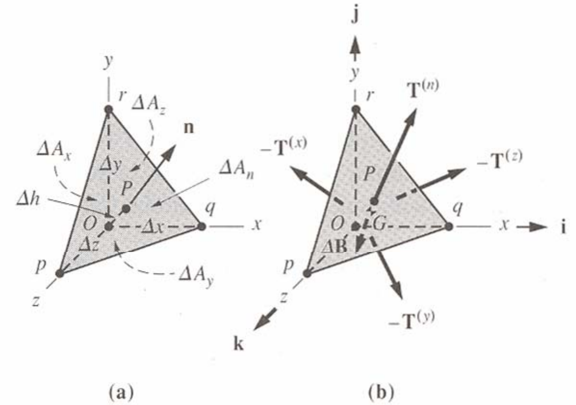


Figure 3.2.4 (a) Small tetrahedron *Opqr* with the inclined cutting plane through *P* as its base. (b) Free-body diagram showing the surface tractions and body force density.

$$\mathbf{T}^{(n)}\Delta A_n - \mathbf{T}^{(x)}\Delta A_x - \mathbf{T}^{(y)}\Delta A_y - \mathbf{T}^{(z)}\Delta A_z + \Delta\mathbf{B} = 0$$

$$\Delta A_x = n_x \Delta A_n \quad \Delta A_y = n_y \Delta A_n \quad \Delta A_z = n_z \Delta A_n$$

$$\mathbf{T}^{(n)} - \mathbf{T}^{(x)} n_x - \mathbf{T}^{(y)} n_y - \mathbf{T}^{(z)} n_z + \mathbf{b}\frac{\Delta h}{3} = 0$$

$$\mathbf{T}^{(n)} = \mathbf{T}^{(x)} n_x + \mathbf{T}^{(y)} n_y + \mathbf{T}^{(z)} n_z$$

$$\mathbf{T}^{(n)} = T_x^{(n)}\mathbf{i} + T_y^{(n)}\mathbf{j} + T_z^{(n)}\mathbf{k}$$

- Definition of Cauchy Stress

$$T_x^{(n)} = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z$$
$$T_y^{(n)} = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z$$
$$T_z^{(n)} = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$

3.2 Stress

The sign convention for stress and matrix expression

$$
[\sigma] = \begin{array}{c} \\ \\ \end{array}
\begin{array}{ccc|c}
x\ \text{plane} & y\ \text{plane} & z\ \text{plane} & \\
\sigma_x & \tau_{yx} & \tau_{zx} & x\ \text{direction} \\
\tau_{xy} & \sigma_y & \tau_{zy} & y\ \text{direction} \\
\tau_{xz} & \tau_{yz} & \sigma_z & z\ \text{direction}
\end{array}
$$

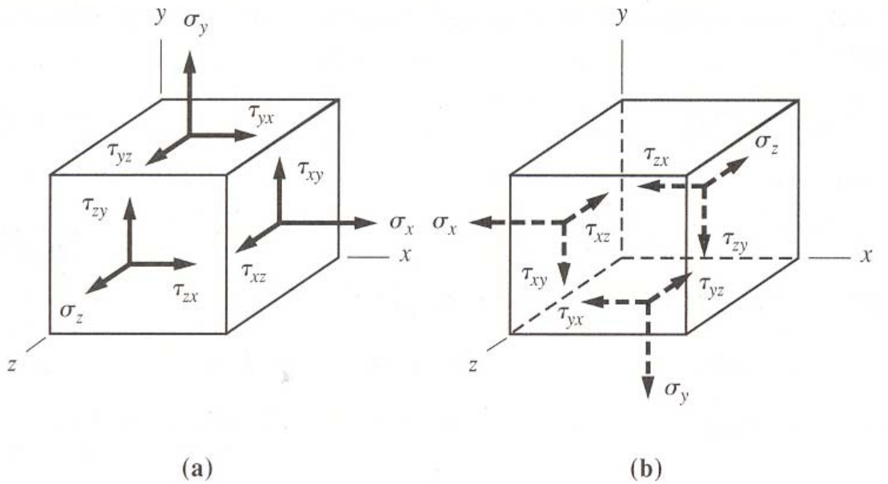


Figure 3.2.5 State of stress on a differential cube at a point. Positive components of stress on: (a) the front, positively-oriented surfaces, and (b) the rear, negatively-oriented surfaces.

3.2 Stress

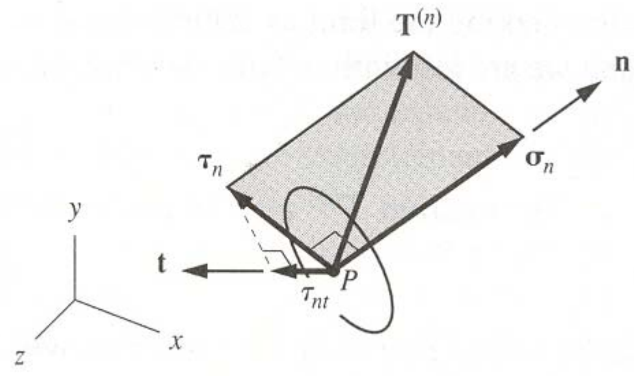


Figure 3.2.6 An in-plane component τ_{nt} of the shear traction on a plane.

Normal stress on a plane with unit normal **n**

$$\sigma_n = \mathbf{T}^{(n)} \cdot \mathbf{n} = T_x^{(n)} n_x + T_y^{(n)} n_y + T_z^{(n)} n_z$$

Shear component of the Traction

$$\boldsymbol{\tau}_n = \mathbf{T}^{(n)} - \boldsymbol{\sigma}_n$$

$$\tau_{nt} = \boldsymbol{\tau}_n \cdot \mathbf{t} = (\mathbf{T}^{(n)} - \boldsymbol{\sigma}_n) \cdot \mathbf{t}$$

$$\tau_{nt} = \mathbf{T}^{(n)} \cdot \mathbf{t} = T_x^{(n)} t_x + T_y^{(n)} t_y + T_z^{(n)} t_z$$

The compomemt $T^{(n)}$ are obtained from Cauchy's equation. So……

$$\boxed{\begin{aligned} \sigma_n &= \sigma_x n_x^2 + \sigma_y n_y^2 + \sigma_z n_z^2 + 2\tau_{xy} n_x n_y + 2\tau_{xz} n_x n_z + 2\tau_{yz} n_y n_z \\ \tau_{nt} &= \sigma_x n_x t_x + \sigma_y n_y t_y + \sigma_z n_z t_z + \tau_{xy}(n_x t_y + t_x n_y) \\ &\quad + \tau_{xz}(n_x t_z + t_x n_z) + \tau_{yz}(n_y t_z + t_y n_z) \end{aligned}} \quad (\mathbf{n} \cdot \mathbf{t} = 0)$$

$$\tau_n = \sqrt{\boldsymbol{\tau}_n \cdot \boldsymbol{\tau}_n} = \sqrt{T^{(n)^2} - \sigma_n^2}$$

3.3 Equilibrium

Moment Equilibrium

the moments about the center of mass of the G

$$\sum \mathbf{M}_G = 0$$

$$\left(\frac{dx}{2}\mathbf{i}\right) \times \mathbf{T}^{(x)}(x+dx, \bar{y}, \bar{z})dydz + \left(-\frac{dx}{2}\mathbf{i}\right) \times [-\mathbf{T}^{(x)}(x, \bar{y}, \bar{z})dydz]$$
$$+\left(\frac{dy}{2}\mathbf{j}\right) \times \mathbf{T}^{(y)}(\bar{x}, y+dy, \bar{z})dxdz + \left(-\frac{dy}{2}\mathbf{j}\right) \times [-\mathbf{T}^{(y)}(\bar{x}, y, \bar{z})dxdz]$$
$$+\left(\frac{dz}{2}\mathbf{k}\right) \times \mathbf{T}^{(z)}(\bar{x}, \bar{y}, z+dz)dxdy$$
$$+\left(\frac{dz}{2}\mathbf{k}\right) \times [-\mathbf{T}^{(z)}(\bar{x}, \bar{y}, z)dxdy] = 0$$

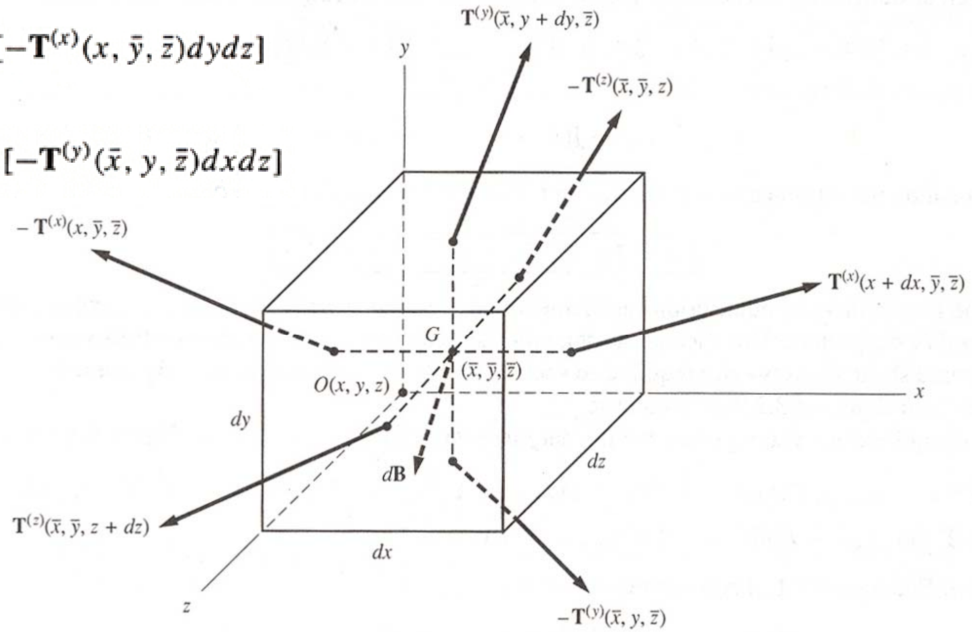


Figure 3.3.1 Variation of tractions at a point. The coordinates of the corner O are (x, y, z), while those of the center of mass G of the cube are $(\bar{x}, \bar{y}, \bar{z})$.

3.3 Equilibrium

From Taylor Series Expansion

$$f(x+dx, y+dv, z+dz) = f(x,y,z) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

We write again

$$\left(\frac{dx}{2}\mathbf{i}\right) \times \left(\mathbf{T}^{(x)} + \frac{1}{2}\frac{\partial \mathbf{T}^{(x)}}{\partial x}dx\right)dydz + \left(\frac{dy}{2}\mathbf{j}\right) \times \left(\mathbf{T}^{(y)} + \frac{1}{2}\frac{\partial \mathbf{T}^{(y)}}{\partial y}dy\right)dxdz + \left(\frac{dz}{2}\mathbf{k}\right) \times$$

$$\left(\mathbf{T}^{(z)} + \frac{\partial \mathbf{T}^{(z)}}{\partial z}dz\right)dxdy + \left(\frac{dx}{2}\mathbf{i}\right) \times \mathbf{T}^{(x)}dydz + \left(\frac{dy}{2}\mathbf{j}\right) \times \mathbf{T}^{(y)}dxdz + \left(\frac{dz}{2}\mathbf{k}\right) \times \mathbf{T}^{(z)}dxdy = 0$$

$$\mathbf{i} \times \mathbf{T}^{(x)}dxdydz + \mathbf{j} \times \mathbf{T}^{(y)}dxdydz + \mathbf{k} \times \mathbf{T}^{(z)}dxdydz + \cdots = 0$$

Ignore high order equation ('…'part , 4th order), and

$$\mathbf{i}\times(\sigma_x\mathbf{i}+\tau_{xy}\mathbf{j}+\tau_{xz}\mathbf{k})+\mathbf{j}\times(\tau_{yx}\mathbf{i}+\sigma_y\mathbf{j}+\tau_{yz}\mathbf{k})+\mathbf{k}\times(\tau_{zx}\mathbf{i}+\tau_{zy}\mathbf{j}+\sigma_z\mathbf{k})=\mathbf{0}$$

$$\mathbf{i}(\tau_{yz}-\tau_{zy})+\mathbf{j}(\tau_{zx}-\tau_{xz})+\mathbf{k}(\tau_{xy}-\tau_{yx})=0$$

$$\boxed{\tau_{yx}=\tau_{xy} \qquad \tau_{zx}=\tau_{xz} \qquad \tau_{zy}=\tau_{yz}}$$

3.3 Equilibrium

Force Equilibrium

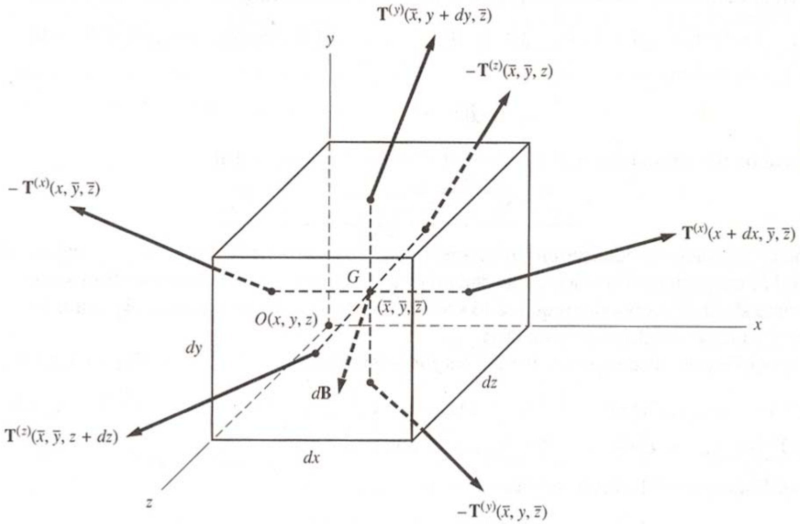


Figure 3.3.1 Variation of tractions at a point. The coordinates of the corner O are (x, y, z), while those of the center of mass G of the cube are $(\bar{x}, \bar{y}, \bar{z})$.

$$\mathbf{T}^{(x)}(x+dx, y, z)dydz - \mathbf{T}^{(x)}(x, y, z)dydz$$
$$+\mathbf{T}^{(y)}(x, y+dy, z)dxdz - \mathbf{T}^{(y)}(x, y, z)dxdz$$
$$+\mathbf{T}^{(z)}(x, y, z+dz)dxdy - \mathbf{T}^{(z)}(x, y, z)dxdy + d\mathbf{B} = 0$$

$$\frac{\partial \mathbf{T}^{(x)}}{\partial x}dxdydz + \frac{\partial \mathbf{T}^{(y)}}{\partial y}dydxdz + \frac{\partial \mathbf{T}^{(z)}}{\partial z}dzdxdy + d\mathbf{B} = 0$$

Body force density

$$\mathbf{b} = \frac{d\mathbf{B}}{dV} = \frac{d\mathbf{B}}{dxdydz}$$

$$\frac{\partial \mathbf{T}^{(x)}}{\partial x} + \frac{\partial \mathbf{T}^{(y)}}{\partial y} + \frac{\partial \mathbf{T}^{(z)}}{\partial z} + \mathbf{b} = 0 \quad \Rightarrow$$

$$\boxed{\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + b_y &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z &= 0
\end{aligned}}$$

3.4 Stress Transformation

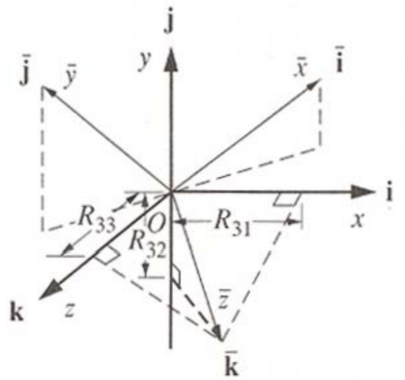


Figure 3.4.1 Two sets of cartesian reference axes, xyz and $\bar{x}\bar{y}\bar{z}$.

$$\bar{\mathbf{i}} = R_{11}\mathbf{i} + R_{12}\mathbf{j} + R_{13}\mathbf{k}$$
$$\bar{\mathbf{j}} = R_{21}\mathbf{i} + R_{22}\mathbf{j} + R_{23}\mathbf{k}$$
$$\bar{\mathbf{k}} = R_{31}\mathbf{i} + R_{32}\mathbf{j} + R_{33}\mathbf{k}$$

$$\Rightarrow \begin{pmatrix} \bar{\mathbf{i}} \\ \bar{\mathbf{j}} \\ \bar{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}$$

Rotational Matrix R is Orthogonal Matrix

$$\Rightarrow \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$$

$$R_{11}^2 + R_{21}^2 + R_{31}^2 = R_{12}^2 + R_{22}^2 + R_{32}^2 = R_{13}^2 + R_{23}^2 + R_{33}^2 = 1$$
$$R_{11}R_{12} + R_{21}R_{22} + R_{31}R_{32} = R_{11}R_{13} + R_{21}R_{23} + R_{31}R_{33} = R_{12}R_{13} + R_{22}R_{23} + R_{32}R_{33} = 0$$

$$\begin{aligned} \sigma_n &= \sigma_x n_x^2 + \sigma_y n_y^2 + \sigma_z n_z^2 + 2\tau_{xy} n_x n_y + 2\tau_{xz} n_x n_z + 2\tau_{yz} n_y n_z \\ \tau_{nt} &= \sigma_x n_x t_x + \sigma_y n_y t_y + \sigma_z n_z t_z + \tau_{xy}(n_x t_y + t_x n_y) \\ &\quad + \tau_{xz}(n_x t_z + t_x n_z) + \tau_{yz}(n_y t_z + t_y n_z) \end{aligned}$$

$$(\mathbf{n} \cdot \mathbf{t} = 0)$$

$$\begin{aligned} \bar{\sigma}_x &= R_{11}^2\sigma_x + R_{12}^2\sigma_y + R_{13}^2\sigma_z + 2R_{11}R_{12}\tau_{xy} + 2R_{11}R_{13}\tau_{xz} + 2R_{12}R_{13}\tau_{yz} \\ \bar{\sigma}_y &= R_{21}^2\sigma_x + R_{22}^2\sigma_y + R_{23}^2\sigma_z + 2R_{21}R_{22}\tau_{xy} + 2R_{21}R_{23}\tau_{xz} + 2R_{22}R_{23}\tau_{yz} \\ \bar{\sigma}_z &= R_{31}^2\sigma_x + R_{32}^2\sigma_y + R_{33}^2\sigma_z + 2R_{31}R_{32}\tau_{xy} + 2R_{31}R_{33}\tau_{xz} + 2R_{32}R_{33}\tau_{yz} \\ \bar{\tau}_{xy} &= R_{11}R_{21}\sigma_x + R_{12}R_{22}\sigma_y + R_{13}R_{23}\sigma_z + (R_{11}R_{22} + R_{12}R_{21})\,\tau_{xy} + (R_{11}R_{23} + R_{13}R_{21})\,\tau_{xz} + (R_{12}R_{23} + R_{13}R_{22})\,\tau_{yz} \\ \bar{\tau}_{xz} &= R_{11}R_{31}\sigma_x + R_{12}R_{32}\sigma_y + R_{13}R_{33}\sigma_z + (R_{11}R_{32} + R_{12}R_{31})\,\tau_{xy} + (R_{11}R_{33} + R_{13}R_{31})\,\tau_{xz} + (R_{12}R_{33} + R_{13}R_{32})\,\tau_{yz} \\ \bar{\tau}_{yz} &= R_{21}R_{31}\sigma_x + R_{22}R_{32}\sigma_y + R_{23}R_{33}\sigma_z + (R_{21}R_{32} + R_{22}R_{31})\,\tau_{xy} + (R_{21}R_{33} + R_{23}R_{31})\,\tau_{xz} + (R_{22}R_{33} + R_{23}R_{32})\,\tau_{yz} \end{aligned}$$

3.4 Stress Transformation

Example 3.4.1

Show that the hydrostatic stress $\frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$ is invariant under a coordinate transformation.

$$\begin{aligned}\bar{\sigma}_x + \bar{\sigma}_y + \bar{\sigma}_z = {} & (R_{11}^2 + R_{21}^2 + R_{31}^2)\sigma_x + (R_{12}^2 + R_{22}^2 + R_{32}^2)\sigma_y + (R_{13}^2 + R_{23}^2 + R_{33}^2)\sigma_z \\ & + 2(R_{11}R_{12} + R_{21}R_{22} + R_{31}R_{32})\tau_{xy} + 2(R_{11}R_{13} + R_{21}R_{23} + R_{31}R_{33})\tau_{xz} \\ & + 2(R_{12}R_{13} + R_{22}R_{23} + R_{32}R_{33})\tau_{yz}\end{aligned}$$

By virtue of Equations 3.4.4, this becomes

$$\bar{\sigma}_x + \bar{\sigma}_y + \bar{\sigma}_z = (1)\sigma_x + (1)\sigma_y + (1)\sigma_z + 2(0)\tau_{xy} + 2(0)\tau_{xz} + 2(0)\tau_{yz}$$

so that

$$\bar{\sigma}_x + \bar{\sigma}_y + \bar{\sigma}_z = \sigma_x + \sigma_y + \sigma_z$$

3.4 Stress Transformaion

Example 3.4.2

Let m and n be the unit normals to two planes through a point, and $\mathbf{T}^{(m)}$ and $\mathbf{T}^{(n)}$ be the tractions on those planes.

Show that $\mathbf{T}^{(n)} \cdot \mathbf{m} = \mathbf{T}^{(m)} \cdot \mathbf{n}$

$$\mathbf{T}^{(n)} \cdot \mathbf{m} = T_x^{(n)} m_x + T_y^{(n)} m_y + T_z^{(n)} m_z = (\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z) m_x$$
$$+ (\tau_{yx} n_x + \sigma_y n_y + \tau_{yz} n_z) m_y + (\tau_{zx} n_x + \tau_{zy} n_y + \sigma_z n_z) m_z$$

$$\mathbf{T}^{(n)} \cdot \mathbf{m} = (\sigma_x m_x + \tau_{xy} m_y + \tau_{xz} m_z) n_x$$
$$+ (\tau_{yx} m_x + \sigma_y m_y + \tau_{yz} m_z) n_y + (\tau_{zx} m_x + \tau_{zy} m_y + \sigma_z m_z) n_z$$

With the Cauchy formulas,

$$\mathbf{T}^{(n)} \cdot \mathbf{m} = T_x^{(m)} n_x + T_y^{(m)} n_y + T_z^{(m)} n_z$$

$$\mathbf{T}^{(n)} \cdot \mathbf{m} = \mathbf{T}^{(m)} \cdot \mathbf{n}$$

3.5. Principal Stress

Principal Stress : Normal stress when Shear Stress on the surface is Zero

$$\mathbf{T}^{(n)} = T_x^{(n)}\mathbf{i} + T_y^{(n)}\mathbf{j} + T_z^{(n)}\mathbf{k} = \sigma\mathbf{n} = \sigma(n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k})$$

$$\begin{aligned} \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z &= \sigma n_x \\ \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z &= \sigma n_y \\ \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z &= \sigma n_z \end{aligned}$$

$$\begin{aligned} (\sigma_x - \sigma) n_x + \tau_{xy} n_y + \tau_{xz} n_z &= 0 \\ \tau_{xy} n_x + (\sigma_y - \sigma) n_y + \tau_{yz} n_z &= 0 \\ \tau_{xz} n_x + \tau_{yz} n_y + (\sigma_z - \sigma) n_z &= 0 \end{aligned}$$

$n_x = 0, n_y = 0, n_z = 0$ is not allowed.

eigenvalue Problem

$$\det\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{bmatrix} = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \det\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} + \det\begin{bmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{bmatrix} + \det\begin{bmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{bmatrix}$$

$$I_3 = \det\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

3.5. Principal Stress

- The three ***I***'s are called the three *invariants* of stress tensor , because they are unaltered by coordinate transformation.
- The principal stresses: the three roots of the characteristic equations

$$\sigma_1 = \frac{I_1}{3} + \frac{2}{3}\sqrt{I_1^2 - 3I_2}\cos\left(\frac{\alpha}{3}\right)$$

$$\sigma_2 = \frac{I_1}{3} + \frac{2}{3}\sqrt{I_1^2 - 3I_2}\cos\left(\frac{\alpha}{3} + \frac{2\pi}{3}\right)$$

$$\sigma_3 = \frac{I_1}{3} + \frac{2}{3}\sqrt{I_1^2 - 3I_2}\cos\left(\frac{\alpha}{3} + \frac{4\pi}{3}\right)$$

$$\alpha = \cos^{-1}\left[\frac{2I_1^3 - 9I_1I_2 + 27I_3}{2(I_1^2 - 3I_2)^{3/2}}\right]$$

3.5. Principal Stress

From the conclusion of Ex 3.4.2

$$\mathbf{T}^{(i)} \cdot \mathbf{n}^{(j)} = \mathbf{T}^{(j)} \cdot \mathbf{n}^{(i)}$$

$$\sigma_i \mathbf{n}^{(i)} \cdot \mathbf{n}^{(j)} = \sigma_j \mathbf{n}^{(j)} \cdot \mathbf{n}^{(i)} \qquad \text{or} \qquad (\sigma_i - \sigma_j)\mathbf{n}^{(i)} \cdot \mathbf{n}^{(j)} = 0$$

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

If $\sigma_i \neq \sigma_j \rightarrow n_i \cdot n_j = 0$

<u>Principal directions</u>

Corresponding to

unique principal stresses

<u>are orthogonal</u>

$$\boxed{\begin{aligned} \sigma_n &= \sigma_1 n_x^2 + \sigma_2 n_y^2 + \sigma_3 n_z^2 \\ n_x^2 &= 1 - n_y^2 - n_z^2 \\ \sigma_n &= (\sigma_2 - \sigma_1) n_y^2 + (\sigma_3 - \sigma_1) n_z^2 + \sigma_1 \end{aligned}}$$

$$\frac{\partial \sigma_n}{\partial n_y} = 2(\sigma_2 - \sigma_1) n_y \qquad \frac{\partial \sigma_n}{\partial n_z} = 2(\sigma_3 - \sigma_1) n_z$$

$$\sigma_2 \neq \sigma_1, \sigma_3 \neq \sigma_1 \Rightarrow \mathrm{n}_y = n_z = 0, n_x = 1$$

Three extreme values of shear stress

$$\tau_1 = \frac{|\sigma_1 - \sigma_2|}{2} \qquad \tau_2 = \frac{|\sigma_1 - \sigma_3|}{2} \qquad \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}$$

$$\tau_{\max} = \frac{1}{2} |\sigma_{\max} - \sigma_{\min}|$$

3.5. Principal Stress

Example 3.5.1

Find the principal stresses, principal normals, and the maximum shear stress if

$$[\boldsymbol{\sigma}] = \begin{bmatrix} -50 & 50 & -50 \\ 50 & 50 & 100 \\ -50 & 100 & 150 \end{bmatrix} \text{ (MPa)}$$

3.6. Plane Stress

For thin sheet and plates of uniform thickness, We assume that the stress components are confined to a plane => $\sigma_z=\tau_{xz}=\tau_{yz}=0$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0 \end{aligned}}$$

$$\boxed{\begin{aligned} T_x^{(n)} &= \sigma_x n_x + \tau_{xy} n_y \\ T_y^{(n)} &= \tau_{xy} n_x + \sigma_y n_y \end{aligned}}$$

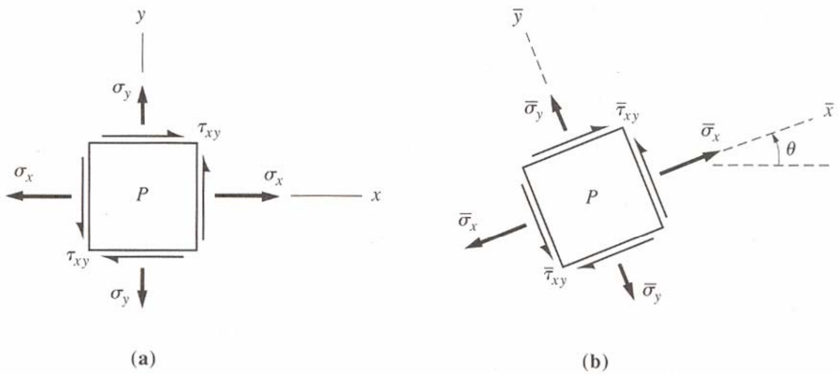


Figure 3.6.1 Positive components of plane stress at a point P in two different cartesian coordinate systems.

$$\begin{array}{llll} \bar{x}: & R_{11} = \cos\theta & R_{12} = \sin\theta & R_{13} = 0 \\ \bar{y}: & R_{21} = -\sin\theta & R_{22} = \cos\theta & R_{23} = 0 \\ \bar{z}: & R_{31} = 0 & R_{32} = 0 & R_{33} = 1 \end{array}$$

$$\boxed{\begin{aligned} \bar{\sigma}_x &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta \\ \bar{\sigma}_y &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta \\ \bar{\tau}_{xy} &= (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \end{aligned}}$$

3.6. Plane Stress

$$\bar{\sigma}_x = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$\bar{\sigma}_y = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$
$$\bar{\tau}_{xy} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad 2\sin\theta\cos\theta = \sin 2\theta$$

$$\bar{\sigma}_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\bar{\sigma}_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$\bar{\tau}_{xy} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

3.6. Plane Stress

- Mohr's circle

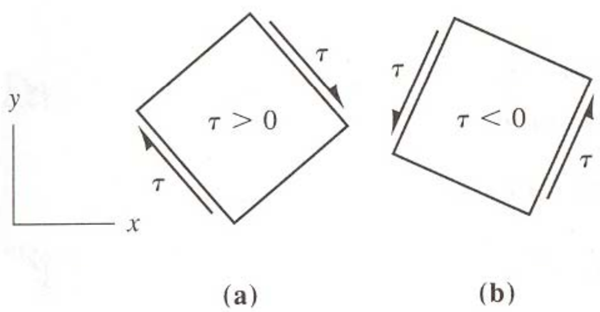


Figure 3.6.2 Shear stress sign convention for Mohr's circle only.

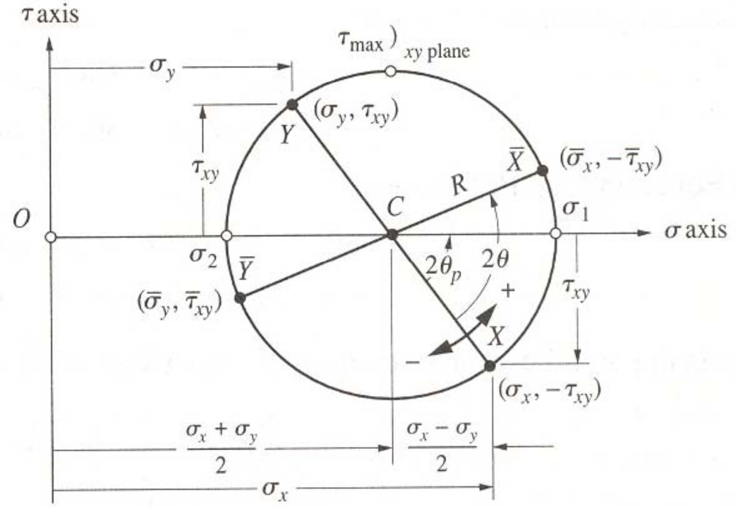


Figure 3.6.3 Mohr's circle for plane stress in the *xy* plane.

$$OC = \frac{\sigma_x + \sigma_y}{2}$$

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

$$R = \sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}$$

Principal Stress: $\sigma_1 = OC + R \qquad \sigma_2 = OC - R$

Stress Components of rotated element : $\bar{\sigma}_x = OC + R\cos(2\theta - 2\theta_p)$

$$-\bar{\tau}_{xy} = R\sin(2\theta - 2\theta_p)$$

$$R\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2} \qquad R\sin\theta_p = \tau_{xy}$$

3.6. Plane Stress

Example 3.6.1

A state of plane stress is represented on the element in Figure 3.6.4. Use a sketch of Mohr's circle to find:

(a) The principal stresses and principal directions

(b) The state of maximum in-plane shear stress

(c) The stress components on an element rotated 50 degrees counterclockwise

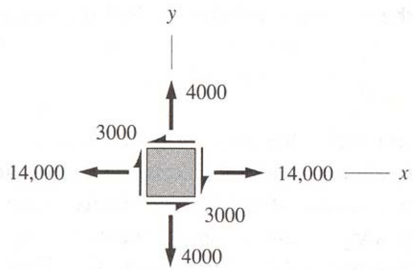


Figure 3.6.4 State of plane stress at a point: $\sigma_x = 14{,}000$ psi, $\sigma_y = 4{,}000$ psi, $\tau_{xy} = -3{,}000$ psi.

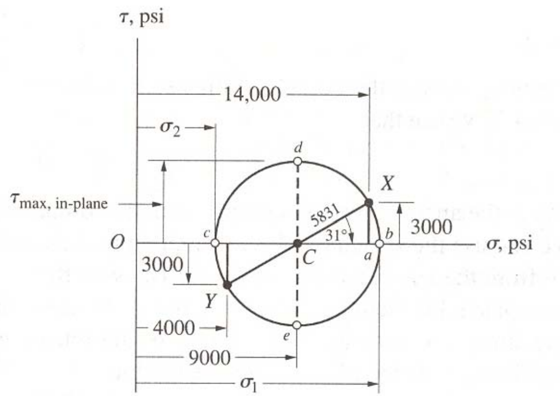


Figure 3.6.5 Mohr's circle for the state of stress in Figure 3.6.4.

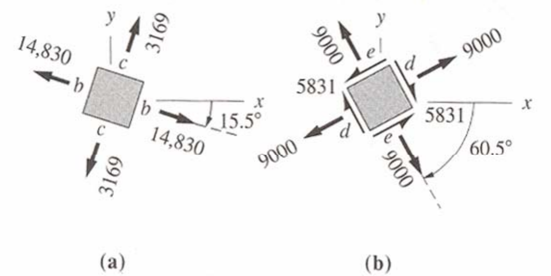


Figure 3.6.6 (a) State of principal stress (psi). (b) Planes of maximum shear (psi).

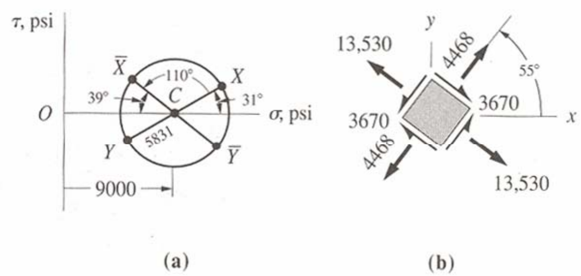


Figure 3.6.7 (a) Mohr's circle of Figure 3.6.5, used here to obtain the stress components in (b).

3.7. Strain

Strain : a measure of relative deformation

- we assume the deformation is so small,
- the changes in an object is ignored
- we assume the loads remain fixed in location and direction

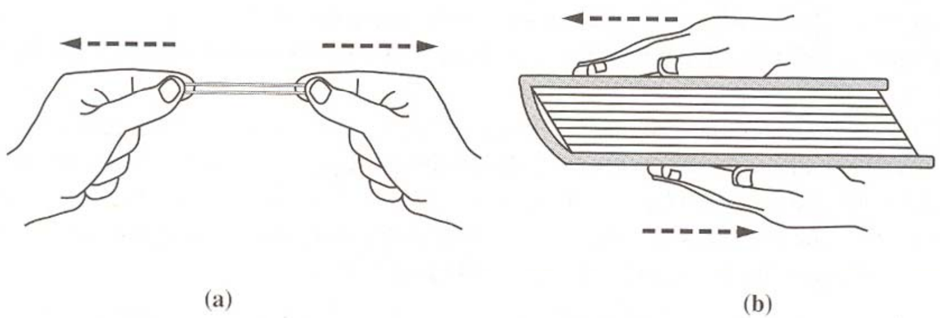


Figure 3.7.1 (a) Pulling a rubber band changes its length. (b) Shearing a book changes its shape.

3.7. Strain

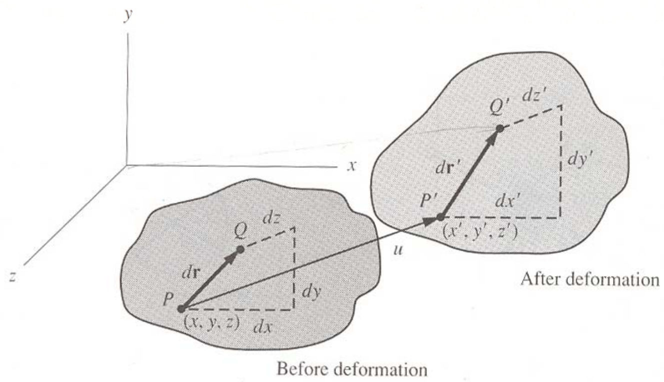


Figure 3.7.2 Directed material line segments of differential length, before and after deformation in the plane.

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

$$|d\mathbf{r}| = ds = \sqrt{dx^2 + dy^2 + dx^2}$$

$$\mathbf{n} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}$$

$$\mathbf{u} = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$$

$$x' = x + u(x, y, z) \quad y' = y + v(x, y, z) \quad z' = z + w(x, y, z)$$

$$d\mathbf{r}' = dx'\mathbf{i} + dy'\mathbf{j} + dz'\mathbf{k}$$

$$|d\mathbf{r}'| = ds' = \sqrt{dx'^2 + dy'^2 + dz'^2}$$

$$dx' = dx + du \qquad dy' = dy + dv \qquad dz' = dz + dw$$

3.7. Strain

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

$$dx' = dx + du$$

$$dy' = dy + dv$$

$$dz' = dz + dw$$

$$dx' = dx + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$dy' = dy + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

$$dz' = dy + \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

3.7.1 Normal Strain

The *normal strain* ε_n is the ratio of the change of its length to its original length

$$\varepsilon_n = \frac{ds' - ds}{ds} = \frac{ds'}{ds} - 1$$

$$ds' = \sqrt{ds^2 + 2\frac{\partial u}{\partial x}dx^2 + 2\frac{\partial v}{\partial y}dy^2 + 2\frac{\partial w}{\partial z}dz^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)dxdy + 2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dxdz + 2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)dydz}$$

$$\frac{ds'}{ds} = \sqrt{1 + 2\frac{\partial u}{\partial x}n_x^2 + 2\frac{\partial v}{\partial y}n_y^2 + 2\frac{\partial w}{\partial z}n_z^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)n_x n_y + 2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)n_x n_z + 2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)n_y n_z}$$

$$\frac{ds'}{ds} = 1 + \frac{1}{2}\left[2\frac{\partial u}{\partial x}n_x^2 + 2\frac{\partial v}{\partial y}v + 2\frac{\partial w}{\partial z}n_z^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)n_x n_y + 2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)n_x n_z + 2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)n_y n_z\right]$$

$$\varepsilon_n = \frac{\partial u}{\partial x}n_x^2 + \frac{\partial v}{\partial y}n_y^2 + \frac{\partial w}{\partial z}n_z^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)n_x n_y$$
$$+ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)n_x n_z + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)n_y n_z$$

3.7.1 Normal Strain

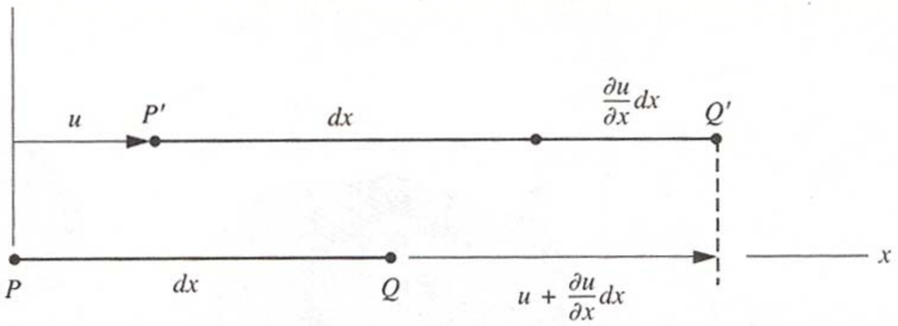


Figure 3.7.3 Normal strain in the x direction.

for ε_x, $n_x = 1, n_y = 0, n_z = 0$ and in similary

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_y = \frac{\partial w}{\partial z}$$

$$\varepsilon_x = \frac{[dx + (\partial u / \partial x)dx] - dx}{dx} = \frac{\partial u}{\partial x}$$

3.7.2 Shear Strain

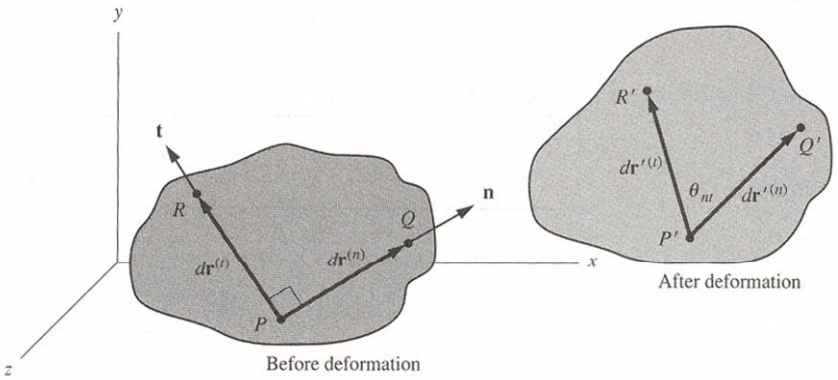


Figure 3.7.4 Change in angle between initially orthogonal directed differential line segments at a point.

$$d\mathbf{r}^{(n)} = dx^{(n)}\mathbf{i} + dy^{(n)}\mathbf{j} + dz^{(n)}\mathbf{k}$$

$$d\mathbf{r}^{(t)} = dx^{(t)}\mathbf{i} + dy^{(t)}\mathbf{j} + dz^{(t)}\mathbf{k}$$

$$\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k} = \frac{dx^{(n)}}{ds^{(n)}}\mathbf{i} + \frac{dy^{(n)}}{ds^{(n)}}\mathbf{j} + \frac{dz^{(n)}}{ds^{(n)}}\mathbf{k}$$

$$\mathbf{t} = t_x\mathbf{i} + t_y\mathbf{j} + t_z\mathbf{k} = \frac{dx^{(t)}}{ds^{(t)}}\mathbf{i} + \frac{dy^{(t)}}{ds^{(t)}}\mathbf{j} + \frac{dz^{(t)}}{ds^{(t)}}\mathbf{k}$$

$$dx^{(n)}dx^{(t)} + dy^{(n)}dy^{(t)} + dz^{(n)}dz^{(t)} = 0$$

$$d\mathbf{r}'^{(n)} \cdot d\mathbf{r}'^{(t)} = ds'^{(n)}ds'^{(t)} \cos\left(\frac{\pi}{2} - \gamma_{nt}\right) = ds'^{(n)}ds'^{(t)} \sin\gamma_{nt}$$

$$\gamma_{nt} = \frac{d\mathbf{r}'^{(n)} \cdot d\mathbf{r}'^{(t)}}{ds'^{(n)}ds'^{(t)}}$$

$$\begin{aligned} d\mathbf{r}'^{(n)} \cdot d\mathbf{r}'^{(t)} &= (dx'^{(n)}\mathbf{i} + dy'^{(n)}\mathbf{j} + dz'^{(n)}\mathbf{k}) \cdot (dx'^{(t)}\mathbf{i} + dy'^{(t)}\mathbf{j} + dz'^{(t)}\mathbf{k}) \\ &= dx'^{(n)}dx'^{(t)} + dy'^{(n)}dy'^{(t)} + dz'^{(n)}dz'^{(t)} \end{aligned}$$

$$\begin{aligned} d\mathbf{r}'^{(n)} \cdot d\mathbf{r}'^{(t)} = &\left(dx^{(n)} + \frac{\partial u}{\partial x}dx^{(n)} + \frac{\partial u}{\partial y}dy^{(n)} + \frac{\partial u}{\partial z}dz^{(n)}\right)\left(dx^{(t)} + \frac{\partial u}{\partial x}dx^{(t)} + \frac{\partial u}{\partial y}dy^{(t)} + \frac{\partial u}{\partial z}dz^{(t)}\right) \\ &+ \left(dy^{(n)} + \frac{\partial v}{\partial x}dx^{(n)} + \frac{\partial v}{\partial y}dy^{(n)} + \frac{\partial v}{\partial z}dz^{(n)}\right)\left(dy^{(t)} + \frac{\partial v}{\partial x}dx^{(t)} + \frac{\partial v}{\partial y}dy^{(t)} + \frac{\partial v}{\partial z}dz^{(t)}\right) \\ &+ \left(dz^{(n)} + \frac{\partial w}{\partial x}dx^{(n)} + \frac{\partial w}{\partial y}dy^{(n)} + \frac{\partial w}{\partial z}dz^{(n)}\right)\left(dz^{(t)} + \frac{\partial w}{\partial x}dx^{(t)} + \frac{\partial w}{\partial y}dy^{(t)} + \frac{\partial w}{\partial z}dz^{(t)}\right) \end{aligned}$$

3.7.2 Shear Strain

$$d\mathbf{r}'^{(n)}\cdot d\mathbf{r}'^{(t)} = dx^{(n)}dx^{(t)} + dy^{(n)}dy^{(t)} + dz^{(n)}dz^{(t)} + 2\frac{\partial u}{\partial x}dx^{(n)}dx^{(t)} + 2\frac{\partial v}{\partial y}dy^{(n)}dy^{(t)}$$
$$+2\frac{\partial w}{\partial z}dz^{(n)}dz^{(t)} + \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)(dx^{(n)}dy^{(t)} + dx^{(t)}dy^{(n)})$$
$$+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)(dx^{(n)}dz^{(t)} + dx^{(t)}dz^{(n)}) + \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)(dy^{(n)}dz^{(t)} + dy^{(t)}dz^{(n)})$$

$$\gamma_{nt} = 2\frac{\partial u}{\partial x}\frac{dx^{(n)}}{ds^{(n)}}\frac{dx^{(t)}}{ds^{(t)}} + 2\frac{\partial v}{\partial y}\frac{dy^{(n)}}{ds^{(n)}}\frac{dy^{(t)}}{ds^{(t)}} + 2\frac{\partial w}{\partial z}\frac{dz^{(n)}}{ds^{(n)}}\frac{dz^{(t)}}{ds^{(t)}} + \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{dx^{(n)}}{ds^{(n)}}\frac{dy^{(t)}}{ds^{(t)}} + \frac{dx^{(t)}}{ds^{(t)}}\frac{dy^{(n)}}{ds^{(n)}}\right)$$
$$+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\left(\frac{dx^{(n)}}{ds^{(n)}}\frac{dz^{(t)}}{ds^{(t)}} + \frac{dx^{(t)}}{ds^{(t)}}\frac{dz^{(n)}}{ds^{(n)}}\right) + \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\left(\frac{dy^{(n)}}{ds^{(n)}}\frac{dz^{(t)}}{ds^{(t)}} + \frac{dy^{(t)}}{ds^{(t)}}\frac{dz^{(n)}}{ds^{(n)}}\right)$$

$$\gamma_{nt} = 2\frac{\partial u}{\partial x}n_x t_x + 2\frac{\partial v}{\partial y}n_y t_y + 2\frac{\partial w}{\partial z}n_z t_z + \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)(n_x t_y + t_x n_y)$$
$$+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)(n_x t_z + t_x n_z) + \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)(n_y t_z + t_y n_z)$$

- strain-displacement relation

$$\boxed{\begin{array}{lll} \varepsilon_x = \dfrac{\partial u}{\partial x} & \varepsilon_y = \dfrac{\partial v}{\partial y} & \varepsilon_z = \dfrac{\partial w}{\partial z} \\ \gamma_{xy} = \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} & \gamma_{xz} = \dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x} & \gamma_{yz} = \dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y} \end{array}}$$

$$\gamma_{xy} = \gamma_{yx} \qquad \gamma_{xz} = \gamma_{zx} \qquad \gamma_{yz} = \gamma_{zy}$$

3.7.2 Shear Strain

- strain transformation formulas

$$\boxed{\begin{aligned}\varepsilon_n &= \varepsilon_x n_x^2 + \varepsilon_y n_y^2 + \varepsilon_z n_z^2 + 2\left(\frac{\gamma_{xy}}{2}\right) n_x n_y + 2\left(\frac{\gamma_{xz}}{2}\right) n_x n_z + 2\left(\frac{\gamma_{yz}}{2}\right) n_y n_z \\ \frac{\gamma_{nt}}{2} &= \varepsilon_x n_x t_x + \varepsilon_y n_y t_y + \varepsilon_z n_z t_z + \left(\frac{\gamma_{xy}}{2}\right)(n_x t_y + t_x n_y) \\ &\quad + \left(\frac{\gamma_{xz}}{2}\right)(n_x t_z + t_x n_z) + \left(\frac{\gamma_{yz}}{2}\right)(n_y t_z + t_y n_z)\end{aligned}} \qquad (\mathbf{n}\cdot\mathbf{t} = 0)$$

$$\begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z \end{bmatrix}$$

$$\alpha = \tan^{-1}\frac{(\partial v/\partial x)\,dx}{dx} \cong \frac{\partial v}{\partial x} \qquad \beta = \tan^{-1}\frac{(\partial u/\partial y)\,dy}{dy} \cong \frac{\partial u}{\partial y}$$

$$\gamma_{xy} = \alpha + \beta = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

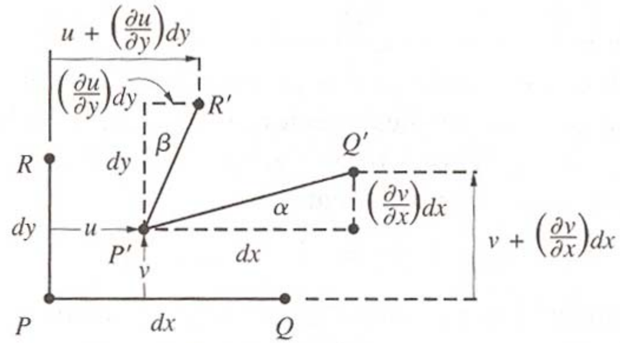


Figure 3.7.5 Shear strain between the x and y directions.

is engineering shear strain.

We define shear strain to be one-half the decrease in angle. $\varepsilon_{nt} = \dfrac{\gamma_{nt}}{2}$

$$\begin{aligned}\varepsilon_n &= \varepsilon_x n_x^2 + \varepsilon_y n_y^2 + \varepsilon_z n_z^2 + \varepsilon_{xy} n_x n_y + \varepsilon_{xz} n_x n_z + \varepsilon_{yz} n_y n_z \\ \varepsilon_{nt} &= \varepsilon_x n_x t_x + \varepsilon_y n_y t_y + \varepsilon_z n_z t_z + \varepsilon_{xy}(n_x t_y + t_x n_y) \\ &\quad + \varepsilon_{xz}(n_x t_z + t_x n_z) + \varepsilon_{yz}(n_y t_z + t_y n_z)\end{aligned} \qquad (\mathbf{n}\cdot\mathbf{t} = 0)$$

3.7.2 Shear Strain

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix}$$

$$\frac{\varepsilon_1 - \varepsilon_2}{2} \qquad \frac{\varepsilon_1 - \varepsilon_3}{2} \qquad \frac{\varepsilon_2 - \varepsilon_3}{2}$$

$$\text{Max. shear strain} = \frac{|\varepsilon_{\max} - \varepsilon_{\min}|}{2}$$

$$\gamma_{\max} = |\varepsilon_{\max} - \varepsilon_{\min}|$$

3.8 Volumetric Strain

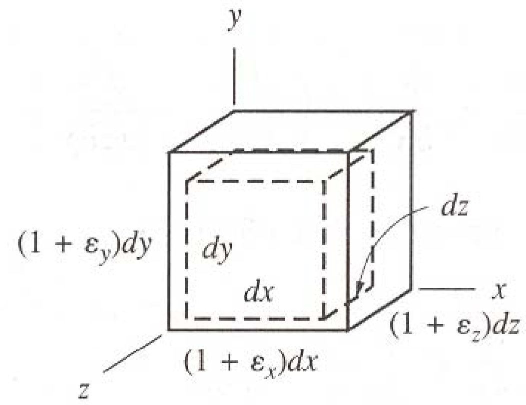


Figure 3.8.1 Change in volume of a differential element.

$$dV_0 = dxdydz$$

$$dV = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)dV_0$$

$$e = \frac{\text{change in volume}}{\text{original volume}} = \frac{dV - dV_o}{dV_o} = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

3.9 Compatibility Conditions

The stratins in a solid must be consistent or compatible with displacements of the solid.

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial^2}{\partial y \partial z}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^3 u}{\partial y \partial z \partial x} = \frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right)$$

$$\frac{\partial^2 u}{\partial y \partial z} = \left(\frac{1}{2}\right)\left(\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z \partial y}\right)$$

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{1}{2}\left(\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z \partial y}\right)\right]$$

By adding and subtracting $\partial^2 v / \partial x \partial z$, we can write $\partial^2 u / \partial y \partial z$ as

$$\begin{aligned} \frac{\partial^2 u}{\partial y \partial z} &= \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 v}{\partial z \partial x} \\ &= \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{\partial^2 v}{\partial z \partial x} = \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial^2 v}{\partial z \partial x} \end{aligned}$$

3.9 Compatibility Conditions

$$\frac{\partial^2 u}{\partial z \partial y} = \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y \partial x}$$
$$= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial \gamma_{xz}}{\partial z} - \frac{\partial^2 w}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z \partial y} = \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$
$$= \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right)$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x \partial y \partial x}$$
$$= \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 \partial y} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}$$

3.9 Compatibility Conditions

6 compatibility equations

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} \qquad \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} \qquad \frac{\partial^2 \gamma_{xz}}{\partial z \partial x} = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y}\right)$$

3.10 Plane Strain

Plane strain : deformation is confined to the xy plane

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

$$\begin{bmatrix} \varepsilon_x & \gamma_{xy} & 0 \\ \gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

strain – displacement relationships

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}$$

strain transformation relationships

$$\begin{aligned} \bar{\varepsilon}_x &= \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta \\ \bar{\varepsilon}_y &= \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - \gamma_{xy} \sin\theta \cos\theta \\ \bar{\gamma}_{xy} &= 2\left(\varepsilon_y - \varepsilon_x\right) \sin\theta \cos\theta + \gamma_{xy} \left(\cos^2\theta - \sin^2\theta\right) \end{aligned}$$

compatibility condition

$$\boxed{\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}}$$

3.10 Plane Strain

Example 3.10.1

Consider the following plane strain fields: (a) $\varepsilon_x = 0,\ \varepsilon_y = 0,\ \gamma_{xy} = 10^{-4}x$, and (b) $\varepsilon_x = 0,\ \varepsilon_y = 0,$

For both fields, the units are centimeters. Sketch the accompanying displacement fields by applying the strains to the initially square elements of the coarse four-by-four grid of Figure 3.10.1

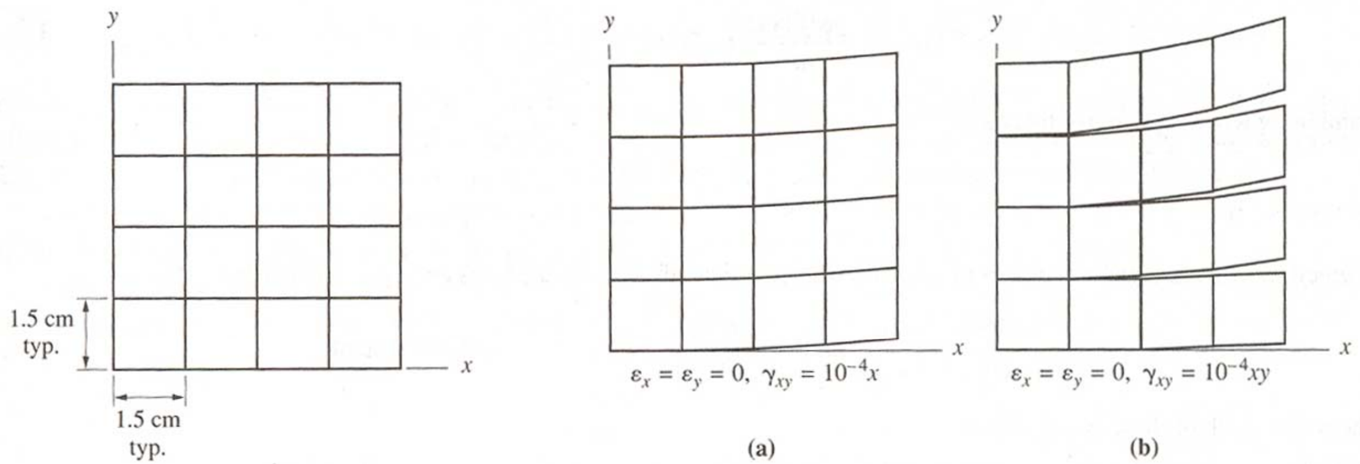


Figure 3.10.1 Square region divided into sixteen elements to which given strains are to be applied.

Figure 3.10.2 Deformation of the grid of initially square elements. (a) Compatible strain field. (b) Noncompatible strain field.

3.10 Plane Strain

Example 3.10.2

Calculate the displacements corresponding to the following two-dimensional plane strain field:

$$\varepsilon_x = a\left(-3x^2 + 7y^2\right) \qquad \varepsilon_y = a\left(x^2 - 5y^2\right) \qquad \gamma_{xy} = 16axy$$

Where a is a nonzero constant.

3.10 Plane Strain

Example 3.10.3

Find the displacements corresponding to the following strains

$$\varepsilon_x = a\left(x^2 + y^2\right) \qquad \varepsilon_y = a\left(x^2 + y^2\right) \qquad \gamma_{xy} = -8axy$$

Where a is a nonzero constant.

3.11 Stress-Strain Equations: Isotropic Elastic Materials

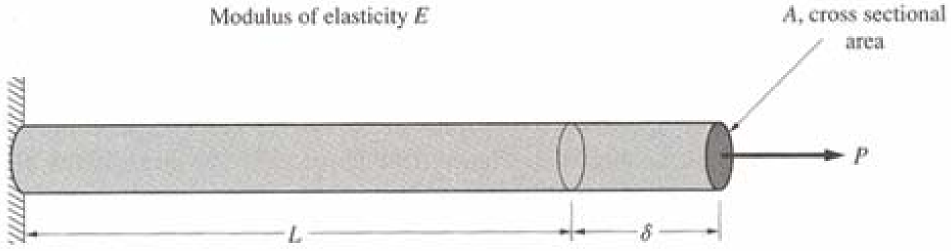


Figure 3.11.1 Deformation of a uniform elastic rod under axial load.

$$\varepsilon = \frac{\delta}{L} \qquad \sigma = \frac{P}{A}$$

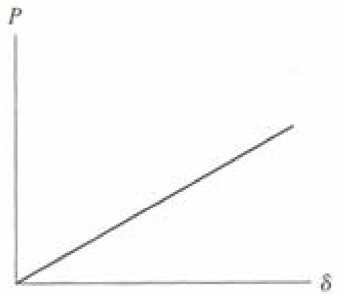


Figure 3.11.2 Linear load versus deflection behavior of an elastic rod.

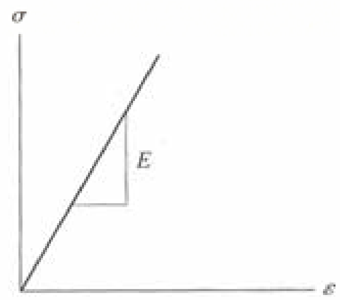
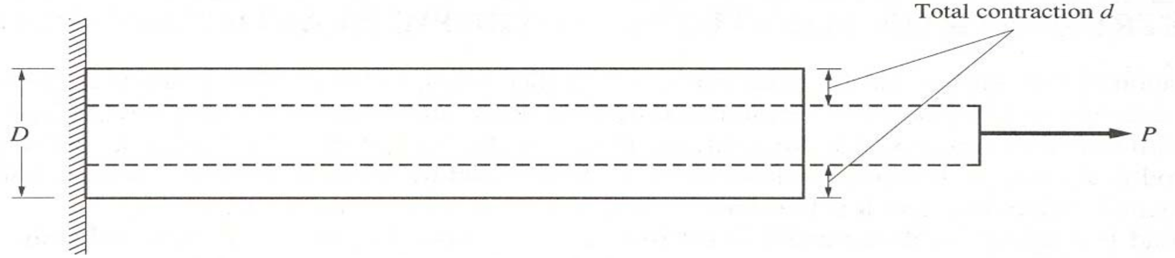


Figure 3.11.3 Stress-strain curve for a linearly elastic material.

The slope of the stress-strain diagram is the *modulus of elasticity*, *Young's modulus*

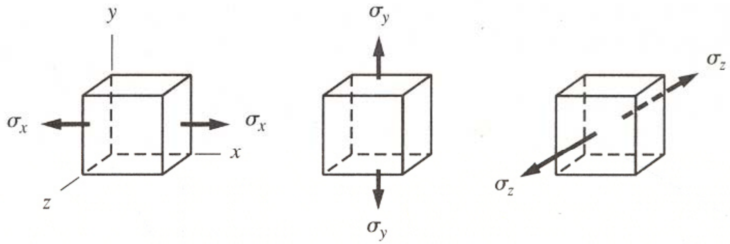
3.11 Stress-Strain Equations: Isotropic Elastic Materials

Poisson effect : a tensile test specimen stretched in the axial direction contracts laterally; if axially compressed, it expands laterally.



$$\varepsilon_{\perp} = -\frac{d}{D}$$

$$\nu = -\frac{\varepsilon_{\perp}}{\varepsilon}$$



$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \varepsilon_y = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E} \qquad \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

(a)

$$\varepsilon_x = -\nu\varepsilon_y = -\nu\frac{\sigma_y}{E} \qquad \varepsilon_y = \frac{\sigma_y}{E} \qquad \varepsilon_z = -\nu\varepsilon_y = -\nu\frac{\sigma_y}{E}$$

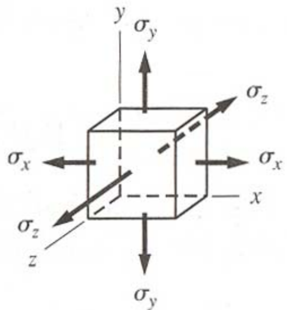
(b)

$$\varepsilon_x = -\nu\varepsilon_z = -\nu\frac{\sigma_z}{E} \qquad \varepsilon_y = -\nu\varepsilon_z = -\nu\frac{\sigma_z}{E} \qquad \varepsilon_z = \frac{\sigma_z}{E}$$

(c)

Figure 3.11.5 Normal strains accompanying uniaxial stress in the x, y, and z directions.

Principle of superposition



$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

Figure 3.11.6 Normal strains accompanying a triaxial state of stress.

3.11 Stress-Strain Equations: Isotropic Elastic Materials

$\tau_{xy} = G\gamma_{xy}$ G: shear modulus

Suppose a state of plane stress

$\sigma_x = \sigma_y = 0 \qquad \tau_{xy} = \tau$

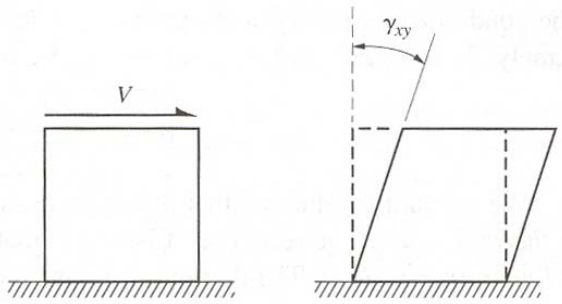


Figure 3.11.7 Pure shear loading.

$$\varepsilon_x = \varepsilon_y = 0 \qquad \gamma_{xy} = \frac{\tau}{G}$$

at 45 degree from xy axes,

$$\bar{\sigma}_x = (0)\cos^2 45 + (0)\sin^2 45 + 2\tau \sin 45 \cos 45 = \tau$$

$$\bar{\sigma}_y = (0)\sin^2 45 + (0)\cos^2 45 - 2\tau \sin 45 \cos 45 = -\tau$$

$$\bar{\tau}_{xy} = (0)\sin 45 \cos 45 + \tau(\cos^2 45 - \sin^2 45) = 0$$

since the material is isotropic

$$\bar{\varepsilon}_x = \frac{\tau}{E} - \nu\frac{(-\tau)}{E} = \frac{1+\nu}{E}\tau \qquad \bar{\varepsilon}_y = \frac{(-\tau)}{E} - \nu\frac{\tau}{E} = -\frac{1+\nu}{E}\tau \qquad \bar{\gamma}_{xy} = 0$$

$$\bar{\varepsilon}_x = (0)\cos^2 45 + (0)\sin^2 45 + \left(\frac{\tau}{2G}\right)\sin 45 \cos 45 = \frac{\tau}{2G}$$

$$\bar{\varepsilon}_y = (0)\sin^2 45 + (0)\cos^2 45 - \left(\frac{\tau}{2G}\right)\sin 45 \cos 45 = -\frac{\tau}{2G}$$

$$\bar{\gamma}_{xy} = 2(0)\sin 45 \cos 45 + \left(\frac{\tau}{2G}\right)(\cos^2 45 - \sin^2 45) = 0$$

$$\bar{\varepsilon}_x = \tau/_{2G} \text{ and } \bar{\varepsilon}_x = (1+\nu)\tau/_{E}$$

$$G = \frac{E}{2(1+\nu)}$$

3.11 Stress-Strain Equations: Isotropic Elastic Materials

Thermal strain

$$\varepsilon_T = \alpha T$$

The strain-stress equations

$$
\begin{aligned}
\varepsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} + \alpha T & \gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\varepsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} + \alpha T & \gamma_{xz} &= \frac{\tau_{xz}}{G} \\
\varepsilon_z &= -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} + \alpha T & \gamma_{yz} &= \frac{\tau_{yz}}{G}
\end{aligned}
$$

The stress- strain relationships

$$
\begin{aligned}
\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\left(\varepsilon_y + \varepsilon_z\right)\right] - \frac{E\alpha T}{1-2\nu} & \tau_{xy} &= G\gamma_{xy} \\
\sigma_y &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu\left(\varepsilon_x + \varepsilon_z\right)\right] - \frac{E\alpha T}{1-2\nu} & \tau_{xz} &= G\gamma_{xz} \\
\sigma_z &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu\left(\varepsilon_x + \varepsilon_y\right)\right] - \frac{E\alpha T}{1-2\nu} & \tau_{yz} &= G\gamma_{yz}
\end{aligned}
$$

The dilatation

$$e = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) + 3\alpha T$$

$$e = \frac{p}{K} + 3\alpha T \qquad p = K(e-3\alpha T)$$

$$K = \frac{E}{3(1-2\nu)}$$: the bulk modulus of elasticity

3.12 The Plane Stress Problem

For plane stress in the xy plane, $\sigma z = \tau xz = \tau yz = 0$

The strain-stress equations

$$\begin{aligned}
\varepsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \alpha T \\
\varepsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} + \alpha T \\
\varepsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y) + \alpha T \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned}$$

$$\varepsilon_z = \frac{-[\nu(\varepsilon_x + \varepsilon_y) + (1+\nu)\alpha T]}{(1-\nu)}$$

The stress-strain relationships

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) - \frac{E\alpha T}{1-\nu} \\
\sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) - \frac{E\alpha T}{1-\nu} \\
\tau_{xy} &= G\gamma_{xy}
\end{aligned}$$

The equilibrium equations in terms of the displacements

$$2\frac{\partial^2 u}{\partial x^2} + (1-\nu)\frac{\partial^2 u}{\partial y^2} + (1+\nu)\frac{\partial^2 v}{\partial x \partial y} + 2\frac{1-\nu^2}{E}b_x = 2\alpha\frac{\partial T}{\partial x}$$

$$2\frac{\partial^2 v}{\partial y^2} + (1-\nu)\frac{\partial^2 v}{\partial x^2} + (1+\nu)\frac{\partial^2 u}{\partial x \partial y} + 2\frac{1-\nu^2}{E}b_y = 2\alpha\frac{\partial T}{\partial y}$$

3.12 The Plane Stress Problem

the stress compatibility equation

$$\frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial x^2} - \nu\left(\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2}\right) - 2(1+\nu)\frac{\partial^2\tau_{xy}}{\partial x\partial y} = -E\alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial\tau_{xy}}{\partial y} = -\frac{\partial\sigma_x}{\partial x} - b_x \qquad \frac{\partial\tau_{xy}}{\partial x} = -\frac{\partial\sigma_y}{\partial y} - b_y$$

$$2\frac{\partial^2\tau_{xy}}{\partial x\partial y} = -\frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial^2\sigma_y}{\partial y^2} - \frac{\partial b_x}{\partial x} - \frac{\partial b_y}{\partial y}$$

$$\nabla^2(\sigma_x + \sigma_y) = -E\alpha\nabla^2 T - (1+\nu)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right)$$

Three stresses:	$\sigma_x,\ \sigma_y,\ \tau_{xy}$	Two equilibrium equations :	Equations 3.6.2
Three strains:	$\varepsilon_x,\ \varepsilon_y,\ \gamma_{xy}$	Three strain–displacement equations:	Equations 3.10.2
Two displacements:	$u,\ v$	Three stress–strain equations:	Equations 3.12.2

3.12 The Plane Stress Problem

We ca satisfy the equilibrium equations [3.6.2] by introducing the Airy stress function such that

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \qquad \tau_x = -\frac{\partial^2 \phi}{\partial x \partial y}$$

We have the 4th order PDE without the body force and the temperature loading,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}\right) = 0$$

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

3.12 The Plane Stress Problem

Example 3.12.1

Consider the following plane stress field:

$$\sigma_x = -2\bar{\tau} h_1 h_2 \frac{g(x)}{h(y)^3} \qquad \sigma_y = 0 \qquad \tau_{xy} = \bar{\tau}\frac{h_1 h_2}{h(y)^2}$$

where

$$g(x) = (h_2 - h_1)\frac{x}{l} - h_2\frac{x_1}{l} \qquad h(y) = h_2 - (h_2 - h_1)\frac{y}{l}$$

Show that these are the stress components within the trapezoidal shear panel of section 2.5, which is illustrated in Figure 3.12.1

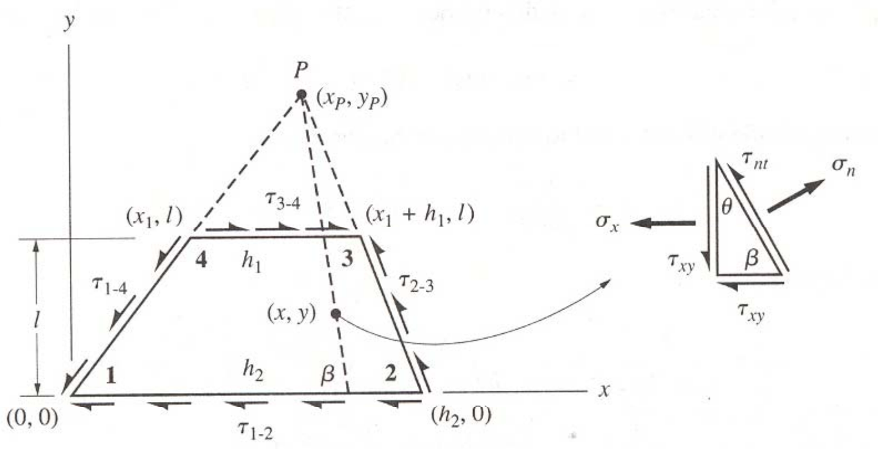


Figure 3.12.1 Trapezoidal plane stress region.

3.12 The Plane Stress Problem

Example 3.12.2

Show that the plane stress field assumed for the trapezoidal shear panel of section 2.5 satisfies equilibrium but is compatible only if the panel is a parallelogram. Assume zero body forces and no thermal strain.

3.12 The Plane Stress Problem

Example 3.12.3

The materially isotropic beam of Figure 3.12.2 has a uniform shear traction applied to its upper surface. Use the Airy stress function to find the stresses in the beam and the displacements of point A at the free end. Assume $2h < 10L.$

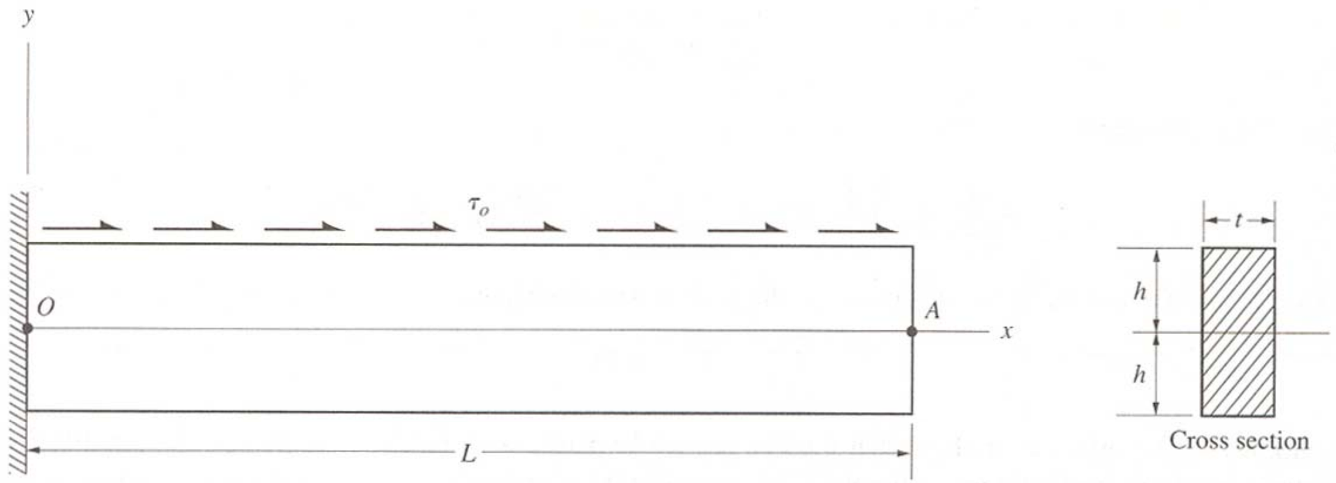


Figure 3.12.2 Beam with uniform shear traction on upper surface.

3.13 Saint-Venant's Principle

In 1855, B. de Saint-Venant proposed a principle that can be stated as follows:

If some distribution of forces acting on a portion of the surface of a body is replaced by a different distribution of forces acting on the same portion of the body, then the effects of two different distributions on the parts of the body sufficiently far removed from the region of application of the forces are essentially the same, provided that the two distributions of forces are statically equivalent.

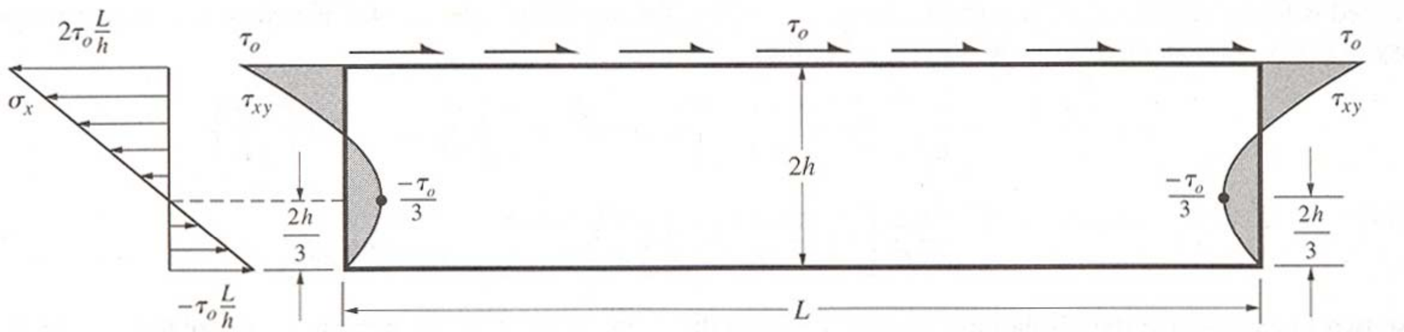


Figure 3.13.1 Surface tractions on the beam of Figure 4.6.1.

3.14 Strain Energy

Assuming constant, uniform temperature

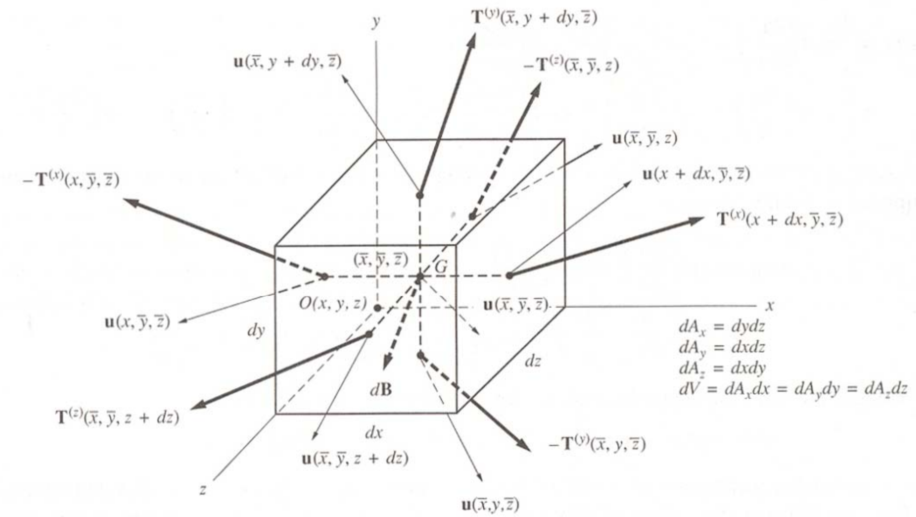


Figure 3.14.1 Surface tractions, body force, and the displacement vectors and their points of application on a differential material element.

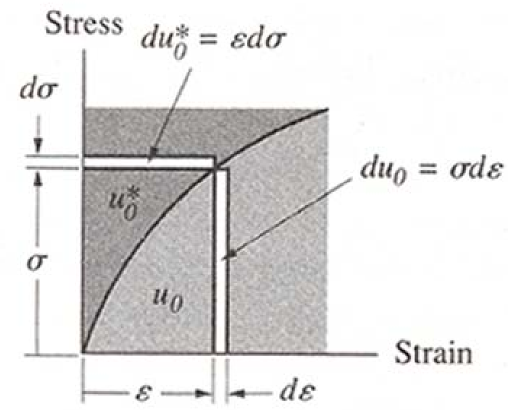


Figure 3.14.2 Elastic stress–strain relationship showing strain energy density (u_o) and complementary strain energy (u_o^*).

Work done By a force **F** is

$$dW_0 = \sum \mathbf{F} \cdot d\mathbf{u}$$

$$\begin{aligned} dW_o = {} & \mathbf{T}^{(x)}(x + dx, \bar{y}, \bar{z}) dA_x \cdot d\mathbf{u}(x + dx, \bar{y}, \bar{z}) - \mathbf{T}^{(x)}(x, \bar{y}, \bar{z})\, dA_x \cdot d\mathbf{u}(x, \bar{y}, \bar{z}) \\ & + \mathbf{T}^{(y)}(\bar{x}, y + dy, \bar{z}) dA_y \cdot d\mathbf{u}(\bar{x}, y + dy, \bar{z}) - \mathbf{T}^{(y)}(\bar{x}, y, \bar{z}) dA_y \cdot d\mathbf{u}(\bar{x}, y, \bar{z}) \\ & + \mathbf{T}^{(z)}(\bar{x}, \bar{y}, z + dz) dA_z \cdot d\mathbf{u}(\bar{x}, \bar{y}, z + dz) - \mathbf{T}^{(z)}(\bar{x}, \bar{y}, z) dA_z \cdot d\mathbf{u}(\bar{x}, \bar{y}, z) + d\mathbf{B} \cdot d\mathbf{u}(\bar{x}, \bar{y}, \bar{z}) \end{aligned}$$

3.14 Strain Energy

$$
\begin{aligned}
dW_o = &\left(\mathbf{T}^{(x)} + \frac{\partial \mathbf{T}^{(x)}}{\partial x} dx\right) dA_x \cdot d\left(\mathbf{u} + \frac{\partial \mathbf{u}}{\partial x} dx\right) - \mathbf{T}^{(x)} dA_x \cdot d\mathbf{u} \\
&+ \left(\mathbf{T}^{(y)} + \frac{\partial \mathbf{T}^{(y)}}{\partial y} dy\right) dA_y \cdot d\left(\mathbf{u} + \frac{\partial \mathbf{u}}{\partial y} dy\right) - \mathbf{T}^{(y)} dA_y \cdot d\mathbf{u} \\
&+ \left(\mathbf{T}^{(z)} + \frac{\partial \mathbf{T}^{(z)}}{\partial z} dz\right) dA_z \cdot d\left(\mathbf{u} + \frac{\partial \mathbf{u}}{\partial z} dz\right) - \mathbf{T}^{(z)} dA_z \cdot d\mathbf{u} + d\mathbf{B} \cdot d\mathbf{u}
\end{aligned}
$$

$$
dW_o = \left(\frac{\partial \mathbf{T}^{(x)}}{\partial x} + \frac{\partial \mathbf{T}^{(y)}}{\partial y} + \frac{\partial \mathbf{T}^{(z)}}{\partial z} + \mathbf{b}\right) \cdot d\mathbf{u} dV + \left[\mathbf{T}^{(x)} \cdot d\left(\frac{\partial \mathbf{u}}{\partial x}\right) + \mathbf{T}^{(y)} \cdot d\left(\frac{\partial \mathbf{u}}{\partial y}\right) + \mathbf{T}^{(z)} \cdot d\left(\frac{\partial \mathbf{u}}{\partial z}\right)\right] dV + \cdot
$$

Assuming the element is in equilibrium, the first term will be zero

$$
\frac{\partial \mathbf{T}^{(x)}}{\partial x} + \frac{\partial \mathbf{T}^{(y)}}{\partial y} + \frac{\partial \mathbf{T}^{(z)}}{\partial z} + \mathbf{b} = 0 \quad \text{(eq 3.3.3)}
$$

the remain terms will be

$$
\begin{aligned}
dw_o = &\left[\sigma_x d\left(\frac{\partial u}{\partial x}\right) + \tau_{xy} d\left(\frac{\partial v}{\partial x}\right) + \tau_{xz} d\left(\frac{\partial w}{\partial x}\right)\right] \\
&+ \left[\tau_{yx} d\left(\frac{\partial u}{\partial y}\right) + \sigma_y d\left(\frac{\partial v}{\partial y}\right) + \tau_{yz} d\left(\frac{\partial w}{\partial y}\right)\right] + \left[\tau_{zx} d\left(\frac{\partial u}{\partial z}\right) + \tau_{zy} d\left(\frac{\partial v}{\partial z}\right) + \sigma_z d\left(\frac{\partial w}{\partial z}\right)\right]
\end{aligned}
$$

3.14 Strain Energy

$$dw_o = \sigma_x d\left(\frac{\partial u}{\partial x}\right) + \sigma_y d\left(\frac{\partial v}{\partial y}\right) + \sigma_z d\left(\frac{\partial w}{\partial z}\right)$$
$$+ \tau_{xy} d\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + \tau_{xz} d\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tau_{yz} d\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

use strain-displacement relationship

$$dw_0 = \sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z + \tau_{xy} d\gamma_{xy} + \tau_{yz} d\gamma_{yz} + \tau_{xz} d\gamma_{xz}$$

In simmilary, the increment of complementary work is

$$dw_o^* = \varepsilon_x d\sigma_x + \varepsilon_y d\sigma_y + \varepsilon_z d\sigma_z + \gamma_{xy} d\tau_{xy} + \gamma_{xz} d\tau_{xz} + \gamma_{yz} d\tau_{yz}$$

all of the work done on an elastic body is

stored as internal energy, or *strain energy*

•strain energy density

$$u_o = u_o(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$$

$$U = \iiint_V u_o dV \qquad du_o = dw_o$$

$$du_o = \frac{\partial u_o}{\partial \varepsilon_x} d\varepsilon_x + \frac{\partial u_o}{\partial \varepsilon_y} d\varepsilon_y + \frac{\partial u_o}{\partial \varepsilon_z} d\varepsilon_z + \frac{\partial u_o}{\partial \gamma_{xy}} d\gamma_{xy} + \frac{\partial u_o}{\partial \gamma_{xz}} d\gamma_{xz} + \frac{\partial u_o}{\partial \gamma_{yz}} d\gamma_{yz}$$

3.14 Strain Energy

$$\left(\sigma_x - \frac{\partial u_o}{\partial \varepsilon_x}\right)d\varepsilon_x + \left(\sigma_y - \frac{\partial u_o}{\partial \varepsilon_y}\right)d\varepsilon_y + \left(\sigma_z - \frac{\partial u_o}{\partial \varepsilon_z}\right)d\varepsilon_z$$
$$+ \left(\tau_{xy} - \frac{\partial u_o}{\partial \gamma_{xy}}\right)d\gamma_{xy} + \left(\tau_{xz} - \frac{\partial u_o}{\partial \gamma_{xz}}\right)d\gamma_{xz} + \left(\tau_{yz} - \frac{\partial u_o}{\partial \gamma_{yz}}\right)d\gamma_{yz} = 0$$

$$\sigma_x = \frac{\partial u_o}{\partial \varepsilon_x} \qquad \sigma_y = \frac{\partial u_o}{\partial \varepsilon_y} \qquad \sigma_z = \frac{\partial u_o}{\partial \varepsilon_z}$$
$$\tau_{xy} = \frac{\partial u_o}{\partial \gamma_{xy}} \qquad \tau_{xz} = \frac{\partial u_o}{\partial \gamma_{xz}} \qquad \tau_{yz} = \frac{\partial u_o}{\partial \gamma_{yz}}$$

•Complementary strain energy density

$$u_o^* = \sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz} - u_o$$

$$u_o^* = u_o^*(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$$

$$du_o^* = d(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}) - dw_o$$

$$du_o^* = dw_o^*$$

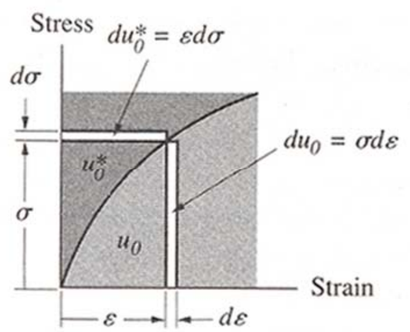


Figure 3.14.2 Elastic stress–strain relationship showing strain energy density (u_o) and complementary strain energy (u_o^*).

3.14 Strain Energy

$$\varepsilon_x = \frac{\partial u_o^*}{\partial \sigma_x} \qquad \varepsilon_y = \frac{\partial u_o^*}{\partial \sigma_y} \qquad \varepsilon_z = \frac{\partial u_o^*}{\partial \sigma_z}$$

$$\gamma_{xy} = \frac{\partial u_o^*}{\partial \tau_{xy}} \qquad \gamma_{xz} = \frac{\partial u_o^*}{\partial \tau_{xz}} \qquad \gamma_{yz} = \frac{\partial u_o^*}{\partial \tau_{yz}}$$

$$U^* = \iiint_V u_o^* dV$$

for linear elastic materials,

$$\boxed{u_o = u_o^* = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})}$$

3.14 Strain Energy

Example 3.14.1

Show that the strain-stress equations, Equations 3.11.4 (with T=0), follow from substituting the strain energy density, Equation 3.14.14, into Equation 3.14.12.

$$\boxed{\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} + \alpha T & \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \varepsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} + \alpha T & \gamma_{xz} &= \frac{\tau_{xz}}{G} \\ \varepsilon_z &= -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} + \alpha T & \gamma_{yz} &= \frac{\tau_{yz}}{G} \end{aligned}}$$

Eq. 3.11.4

$$\varepsilon_x = \frac{\partial u_o^*}{\partial \sigma_x} \qquad \varepsilon_y = \frac{\partial u_o^*}{\partial \sigma_y} \qquad \varepsilon_z = \frac{\partial u_o^*}{\partial \sigma_z}$$

$$\gamma_{xy} = \frac{\partial u_o^*}{\partial \tau_{xy}} \qquad \gamma_{xz} = \frac{\partial u_o^*}{\partial \tau_{xz}} \qquad \gamma_{yz} = \frac{\partial u_o^*}{\partial \tau_{yz}}$$

Eq. 3.14.12

$$u_o = u_o^* = \frac{1}{2}\frac{E}{(1+\nu)(1\text{–}2\nu)}\left[(1-\nu)(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + 2\nu(\varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z)\right] + \frac{1}{2}G(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2)$$

3.15 Static Failure Theories

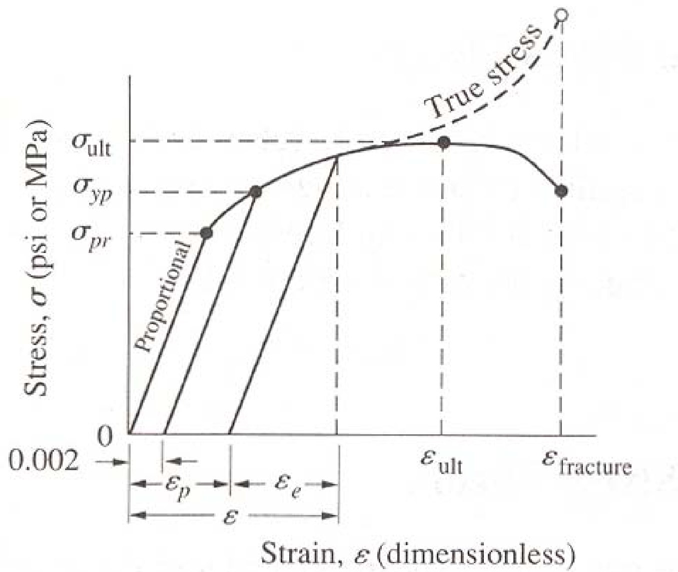


Figure 3.15.1 A typical tensile stress–strain diagram for ductile metal.

3.15 Static Failure Theories

3.15.1 Maximum Normal Stress Theory

$$\sigma_{\max} = \sigma_{yp}$$

3.15.2 Maximum Shear Stress Theory

$$2\tau_{\max} = \sigma_{yp}$$

$$\sigma_{\max} - \sigma_{\min} = \sigma_{yp}$$

3.15 Static Failure Theories

3.15.3 Maximum Distortion Energy Theory

$$\sigma_x = \sigma_x' + p \qquad \sigma_y = \sigma_y' + p \qquad \sigma_z = \sigma_z' + p$$ (p: the hydrostatic stress)

Substituting the normal stress eqs in strain energy density formula,

$$u_o = u_v + u_d$$

- The strain energy density due to volume change

$$u_v = \frac{1}{2K} p^2$$

- The distrotion strain energy density

$$u_d = \frac{1}{2E}\left(\sigma_x'^2 + \sigma_y'^2 + \sigma_z'^2\right) - \frac{\nu}{E}\left(\sigma_x'\sigma_y' + \sigma_x'\sigma_z' + \sigma_y'\sigma_z'\right) + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)$$

From the hydrostratic stress $p = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$

$$\sigma_x' = \tfrac{2}{3}\sigma_x - \tfrac{1}{3}(\sigma_y + \sigma_z) \qquad \sigma_y' = \tfrac{2}{3}\sigma_y - \tfrac{1}{3}(\sigma_x + \sigma_z) \qquad \sigma_z' = \tfrac{2}{3}\sigma_z - \tfrac{1}{3}(\sigma_x + \sigma_y)$$

$$u_d = \frac{1+\nu}{3E}\left\{\frac{1}{2}\left[(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2\right] + 3\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)\right\}$$

3.15 Static Failure Theories

3.15.3 Maximum Distortion Energy Theory

In a uniaxial tension test, the only nonzero stress is σ_x

the distortion strain energy density when yield occurs $(\sigma_x = \sigma_{yp})$ is

$$u_d = \frac{1+\nu}{3E}\sigma_{yp}^2$$

According to the maximum distortion energy theory of failure

the distortion strain energy density

⇨ the distortion strain energy density the yield point of a tensile test

$$\frac{1+\nu}{3E}\left\{\frac{1}{2}\left[(\sigma_x-\sigma_y)^2+(\sigma_x-\sigma_z)^2+(\sigma_y-\sigma_z)^2\right]+3\left(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2\right)\right\} = \frac{1+\nu}{3E}\sigma_{yp}^2$$

The von Mises stress

$$\sigma_{VM} = \sqrt{\frac{1}{2}\left[(\sigma_x-\sigma_y)^2+(\sigma_x-\sigma_z)^2+(\sigma_y-\sigma_z)^2\right]+3\left(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2\right)}$$

from the maximum distortion energy theory, the failure criterion is

$$\sigma_{VM} = \sigma_{yp}$$

$$\sigma_{VM} = \sqrt{\frac{1}{2}\left[(\sigma_{\max}-\sigma_{\min})^2+(\sigma_{\max}-\sigma_{\text{int}})^2+(\sigma_{\text{int}}-\sigma_{\min})^2\right]}$$

3.15 Static Failure Theories

3.15.3 Maximum Distortion Energy Theory

$$\sigma_{VM} = \sqrt{\frac{1}{2}\left[(\sigma_{\max} - \sigma_{\min})^2 + (\sigma_{\max} - \sigma_{\text{int}})^2 + (\sigma_{\text{int}} - \sigma_{\min})^2\right]}$$

$$\sigma_{VM} = \frac{1}{\sqrt{2}}(\sigma_{\max} - \sigma_{\min})\sqrt{1 + \left(\frac{\sigma_{\max} - \sigma_{\text{int}}}{\sigma_{\max} - \sigma_{\min}}\right)^2 + \left(\frac{\sigma_{\text{int}} - \sigma_{\min}}{\sigma_{\max} - \sigma_{\min}}\right)^2}$$

from this we can deduce that

$$0.866(\sigma_{\max} - \sigma_{\min}) \le \sigma_{VM} \le (\sigma_{\max} - \sigma_{\min})$$

$$\sigma_{VM} = \sqrt{(\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y) + 3\tau_{xy}^2}$$

3.15 Static Failure Theories

Example 3.15.1

The state of stress at a point is

$$\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

Find the maximum allowable value of p in terms of the yield stress σ_{yp}, according to each of the three failure theories presented.

3.15 Static Failure Theories

Example 3.15.2

The state of stress at a point is

$$\begin{bmatrix} -p & \tau & \tau \\ \tau & -p & \tau \\ \tau & \tau & -p \end{bmatrix}$$

Where $p > 0$ and $\tau > 0$. Find the maximum allowable values of p and τ, according to each of the three failure criteria. Assume the compressive and tensile yield stresses are identical.

3.15 Static Failure Theories

Example 3.15.2

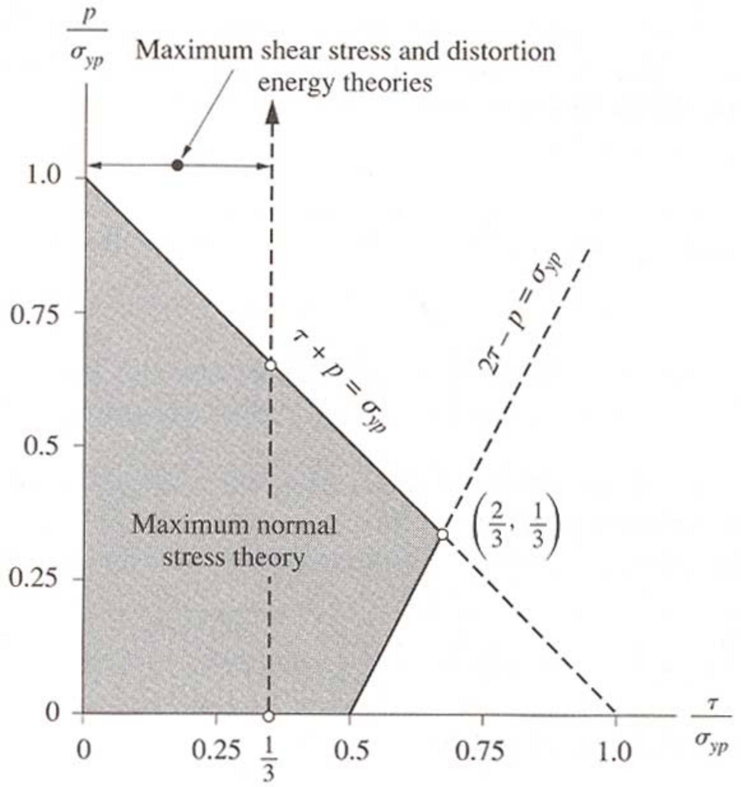


Figure 3.15.2 The range of allowable values for τ and p for the state of stress in Figure 3.15.1.

3.16 Margin of Safety

***allowable stress*: the limits to which the material can be stressed without damage**

***required stress:* the stress calculated for the maximum service load condition**

$$MS = \frac{\text{Excess strength}}{\text{Required strength}} = \frac{\sigma_{\text{allowable}} - \sigma_{\text{required}}}{\sigma_{\text{required}}}$$

$$MS = \frac{\sigma_{\text{allowable}}}{\sigma_{\text{required}}} - 1$$

A safety factor of 1.5 is applied to the limit loads to define the *ultimate loads*

An airplane is designed to survive these ultimate load conditions

3.16 Margin of Safety

A requirement of aircraft design is that all structural members satisfy three basic condition

1. *the stresses accompanying limit loads must not cause plastic deformation*

$$\sigma_{\text{limit loads}} < \sigma_{yp}$$

$$MS)_{\text{yield}} = \frac{\sigma_{yp} - \sigma_{\text{limit load}}}{\sigma_{\text{limit load}}} = \frac{\sigma_{yp}}{\sigma_{\text{limit load}}} - 1$$

2. *the stresses due to ultimate load conditions must not exceed the ultimate strength .*

$\sigma_{\text{ultimate loads}} < \sigma_{ult}$ which implies that $\sigma_{\text{limit loads}} < \dfrac{\sigma_{ult}}{1.5}$

$$MS)_{\text{ult load}} = \frac{\sigma_{ult} - \sigma_{\text{ultimate load}}}{\sigma_{\text{ultimate load}}} = \frac{\sigma_{ult}}{\sigma_{\text{ultimate load}}} - 1 = \frac{\sigma_{ult}}{1.5\sigma_{\text{limit load}}} - 1$$

3. *the limit load stresses must not exceed the buckling strength*

$$\sigma_{\text{limit loads}} < \sigma_{cr}$$

$$MS)_{\text{buckling}} = \frac{\sigma_{cr} - \sigma_{\text{limit load}}}{\sigma_{\text{limit load}}} = \frac{\sigma_{cr}}{\sigma_{\text{limit load}}} - 1$$

3.16 Margin of Safety

Example 3.16.1

The state of stress at the most critical point of a structure is

$$\begin{bmatrix} 10,000 & 5000 & -6000 \\ 5000 & 15,000 & 8000 \\ -6000 & 8000 & 4000 \end{bmatrix} \text{(psi)}$$

The material yield stress is 25,000 psi. Calculate the margin of safety based on:

(a) Distortion energy

(b) Maximum shear stress theory

(c) Maximum normal stress theory

3.18 Stress Concentration and Fatigue

$$\sigma_x = \frac{P}{A} \qquad \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$

$$\sigma_{nom} = \frac{P}{A_{net}} = \frac{P}{A(1 - \frac{d}{w})}$$

$$K_t = \frac{\sigma_{max}}{\sigma_{nom}}$$

$$K_t = 1 + 2\frac{a}{b}$$

$$K_t = 1 + 2\sqrt{\frac{a}{\rho}}$$

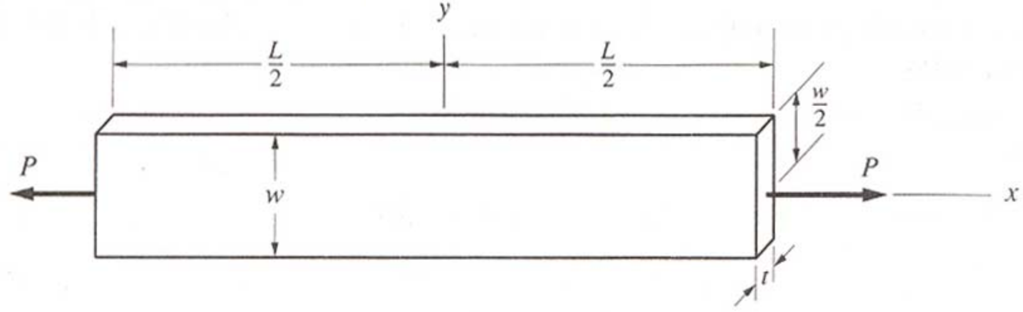


Figure 3.18.1 Flat bar of width *w* and thickness *t* in uniaxial tension.

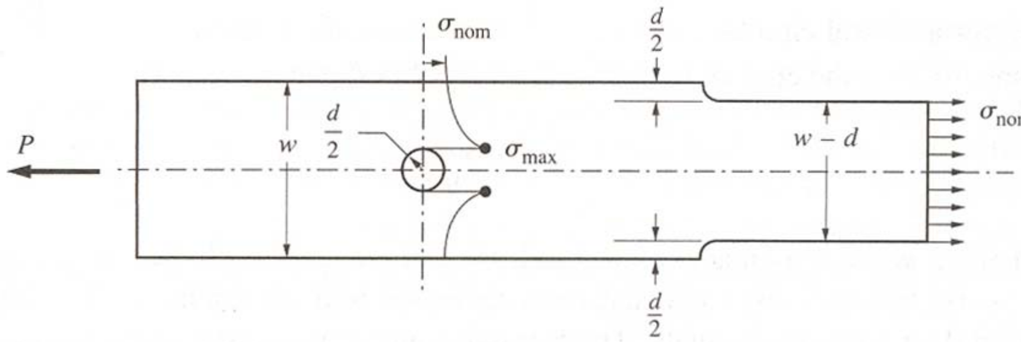


Figure 3.18.2 Small, central hole in a flat bar in uniaxial tension.

3.18 Stress Concentration and Fatigue

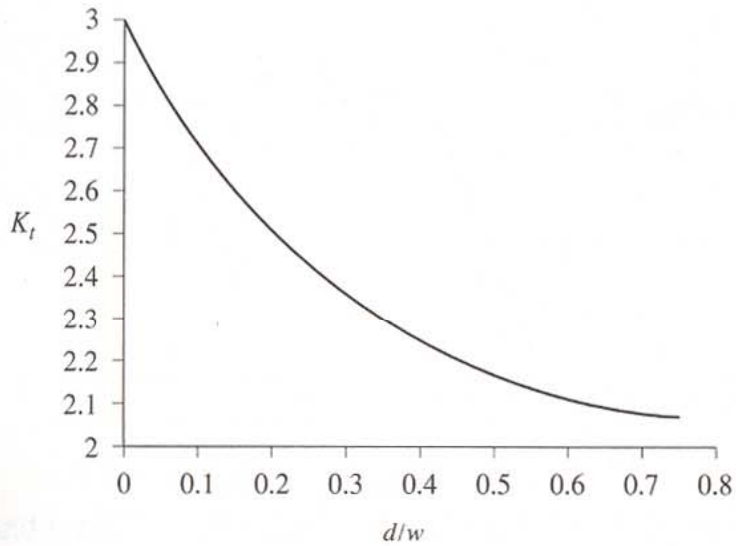


Figure 3.18.3 Stress concentration factor K_t for a central hole of diameter d in a thin, flat bar of width w in uniaxial tension (Figure 3.18.2).

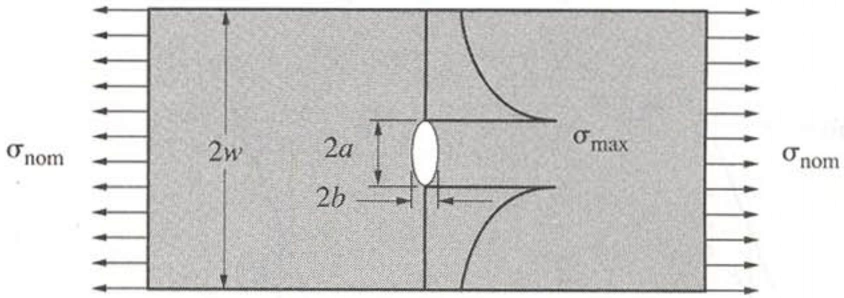


Figure 3.18.4 Stress concentration at the ends of an elliptical hole.

3.18 Stress Concentration and Fatigue

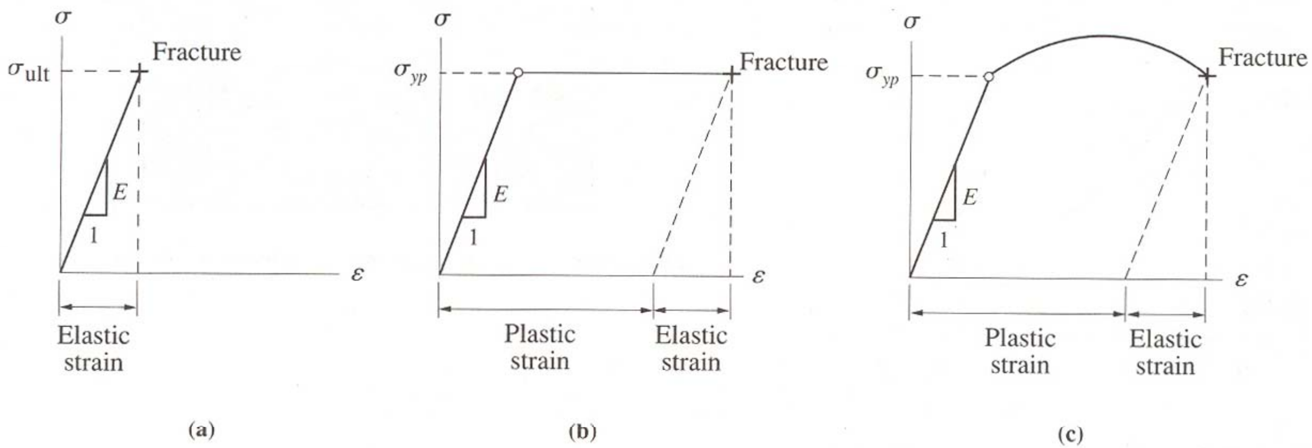


Figure 3.18.5 (a) Elastic, perfectly-brittle material. (b) Elastic, perfectly-plastic material. (E is Young's modulus.) (c) Ductile

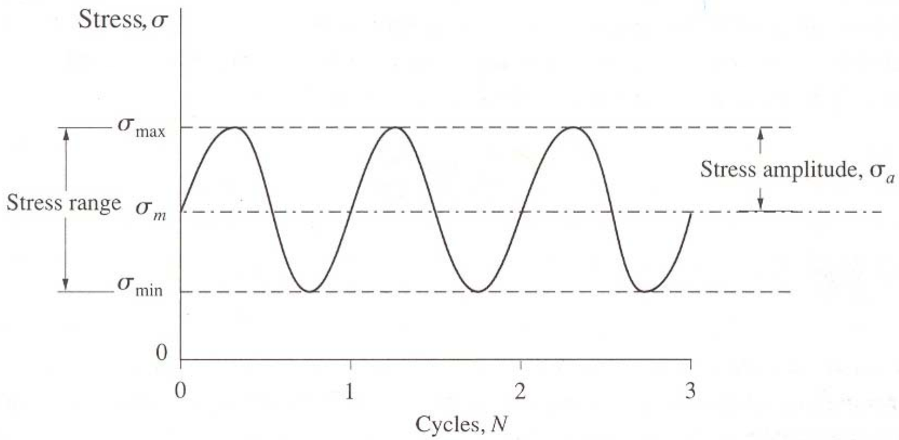


Figure 3.18.6 Typical fatigue loading (load control). σ_m is the mean stress.

3.18 Stress Concentration and Fatigue

The fatigue properties of metallic materials used in aerospace vehicles appear in *MIL-HDBK-5* in the form of *S-N* curve or *e-N* diagram.

In actual service, a part may experience cyclic loading at a variety of stress amplitudes. One means of predicting fatigue life for variable amplitude loadings is the *Palmgren-Miner* Method.

$$D = \sum_{i=1}^{k} \frac{n_i}{N_i}$$

A fatigue crack forms when D=1.0

3.18 Stress Concentration and Fatigue

Example 3.18.1

An unnotched, previously unstressed part made of 2024-T4 aluminum alloy is subjected to an alternating stress σ_a of 30 ksi and a variable mean stress as follows:

$$
\begin{aligned}
\sigma_m &= 0 \text{ ksi for } 30,000 \text{ cycles} \\
\sigma_m &= 10 \text{ ksi for } 20,000 \text{ cycles} \\
\sigma_m &= 20 \text{ ksi for } 10,000 \text{ cycles} \\
\sigma_m &= 30 \text{ ksi for } 7000 \text{ cycles} \\
\sigma_m &= 40 \text{ ksi for } 3000 \text{ cycles} \\
\sigma_m &= 50 \text{ ksi for } 1000 \text{ cycles}
\end{aligned}
$$

According to Palmgren-Miner, what percentage of the fatigue life of the part remains? What is the remaining fatigue life for an alternating stress of 45 ksi together with a mean stress of 0 ksi?

3.18 Stress Concentration and Fatigue

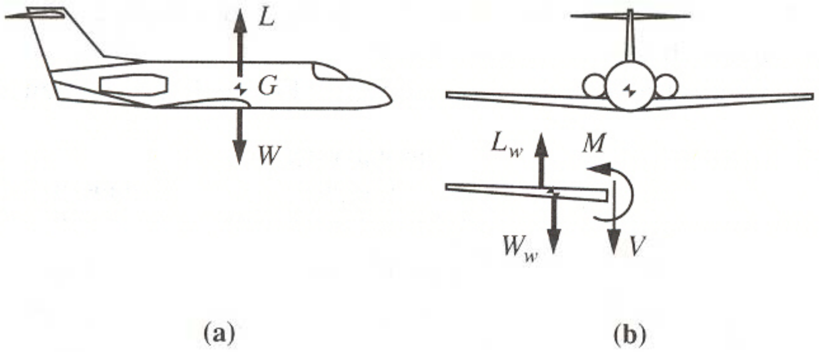


Figure 3.18.8 (a) Lift balances, weight in straight and level flight. (b) When $n = 1$, the shear and bending moment at the wing root are in equilibrium with the resultant lift L_w and weight W_w of the wing.

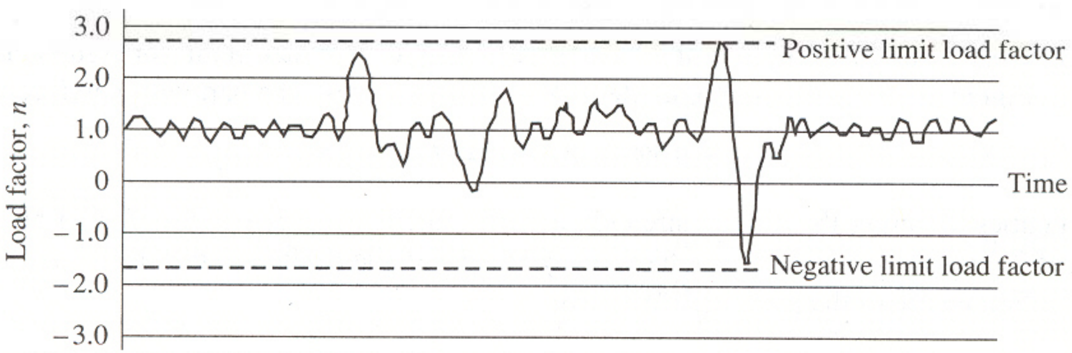


Figure 3.18.9 Load factor history.

Exceeding the limit load factors is likely to damage the airframe.

3.18 Stress Concentration and Fatigue

Load factor: n = L/W

Level flight: n = 1.

The symbol n+ stands for load factors great than 1.0 and n- represents load factors less than 1.0.

If load factor fluctuate between n+ and n- the min. and max. stresses may be written by

$$\sigma_{\max} = n^{+}\sigma_0$$

$$\sigma_{\min} = n^{-}\sigma_0$$

$$\sigma_a = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{n^{+} - n^{-}}{2}\sigma_0$$

$$\sigma_m = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{n^{+} + n^{-}}{2}\sigma_0$$

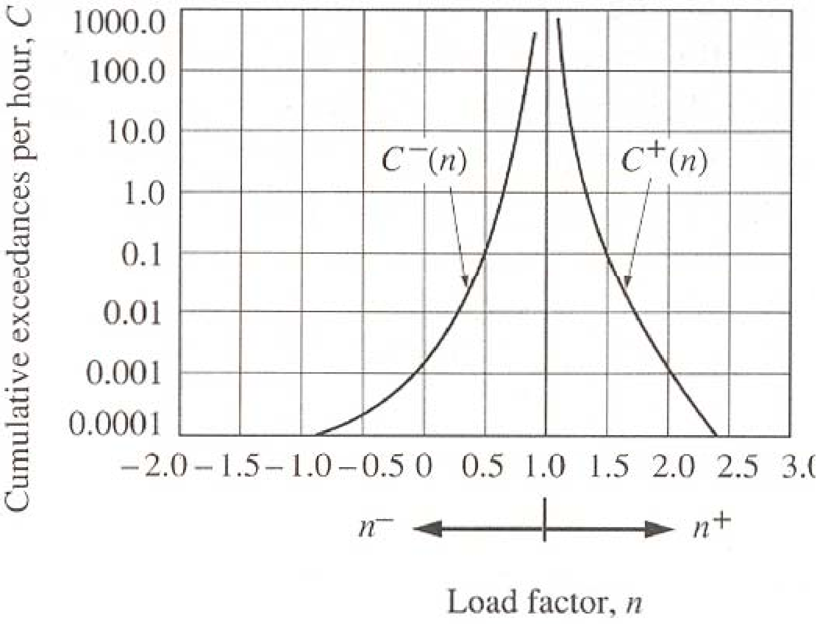


Figure 3.18.10 Typical maneuvering or gust load spectrum for a given type of aircra a specified cruise speed, and prescribed limit load factors.

3.18 Stress Concentration and Fatigue

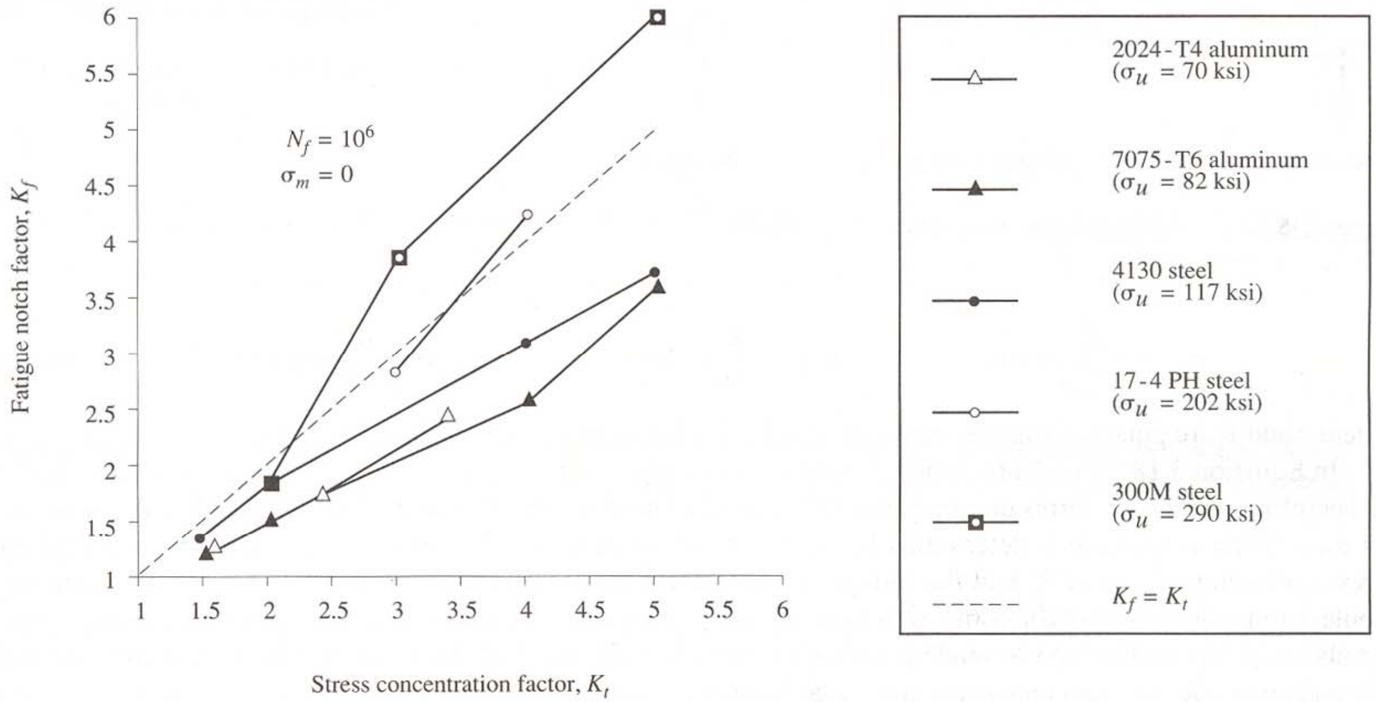


Figure 3.18.12 Fatigue notch factor versus stress concentration factor for several aluminum and steel alloys, using room-temperature data presented in *MIL-HDBK–5G* for fully-reversed loading and a fatigue life of 10^6 cycles. (σ_u is the ultimate static strength.)

3.18 Stress Concentration and Fatigue

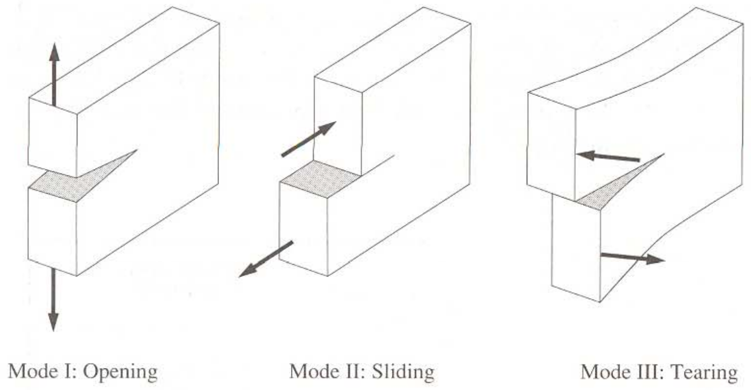


Figure 3.18.13 The three basic modes of loading a crack.

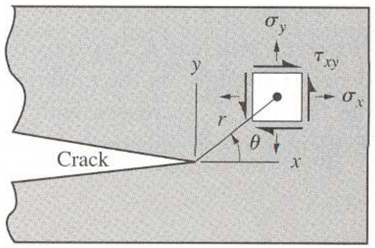


Figure 3.18.14 Stress field at a crack tip.

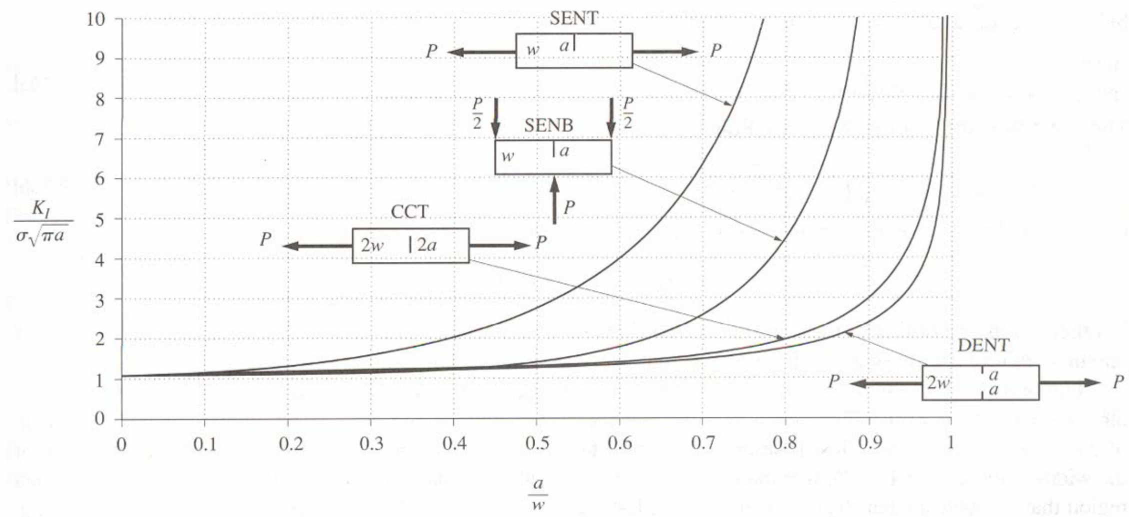


Figure 3.18.15 Mode I stress intensity factors for *single-edge notched tension* (SENT), *double-edge notched tension* (DENT), *single-edge notched bending* (SENB), and *center-cracked tension* (CCT) specimens. (a gives the flaw size and w relates to the specimen width.)

3.18 Stress Concentration and Fatigue

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

$$\varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y)$$

$$\gamma_{xz} = \gamma_{yz} = \varepsilon_z = 0$$

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\sigma_{VM})_{\text{plane strain}} = \sqrt{\left(\sigma_x - \sigma_y\right)^2 + \sigma_x\sigma_y + 3\tau_{xy}^2 - \nu\,(1-\nu)\,\left(\sigma_x + \sigma_y\right)^2} \qquad \text{plane strain}$$

$$\sigma_{VM})_{\text{plane stress}} = \sqrt{\left(\sigma_x - \sigma_y\right)^2 + \sigma_x\sigma_y + 3\tau_{xy}^2} \qquad \text{plane strain}$$

3.18 Stress Concentration and Fatigue

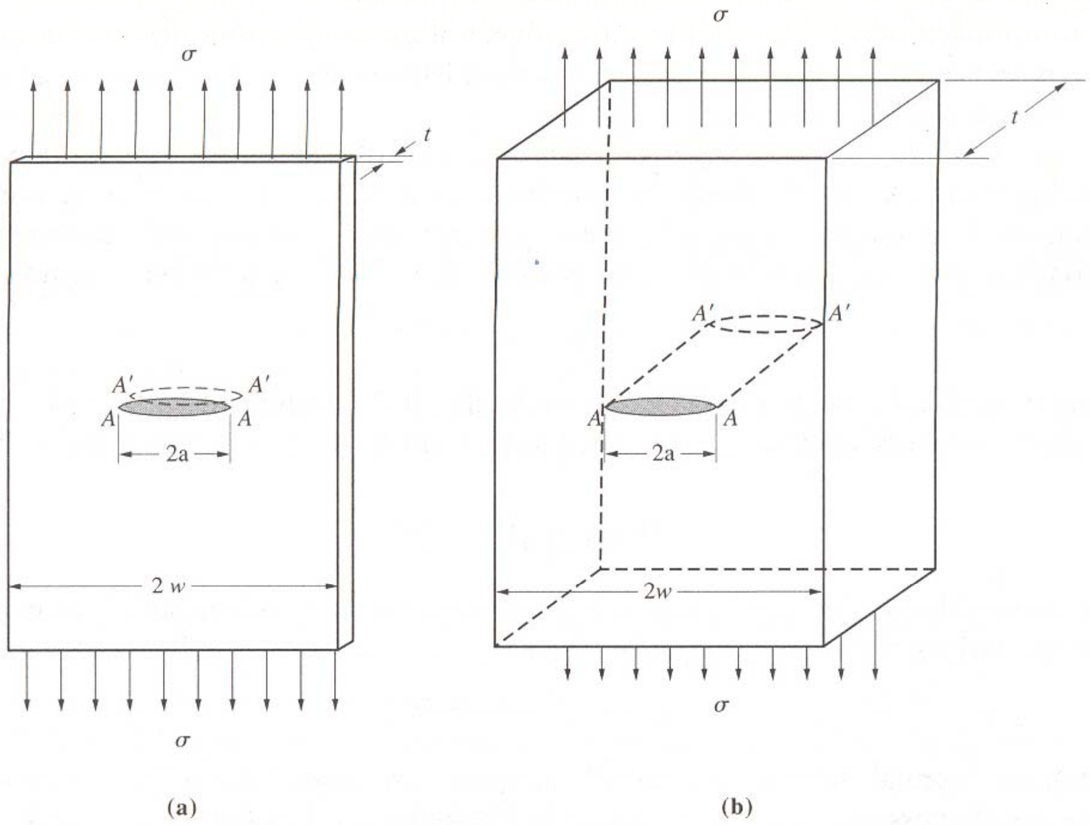


Figure 3.18.16 (a) Thin center-cracked specimen in which the crack tip A–A' is in plane stress throughout the thickness t ($t \ll 2w$). (b) Thick specimen in which the bulk of the crack tip A–A' is in triaxial stress (plane strain).

3.18 Stress Concentration and Fatigue

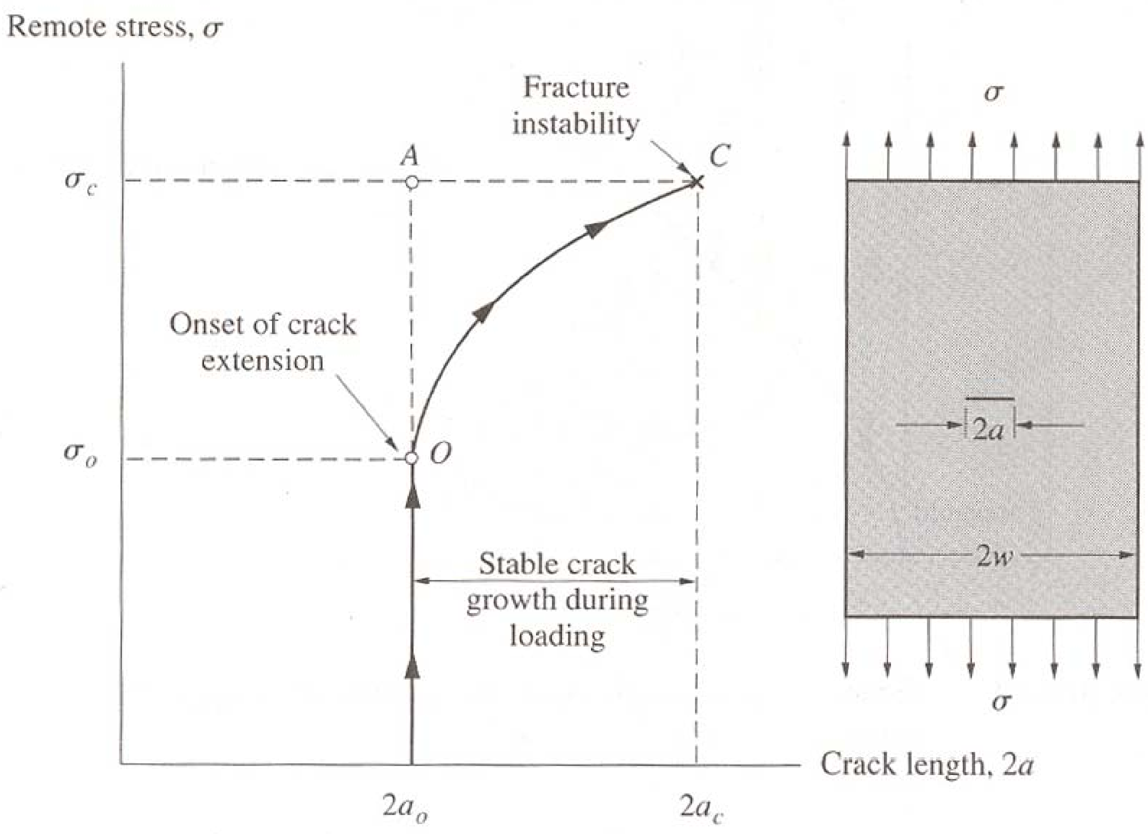


Figure 3.18.17 Crack growth curve for a thin sheet (plane stress fracture).

Source: *MIL-HDBK–5G.*