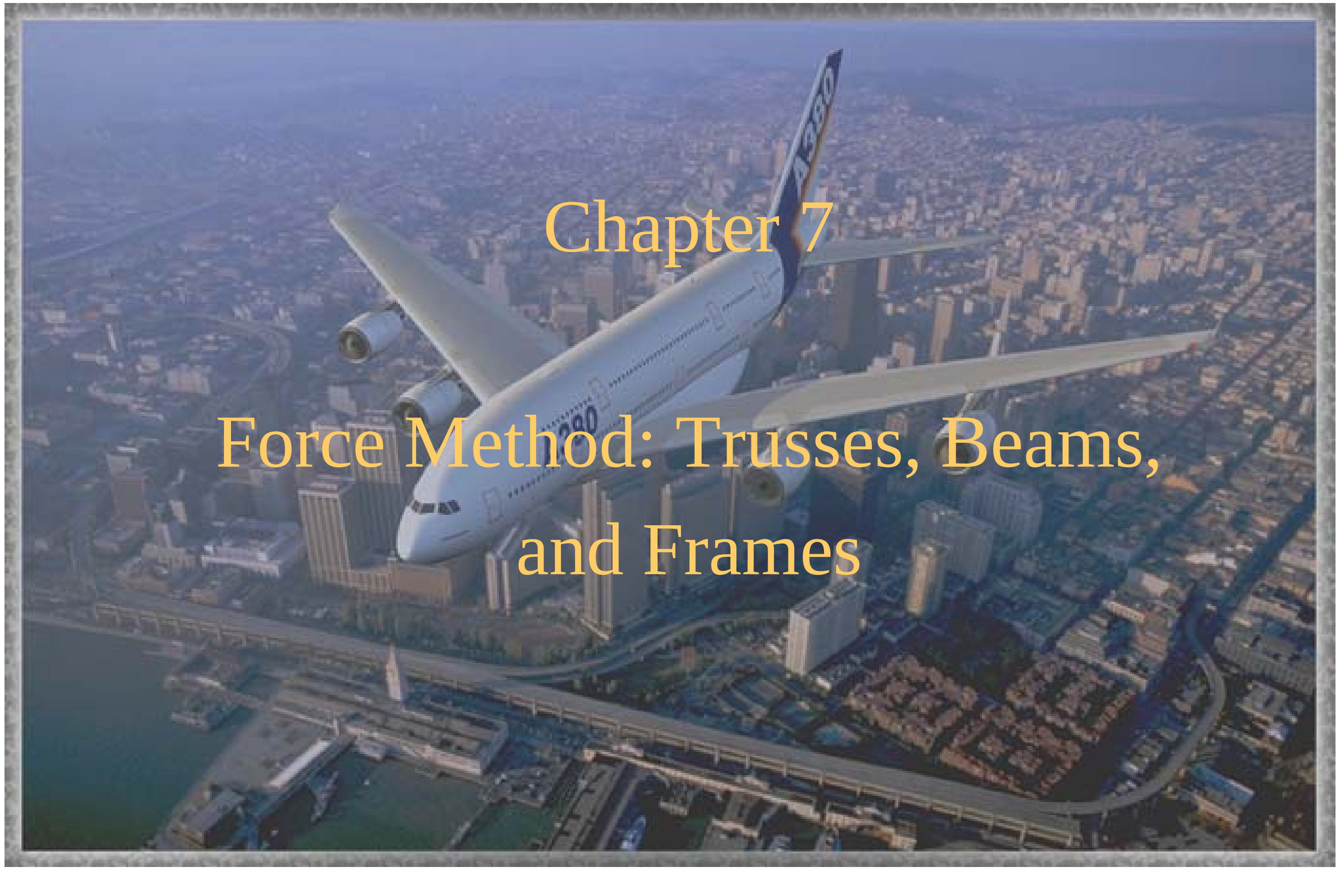
Aircraft Structural Analysis



Chapter 7

Force Method: Trusses, Beams, and Frames

7.1 Introduction

The *force method* of analyzing a structure begins with the use of statics to obtain the equilibrium equations, which relate the unknown forces to the known forces.

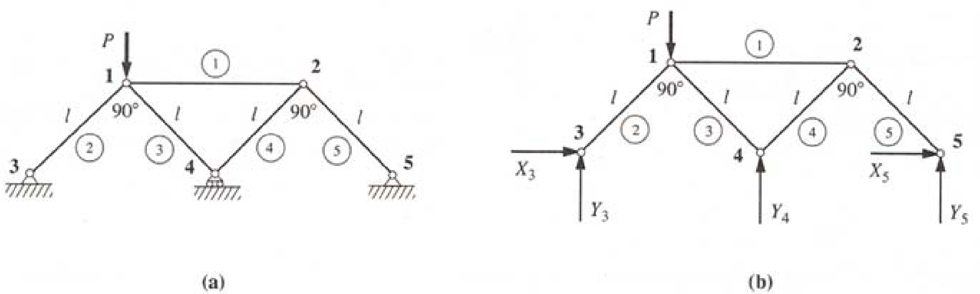
- the known forces : the applied loads

- the unknowns : the reactions at the supports and the internal member loads

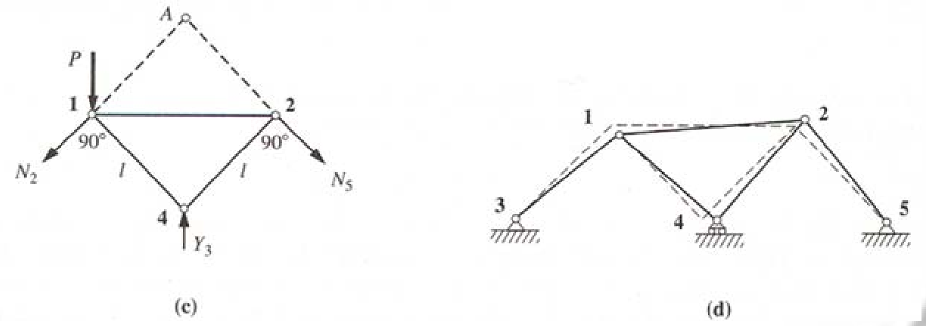
- **No. of unknowns = No. of equilibrium equations : statically determinate**
 ; as far as the forces are concerned, the problem is finished once equations are solved.

We must be sure that **the structure is stable** before seeking the solution of the equilibrium equations; if it is not, a solution is not possible.

7.1 Introduction



(a) plane truss (b) Free-body diagram showing the support reactions.



(c) Free-body diagram of the portion of the truss spanning nodes 1,2, and 4. (d) illustration of the kinematic instability.

unknowns : 5 support reactions

+ 5 member forces = 10

equilibrium eqs. = 5 node * 2

= 10

Statically Determinate.

But the structure is unstable.

In figure (c) there exist moment about point A.

7.1 Introduction

Once the forces have been determined, we can find the displacements at selected points on the structure using *the principle of complementary virtual work*. This requires bringing the structure's material properties and cross-sectional details into the picture. All real structures are flexible to one degree or another; there is no such ting as a perfectly rigid body. The degree of flexibility allowed is part of the design process, and deflection analysis is required to ensure that static displacements remain within the limits prescribed. Calculating displacements is therefore fundamental to the analysis of structural dynamics and structural stability.

Statically indeterminate structures are those for which the methods of statics alone are not sufficient for calculating the internal loads and external reactions. A structure of this type has more than the minimum number of members and/or supports required for it to sustain a given load without collapsing or moving off as a rigid body. The excess members and supports are called *redundants*. Highly redundant structures, such as aircraft, provide a variety of internal load path options. Should a given redundant member fail for some reason, the remaining ones will continue to provide the means of carrying the load.

7.1 Introduction

- Simple example of a redundant structure

• One of the springs is redundant, only a single spring is needed to transmit the load *P* to the wall. => the redundant : member spring 2

• Applying statics to the free-body diagram in part (b) of the figure, $P_1 = P - P_2$

• We must also take into consideration the deformation of the structure and invoke a compatibility condition. If s_1 is the stretch of spring 1 and s_2 that of spring 2, then compatibility simply means that $s_1 = s_2$

• From Hook's Law $s_1 = \dfrac{P_1}{k_1} \qquad s_2 = \dfrac{P_2}{k_2}$

$$P_2 = \frac{k_2}{k_1} P_1 \qquad P_1 = \frac{k_1}{k_1 + k_2} P \qquad P_2 = \frac{k_2}{k_1 + k_2} P$$

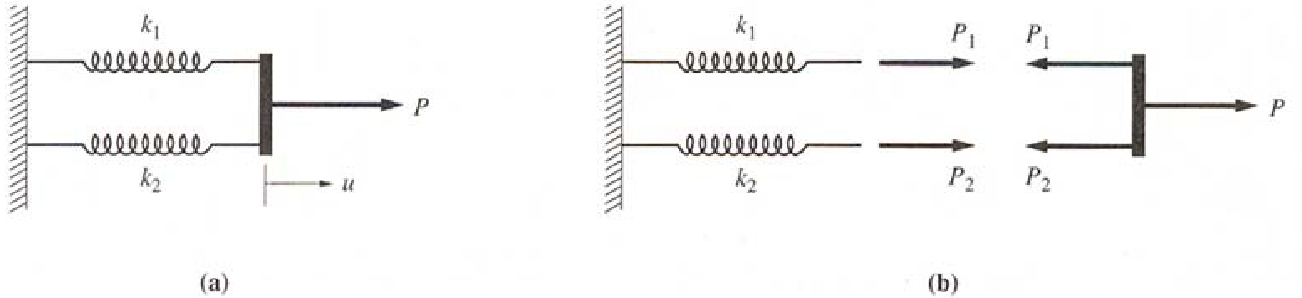


Figure 7.1.2 (a) Parallel (redundant) elastic spring structure. (b) Free-body diagrams showing internal loads.

7.1 Introduction

The portion of the applied load carried by a spring depends on its relative stiffness. The large part of the load is borne by the stiffest spring. Obviously, as the spring rake k_1 goes to zero, so does P1, and the other spring must absorb all of the load. If $k_1=k_2$, then $P_1=P_2=P/2$: both springs share the load equally.

Adding more parallel springs to the assembly would not limit our ability to find all of the internal loads using this procedure, which is a simple example of the force method. In more complex structures, the principle of complementary virtual work is used to enforce compatibility.

In this chapter, we will apply the force method to *the analysis of skeletal or "stick-like" structures*, using the principle of complementary virtual work to find displacements and forces in statically determinte and indeterminate trusses, beams, and frames.

7.2 RODS: Complementary Virtual Work

- a slender bar, possibly slightly tapered

- *transmit loads parallel to its long axis*

- The load resultant N at any section must pass through the centroid of that section

- carry an intermediate load p distributed along its axis, such as when it is attached to shear panels in a stiffened web structure.

- There may also be a variable temperature change T from ambient along the rod, producing thermal strain.

- Since the axial load N acts at the centroid, the stress $\boldsymbol{\sigma}_x$ is uniform across each section and $\boldsymbol{\sigma}_x$ is the only nonzero stress component; likewise $\boldsymbol{\delta\sigma}_x$ is the only virtual stress component.

7.2 RODS: Complementary Virtual Work

$$\delta W^*_{\text{int}} = \iiint_V \delta\sigma_x \varepsilon_x dV$$

$$= \int_0^L \delta\sigma_x \left(\frac{\sigma_x}{E} + \alpha T\right)(A dx)$$

$$= \int_0^L \left(\frac{\delta N}{A}\right)\left(\frac{N}{AE}\right)(A dx) + \int_0^L \left(\frac{\delta N}{A}\right)(\alpha T)(A dx)$$

$$\delta W^*_{\text{int}} = \int_0^L \frac{N \delta N}{AE} dx + \int_0^L (\alpha T) \delta N \, dx$$

In a truss structure, constant cross section and no distributed axial load

$$\boxed{\delta W^*_{\text{int}} = \frac{NL}{AE}\delta N + \left(\int_0^L (\alpha T)\, dx\right)\delta P} \quad \text{truss member}$$

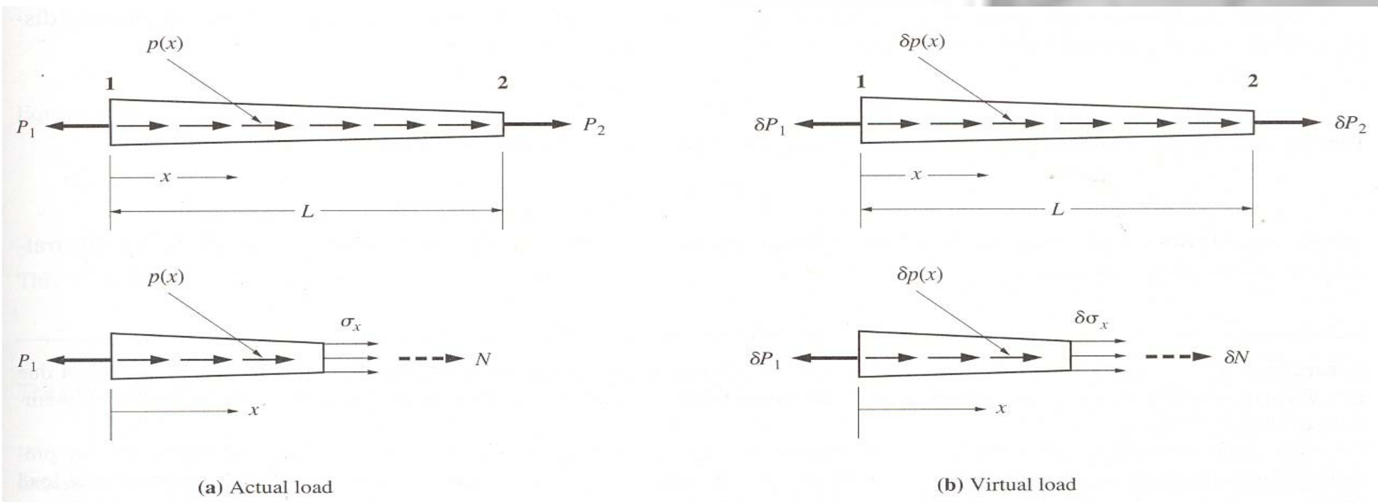


Figure 7.2.1 The rod element.

7.3 Truss Deflections Using Complementary Virtual Work

To find the displacement at a given node of a statically determinate truss

1. calculate the internal forces $N^{(e)}$ in each truss member e due to the actual applied loads.
2. remove all of the true loads and apply a fictitious force δQ to the given node, in the direction of the desired displacement component.
3. solve for the resultant virtual forces $\delta N^{(e)}$ throughout the truss.
4. $$\delta W^*_{\text{int}} = \sum_{e=1}^{\text{no. of rods}} \left\{ \frac{L^{(e)} N^{(e)}}{A^{(e)} E^{(e)}} + \int_0^{L^{(e)}} \left(\alpha^{(e)} T^{(e)}\right) ds \right\} \delta N^{(e)}$$
5. Recall that the complementary virtual work of the virtual force δQ is the product of δQ and the actual displace $T^{(e)}(s)$ in the direction of the virtual force or
$$\delta W^*_{\text{ext}} = q\delta Q$$
6. Finally, we equate the internal and external complementary virtual work, as follows:
$$\delta W^*_{\text{ext}} = \delta W^*_{\text{int}}$$

7.3 Truss Deflections Using Complementary Virtual Work

Example 7.3.1 For the statically determinate truss in Figure 7.3.1, calculate the horizontal displacement u_4 at node 4 due to a vertical load P at node 1, using the principle of complementary virtual work. The Axial rigidity AE is the same for all members of the structure.

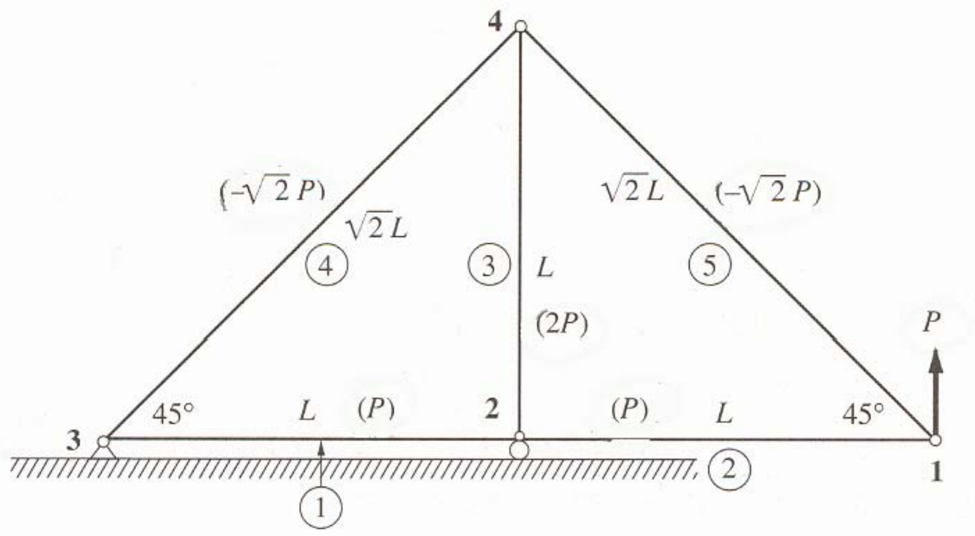


Figure 7.3.1 True load P at node 1 and the corresponding member forces (in parentheses).

7.3 Truss Deflections Using Complementary Virtual Work

We can determine the member forces resulting from the vertical force P at node 1 using statics, and the results are presented alongside each member in Figure 8.3.4. Since we are seeking the horizontal movement of node 4, we remove the load from node 1 and apply a virtual force δQ acting horizontally at node 4. The corresponding member loads are shown in parentheses in Figure 7.3.2.

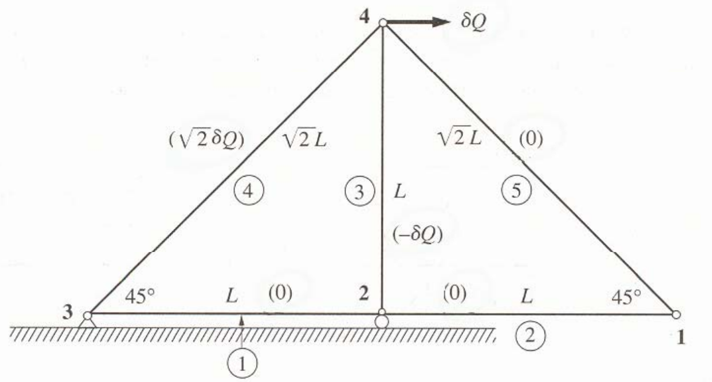


Figure 7.3.2 Virtual applied load and corresponding virtual member loads (in parentheses).

$$u_4\delta Q = \overbrace{\frac{(P)\,L}{AE}(0)}^{\text{rod 1}} + \overbrace{\frac{(P)\,L}{AE}(0)}^{\text{rod 2}} + \overbrace{\frac{(2P)\,L}{AE}(-\delta Q)}^{\text{rod 3}} + \overbrace{\frac{\left(-\sqrt{2}P\right)\sqrt{2}L}{AE}\left(\sqrt{2}\delta Q\right)}^{\text{rod 4}} + \overbrace{\frac{\left(-\sqrt{2}P\right)\sqrt{2}L}{AE}(0)}^{\text{rod 5}}$$

$$u_4 = -2\left(1+\sqrt{2}\right)\frac{PL}{AE}$$

The minus sign means that the displacement is to the left, in the direction opposite to that of δQ.

7.3 Truss Deflections Using Complementary Virtual Work

Example 7.3.2 For the truss of the previous example, loaded as shown in Figure 7,3,1, Calculate the rotation of member 5.

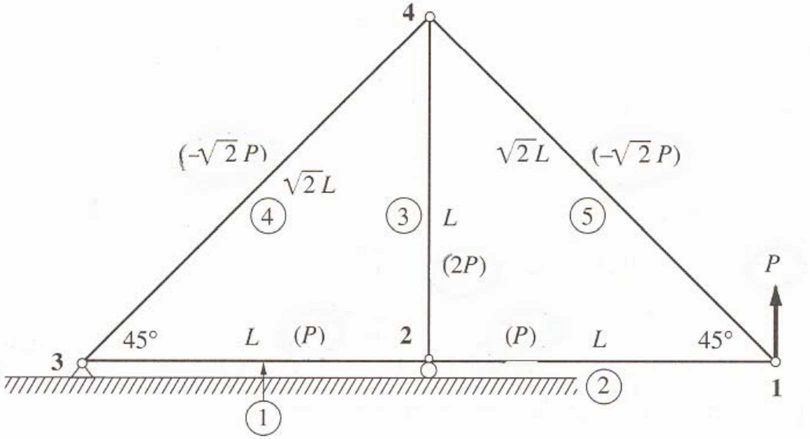


Figure 7.3.1 True load P at node 1 and the corresponding member forces (in parentheses).

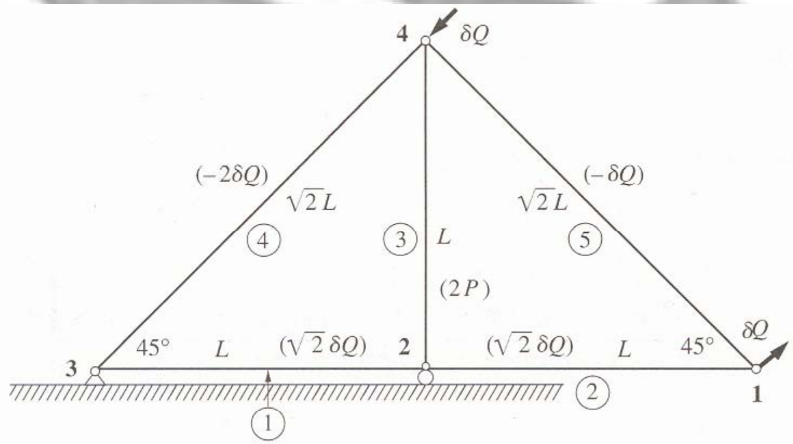


Figure 7.3.3 Virtual loads required to find the rotation of member 5.

7.3 Truss Deflections Using Complementary Virtual Work

The complementary virtual work associated with a (small) rotation $\theta^{(5)}$ of member 5 is $\theta^{(5)}\delta C$, where δC is a virtual couple in the plane of rotation. We cannot apply a point couple to rod 5, because it is a two-force member, and loads can only be applied to it at its ends. Therefore, we apply a pair of equal but opposite virtual forces δQ to nodes 1 and 4, as shown in Figure 7.3.3, which also shows the resulting member loads, found by using statics. The moment of the couple formed by the virtual forces is $\delta Q \times \sqrt{2}L$, counterclockwise. Therefore, from the principle of complementary virtual work, we have

$$\theta^{(5)}\left(\sqrt{2}L\delta Q\right) = \overbrace{\frac{(P)\,L}{AE}\left(\sqrt{2}\delta Q\right)}^{\text{rod 1}} + \overbrace{\frac{(P)\,L}{AE}\left(\sqrt{2}\delta Q\right)}^{\text{rod 2}} + \overbrace{\frac{(2P)\,L}{AE}\left(\sqrt{2}\delta Q\right)}^{\text{rod 3}}$$
$$+ \overbrace{\frac{\left(-\sqrt{2}P\right)\left(\sqrt{2}L\right)}{AE}(-2\delta Q)}^{\text{rod 4}} + \overbrace{\frac{\left(-\sqrt{2}P\right)\left(\sqrt{2}L\right)}{AE}(-\delta Q)}^{\text{rod 5}}$$

Upon simplification, we get

$$\theta^{(5)}\left(\sqrt{2}L\delta Q\right) = \left(6 + 4\sqrt{2}\right)\frac{PL}{AE}\delta Q$$

so that

$$\theta^{(5)} = 8.243\frac{P}{AE}$$

The rotation of the rod is counterclockwise, in the same direction as the virtual couple.

7.3 Truss Deflections Using Complementary Virtual Work

Example 7.3.3 The Truss in Figure 7.3.4a is not loaded, but member 1 is heated to a uniform temperature T above ambient and member 2 is heated to 2T. The temperature in the other rods increases linearly from T to 2T. If the axial rigidity AE and the thermal expansion coefficient α are the same for all members, calculate the displacement u_1 of node 1, using the principle of complementary virtual work.

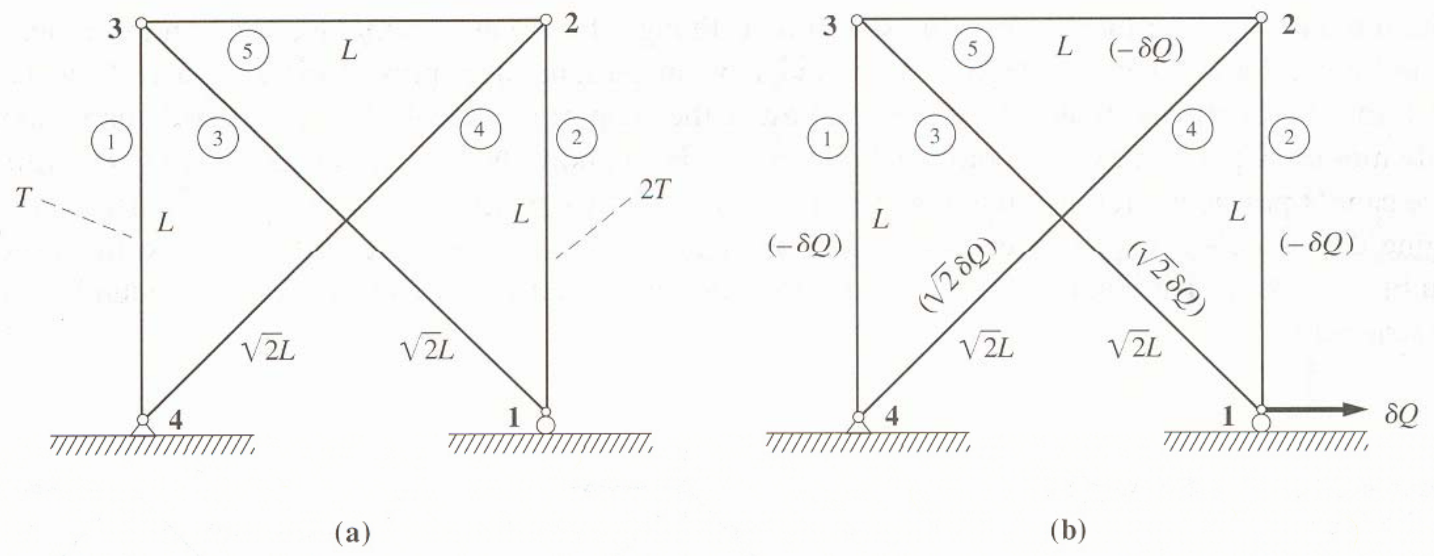


Figure 7.3.4 (a) Unloaded truss with thermal strain. (b) Virtual load at node 1 and the corresponding member forces.

7.3 Truss Deflections Using Complementary Virtual Work

The linear temperature variation in rods 3 and 4 is given by $T\left[1+\left(s/\sqrt{2}L\right)\right]$, while that in rod 5 is $T\left[1+(s/L)\right]$. To find the displacement at node 1, we remove the thermal loading and apply the virtual force δQ, which is shown in Figure 7.3.4b along with the corresponding member loads, found by using statics. Even though the true loads are zero, we still have a thermal strain term in the virtual work expression, Equation 7.3.1. Therefore, by the principle of complementary virtual work, we have

$$u_1\delta Q = \overbrace{(\alpha TL)(-\delta Q)}^{\text{rod 1}} + \overbrace{(2\alpha TL)(-\delta Q)}^{\text{rod 2}}$$
$$+ \overbrace{\left[\int_0^{\sqrt{2}L} \alpha T\left(1+\frac{s}{\sqrt{2}L}\right) ds\right]\left(\sqrt{2}\delta Q\right)}^{\text{rod 3}} + \overbrace{\left[\int_0^{\sqrt{2}L} \alpha T\left(1+\frac{s}{\sqrt{2}L}\right) ds\right]\left(\sqrt{2}\delta Q\right)}^{\text{rod 4}}$$
$$+ \overbrace{\left[\int_0^{L} \alpha T\left(1+\frac{s}{L}\right) ds\right](-\delta Q)}^{\text{rod 5}}$$

This reduces to

$$u_1\delta Q = -\alpha TL\delta Q - 2\alpha TL\delta Q + 3\alpha TL\delta Q + 3\alpha TL\delta Q - \frac{3}{2}\alpha TL\delta Q$$

so that we finally obtain a rightward displacement of node 1 in the following amount

$$u_1 = \frac{3}{2}\alpha TL$$

7.4 Statically Indeterminate Trusses

- For indeterminate structures,

  the number of unknown forces > the number of equations available from statics

  => The differences are the number of *redundant* forces

- the first step towards calculating the forces in an indeterminate structure is to single out the redundant members and/or supports and, on a sketch, show their effect on the structure through the loads they exert. In other words,

  <u>We cut through each redundant member, revealing the force within it</u>

  <u>and applying that force to the structure as though it were an external load.</u>

7.4 Statically Indeterminate Trusses

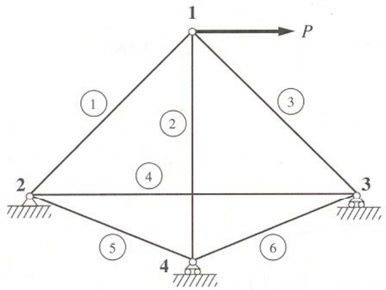


Figure 7.4.1 Doubly-redundant truss.

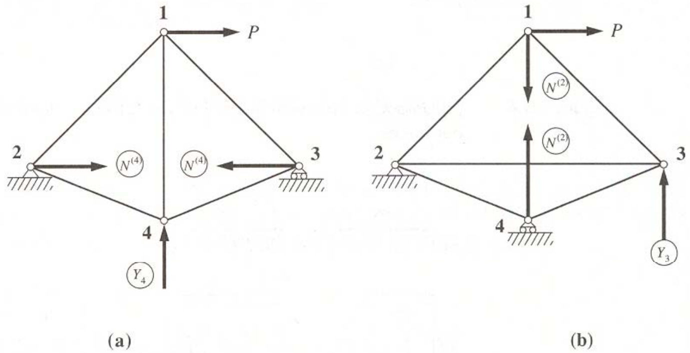


Figure 7.4.2 Possible choices of redundants in the truss of Figure 7.4.1.

This has two degrees of redundancy: an extra member plus an extra support. If we take away member 4 the truss remains stable. If we also take away the support at node 4, the truss is still unable to undergo rigid-body motion. Upon removal of both the member and the supports, the truss becomes statically determinate. **The statically determinate structure that remains after cutting away the redundant i called the base stru**

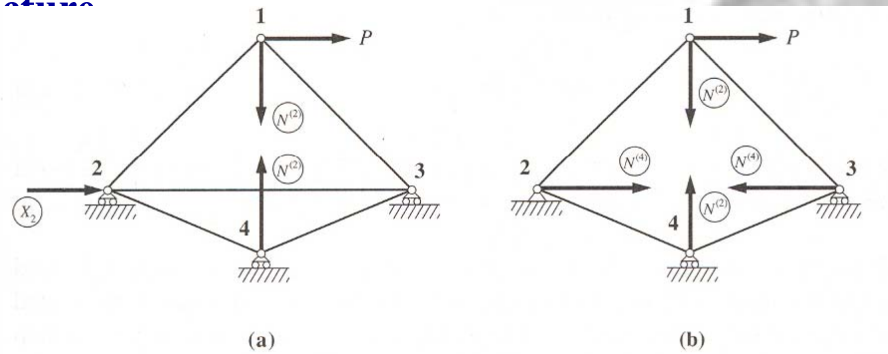


Figure 7.4.3 Invalid choices of base structure for the indeterminate truss in Figure 7.4.1.

7.4 Statically Indeterminate Trusses

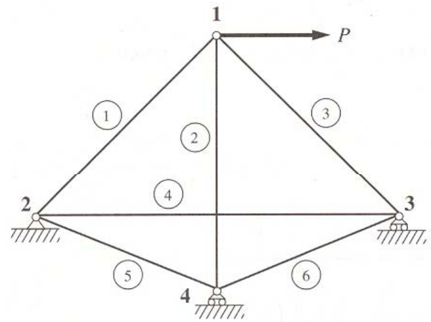


Figure 7.4.1 Doubly-redundant truss.

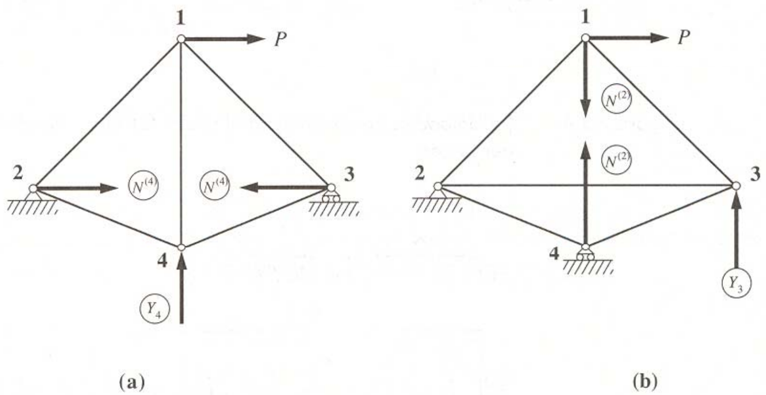


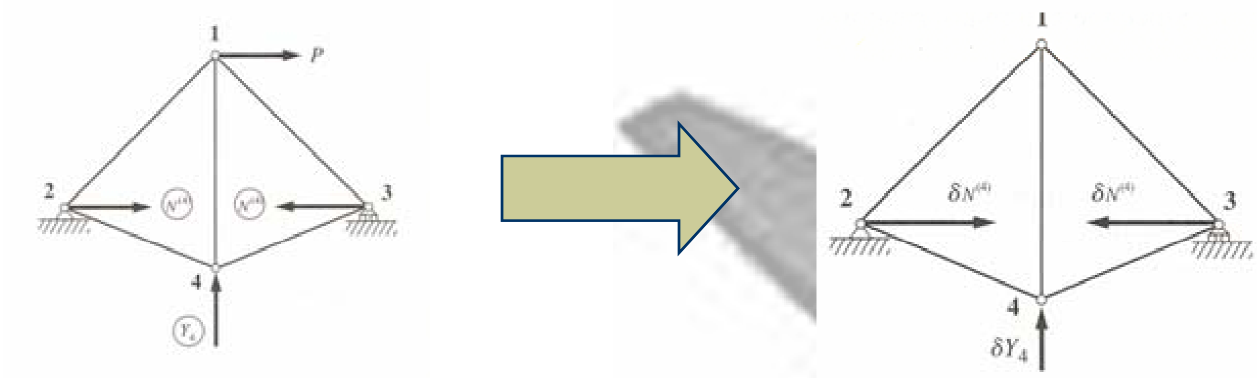
Figure 7.4.2 Possible choices of redundants in the truss of Figure 7.4.1.

Assuming member 4 is in tension, it exerts the force $N^{(4)}$ on nodes 2 and 3. The roller support at node 4 applies a vertical force Y_4 to the node. We assume this force to be directed upwards. The truss in Figure 7.4.2a appears to be a statically determinate truss acted on by four external forces.

<u>Using statics, we can now solve for the forces throughout the base structure, in terms of P, $N^{(4)}$, and Y_4.</u>

Figure7.4.2b shows an alternate choice of base structure. Indeed, there are several other possibilities, all of them equally valid. In any case, the base structure we end up with must be stable and properly supported.

7.4 Statically Indeterminate Trusses



After settling on a proper base structure, we draw its free-body diagram and calculate the axial load $N^{(e)}$ in each member e in terms of the applied load P and the redundants $N^{(4)}$ and Y_4. We then <u>remove the true loads from the truss and replace the redundant loads by virtual loads.</u> Using statics, <u>we calculate the resulting virtual internal load $\delta N^{(4)}$ in each member of the truss.</u> With the real and virtual loads thus determined, we can write the complementary internal virtual work $\delta W_{\text{int}}^{*(e)}$ for each member of the original structure. We sum them up to obtain $\delta W_{\text{int}}^{*}$ for the whole truss, as follows:

$$\delta W_{\text{int}}^{*} = \sum_e \delta W_{\text{int}}^{*(e)} = \sum_e \left(\frac{L}{AE}\right)^{(e)} N^{(e)} \delta N^{(e)}$$

According to the principle of complementary virtual work, <u>the external and internal complementary virtual works must always be the same</u>. Therefore, for the truss, we have

$$\delta W_{\text{ext}}^{*} = \sum_e \left(\frac{L}{AE}\right)^{(e)} N^{(e)} \delta N^{(e)}$$

7.4 Statically Indeterminate Trusses

The external complementary virtual work δW^*_{ext} is that of the virtual forces in Figure 7.4.4 acting through the real displacements of their points of application. The complementary virtual work of is δY_4 is $\delta Y_4 \times v_4$, where v_4 is these true vertical component of displacement of node 4. But $v_4 = 0$ because the support at node 4 presumably prevents vertical motion. The complementary virtual work $\delta N^{(4)}$ is also zero, as can be seen by studying Figure 7.4.5

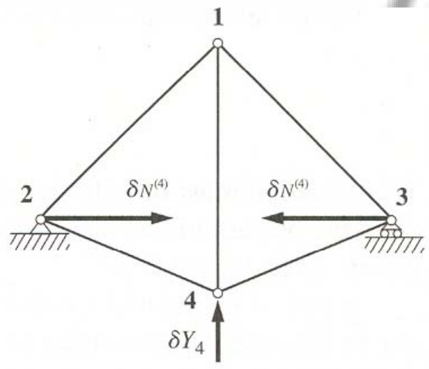


Figure 7.4.4 The base truss of Figure 7.4.2a subjected only to the virtual redundant loads.

Remember that we do not physically cut the redundant members out of the structure. The redundant forces shown on the base structure in Figure 7.4.4 occur at an imaginary cut taken at some point along the member, such as point *a* in Figure 7.4.5. The complementary virtual work of these virtual forces is

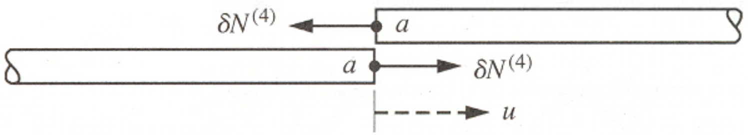


Figure 7.4.5 Virtual member loads and true displacement at the point of the imaginary cut.

$$\delta W^* = \delta N^{(4)} \times u_{\text{left}} + \delta N^{(4)} \times u_{\text{right}}$$

$$\delta W^* = \delta N^{(4)} \times u + \delta N^{(4)} \times (-u) = 0$$

7.4 Statically Indeterminate Trusses

This same argument can be used to show that the complementary virtual work of redundant member forces is zero not only for truss elements but for members of any other type of structure, as well.

Since the external virtual work is zero,

$$\sum_e \left(\frac{L}{AE}\right)^{(e)} N^{(e)} \delta N^{(e)} = 0$$

This equation must be valid for any choice of the two virtual loads δY_4 and $\delta N^{(4)}$ Therefore, above equation will yield the two additional equations needed to solve for all of the forces in the indeterminate truss.

It should be pointed out that although the displacements at a structure's supports are usually zero, this need not be the case. For example, node 4 of the truss in Figure 7.4.1 might well have been given a specified upward displacement v_4. In that case, the complementary virtual work of δY_4 would be $\delta Y_4 \times v_4$ instead of zero, which would appear on the right of above equation. The point if that v_4 is a known nonzero quantity, so that the complementary virtual work equality,

$$\sum_e \left(\frac{L}{AE}\right)^{(e)} N^{(e)} \delta N^{(e)} = v_4 \delta Y_4$$

7.4 Statically Indeterminate Trusses

Example 7.4.1 Calculate the internal forces in the Figure 7.4.6, using the principle of complementary virtual work.

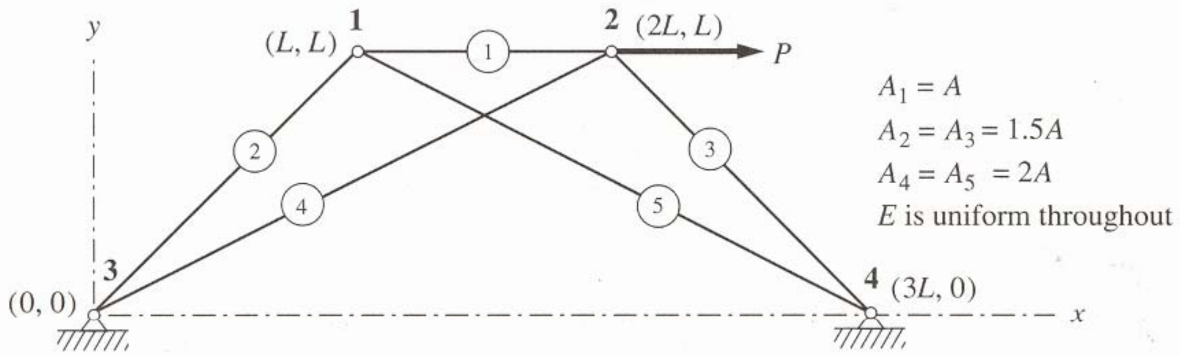


Figure 7.4.6 Truss with single degree of indeterminacy.

7.4 Statically Indeterminate Trusses

Let us choose the horizontal support at node 4 as the redundant, so that our base structure is as shown in Figure 7.4.7. The support reactions are immediately obtained from statics. Using the joint method, we can find the member loads, as shown in the figure, in terms of the applied load P and the redundant reaction X_4. In Figure 7.4.8, we remove all of the true loads on the truss, apply only a virtual redundant load δX_4, and calculate the virtual internal loads shown.

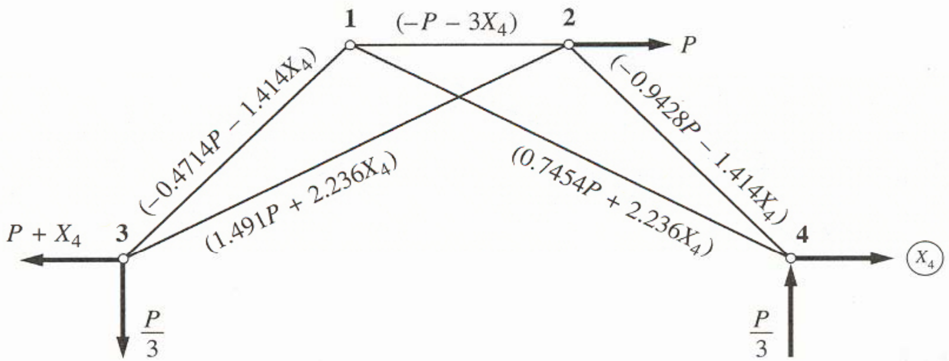


Figure 7.4.7 Base structure for the indeterminate truss, showing the computed true reactions and member loads.

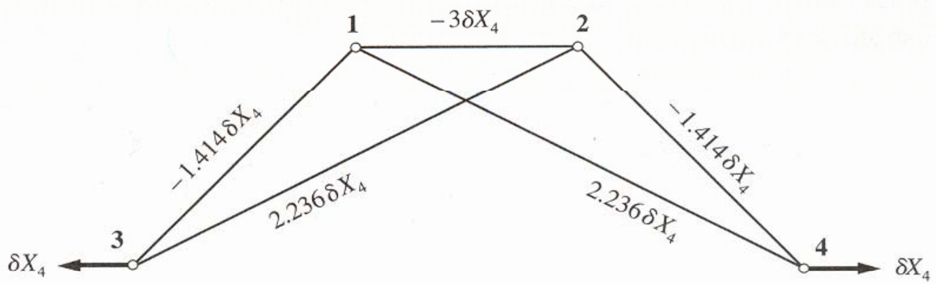


Figure 7.4.8 Virtual loading of the base truss.

7.4 Statically Indeterminate Trusses

According to the principle of complementary virtual work, $\delta W^*_{\text{int}} = \delta W^*_{\text{ext}}$, so that in this case,

$$\left(\frac{L}{AE}\right)^{(e)} N^{(e)} \delta N^{(e)} = u_4 \delta X_4 = (0)\, \delta X_4 = 0$$

since the horizontal displacement u_4 at node 4 is constrained to zero by the pin support. Expanding the summation, using the data shown in Figure 7.4.7, and noting that $L^{(1)} = L$, $L^{(2)} = L^{(3)} = 1.414L$, and $L^{(4)} = L^{(5)} = 2.236L$, we get

$$\overbrace{\frac{L}{AE}(P - 3X_4)(-3\delta X_4)}^{\text{element 1}} + \overbrace{\frac{1.414L}{1.5AE}(-0.4714P - 1.414X_4)(-1.414\delta X_4)}^{\text{element 2}} + \overbrace{\frac{1.414L}{1.5AE}(-0.9428P - 1.414X_4)(-1.414\delta X_4)}^{\text{element 3}}$$

$$+ \overbrace{\frac{2.236L}{2AE}(1.491P + 2.236X_4)(2.236\delta X_4)}^{\text{element 4}} + \overbrace{\frac{2.236L}{2AE}(0.7454P + 2.236X_4)(2.236\delta X_4)}^{\text{element 5}} = 0$$

Collecting terms and simplifying, we get

$$\left(10.47\frac{PL}{AE} + 23.95\frac{X_4 L}{AE}\right)\delta X_4 = 0 \qquad \text{[a]}$$

which means that

$$X_4 = -\frac{10.47}{23.95}P = -0.4374P$$

The redundant load X_4 has thus been found in terms of the applied load P. Substituting X_4 into Figure 7.4.7, we obtain the results shown in Figure 7.4.9. Observe that the actual direction of X_3 turned out to be opposite to what we assumed.

7.4 Statically Indeterminate Trusses

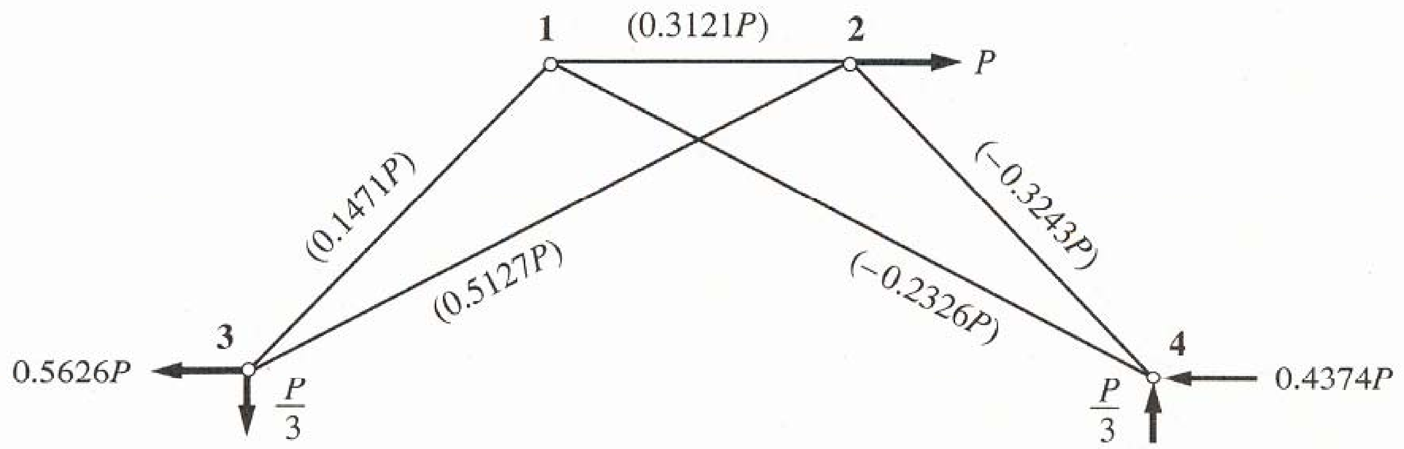


Figure 7.4.9 Reactions and member forces for the truss of Figure 7.4.6

7.4 Statically Indeterminate Trusses

Example 7.4.2 Solve the problem in the previous example by selecting member 1 of the truss as the redundant.

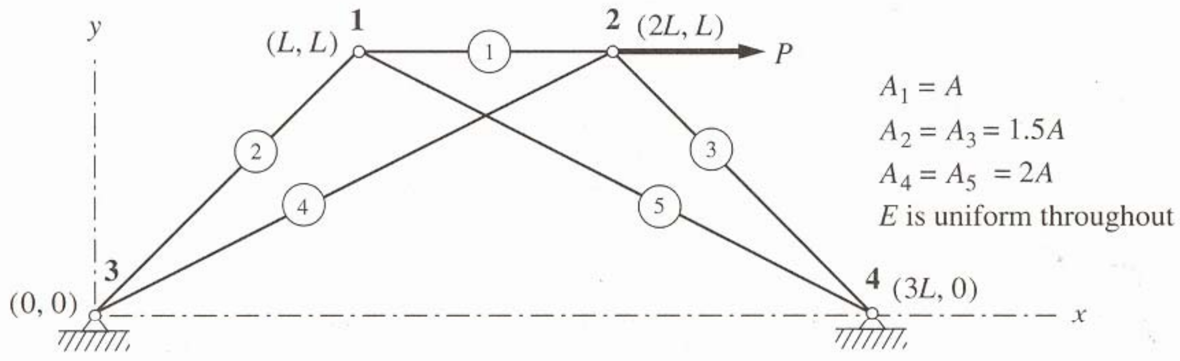


Figure 7.4.6 Truss with single degree of indeterminacy.

7.4 Statically Indeterminate Trusses

With the horizontal member 1 of the truss in Figure 7.4.6 chosen as the redundant, the base structure becomes that shown in Figure 7.4.10, and statics yields the true reactions and member loads, also shown. Removing the true loads and applying just a virtual redundant load yields the situation illustrated in Figure 7.4.11.

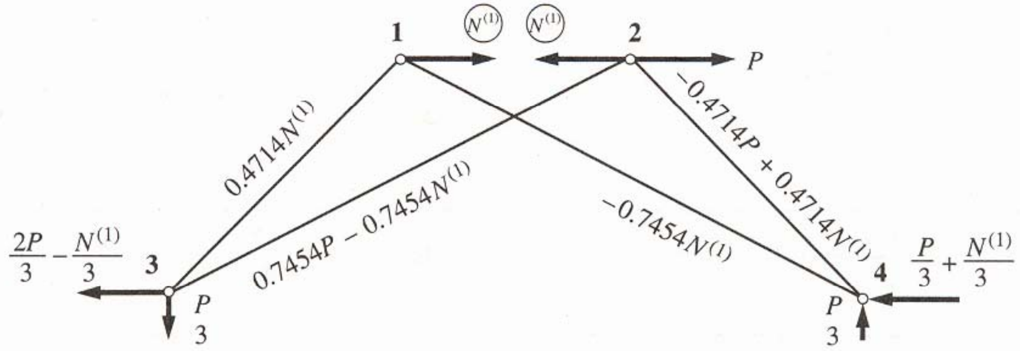


Figure 7.4.10 Base structure of the truss in Figure 7.4.6 when member 1 is selected as the redundant.

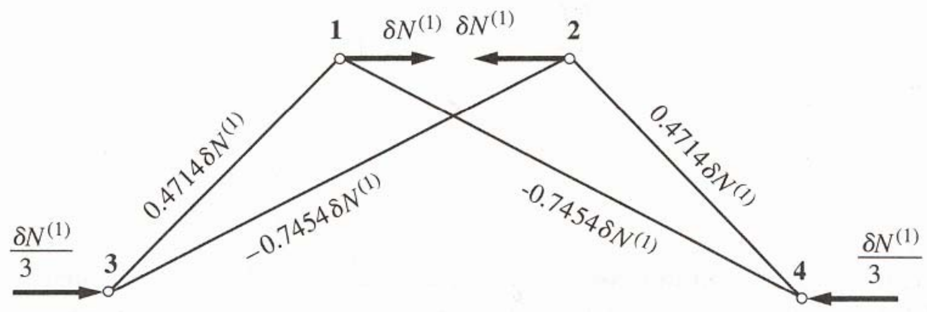


Figure 7.4.11 Virtual loads due to virtual redundant $\delta N^{(1)}$.

7.4 Statically Indeterminate Trusses

Since the redundant is an internal force, the external complementary virtual work is zero. Therefore, according to the principle of complementary virtual work, the internal complementary virtual work δW^*_{int} must also be zero. That is,

$$\overbrace{\frac{L}{AE}N^{(1)}\delta N^{(1)}}^{\text{element 1}} + \overbrace{\frac{1.414L}{1.5AE}(0.4714N^{(1)})(0.4714\delta N^{(1)})}^{\text{element 2}} + \overbrace{\frac{1.414L}{1.5AE}(-0.4714P + 0.4714N^{(1)})(0.4714\delta N^{(1)})}^{\text{element 3}}$$

$$+ \overbrace{\frac{2.236L}{2AE}(0.7454P - 0.7454N^{(1)})(-0.7454\delta N^{(1)})}^{\text{element 4}} + \overbrace{\frac{2.236L}{2AE}(-0.7454N^{(1)})(-0.7454\delta N^{(1)})}^{\text{element 5}} = 0$$

or

$$\frac{L}{AE}(2.661N^{(1)} - 0.8307P)\delta N^{(1)} = 0$$

so that the redundant load, in terms of P, is

$$N^{(1)} = \frac{0.8307}{2.661}P = -0.3121P$$

We can substitute $N^{(1)}$ into Figure 7.4.10 to obtain the values for all the other initially unknown loads, which are identical to those in Figure 7.4.9, as the reader can verify.

7.4 Statically Indeterminate Trusses

Example 7.4.3 Calculate the vertical component of the displacement of node 1 of the staticallty indeterminate truss of examples 7.4.1 and 7.4.2.

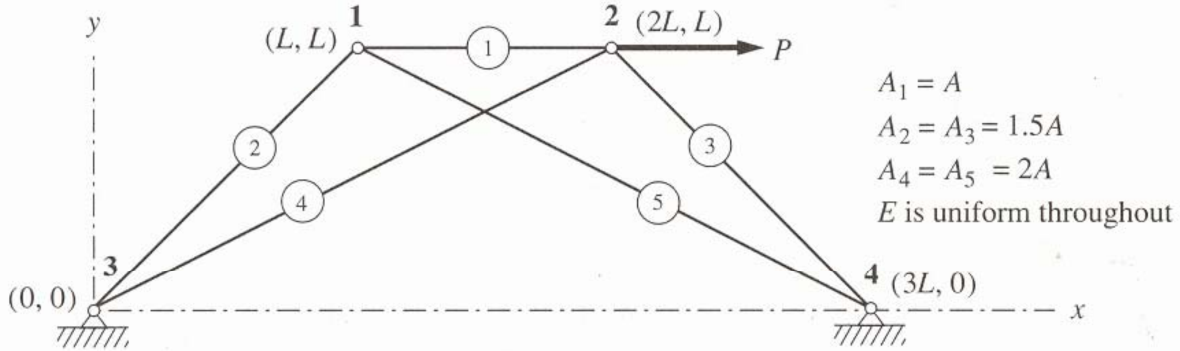


Figure 7.4.6 Truss with single degree of indeterminacy.

7.4 Statically Indeterminate Trusses

For convenience, the structure and its computed loads are reproduced in Figure 7.4.12.

To calculate the vertical displacement of node 1, we apply a vertical virtual force to that point of the truss. That virtual load can be supported by any stable, statically determinate substructure of the complete truss. For example, we can imagine δP to be supported entirely by members 2 and 5, as illustrated in Figure 7.4.13, which also shows the rod loads required for node 1 to be in equilibrium. The virtual forces throughout the rest of the truss may be assumed to be zero. Therefore, according to the principle of complementary virtual work, we have

$$v_1 \delta P = \frac{N^{(2)} L^{(2)}}{A^{(2)} E^{(2)}} \delta N^{(2)} + \frac{N^{(5)} L^{(5)}}{A^{(5)} E^{(5)}} \delta N^{(5)} = \frac{(0.1471P)(1.414L)}{1.5AE}(0.9428\delta P) + \frac{(-0.2327P)(2.236L)}{2AE}(0.7454\delta P)$$

$$= -0.06310 \frac{PL}{AE} \delta P$$

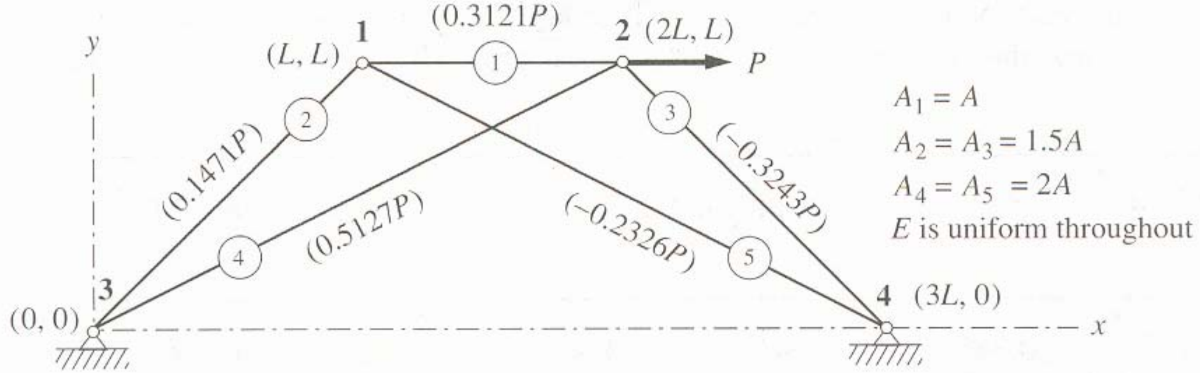


Figure 7.4.12 Computed member loads (in parentheses) in the statically indeterminate truss of the previous two examples.

7.4 Statically Indeterminate Trusses

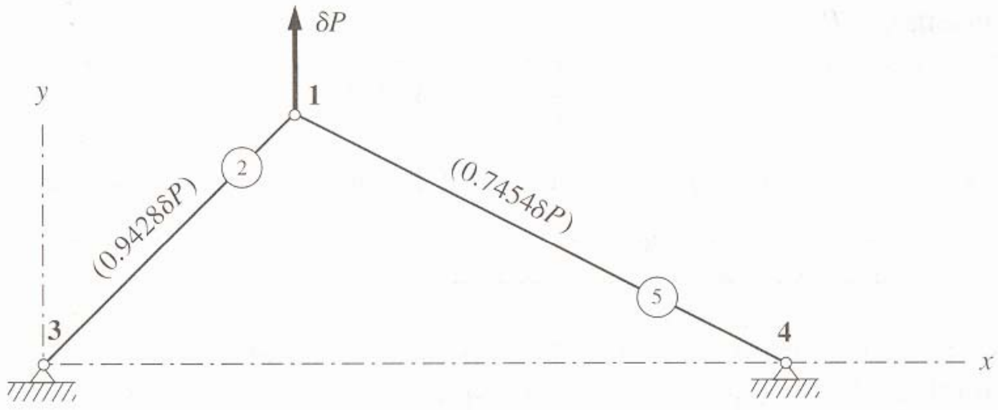


Figure 7.4.13 Statically determinate substructure of the truss containing node 1.

The vertical displacement of the node is then

$$v_1 = -0.06310\frac{PL}{AE} \qquad [a]$$

7.4 Statically Indeterminate Trusses

The minus sign means that the displacement is downward, in the direction opposite to that of the virtual load δP.

To illustrate the fact that *any* statically determinate substructure containing node 1 may be used, let us choose the one shown in Figure 7.4.14. Analysis of the equilibrium of nodes 1 and 2 yields the internal virtual forces written alongside each rod. The virtual forces throughout the remainder of the complete truss are zero. Once again, the principle of complementary virtual work requires that

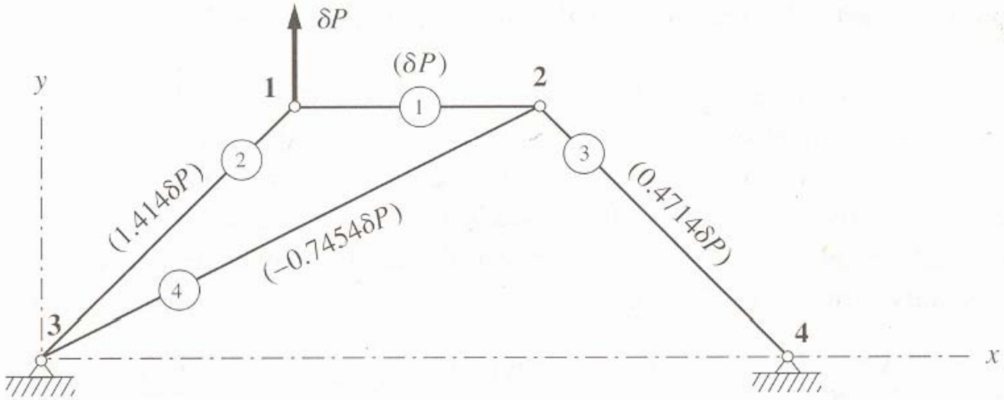


Figure 7.4.14 Alternative choice of statically determinate substructure to support the virtual load at node 1.

7.4 Statically Indeterminate Trusses

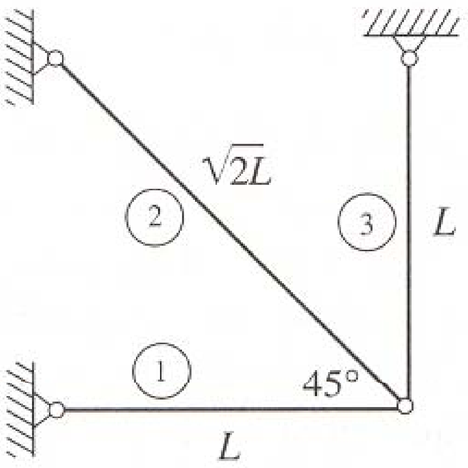
$$
\begin{aligned}
v_1 \delta P &= \frac{N^{(1)}L^{(1)}}{A^{(1)}E^{(1)}}\delta N^{(1)} + \frac{N^{(2)}L^{(2)}}{A^{(2)}E^{(2)}}\delta N^{(2)} + \frac{N^{(3)}L^{(3)}}{A^{(3)}E^{(3)}}\delta N^{(3)} + \frac{N^{(4)}L^{(4)}}{A^{(4)}E^{(4)}}\delta N^{(4)} \\
&= \overbrace{\frac{(0.3121P)(L)}{AE}(\delta P)}^{\text{rod 1}} + \overbrace{\frac{(0.1471P)(1.414L)}{(1.5A)E}(1.414\delta P)}^{\text{rod 2}} \\
&\quad + \overbrace{\frac{(-0.3243P)(1.414L)}{(1.5A)E}(0.4714\delta P)}^{\text{rod 3}} + \overbrace{\frac{(0.5127P)(2.236L)}{(2A)E}(-0.7454\delta P)}^{\text{rod 4}} \\
&= -0.06308\frac{PL}{AE}\delta P
\end{aligned}
$$

so that, as before,

$$
v_1 = -0.06310\frac{PL}{AE} \qquad \text{[b]}
$$

7.4 Statically Indeterminate Trusses

Example 7.4.4 Each member of the indeterminate truss in Figure 7l4l15 undergoes a uniform temperature rise from ambient, in which state the truss is unstressed. Assuming the axial rigidity AE and the thermal expansion coefficient are the same for all of the members, find the internal loads.



A, E, α uniform throughout
Temperature change in each rod: T_1, T_2, T_3

Figure 7.4.15 Thermally-loaded indeterminate truss.

7.4 Statically Indeterminate Trusses

Select member 2 as the redundant, so that the base structure is as shown in Figure 7.4.16a, along with the true loads. Removing the true loads and applying a virtual redundant load yields the same picture, Figure 7.4.16b. Since the supports are immobile, the external complementary virtual work is zero. Therefore, $\delta W^*_{\text{int}} = 0$, and from Equation 7.2.2,

$$\delta W^*_{\text{int}} = \sum_{e=1}^{3} \delta W^{*(e)}_{\text{int}} = \sum_{e=1}^{3} \left(\frac{N^{(e)}}{A^{(e)}E^{(e)}} + \alpha^{(e)} T^{(e)} \right) L^{(e)} \delta N^{(e)} = 0$$

$$\overbrace{\left(\frac{-\frac{\sqrt{2}}{2} N^{(2)}}{AE} + \alpha T^{(1)} \right) L \left(-\frac{\sqrt{2}}{2} \delta N^{(2)} \right)}^{\text{element 1}} + \overbrace{\left(\frac{N^{(2)}}{AE} + \alpha T^{(2)} \right) \sqrt{2} L \delta N^{(2)}}^{\text{element 2}} + \overbrace{\left(\frac{-\frac{\sqrt{2}}{2} N^{(2)}}{AE} + \alpha T^{(3)} \right) L \left(-\frac{\sqrt{2}}{2} \delta N^{(2)} \right)}^{\text{element 3}} = 0$$

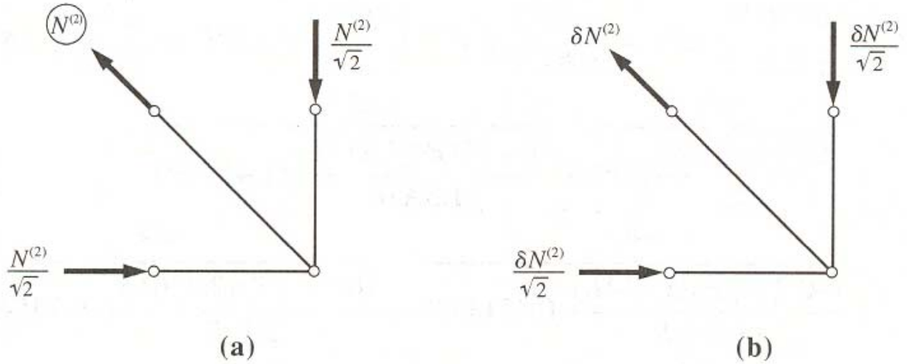


Figure 7.4.16 (a) Base truss showing the redundant load and the reactions to it. (b) The virtual loads.

7.4 Statically Indeterminate Trusses

This simplifies to

$$\left\{\left(1+\sqrt{2}\right)\frac{N^{(2)}L}{AE}+\alpha L\left[\sqrt{2}T^{(2)}-\frac{\sqrt{2}}{2}\left(T^{(1)}+T^{(3)}\right)\right]\right\}\delta N^{(2)}=0$$

Solving for $N^{(2)}$, we have

$$N^{(2)}=\frac{\sqrt{2}}{1+\sqrt{2}}AE\alpha\left[\frac{1}{2}\left(T^{(1)}+T^{(3)}\right)-T^{(2)}\right]$$

Then, from Figure 7.4.16a, we get

$$N^{(1)}=N^{(3)}=-\frac{\sqrt{2}}{2}N^{(2)}=-\frac{1}{1+\sqrt{2}}AE\alpha\left[\frac{1}{2}\left(T^{(1)}+T^{(3)}\right)-T^{(2)}\right]$$

If the temperature change is the same for all of the rods ($T^{(1)}=T^{(2)}=T^{(3)}=T$), then $N^{(1)}=N^{(2)}=N^{(3)}=0$, that is, no loads develop within the structure if the heating is uniform. Otherwise, axial loads will accompany temperature changes. For example, if $T^{(2)}=T$ and $T^{(1)}=T^{(3)}=0$, then $P^{(1)}=P^{(3)}=0.4142AE\alpha T$ and $P^{(2)}=-0.5858AE\alpha T$; if $T^{(1)}=T$ and $T^{(2)}=T^{(3)}=0$, then $P^{(1)}=P^{(3)}=-0.2071AE\alpha T$ and $P^{(2)}=-0.2929AE\alpha T$.

7.4 Statically Indeterminate Trusses

Example 7.4.5 Find the forces throughout the truss in Figure 7.4.17, using the principle of complementary virtual work. The axial rigidity AE is the same for all of the members.

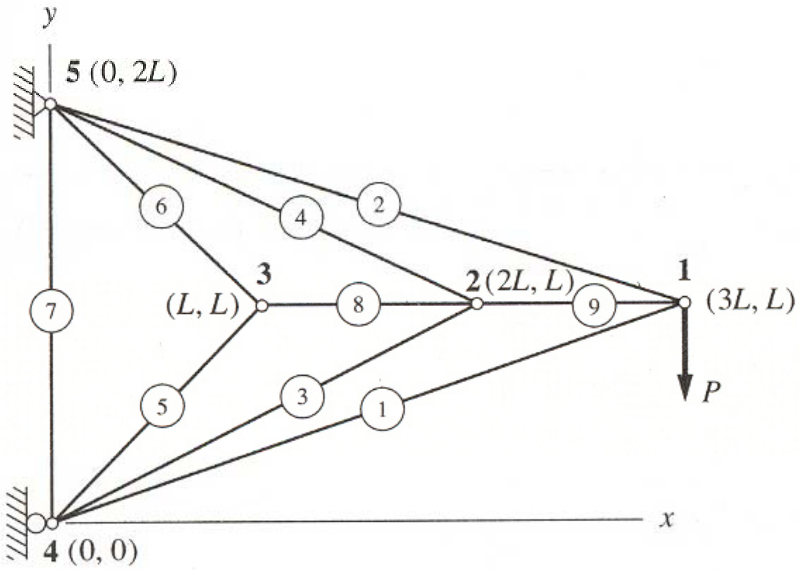


Figure 7.4.17 A truss with two degrees of static indeterminacy.

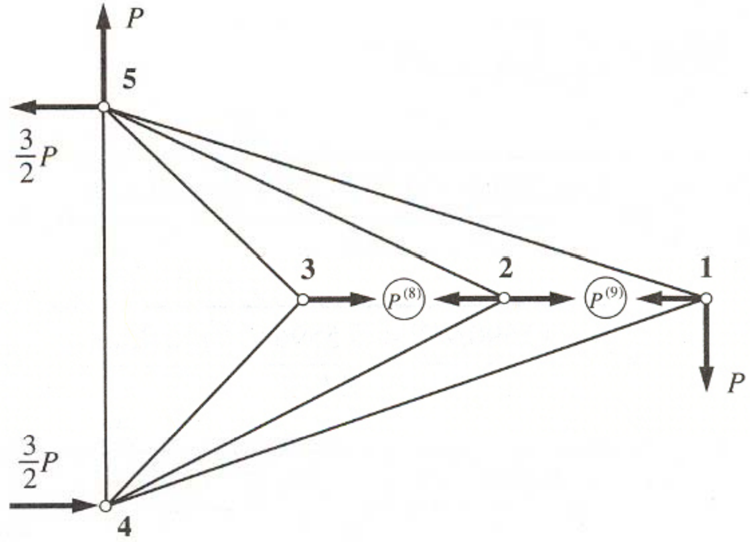


Figure 7.4.18 Statically determinate base structure and the true loading for the truss in Figure 7.4.17.

7.4 Statically Indeterminate Trusses

This truss has two redundants, which we can select as members 8 and 9, in which case the base structure is as shown in Figure 7.4.18. The reactions shown in the figure are obtained from statics, and the joint method of equilibrium analysis yields the member loads in terms of the applied loads P and the redundants $N^{(8)}$ and $N^{(9)}$. These are listed here in Equation a. The virtual loads on the base structure are shown in Figure 7.4.19. The corresponding virtual member loads can be obtained from the true member loads by setting the true applied load P equal to zero and replacing the true redundant member loads $N^{(8)}$ and $N^{(9)}$ by their virtual counterparts, $\delta N^{(8)}$ and $\delta N^{(9)}$, respectively. In Equation a all of the self-equilibrating virtual member loads are listed alongside the true loads.

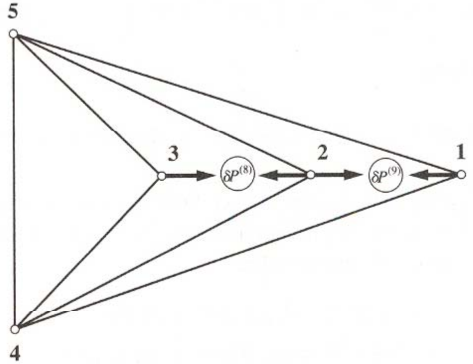


Figure 7.4.19 Virtual loads on the base truss.

True Loads	Virtual Loads
$N^{(1)} = -1.581P - 0.5270N^{(9)}$	$\delta N^{(1)} = -0.5270\delta N^{(9)}$
$N^{(2)} = 1.581P - 0.5270N^{(9)}$	$\delta N^{(2)} = -0.5270\delta N^{(9)}$
$N^{(3)} = -0.5590N^{(8)} + 0.5590N^{(9)}$	$\delta N^{(3)} = -0.5590\delta N^{(8)} + 0.5590\delta N^{(9)}$
$N^{(4)} = -0.5590N^{(8)} + 0.5590N^{(9)}$	$\delta N^{(4)} = -0.5590\delta N^{(8)} - 0.5590\delta N^{(9)}$
$N^{(5)} = 0.7071N^{(8)}$	$\delta N^{(5)} = 0.7071\delta N^{(8)}$
$N^{(6)} = 0.7071N^{(8)}$	$\delta N^{(6)} = 0.7071\delta N^{(8)}$
$N^{(7)} = 0.5P - 0.25N^{(8)} - 0.08333N^{(9)}$	$\delta N^{(7)} = -0.25\delta N^{(8)} - 0.08333\delta N^{(9)}$

[a]

7.4 Statically Indeterminate Trusses

The internal complementary virtual work of the truss is as follows:

$$\delta W^*_{\text{int}} = \sum_{e=1}^{12} \frac{N^{(e)} L^{(e)}}{A^{(e)} E^{(e)}} \delta N^{(e)}$$

$$= \overbrace{\frac{(-1.581P - 0.5270N^{(9)})\,(3.162L)}{AE}(-0.5270\delta N^{(9)})}^{(1)} + \overbrace{\frac{(1.581P - 0.5270N^{(9)})\,(3.162L)}{AE}(-0.5270\delta N^{(9)})}^{(2)}$$

$$+ \overbrace{\frac{(-0.5590N^{(8)} + 0.5590N^{(9)})\,(2.236L)}{AE}(-0.5590\delta N^{(8)} + 0.5590\delta N^{(9)})}^{(3)}$$

$$+ \overbrace{\frac{(-0.5590N^{(8)} + 0.5590N^{(9)})\,(2.236L)}{AE}(-0.5590\delta N^{(8)} + 0.5590\delta N^{(9)})}^{(4)}$$

$$+ \overbrace{\frac{(0.7071N^{(8)})\,(1.414L)}{AE}(0.7071\delta N^{(8)})}^{(5)} + \overbrace{\frac{(0.7071N^{(8)})\,(1.414L)}{AE}(0.7071\delta N^{(8)})}^{(6)}$$

$$+ \overbrace{\frac{(0.5P - 0.25N^{(8)} - 0.08333N^{(9)})\,(2L)}{AE}(-0.25\delta N^{(8)} - 0.08333\delta N^{(9)})}^{(7)} + \overbrace{\frac{N^{(8)}L}{AE}\delta N^{(8)}}^{(8)} + \overbrace{\frac{N^{(9)}L}{AE}\delta N^{(9)}}^{(9)}$$

7.4 Statically Indeterminate Trusses

Collecting terms and simplifying, we arrive at

$$\delta W^*_{\text{int}} = \frac{L}{AE}[(3.937N^{(8)} - 1.356N^{(9)} - 0.25P)\delta N^{(8)} + (-1.356N^{(8)} + 4.168N^{(9)} - 0.08333P)\delta N^{(9)}] \qquad \textbf{[b]}$$

The external complementary virtual work δW^*_{ext} is zero, since both of the redundants are internal loads. Therefore, δW^*_{int} must also be zero, for any choice of $\delta N^{(8)}$ and $\delta N^{(9)}$. This implies that the coefficients of these two virtual loads must vanish. The two equations for the two unknown redundants are therefore

$$\begin{aligned} 3.937N^{(8)} \quad -1.356N^{(9)} &= 0.25P \\ -1.356N^{(8)} + 4.168N^{(9)} &= 0.08333P \end{aligned} \qquad \textbf{[c]}$$

The solution of this system is

$$N^{(8)} = 0.07927P \qquad N^{(9)} = 0.04578P$$

Substituting these two redundant forces into Equation a gives the loads in all of the members, as summarized in Figure 7.4.20.

7.4 Statically Indeterminate Trusses

Example 7.4.6 Use the principle of complementary virtual work to calculate the horizontal component of the displacement at node 1 of the previous example

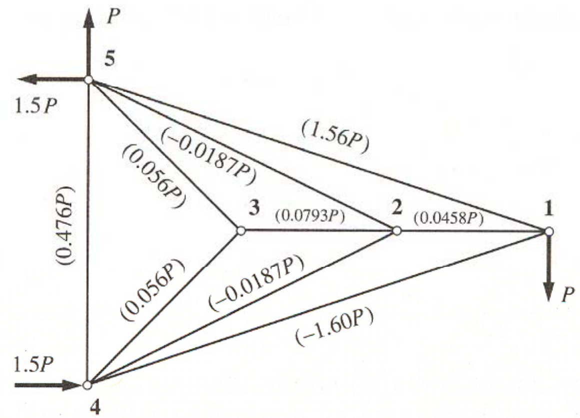


Figure 7.4.20 Computed loads in the truss of Figure 7.4.17.

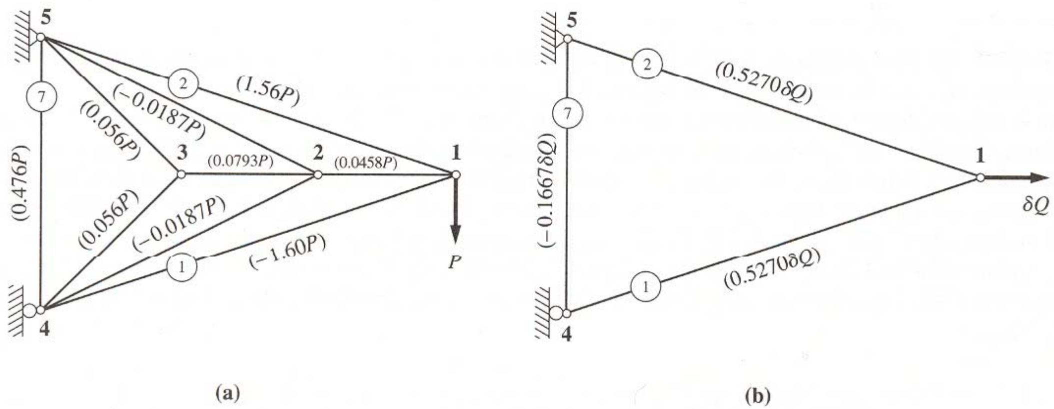


Figure 7.4.21 (a) The truss of Example 7.4.5 and the computed member loads. (b) Statically determinate substructure and the member loads due to a virtual load δQ applied to find the horizontal displacement of node 1.

7.4 Statically Indeterminate Trusses

Figure 7.4.21a shows the results of the analysis in Example 7.4.5. Using the principle of complementary virtual work to find the horizontal component of the displacement at node 1 requires applying a horizontal virtual load at that point, as shown in Figure 7.4.21b. Since the virtual loads need not satisfy compatibility, we can select any stable, statically determinate substructure of the truss to pick up the virtual load δQ. One of several such substructures is shown in Figure 7.4.21b and is comprised of only members 1, 2 and 7, the virtual loads in which are readily found from statics. Since the virtual loads in the rest of the truss are zero, the internal complementary virtual work is given by

$$\begin{aligned}\delta W^*_{\text{int}} &= \frac{N^{(1)}L^{(1)}}{A^{(1)}E^{(1)}}\delta N^{(1)} + \frac{N^{(2)}L^{(2)}}{A^{(2)}E^{(2)}}\delta N^{(2)} + \frac{N^{(7)}L^{(7)}}{A^{(7)}E^{(7)}}\delta N^{(7)} \\ &= \frac{(-1.6053P)(3.162L)}{AE}(0.5270\delta Q) + \frac{(1.557P)(3.162L)}{AE}(0.5270\delta Q) + \frac{(0.4764P)(2L)}{AE}(-0.1667\delta Q)\end{aligned}$$

so that

$$\delta W^*_{\text{int}} = -0.2392\frac{PL}{AE}\delta Q$$

In terms of the true horizontal displacement u_1 and the virtual load δQ, the external complementary virtual work is

$$\delta W^*_{\text{ext}} = u_1\delta Q$$

Since $\delta W^*_{\text{ext}} = \delta W^*_{\text{int}}$, we have

$$u_1 = -0.2392\frac{PL}{AE}$$

and the displacement is to the left.

7.4 Statically Indeterminate Trusses

Example 7.4.7 The truss in Figure 7.4.23 is identical to that in Example 7.4.5. However, in addition to the applied load P, the vertical displacement at the roller support 4 is prescribed to be d. Calculate the member loads.

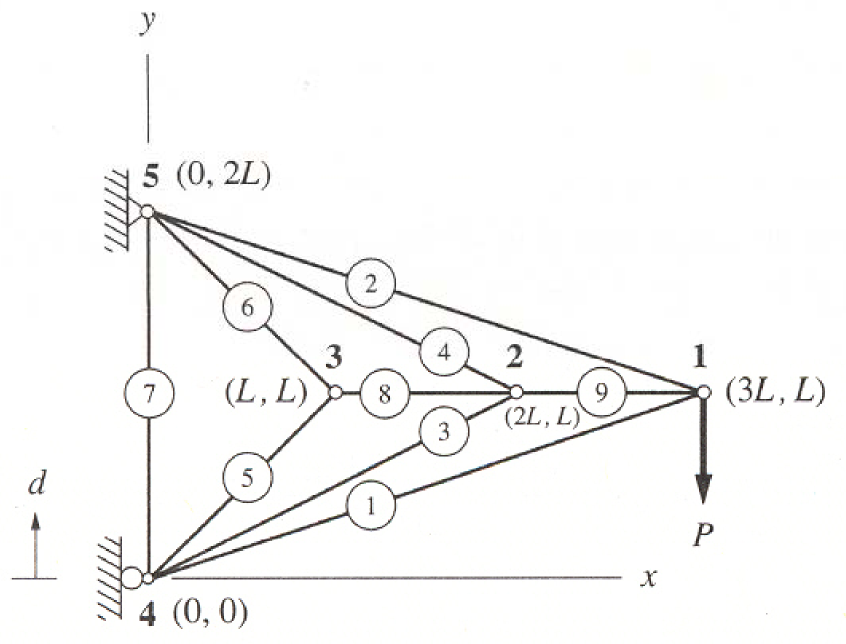


Figure 7.4.23 The truss of Figure 7.4.17, with a prescribed displacement at node 4.

7.4 Statically Indeterminate Trusses

The truss has two degrees of static indeterminacy, and in Example 7.4.5, the loads in members 8 and 9 were selected as the redundants. Since the load at 4 required to produce the specified displacement d is unknown, the degree of static indeterminacy increases from two to three. We will treat the vertical reaction Y_4 at node 4 as the additional redundant, although several other choices are apparent. Figure 7.3.24a shows the member loads, obtained from statics, as functions of the applied load P and the redundants $N^{(8)}$, $N^{(9)}$, and Y_4. Removing the true load P and applying virtual loads in place of the redundants yields the system illustrated in Figure 7.4.24b. Except for member 7, the loads in the rod elements are identical to those found in Example 7.4.5. Therefore, the only difference in the internal complementary virtual work will be that contributed by element 7. Thus,

$$\delta W^*_{\text{int}} = \sum_{i=1}^{9} \delta W^{*\,(i)}_{\text{int}} = \left(\sum_{i=1}^{6} \delta W^{*\,(i)}_{\text{int}} + \delta W^{*\,(8)}_{\text{int}} + \delta W^{*\,(9)}_{\text{int}} \right) + \delta W^{*\,(7)}_{\text{int}} \quad \text{[a]}$$

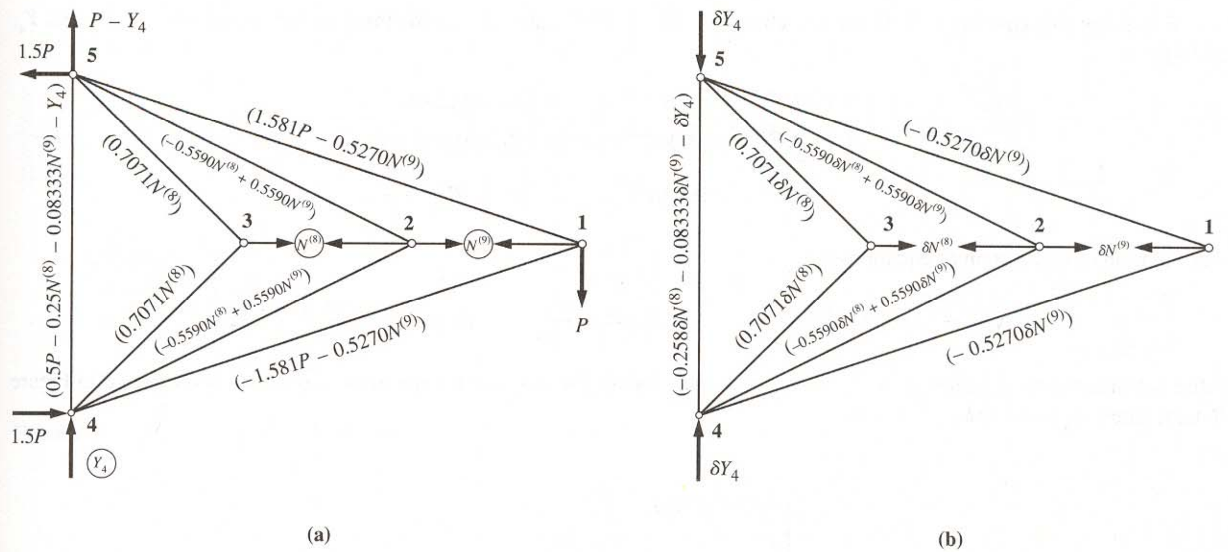


Figure 7.4.24 (a) Member loads in terms of the applied loads and the redundant loads (circled). (b) The virtual loading.

7.4 Statically Indeterminate Trusses

Using the calculations presented in Example 7.4.5, we get

$$\delta W^*_{\text{int}} = \frac{L}{AE}\left[\left(3.812N^{(8)} - 1.398N^{(9)}\right)\delta N^{(8)} + \left(-1.398N^{(8)} + 4.154N^{(9)}\right)\delta N^{(8)}\right] + \delta W^{*\,(7)}_{\text{int}} \quad \text{[b]}$$

Since, in this case, we have

$$\delta W^{*(7)}_{\text{int}} = \frac{N^{(7)}L^{(7)}}{A^{(7)}E^{(7)}}\delta N^{(7)} = \frac{(0.5P - 0.25N^{(8)} - 0.08333N^{(9)} - Y_4)(2L)}{AE}(-0.25\delta N^{(8)} - 0.08333\delta N^{(9)} - \delta Y_4)$$

Equation b yields, after substitution,

$$\delta W^*_{\text{int}} = \frac{L}{AE}\Big[\left(3.937N^{(8)} - 1.356N^{(9)} + 0.5Y_4 - 0.25P\right)\delta N^{(8)}$$
$$+ \left(-1.356N^{(8)} + 4.168N^{(9)} + 0.1667Y_4 - 0.08333P\right)\delta N^{(9)} + \left(0.5N^{(8)} + 0.1667N^{(9)} + 2Y_4 - P\right)\delta Y_4\Big]$$

This expression must be equated to the external complementary virtual work, which is the product of the true prescribed displacement d at node 4 and the external virtual load in the direction of d. That is,

$$\delta W^*_{\text{ext}} = \delta Y_4 \times d$$

Since $\delta W^*_{\text{ext}} = \delta W^*_{\text{int}}$, after rearranging terms, we therefore obtain

$$\frac{L}{AE}\left(3.937N^{(8)} - 1.356N^{(9)} + 0.5Y_4 - 0.25P\right)\delta N^{(8)}$$
$$+ \frac{L}{AE}\left(-1.356N^{(8)} + 4.168N^{(9)} + 0.1667Y_4 - 0.08333P\right)\delta N^{(9)}$$
$$+ \frac{L}{AE}\left(0.5N^{(8)} + 0.1667N^{(9)} + 2Y_4 - P - \frac{AE}{L}d\right)\delta Y_4 = 0$$

7.4 Statically Indeterminate Trusses

Requiring this equality to hold for any choice of $\delta P^{(8)}$, $\delta P^{(9)}$, and δY_4 yields three equations for $N^{(8)}$, $N^{(9)}$, and Y_4, as follows:

$$\begin{aligned}
3.937N^{(8)} - 1.356N^{(9)} + 0.5Y_4 &= 0.25P \\
-1.356N^{(8)} + 4.168N^{(9)} + 0.1667Y_4 &= 0.08333P \\
0.5N^{(8)} + 0.1667N^{(9)} + 2Y_4 &= P + \frac{AE}{L}d
\end{aligned}$$

The solution of this system of equations is

$$N^{(8)} = -0.08320\frac{AEd}{L} \qquad N^{(9)} = 0.04805\frac{AEd}{L} \qquad Y_4 = 0.5248\frac{AEd}{L} + 0.5P$$

After substituting these values of $N^{(8)}$, $N^{(9)}$, and Y_4 into Figure 7.4.24a, we find the truss loads to be as illustrated in Figure 7.4.25, where $F = AEd/L$.

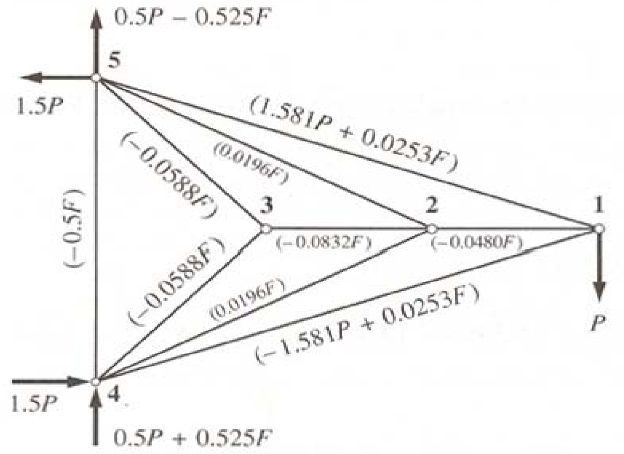


Figure 7.4.25 Member loads and reactions for the truss of Figure 7.4.23. (F=AEd/L.)

7.5 SIMPLE BEAMS: Complementary Virtual Work

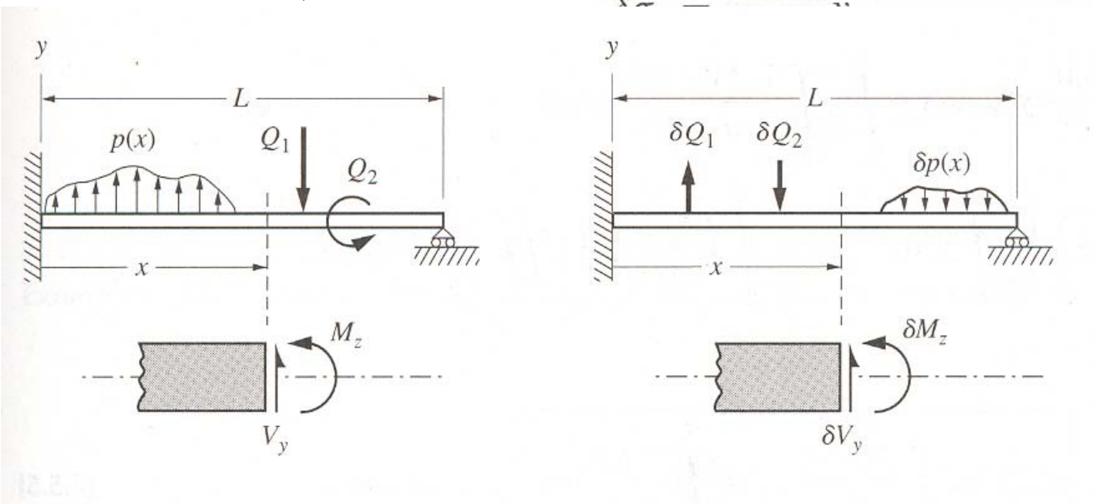
At any point in the cross section of a beam loaded in the *xy* plane, there is a normal stress σ_x due to the bending moment M_z and a shear stress τ_{xy} due to shear load V_y on the section.

Assuming, for simplicity,

- symmetry of the beam cross section => $I_{yz} = 0$
- $M_y = 0$

The normal stress at a distance y from the neutral axis is given by $\sigma_x = -\frac{M_z}{I_z} y$

For a virtual load, this becomes $\delta\sigma_x = -\frac{\delta M_z}{I_z} y$



(a) Actual load (b) Virtual load

7.5 SIMPLE BEAMS: Complementary Virtual Work

The shear stress at distance *y* from the neutral axis is

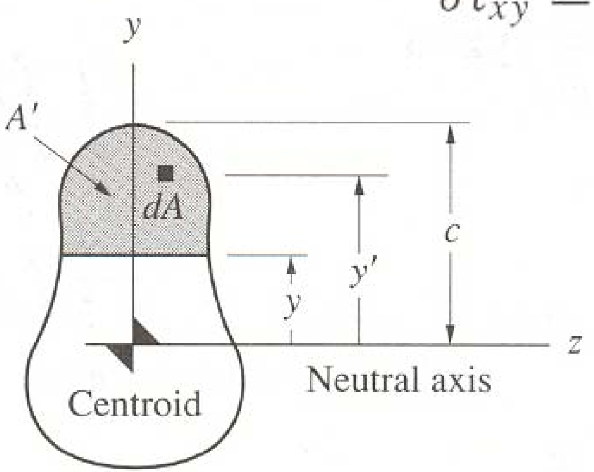
$$\tau_{xy} = \frac{Q_z V_y}{I_z t}$$

Where Q_z is the first moment about the neutral axis of the shaded area A', that is,

$$Q_z = \iint_{A'} y' dA$$

The shear stress arising from a virtual load is given by the same formula,

$$\delta\tau_{xy} = \frac{Q_z \delta V_y}{I_z t}$$



7.5 SIMPLE BEAMS: Complementary Virtual Work

Using these equations, as well as Hooke's Law for isotropic materials

$$
\begin{aligned}
\delta W^*_{\text{int}} &= \iiint\limits_V \left(\delta\sigma_x \varepsilon_x + \delta\tau_{xy}\gamma_{xy}\right) dV \\
&= \iiint\limits_V \left[\delta\sigma_x \left(\frac{\sigma_x}{E}\right) + \delta\tau_{xy}\left(\frac{\tau_{xy}}{G}\right)\right] dV \\
&= \int_0^L \left(\iint\limits_A \frac{\sigma_x \delta\sigma_x}{E} dA\right) dx + \int_0^L \left(\iint\limits_A \frac{\tau_{xy}\delta\tau_{xy}}{G} dA\right) dx \\
&= \int_0^L \iint\limits_A \frac{M_z \delta M_z}{E I_z^2} y^2 dA dx + \int_0^L \iint\limits_A \frac{V_y \delta V_y}{G} \frac{Q_z^2}{I_z^2 t^2} dA dx \\
&= \int_0^L \left(\frac{M_z \delta M_z}{E I_z^2}\right)\left(\iint\limits_A y^2 dA\right) dx + \int_0^L \left(\frac{V_y \delta V_y}{G I_z^2}\right)\left(\iint_A \frac{Q_z^2}{t^2} dA\right) dx
\end{aligned}
$$

$$
\boxed{\delta W^*_{\text{int}} = \underbrace{\int_0^L \frac{M_z \delta M_z}{E I_z} dx}_{\delta W^*_{\text{int,bending}}} + \underbrace{\int_0^L \frac{V_y \delta V_y}{G A_s} dx}_{\delta W^*_{\text{int,shear}}}}
$$

simple beam

7.5 SIMPLE BEAMS: Complementary Virtual Work

- The familiar definition of area moment of inertia, $I_z = \int y^2 dA$,
- A new quantity called the *area effective in shear, A_s* ,

$$A_s = \frac{I_z^2}{\int\limits_A \left(\frac{Q_z}{t}\right)^2 dA}$$

- The product EI_Z is *the flexural rigidity* and GA_s is *the shear rigidity*.

The interpretation of A_s is as follows. Recall that the shear stress distribution over a cross section is not uniform. In the case of a rectangular section, it varies parabolically; for other sections, the distribution is more complex. Suppose we define a *nominal* shear stress $\boldsymbol{\tau} = V_y/A_s$, which is uniformly distributed over the cross section A_s. Let the corresponding shear strain be denoted γ. Using these quantities to calculate the internal complementary virtual work due to shear would yield the following:

$$\delta W^*_{\text{int, shear}} = \iiint\limits_V \gamma \delta \tau dV = \iiint\limits_V \left(\frac{\tau}{G}\right) \delta \tau dV = \int_0^L \iint\limits_{A_s} \left(\frac{V_y}{GA_s}\right)\left(\frac{\delta V_y}{A_s}\right) dA dx = \int_0^L \frac{V_y \delta V_y}{GA_s} dx$$

7.5 SIMPLE BEAMS: Complementary Virtual Work

We can now introduce the form factor

$$k = \frac{A_s}{A}$$

in terms of which the shear component of the complementary internal virtual work can be written as follows:

$$\delta W^*_{\text{int}} = \int_0^L \frac{V_y \delta V_y}{kGA} dx$$

The form factor for a given section is

$$k = \frac{\frac{I_z^2}{A}}{\iint\limits_A \left(\frac{Q_z}{t}\right)^2 dA}$$

7.5 SIMPLE BEAMS: Complementary Virtual Work

Example 7.5.1 Use Equation 7.5.9 to calculate the form factor for the rectangular section illustrated in Figure 7.5.6.

$$k = \frac{\frac{I_z^2}{A}}{\iint\limits_A \left(\frac{Q_z}{t}\right)^2 dA}$$

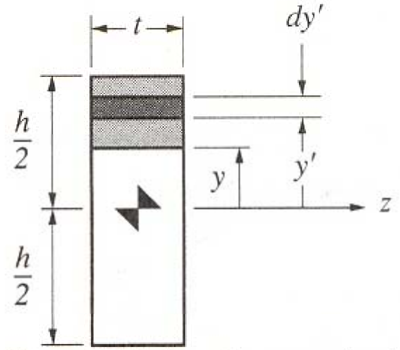


Figure 7.5.6 Rectangular beam cross section.

7.5 SIMPLE BEAMS: Complementary Virtual Work

The area is $A = ht$, the centroidal area moment of inertia is $I_z = \frac{1}{12}th^3$, and Q_z is the first area moment of the cross-hatched area about the z axis, which is

$$Q_z(y) = \iint_A y' dA = \int_y^{\frac{h}{2}} y'\left(t dy'\right) = \frac{1}{2}ty'^2\Big|_y^{\frac{h}{2}} = \frac{t}{2}\left(\frac{h^2}{4} - y^2\right)$$

For the denominator of Equation 7.5.9, we thus have

$$\iint_A \left(\frac{Q_z}{t}\right)^2 dA = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{Q_z}{t}\right)^2 t\, dy = \frac{t}{4}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{h^2}{4} - y^2\right)^2 dy$$

$$= \frac{t}{2}\int_0^{\frac{h}{2}} \left(\frac{h^4}{16} - \frac{h^2y^2}{2} + y^4\right) dy = \frac{t}{2}\left(\frac{h^4 y}{16} - \frac{h^2y^3}{6} + \frac{y^5}{5}\right)\Bigg|_0^{\frac{h}{2}} = \frac{th^5}{120}$$

Substituting this result into Equation 7.5.9 yields

$$k = \frac{I_z^2}{A}\left(\frac{1}{\iint_A \left(\frac{Q_z}{t}\right)^2 dA}\right) = \frac{\left(\frac{1}{12}th^3\right)^2}{th}\left(\frac{1}{\frac{th^5}{120}}\right) = \frac{120}{144} = \frac{5}{6}$$

The form factor for a rectangular section is $5/6$. That is, for purposes of deflection analysis the shear force may be imagined to be uniformly distributed over 83 percent of a rectangular cross section.

7.6 Torsion: Complementary Virtual Work

The internal complementary virtual work for isotropic materials and including only the nonzero stress terms, is

$$\delta W^*_{\text{int}} = \iiint_V (\delta\tau_{xy}\gamma_{xy} + \delta\tau_{xz}\gamma_{xz})\,dV = \iiint_V \left(\delta\tau_{xy}\frac{\tau_{xy}}{G} + \delta\tau_{xz}\frac{\tau_{xz}}{G}\right) dV = \frac{1}{G}\iiint_V (\tau_{xy}\delta\tau_{xy} + \tau_{xz}\delta\tau_{xz})\,dV \quad [7.6.2]$$

where

$$\delta\tau_{xy} = \frac{\delta T}{J}\left(\frac{\partial\psi}{\partial y} - z\right) \qquad \delta\tau_{xz} = \frac{\delta T}{J}\left(\frac{\partial\psi}{\partial z} + y\right) \qquad [7.6.3]$$

Substituting Eq 7.6.1, 7.6.3 into Eq 7.6.2 yields

$$\delta W^*_{\text{int}} = \int_0^L \frac{T\,\delta T}{GJ^2}\left\{\iint_A \left[\left(\frac{\partial\psi}{\partial y} - z\right)^2 + \left(\frac{\partial\psi}{\partial z} + y\right)^2\right] dA\right\} dx \qquad [7.6.4]$$

after expanding the terms of the integrand within the curly brackets and *using the formula for the torsion constant J*

$$\iint_A \left[\left(\frac{\partial\psi}{\partial y} - z\right)^2 + \left(\frac{\partial\psi}{\partial z} + y\right)^2\right] dA = J + \iint_A \left[\left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - z\frac{\partial\psi}{\partial y} + y\frac{\partial\psi}{\partial z}\right] dA$$

7.6 Torsion: Complementary Virtual Work

Solving for J and making use of Equation 4.4.10 leads to the expression

$$J = \iint_A \left[\left(\frac{\partial \psi}{\partial y} - z \right)^2 + \left(\frac{\partial \psi}{\partial z} + y \right)^2 \right] dA - \iint_A \left\{ \frac{\partial}{\partial y}\left[\psi \left(\frac{\partial \psi}{\partial y} - z \right) \right] + \frac{\partial}{\partial z}\left[\psi \left(\frac{\partial \psi}{\partial z} + y \right) \right] \right\} dA$$

By applying the divergence theorem for the plane, the second integral over the arbitrarily shaped cross section A of the bar can be converted into a line integral around the boundary C of A, so that

$$J = \iint_A \left[\left(\frac{\partial \psi}{\partial y} - z \right)^2 + \left(\frac{\partial \psi}{\partial z} + y \right)^2 \right] dA - \oint_C \left[\psi \left(\frac{\partial \psi}{\partial y} - z \right) n_y + \psi \left(\frac{\partial \psi}{\partial z} + y \right) n_z \right] dl$$

According to Eq. 4.4.12, the line integral vanishes and we are left with the following *formula for the torsion constant* as an alternative to equation 4.4.14b:

$$J = \iint_A \left[\left(\frac{\partial \psi}{\partial y} - z \right)^2 + \left(\frac{\partial \psi}{\partial z} + y \right)^2 \right] dA$$

Substituting this into Eq. 7.6.4

y

$$\boxed{\delta W^*_{\text{int}} = \int_0^L \frac{T\,\delta T}{GJ} dx} \quad \text{torsion bar} \qquad [7.6.5]$$

$$\boxed{\delta W^*_{\text{int,torsion}} = \frac{TL}{GJ}\delta T}$$

if the torque is constant over the length of the torsion member,

7.7 Beam And Frame Deflections Using Complementary Virtual Work

Example 7.7.1 Calculate the vertical displacement and the rotation at the left end of

a cantilever beam under a uniformly distributed load, as in Figure 7.7.1,

on due to shear.

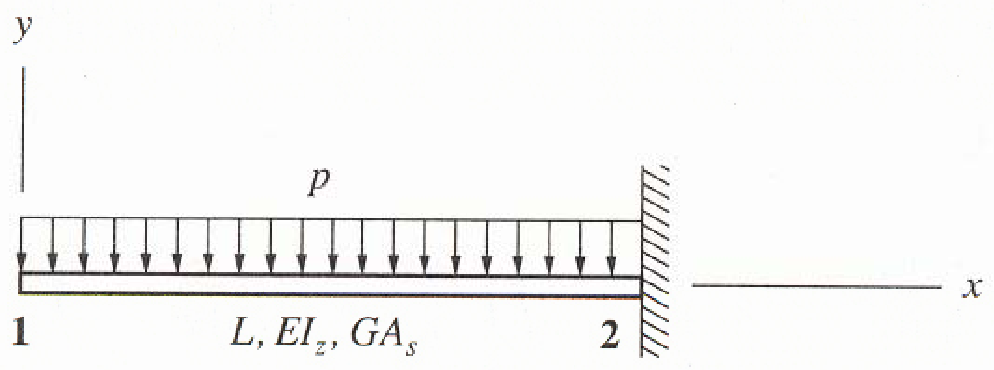


Figure 7.7.1 Uniformly-loaded simple cantilever beam.

7.7 Beam And Frame Deflections Using Complementary Virtual Work

First calculate internal loads (shear and moments)

then calculate virtual shear and bending moments to use eq. 7.5.5.

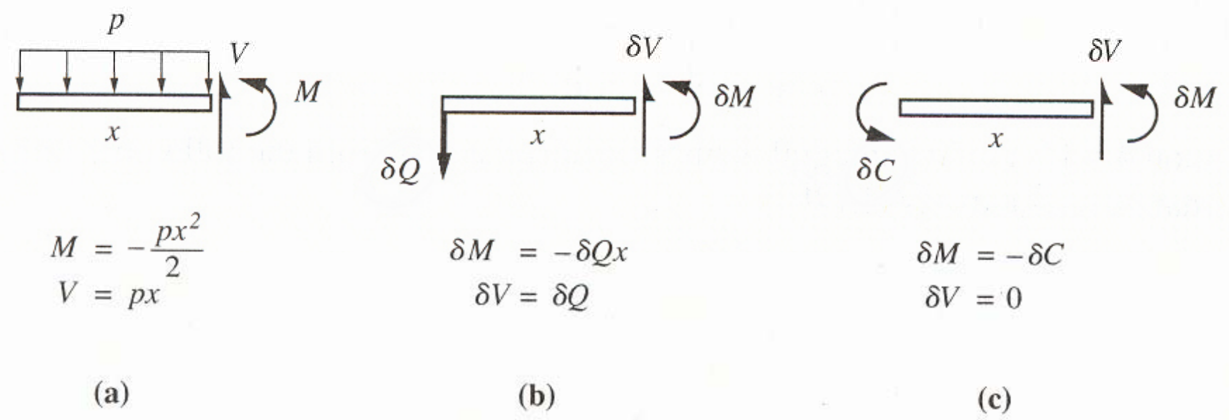


Figure 7.7.2 (a) Shear and bending moment due to the actual load on the beam. (b) and (c) Virtual shear and bending moments due to a virtual point load and point couple, respectively, applied at the free end.

$$\delta W_{\text{int}}^{*} = \int_0^L \frac{M_z \delta M_z}{EI_z} dx + \int_0^L \frac{V_y \delta V_y}{GA_s} dx \qquad (7.5.5)$$

then

$$v_1 \delta Q = \frac{1}{EI_z} \int_0^L \left(-\frac{ps^2}{2}\right)(-\delta Q\, s) ds + \frac{1}{GA_s} \int_0^L (ps)(\delta Q) ds$$

7.7 Beam And Frame Deflections Using Complementary Virtual Work

$$v_1 \delta Q = \frac{pL^4}{8EI_z}\delta Q + \frac{pL^2}{2GA_s}\delta Q$$

therefore, the transverse displacement is

$$v_1 = \underbrace{\frac{pL^4}{8EI_z}}_{\text{due to bending}} + \underbrace{\frac{pL^2}{2GA_s}}_{\text{due to shear}}$$

for the shear modulus: $G = \frac{1}{2}E/(1+\nu)$

set Poisson's ratio $\nu = 0.25$

let the effective area in shear equal to the total cross-section area $A_s = A$

$A = I_z / \rho_z^2$ (ρ_z is the radius of gyration of cross section)

$$v_1 = \frac{pL^4}{8EI_z}\left[1 + 10\left(\frac{\rho_z}{L}\right)^2\right]$$

The ratio of the shear portion of the displacement to the total displacement is

$$\frac{v_{1,\,\text{shear}}}{v_{1,\,\text{total}}} = \frac{10\left(\frac{\rho_z}{L}\right)^2}{1+10\left(\frac{\rho_z}{L}\right)^2} = \frac{1}{1+0.1\left(\frac{L}{\rho_z}\right)^2}$$

L/ρ_z is the *slenderness ratio.* the displacement decreases rapidly with increasing slenderness ratio.

The rotation of the end of the beam

$$\theta_1 \delta C = \frac{1}{EI_z}\int_0^L \left(-\frac{ps^2}{2}\right)(-\delta C)\,ds + \frac{1}{GA_s}\int_0^L (ps)\,(0)\,ds$$

$$\theta_1 = \frac{pL^3}{6EI_z}$$

7.7 Beam And Frame Deflections Using Complementary Virtual Work

Frames are composed of slender, possibly curved, beam-like elements capable of *carrying shear and bending loads, as well as axial loads* like the members of a truss.

Even though bending deflections usually dominate, we may wish to *consider the contribution of axial deformation, as well as shear deformation, in the frame members*. To do so, we must combine Equation 7.2.2 and 7.5.5, so that the internal complementary virtual work of a plane frame member (neglecting thermal strain) is

$$\delta W^*_{\text{int}} = \underbrace{\int_0^L \frac{M_z \delta M_z}{E I_z} ds}_{\textit{bending}} + \underbrace{\int_0^L \frac{V_y \delta V_y}{G A_s} ds}_{\textit{shear}} + \underbrace{\int_0^L \frac{N \delta N}{AE} ds}_{\textit{axial load}} \qquad [7.7.1]$$

where N is the force normal to the cross section of the member. In a curved frame like that of Figure 7.7.5 in witch the depth h of the cross section is much smaller than the radius of curvature *R*, we can use Equation 7.7.1, replacing *ds* with *Rdϕ*

7.7 Beam And Frame Deflections Using Complementary Virtual Work

Example 7.7.3 Find the horizontal displacement of the free end of the statically determinate, thin circular frame (curved beam) shown in Figure 7.7.6

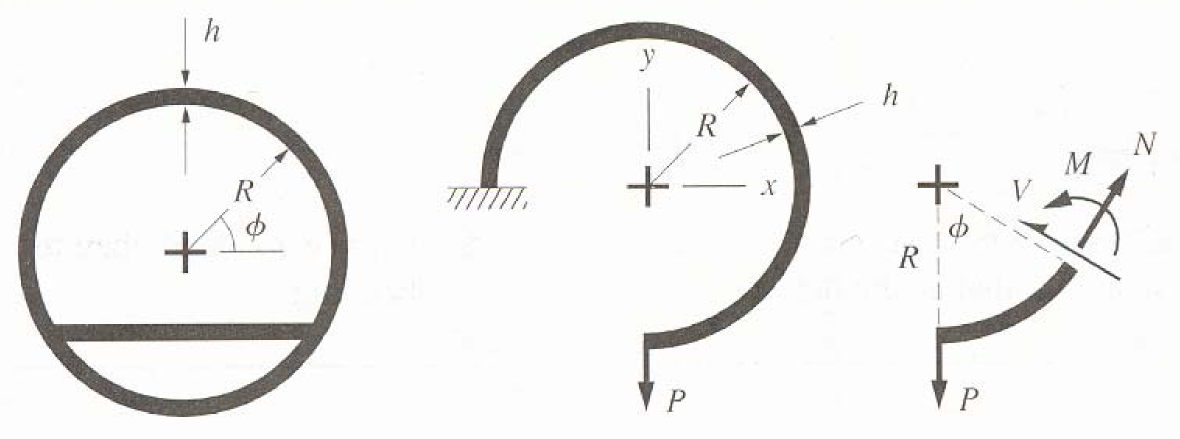
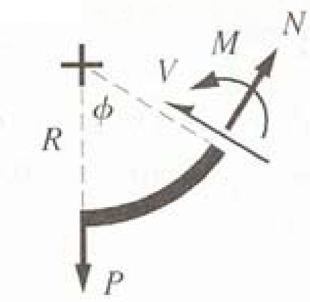


Figure 7.7.5 Circular frame in which $R >> h$.

Figure 7.7.6 Curved beam.

7.7 Beam And Frame Deflections Using Complementary Virtual Work



$$M = -PR\sin\phi$$

$$V = P\cos\phi$$

$$N = P\sin\phi$$

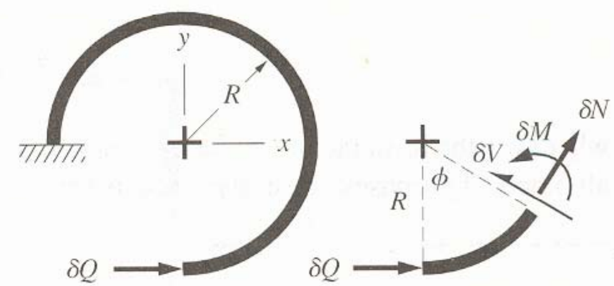


Figure 7.7.7 Virtual load on the circular frame.

$$\delta M = -\delta Q\, R(1-\cos\phi)$$

$$\delta V = \delta Q\, \sin\phi$$

$$\delta N = -\delta Q\, \cos\phi$$

$$\delta W^*_{\text{int}} = \int_0^L \frac{M_z \delta M_z}{EI_z} ds + \int_0^L \frac{V_y \delta V_y}{GA_s} ds + \int_0^L \frac{N\delta N}{AE} ds \qquad [7.7.1]$$

$$\delta W^*_{\text{int}} = \frac{1}{EI_z}\int_0^{\frac{3\pi}{2}} (-PR\sin\phi)\,[-\delta Q\, R\,(1-\cos\phi)]\, Rd\phi$$

$$+\frac{1}{GA_s}\int_0^{\frac{3\pi}{2}} (P\cos\phi)\,(\delta Q\, \sin\phi)\, Rd\phi + \frac{1}{AE}\int_0^{\frac{3\pi}{2}} (P\sin\phi)\,(-\delta Q\, \cos\phi)\, Rd\phi$$

7.7 Beam And Frame Deflections Using Complementary Virtual Work

So that

$$\delta W_{\text{int}}^{*} = \frac{\delta Q}{2}\left(\frac{PR^3}{EI_z} + \frac{PR}{GA_s} - \frac{PR}{AE}\right)$$

Setting this equal to the external complementary virtual work $\delta W_{\text{ext}}^{*} = u\delta Q$

$$u = \frac{PR^3}{2EI_z}\left\{1 + \left[\frac{2(1+\nu)}{k} - 1\right]\left(\frac{\rho_z}{R}\right)^2\right\}$$

7.7 Beam And Frame Deflections Using Complementary Virtual Work

Example 7.7.4 Use the principle of complementary virtual work to calculate the horizontal displacement of point 1 of the statically determinate frame in Figure 7.7.8a. The area, moment of inertia, and material properties are uniform throughout

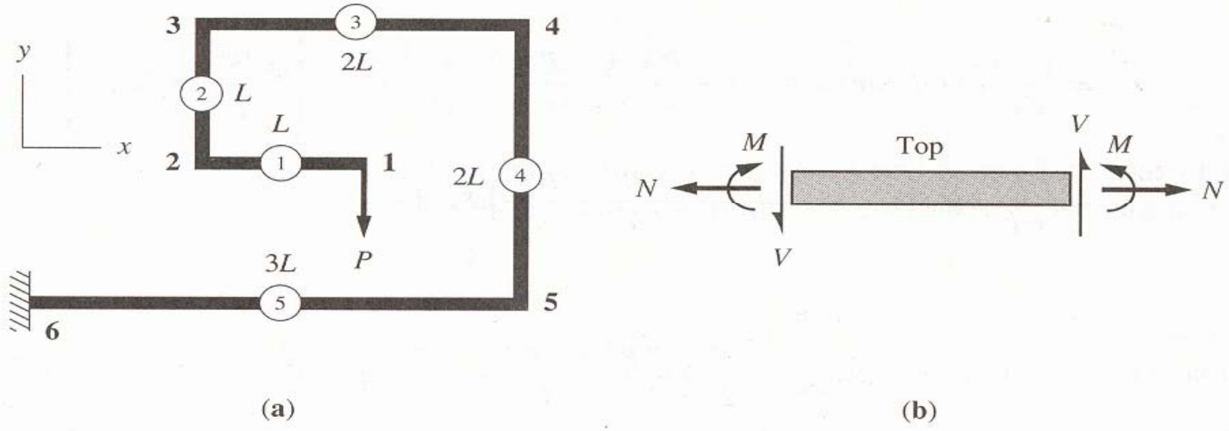


Figure 7.7.8 (a) Frame with load at point 1. (b) Sign convention for internal forces.

7.7 Beam And Frame Deflections Using Complementary Virtual Work

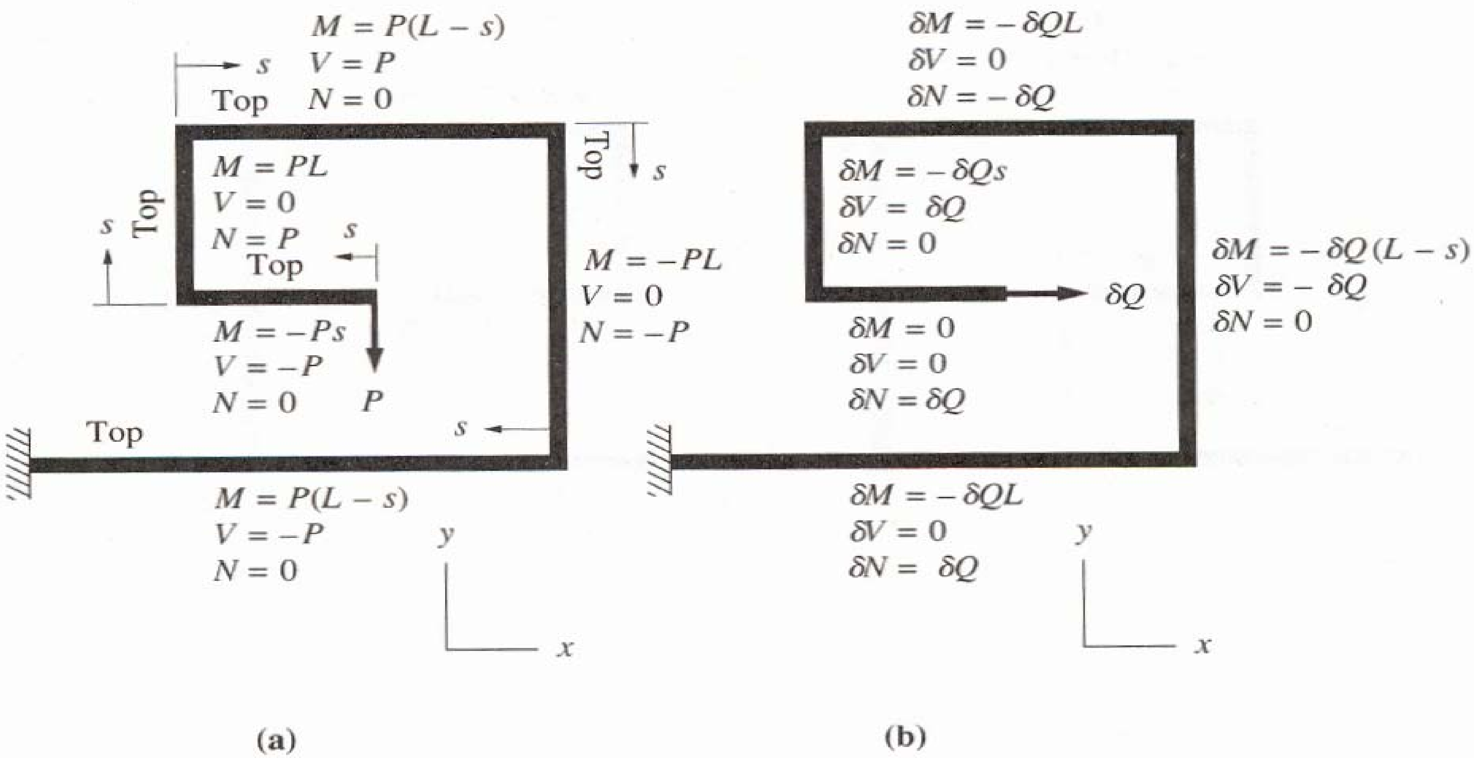


Figure 7.7.9 (a) Internal forces due to actual loading. (b) Internal forces due to virtual loading.

7.7 Beam And Frame Deflections Using Complementary Virtual Work

$$\delta W^*_{\text{int}} = \overbrace{\frac{1}{EI_z}\int_0^L (-Ps)(0)ds + \frac{1}{AE}\int_0^L (0)(\delta Q)ds + \frac{1}{GA_s}\int_0^L (-P)(0)ds}^{\text{beam 1}}$$

$$+ \overbrace{\frac{1}{EI_z}\int_0^L (PL)(-\delta Q\, s)ds + \frac{1}{AE}\int_0^L (P)(0)ds + \frac{1}{GA_s}\int_0^L (0)(\delta Q)ds}^{\text{beam 2}}$$

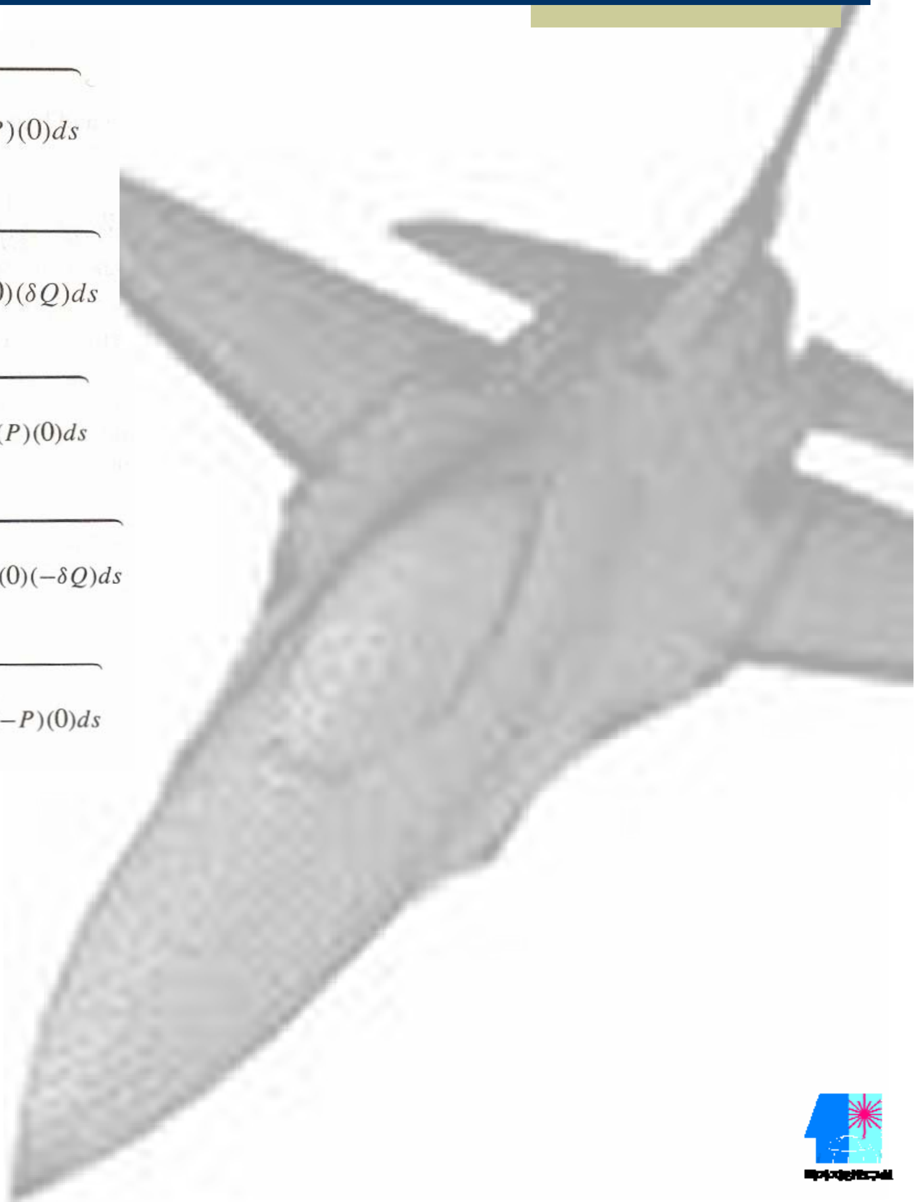
$$+ \overbrace{\frac{1}{EI_z}\int_0^{2L} [P\,(L-s)]\,(-\delta Q\, L)ds + \frac{1}{AE}\int_0^{2L} (0)(-\delta Q)ds + \frac{1}{GA_s}\int_0^{2L} (P)(0)ds}^{\text{beam 3}}$$

$$+ \overbrace{\frac{1}{EI_z}\int_0^{2L} (-PL)\,[-\delta Q(L-s)]\,ds + \frac{1}{AE}\int_0^{2L} (-P)(0)ds + \frac{1}{GA_s}\int_0^{L} (0)(-\delta Q)ds}^{\text{beam 4}}$$

$$+ \overbrace{\frac{1}{EI_z}\int_0^{3L} [P\,(L-s)][-\delta Q\, L]\,ds + \frac{1}{AE}\int_0^{3L} (0)\,(\delta Q)\,ds + \frac{1}{GA_s}\int_0^{3L} (-P)(0)ds}^{\text{beam 5}}$$

$$\delta W^*_{\text{int}} = \delta Q\left[0 + \left(-\frac{PL^3}{2EI_z}\right) + (0) + (0) + \left(\frac{3PL^3}{2EI_z}\right)\right]$$

$$u_1 = \frac{PL^3}{EI_z}$$



7.7 Beam And Frame Deflections Using Complementary Virtual Work

Example 7.7.5 Use the principle of complementary virtual work to calculate the rotation

at point 4 of the frame of the previous example.

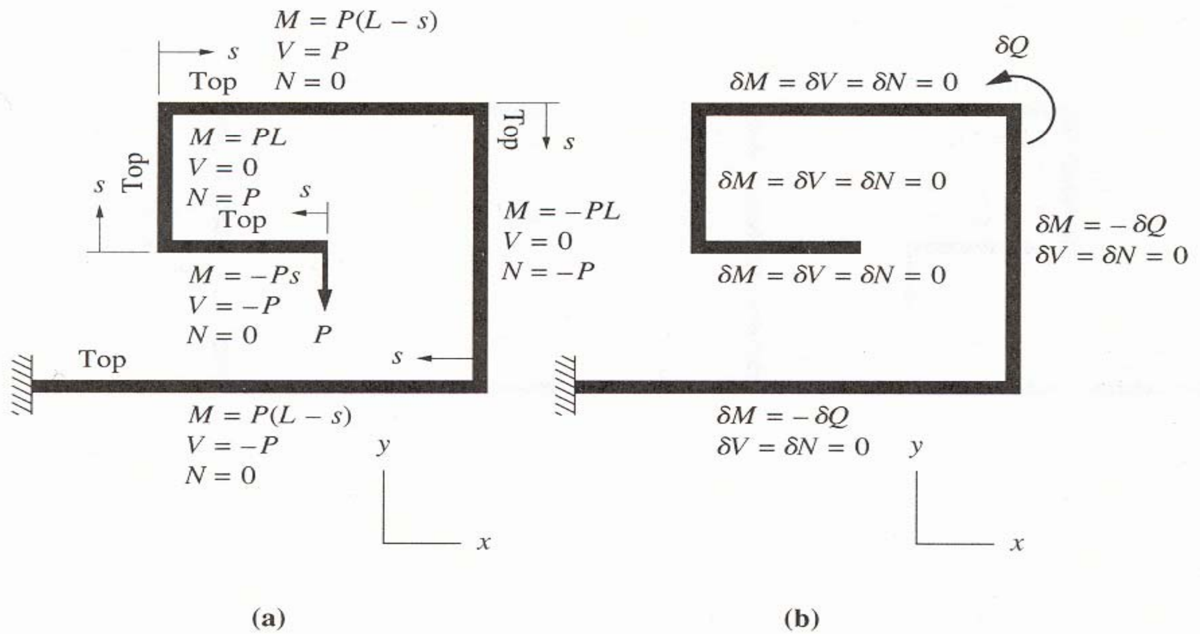


Figure 7.7.10 (a) Internal forces due to the actual loading. (b) Internal forces due to the virtual couple at point 4.

7.7 Beam And Frame Deflections Using Complementary Virtual Work

$$\delta W^*_{\text{int}} = \overbrace{0}^{\text{beam 1}} + \overbrace{0}^{\text{beam 2}} + \overbrace{0}^{\text{beam 3}} + \overbrace{\frac{1}{EI_z}\int_0^{2L}(-PL)(-\delta Q)\,ds}^{\text{beam 4}} + \overbrace{\frac{1}{EI_z}\int_0^{3L}[P\,(L-s)](-\delta Q)\,ds}^{\text{beam 5}}$$

$$\delta W^*_{\text{int}} = \frac{2PL^2}{EI_z}\delta Q + \frac{3PL^2}{2EI_z}\delta Q = \frac{7PL^2}{2EI_z}\delta Q$$

Since the external complementary virtual work is $\theta_{x_4}\delta Q$

$$\delta W^*_{\text{ext}} = \delta W^*_{\text{int}}$$

$$\theta_{x_4} = \frac{7PL^2}{2EI_z}$$

7.8 Statically Indeterminate Beams and Frames

We will use the principle of complementary virtual work to analyze simple, *statically indeterminate beams and frames.*

First, Identify the redundants and treat them as external load applied to a statically determinate base structure.

Then Remove the true loads from the structure and apply virtual loads in the directions of each redundant.

Finally, we equate the external and internal complementary virtual work expressions and solve the resulting equations for the redundant loads, after which all the other loads follow from the analysis of the statically determinate base structure.

7.8 Statically Indeterminate Beams and Frames

Example 7.8.1 Use the principle of complementary virtual work to calculate the reaction EI_z at the left end of the beam in Figure 7.8.1. The flexural rigidity is

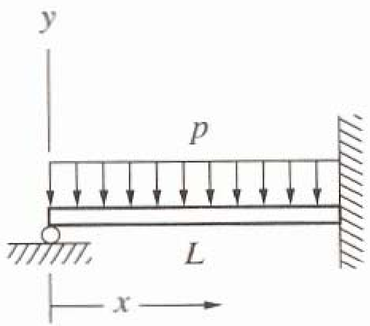


Figure 7.8.1 Singly-redundant beam.

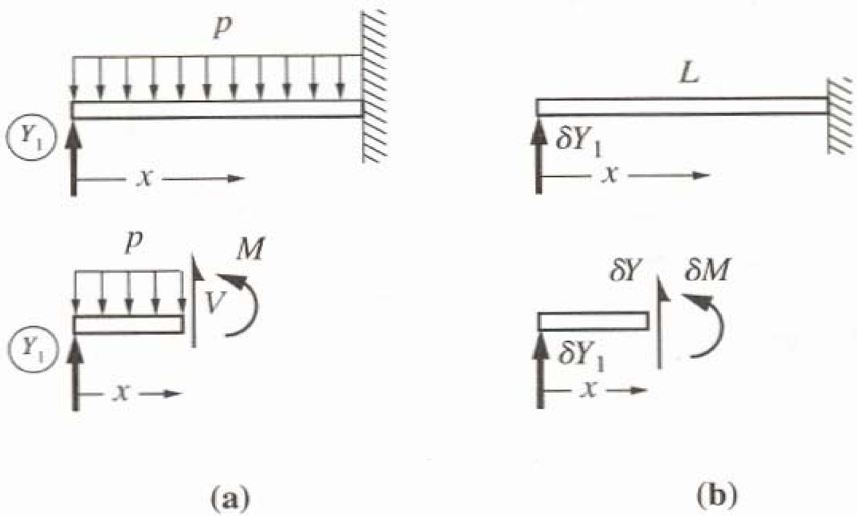


Figure 7.8.2 (a) Statically determinate base structure with the actual applied loads. (b) Base structure with the virtual load only.

7.8 Statically Indeterminate Beams and Frames

From figure (a), The true moment in terms of the true load

$$M = Y_1 x - \frac{px^2}{2}$$

From figure (b), The virtual moment in terms of the virtual load

$$\delta M = \delta Y_1 x$$

The internal complementarty virtual work,

$$\delta W_{\text{int}}^* = \int_0^L \frac{M_z \delta M_z}{EI_z} = \frac{1}{EI_z}\int_0^L \left(Y_1 x - \frac{px^2}{2}\right)(\delta Y_1 x)\,dx = \frac{1}{EI_z}\left(\frac{Y_1 L^3}{3} - \frac{pL^4}{8}\right)\delta Y_1$$

$$\delta W_{\text{int}}^* = \delta W_{\text{ext}}^*$$

However , since the displacement in the direction of Y1 is 0 (Supported)

$$\delta W_{\text{ext}}^* = 0 \qquad \delta W_{\text{int}}^* = 0$$

$$\frac{1}{EI_z}\left(\frac{Y_1 L^3}{3} - \frac{pL^4}{8}\right)\delta Y_1 = 0$$

$$Y_1 = \frac{3}{8} pL$$

7.8 Statically Indeterminate Beams and Frames

Example 7.8.2 Using the principle of complementary virtual work, find the location and magnitude of the maximum bending moment in the clamped–clamped simple beam in Figure 7.8.3. The flexural rigidity EI_z is unifor

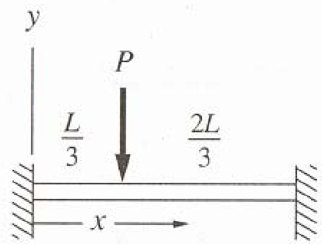


Figure 7.8.3 Doubly-redundant simple beam.

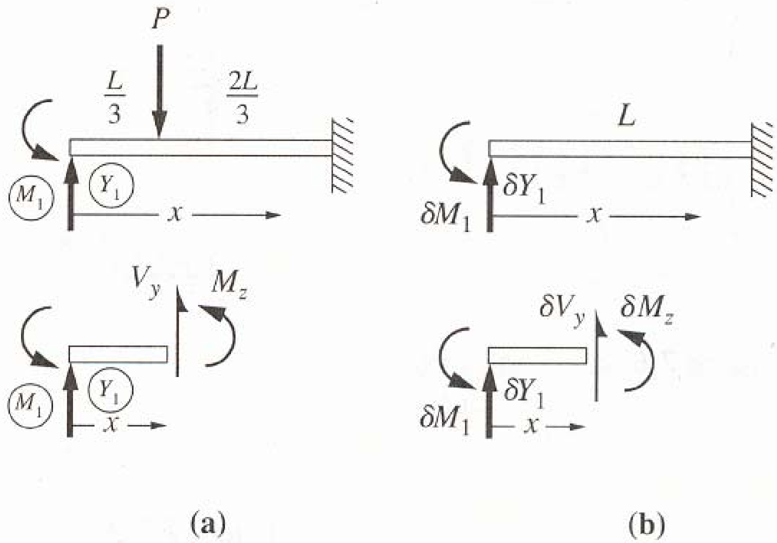


Figure 7.8.4 (a) Statically determinate base structure with the actual applied loads. (b) Base structure with virtual load only.

7.8 Statically Indeterminate Beams and Frames

$$M_z = Y_1 x - M_1 \qquad 0 \le x \le \frac{L}{3}$$

$$M_z = Y_1 x - P\left(x - \frac{L}{3}\right) - M_1 = (Y_1 - P)x + \frac{PL}{3} - M_1 \qquad \frac{L}{3} \le x \le L$$

$$\delta M_z = \delta Y_1 x - \delta M_1 \qquad 0 \le x \le L$$

$$\delta W_{\text{int}}^* = \int_0^L \frac{M_z \delta M_z}{EI_z} dx = \left(\frac{1}{EI_z}\right)\left\{\int_0^{L/3} (Y_1 x - M_1)(\delta Y_1 x - \delta M_1)\,dx + \int_{L/3}^{L} \left[(Y_1 - P)x + \frac{PL}{3} - M_1\right](\delta Y_1 x - \delta M_1)\,dx\right\}$$

$$\delta W_{\text{int}}^* = \frac{1}{EI_z}\left(M_1 L - \frac{1}{2}Y_1 L^2 + \frac{2}{9}PL^2\right)\delta M_1 + \frac{1}{EI_z}\left(-\frac{1}{2}M_1 L^2 + \frac{1}{3}Y_1 L^3 - \frac{14}{81}PL^3\right)\delta Y$$

$$EI_z\left(M_1 L - \frac{1}{2}Y_1 L^2 + \frac{2}{9}PL^2\right)\delta M_1 + EI_z\left(-\frac{1}{2}M_1 L^2 + \frac{1}{3}Y_1 L^3 - \frac{14}{81}PL^3\right)\delta Y_1 = 0$$

$$M_1 = \frac{4}{27}PL \qquad Y_1 = \frac{20}{27}P$$

7.8 Statically Indeterminate Beams and Frames

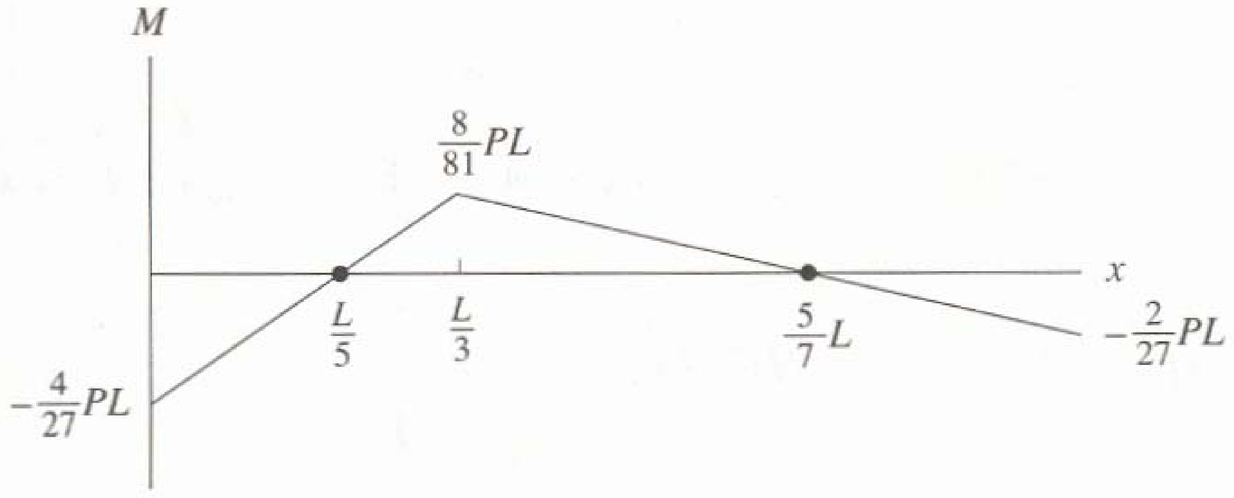


Figure 7.8.5 Bending moment diagram for the beam in Figure 7.8.3.

7.8 Statically Indeterminate Beams and Frames

Example 7.8.3 The simple beam in Figure 7.8.6 is built in at both ends, and there are two intermediate roller supports. The left wall is displaced downwards a prescribed amount d but remains vertical. Neglecting shear and assuming EI is uniform, use the principle of complementary virtual work to

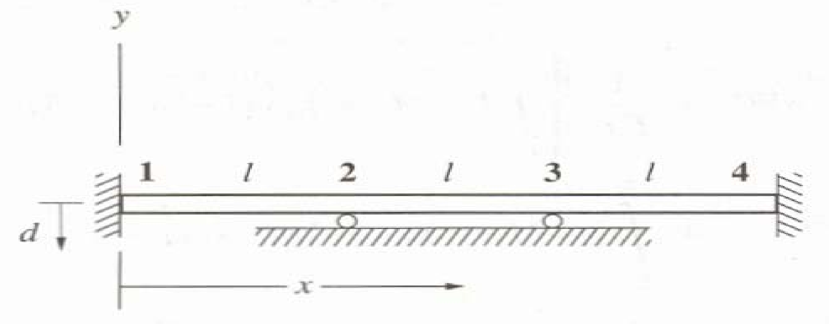


Figure 7.8.6 Simple beam of redundancy four.

supports.

7.8 Statically Indeterminate Beams and Frames

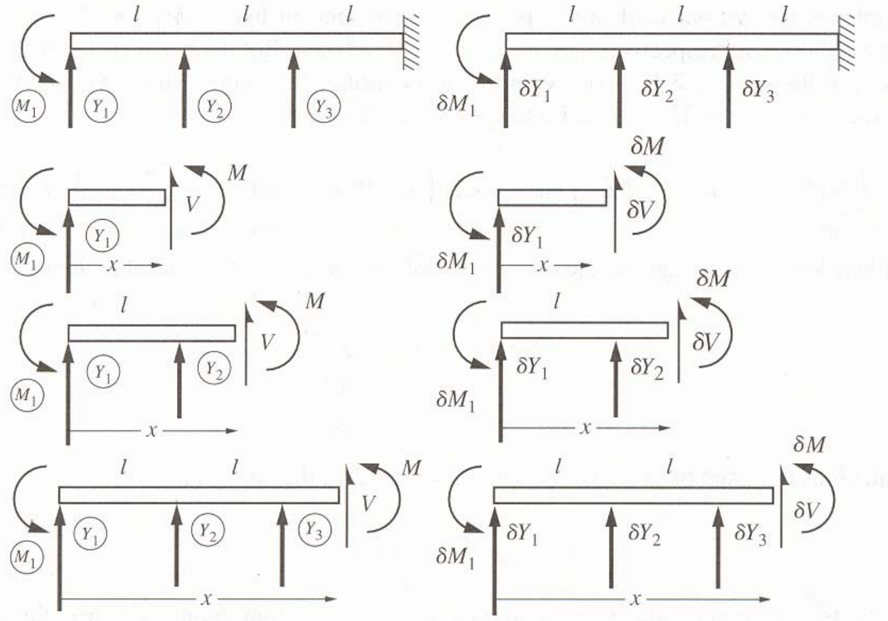


Figure 7.8.7 (a) Statically determinate base beam with the true applied loads, and the intermediate free-body diagrams. (b) Base structure with the four applied virtual loads, and the intermediate free-body diagrams.

For $0 \le x \le l$,	$M_z = -M_1 + Y_1 x$	$\delta M_z = -\delta M_1 + \delta Y_1 x$
For $l \le x \le 2l$,	$M_z = -M_1 + Y_1 x + Y_2(x-l)$	$\delta M_z = -\delta M_1 + \delta Y_1 x + \delta Y_2(x-l)$
For $2l \le x \le 3l$,	$M_z = -M_1 + Y_1 x + Y_2(x-l) + Y_3(x-2l)$	$\delta M_z = -\delta M_1 + \delta Y_1 x + \delta Y_2(x-l) + \delta Y_3(x-2l)$

7.8 Statically Indeterminate Beams and Frames

$$\delta W^*_{\text{int}} = \frac{1}{EI_z}\left[\int_0^l \{[-M_1 + Y_1 x][-\delta M_1 + \delta Y_1 x]\}\,dx + \int_l^{2l} \{[-M_1 + Y_1 x + Y_2(x-l)][-\delta M_1 + \delta Y_1 x + \delta Y_2(x-l)]\}\,dx \right.$$
$$\left. + \int_{2l}^{3l} \{[-M_1 + Y_1 x + Y_2(x-l) + Y_3(x-2l)][-\delta M_1 + \delta Y_1 x + \delta Y_2(x-l) + \delta Y_3(x-2l)]\}\,dx\right]$$

$$\delta W^*_{\text{int}} = \left(\frac{1}{EI_z}\right)\left[\left(3lM_1 - \tfrac{9}{2}l^2 Y_1 - 2l^2 Y_2 - \tfrac{1}{2}l^2 Y_3\right)\delta M_1 + \left(-\tfrac{9}{2}l^2 M_1 + 9l^3 Y_1 + \tfrac{14}{3}l^3 Y_2 + \tfrac{4}{3}l^3 Y_3\right)\delta Y_1\right.$$
$$\left. + \left(-2l^2 M_1 + \tfrac{14}{3}l^3 Y_1 + \tfrac{8}{3}l^3 Y_2 + \tfrac{5}{6}l^3 Y_3\right)\delta Y_2 + \left(-\tfrac{1}{2}l^2 M_1 + \tfrac{4}{3}l^3 Y_1 + \tfrac{5}{6}l^3 Y_2 + \tfrac{1}{3}l^3 Y_3\right)\delta Y_3\right]$$

$$\delta W^*_{\text{ext}} = -v\,\delta Y_1$$

$$\begin{aligned}
3lM_1 - \tfrac{9}{2}lY_1 - 2l^2 Y_2 - \tfrac{1}{2}l^2 Y_3 &= 0\\
-\tfrac{9}{2}lM_1 + 9l^3 Y_1 + \tfrac{14}{3}l^3 Y_2 + \tfrac{4}{3}l^3 Y_3 &= -EI_z d\\
-2l^2 M_1 + \tfrac{14}{3}l^3 Y_1 + \tfrac{8}{3}l^3 Y_2 + \tfrac{5}{6}l^3 Y_3 &= 0\\
-\tfrac{1}{2}l^2 M_1 + \tfrac{4}{3}l^3 Y_1 + \tfrac{5}{6}l^3 Y_2 + \tfrac{1}{3}l^3 Y_3 &= 0
\end{aligned}$$

$$M_1 = -\frac{22EI_z}{5l^2}d \qquad Y_1 = -\frac{36EI_z}{5l^3}d \qquad Y_2 = \frac{54EI_z}{5l^3}d \qquad Y_3 = -\frac{24EI_z}{5l^3}d$$

7.8 Statically Indeterminate Beams and Frames

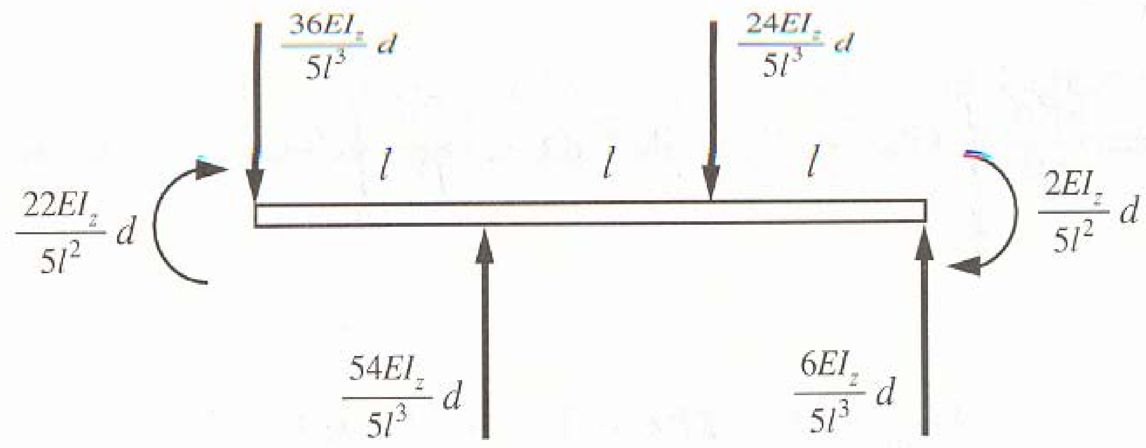


Figure 7.8.8 Support reactions on the beam of Figure 7.8.6.

7.8 Statically Indeterminate Beams and Frames

Example 7.8.4 Calculate the transverse displacement at the midpoint of the indeterminate beam in the previous example.

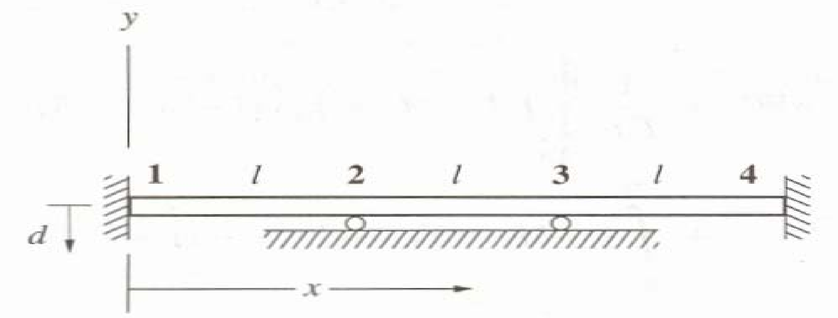


Figure 7.8.6 Simple beam of redundancy four.

7.8 Statically Indeterminate Beams and Frames

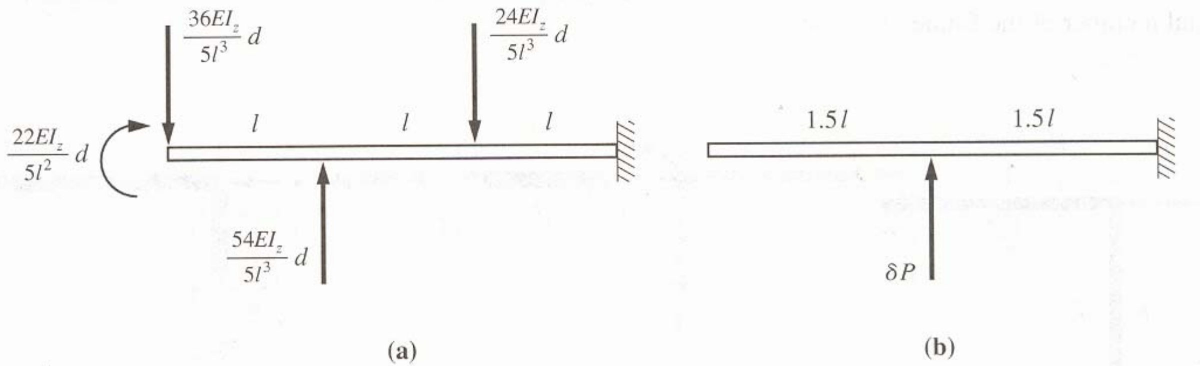


Figure 7.8.9 (a) Actual loads, computed in Example 7.8.3. (b) Virtual load and the selected statically determinate base structure.

$$\delta W^*_{\text{int}} = \frac{1}{EI_z}\int_0^{3l} M\delta M dx = \frac{1}{EI_z}\int_{1.5l}^{3l} M\delta P(x-1.5l)dx$$

$$\delta W^*_{\text{int}} = \frac{1}{EI_z}\int_{\frac{3l}{2}}^{2l}\left[\frac{22EI_z d}{5l^2} - \frac{36EI_z d}{5l^3}x + \frac{54EI_z d}{5l^3}(x-l)\right]\delta P\left(x-\frac{l}{2}\right)dx$$

$$+\frac{1}{EI_z}\int_{2l}^{3l}\left[\frac{22EI_z d}{5l^2} - \frac{36EI_z d}{5l^3}x + \frac{54EI_z d}{5l^3}(x-l) - \frac{24EI_z d}{5l^3}(x-2l)\right]\delta P\left(x-\frac{l}{2}\right)dx$$

7.8 Statically Indeterminate Beams and Frames

$$\delta W^*_{\text{int}} = \frac{\delta P d}{5l^3} \int_{\frac{3l}{2}}^{2l} (18x^2 - 59lx + 48l^2)\, dx + \frac{\delta P d}{l^3} \int_{2l}^{3l} (-6x^2 + 25lx - 24l^2)\, dx$$

$$\delta W^*_{\text{int}} = 0.025\delta P d + 0.100\delta P d = 0.125\delta P d$$

$$\delta W^*_{\text{ext}} = v\delta P$$

$$v = 0.125d.$$

Then, a downward displacement d at the free end beam produces an upward displacement d/8 at its midpoint.

7.8 Statically Indeterminate Beams and Frames

Example 7.8.5 Calculate the reaction at point 1 of the frame in Figure 7.8.10, using the principle of complementary virtual work. Assess the effects of shear and stretching on the result. Assume that the material properties and sectior prope

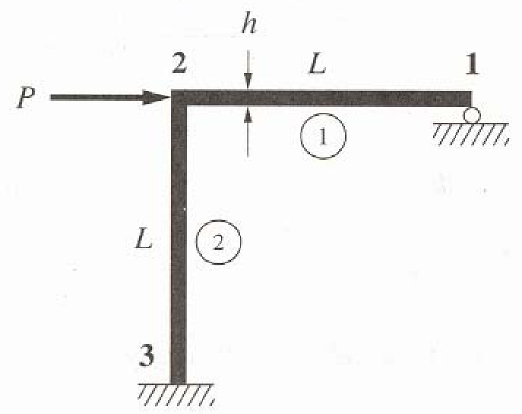


Figure 7.8.10 Singly-redundant frame.

7.8 Statically Indeterminate Beams and Frames

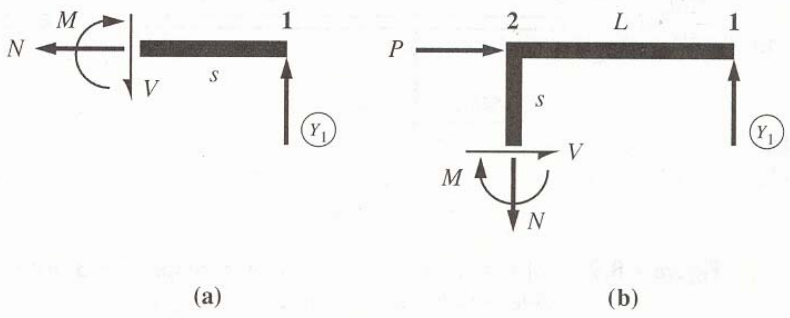


Figure 7.8.11 Free-body diagrams of the statically determinate base structure, showing the redundant reaction Y_1 as an externally applied load.

$$M = Y_1 s$$
$$V = Y_1 \qquad \text{Element 1}$$
$$N = 0$$

$$M = Y_1 L - Ps$$
$$V = -P \qquad \text{Element 2}$$
$$N = Y_1$$

$$\delta M = \delta Y_1 s$$
$$\delta V = \delta Y_1 \qquad \text{Element 1}$$
$$\delta N = 0$$

$$\delta M = \delta Y_1 L$$
$$\delta V = 0 \qquad \text{Element 2}$$
$$\delta N = \delta Y_1$$

7.8 Statically Indeterminate Beams and Frames

$$\delta W^*_{\text{int}} = \delta W^*_{\text{int,bending}} + \delta W^*_{\text{int,stretching}} + \delta W^*_{\text{int,shear}}$$

$$\delta W^*_{\text{int,bending}} = \sum_{e=1}^{2} \int_0^{L^{(e)}} \left(\frac{M\,\delta M}{E I_z}\right)^{(e)} ds$$

$$\delta W^*_{\text{int,stretching}} = \sum_{e=1}^{2} \int_0^{L^{(e)}} \left(\frac{N\,\delta N}{A E}\right)^{(e)} ds$$

$$\delta W^*_{\text{int,shear}} = \sum_{e=1}^{2} \int_0^{L^{(e)}} \left(\frac{V\,\delta V}{G A_s}\right)^{(e)} ds$$

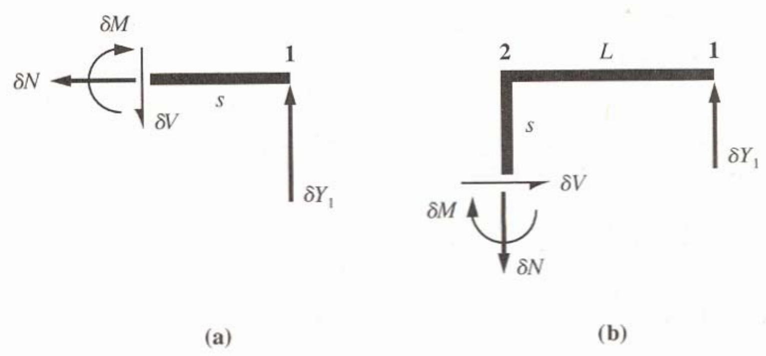


Figure 7.8.12 Free-body diagrams of the base frame with a virtual load applied at 1 in the direction of the redundant.

7.8 Statically Indeterminate Beams and Frames

$$\delta W^*_{\text{int, bending}} = \int_0^{L^{(1)}} \left(\frac{M\delta M}{EI_z}\right)^{(1)} ds + \int_0^{L^{(2)}} \left(\frac{M\delta M}{EI_z}\right)^{(2)} ds = \frac{1}{EI_z}\int_0^L (Y_1 s)(\delta Y_1 s)\, ds + \frac{1}{EI_z}\int_0^L (Y_1 L - Ps)(\delta Y_1 L)\, ds$$

$$\delta W^*_{\text{int, bending}} = \left(\frac{4}{3}\frac{Y_1 L^3}{EI_z} - \frac{1}{2}\frac{PL^3}{EI_z}\right)\delta Y_1$$

$$\delta W^*_{\text{int, shear}} = \int_0^{L^{(1)}} \left(\frac{V\delta V}{GA_s}\right)^{(1)} ds + \int_0^{L^{(2)}} \left(\frac{V\delta V}{GA_s}\right)^{(2)} ds = \frac{1}{GA_s}\int_0^L (Y_1)(\delta Y_1)\, ds + \frac{1}{GA_s}\int_0^L (-P)(0)\, ds$$

$$\delta W^*_{\text{int, shear}} = \frac{Y_1 L}{GA_s}\delta Y_1$$

$$\delta W^*_{\text{int, stretching}} = \int_0^{L^{(1)}} \left(\frac{N\delta N}{AE}\right)^{(1)} ds + \int_0^{L^{(2)}} \left(\frac{N\delta N}{AE}\right)^{(2)} ds = \frac{1}{AE}\int_0^L (0)(0)\, ds + \frac{1}{AE}\int_0^L (Y_1)(\delta Y_1)\, ds$$

$$\delta W^*_{\text{int, stretching}} = \frac{Y_1 L}{AE}\delta Y_1$$

7.8 Statically Indeterminate Beams and Frames

$$\delta W_{\text{int}}^{*} = \left(\frac{4}{3}\frac{Y_1 L^3}{E I_z} - \frac{1}{2}\frac{P L^3}{E I_z}\right)\delta Y_1 + \frac{Y_1 L}{G A_s}\delta Y_1 + \frac{Y_1 L}{E A}\delta Y_1$$

$$\left(\frac{4}{3}\frac{Y_1 L^3}{E I_z} - \frac{1}{2}\frac{P L^3}{E I_z}\right) + \frac{Y_1 L}{G A_s} + \frac{Y_1 L}{E A} = 0$$

$$Y_1 = \frac{3P}{2}\,\frac{G A A_s L^2}{4 G A A_s L^2 + 3 G I_z A_s + 3 E I_z A}$$

$$Y_1 = \frac{3P}{8}\,\frac{1}{1 + \frac{3}{4}(\frac{\rho_z}{L})^2[1 + \frac{2(1+\nu)}{k}]}$$

$$\lim_{\left(\frac{\rho_z}{L}\right)\to 0} \frac{1}{1 + \frac{3}{4}\left(\frac{\rho_z}{L}\right)^2\left[1 + \frac{2(1+\nu)}{k}\right]} = 1$$

$$\frac{1}{1 + \frac{3}{4}\left(\frac{\rho_z}{L}\right)^2\left[1 + \frac{2(1+\nu)}{k}\right]} \cong 0.99$$

7.8 Statically Indeterminate Beams and Frames

Example 7.8.6 A pin-supported semicircular frame supports a horizontal load at point 2

on the top, as shown in Figure 7.8.13. Assuming that the frame can be

treated as a curved, slender beam of constant cross section, use the

princip work to calculate the reactions at the

suppo ocations of the maximum bending

mome leformation.

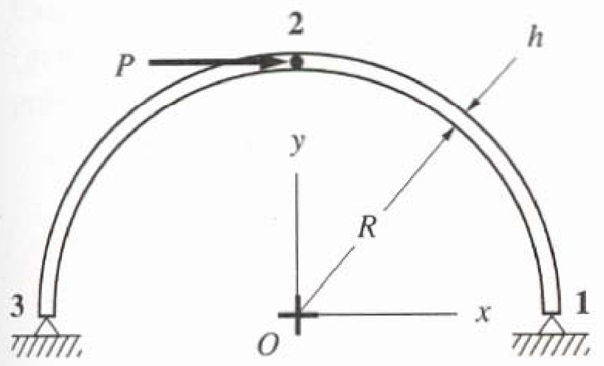


Figure 7.8.13 Semicircular frame joined to the floor by pin supports.

7.8 Statically Indeterminate Beams and Frames

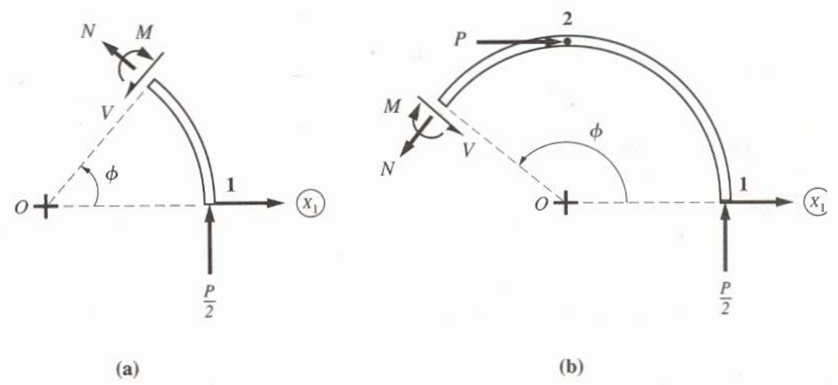


Figure 7.8.14 Free-body diagrams of the statically determinate base frame, showing the horizontal reaction at 1 as the redundant.

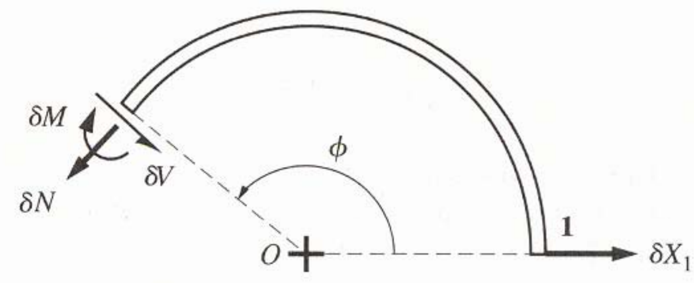


Figure 7.8.15 Free-body diagram of the base frame with true loads removed and a virtual load applied in the direction of the redundant.

For $0 \le \phi \le \frac{\pi}{2}$: $\quad M = X_1 R \sin\phi + \frac{PR}{2}(1 - \cos\phi)$

For $\frac{\pi}{2} \le \phi \le \pi$: $\quad M = (X_1 + P)\, R \sin\phi - \frac{PR}{2}(1 + \cos\phi)$

$$\delta M = \delta X_1 R \sin\phi \qquad 0 \le \phi \le \pi$$

7.8 Statically Indeterminate Beams and Frames

$$\delta W_{\text{int}}^{*} = \int_{0}^{\pi} \frac{M \delta M}{E I_z} (R d\phi)$$

$$\delta W_{\text{int}}^{*} = \int_{0}^{\pi/2} \frac{M \delta M}{E I_z} (R d\phi) + \int_{\pi/2}^{\pi} \frac{M \delta M}{E I_z} (R d\phi)$$

$$= \frac{R}{E I_z} \int_{0}^{\pi/2} \left[X_1 R \sin\phi + \frac{PR}{2} (1 - \cos\phi) \right] [\delta X_1 R \sin\phi]\, d\phi + \frac{R}{E I_z} \int_{\pi/2}^{\pi} \left[(X_1 + P)\, R \sin\phi - \frac{PR}{2} (1 + \cos\phi) \right] (\delta X_1 R \sin\phi)\, d\phi$$

$$\delta W_{\text{int}}^{*} = \frac{R^3}{4 E I_z} \{ [(\pi + 1)\, P + 2 X_1 \pi] \}\, \delta X_1$$

$$X_1 = -\frac{P}{2} (1 + \frac{1}{\pi}) = -0.659 P$$

7.8 Statically Indeterminate Beams and Frames

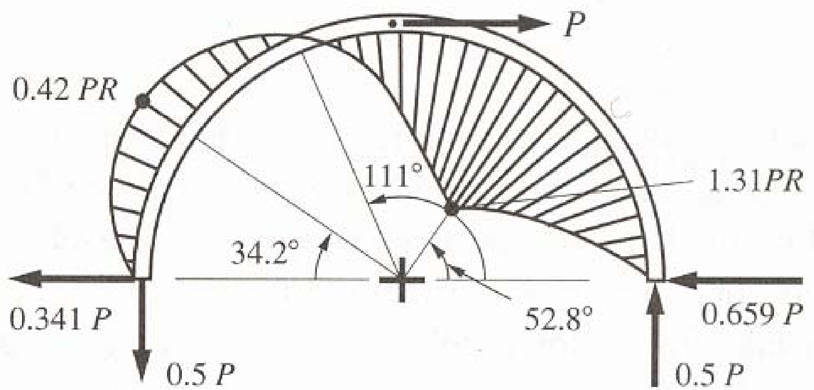


Figure 7.8.16 Computed reactions and bending moment distribution for the frame in Figure 7.8.13.

The bending moment is plotted normal to the frame, on the side in compression.

7.8 Statically Indeterminate Beams and Frames

Example 7.8.7 Use the principle of complementary virtual work to calculate the

maximum bending moment in the frame (grillage) of

Figure 7.8.17.

Assume the member cross sections are solid circles of

radius r and that

uniform throughout, with

Poisson`s ration

nd stretching.

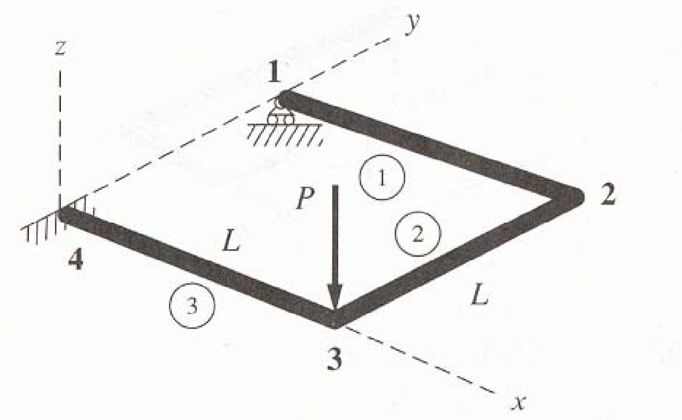


Figure 7.8.17 Singly-redundant grillage.

7.8 Statically Indeterminate Beams and Frames

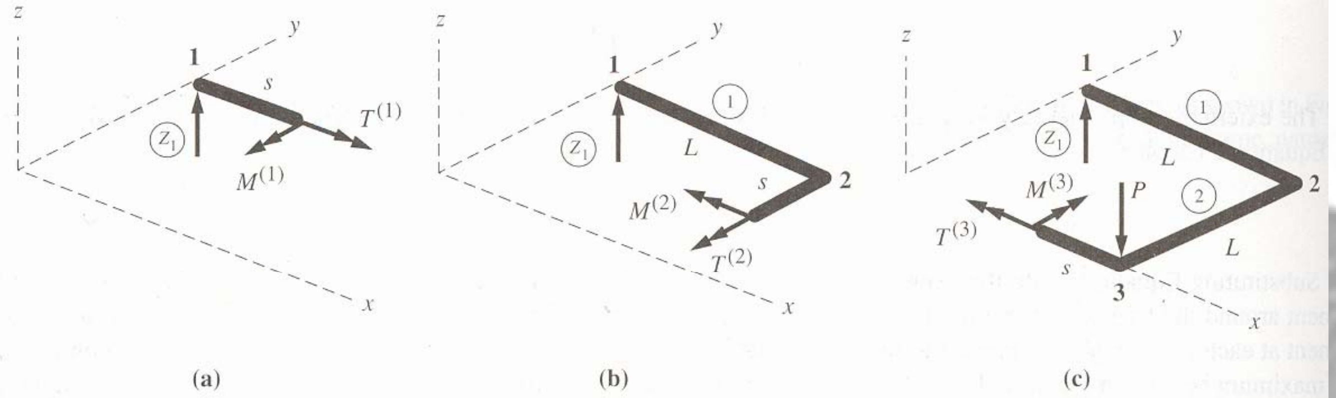


Figure 7.8.18 Free-body diagrams of the true loads on the statically determinate base structure.

Since the effects of shear and extension are neglected, only the bending and twisting couples on each section are shown, for simplicity.

$$M^{(1)} = Z_1 s \qquad T^{(1)} = 0 \qquad \text{Element 1}$$

$$M^{(2)} = Z_1 s \qquad T^{(2)} = Z_1 L \qquad \text{Element 2}$$

$$M^{(3)} = (Z_1 - P)\,s - Z_1 L \qquad T^{(3)} = Z_1 L \qquad \text{Element 3}$$

7.8 Statically Indeterminate Beams and Frames

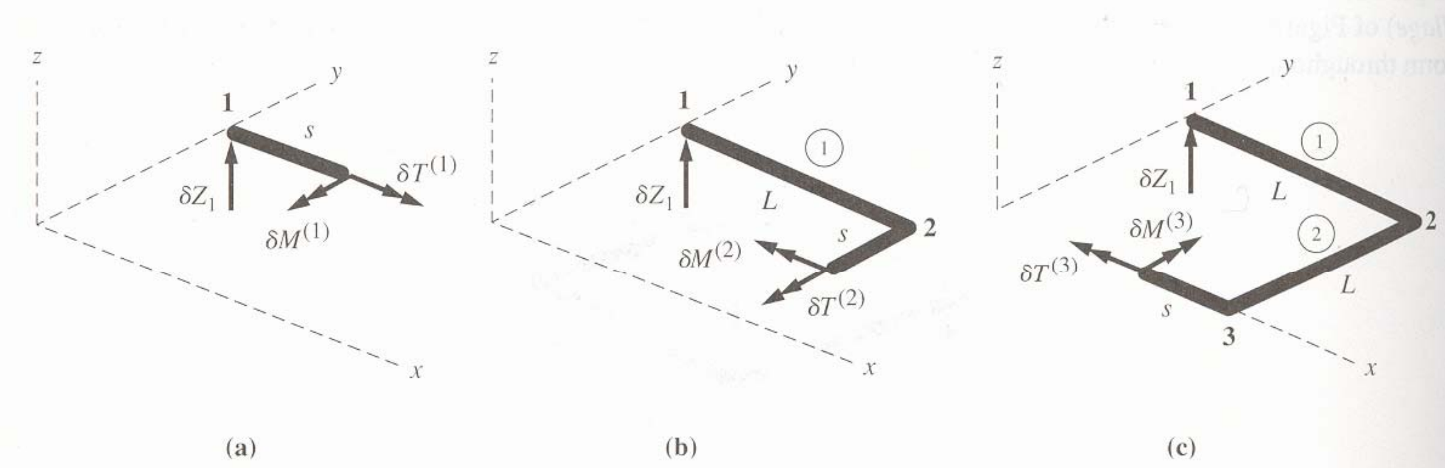


Figure 7.8.19 Virtual load of the base structure.

$\delta M^{(1)} = \delta Z_1 s$ $\delta T^{(1)} = 0$	Element 1
$\delta M^{(2)} = \delta Z_1 s$ $\delta T^{(2)} = \delta Z_1 L$	Element 2
$\delta M^{(3)} = \delta Z_1 (s - L)$ $\delta T^{(3)} = \delta Z_1 L$	Element 3

7.8 Statically Indeterminate Beams and Frames

$$\delta W^*_{\text{int}} = \delta W^*_{\text{int,bending}} + \delta W^*_{\text{int,torsion}}$$

$$\delta W^*_{\text{int,bending}} = \sum_{e=1}^{3} \int_0^{L^{(e)}} \left(\frac{M\,\delta M}{EI}\right)^{(e)} ds$$

$$\delta W^*_{\text{int,bending}} = \frac{1}{EI}\int_0^L (Z_1 s)\,(\delta Z_1 s)\,ds + \frac{1}{EI}\int_0^L (Z_1 s)\,(\delta Z_1 s)\,ds$$
$$+ \frac{1}{EI}\int_0^L [(Z_1 - P)\,s - Z_1 L]\,[\delta Z_1\,(s - L)]\,ds$$

$$\delta W^*_{\text{int,bending}} = \frac{L^3}{EI}\left(\frac{Z_1}{3} + \frac{P}{6}\right)\delta Z_1$$

7.8 Statically Indeterminate Beams and Frames

$$\delta W^*_{\text{int,torsion}} = \sum_{e=1}^{3} \int_0^{L^{(e)}} \left(\frac{T\,\delta T}{GJ} \right)^{(e)} ds$$

$$\delta W^*_{\text{int,torsion}} = \frac{1}{GJ}\int_0^L (0)\,(0)\,ds + \frac{1}{GJ}\int_0^L (Z_1 L)\,(\delta Z_1 L)\,ds + \frac{1}{GJ}\int_0^L (Z_1 L)\,(\delta Z_1 L)\,ds$$

$$\delta W^*_{\text{int,torsion}} = \frac{2 Z_1 L^3}{GJ}\delta Z_1$$

$$\delta W^*_{\text{int}} = \frac{2 Z_1 L^3}{GJ}\delta Z_1 + \frac{L^3}{EI}\left(\frac{Z_1}{3} + \frac{P}{6}\right)\delta Z_1$$

$$Z_1 = -\frac{P}{2}\frac{GJ}{GJ + 6EI} = -\frac{P}{2}\frac{\left[\frac{E}{2(1+\nu)}\right]J}{\left[\frac{E}{2(1+\nu)}\right]J + 6EI} = -\frac{P}{2}\frac{1}{1 + 12\,(1+\nu)\,\frac{I}{J}}$$

$$Z_1 = -\frac{P}{17}$$

7.8 Statically Indeterminate Beams and Frames

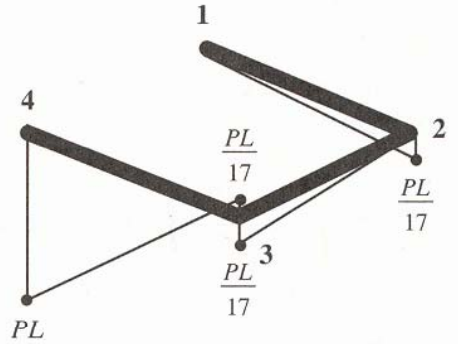


Figure 7.8.20 Bending moment distribution in the grillage of Figure 7.8.17.

The bending moment diagram is plotted on the side of the element in compression.

7.8 Statically Indeterminate Beams and Frames

Example 7.8.8 Use the principle of complementary virtual work to find the magnitude and location of the maximum bending moment in the circular frame in Figure 7.8.21. The material and section properties are uniform

ng.

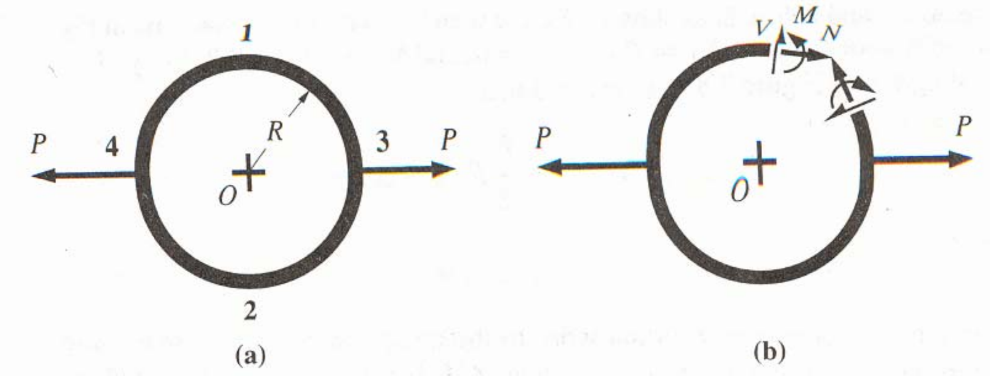


Figure 7.8.21 Circular frame loaded in tension along a diameter.

7.8 Statically Indeterminate Beams and Frames

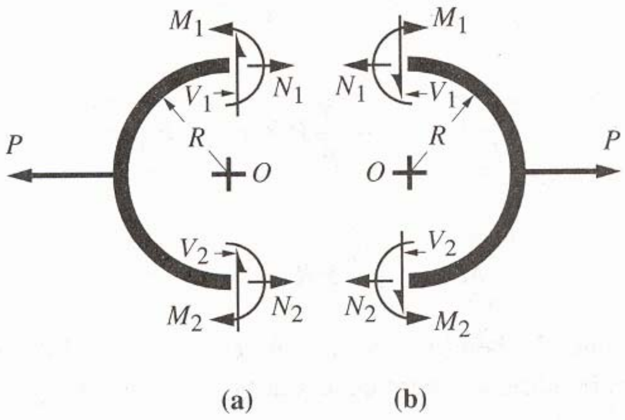


Figure 7.8.22 Free-body diagrams of the frame divided in two along the vertical diameter.

$$V_1 = 0.$$

$$V_2 = 0.$$

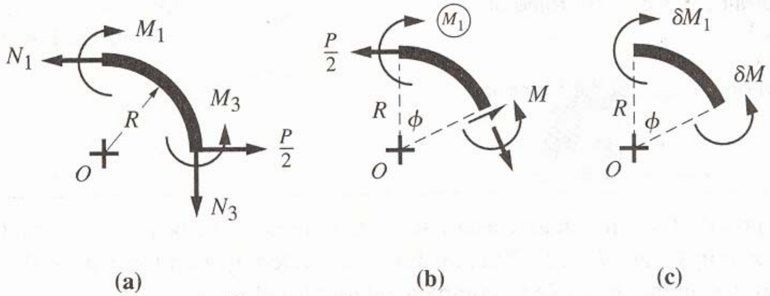


Figure 7.8.23 (a) Free-body diagram of a quarter circle of the ring. (b) Free-body diagram of the sectioned quarter circle showing M_1 as an applied load. (c) Free-body diagram of the sectioned quarter circle with the true loading removed and replaced by the virtual couple δM_1.

7.8 Statically Indeterminate Beams and Frames

From the free-body diagram of Figure 7.8.23b, we find that

$$M = M_1 - \frac{P}{2} R(1 - \cos\phi) \tag{a}$$

and from Figure 7.8.23c, we have

$$\delta M = \delta M_1 \tag{b}$$

Applying the principle of complementary virtual work to the complete circular frame, using symmetry and recognizing that the complementary virtual work of the internal virtual load δM_1 is zero, we have, for bending alone,

$$\delta W^*_{\text{int}} = 4\left[\frac{1}{EI_z}\int_0^{\pi/2}(M\delta M)\,R d\phi\right] = \frac{4R}{EI_z}\int_0^{\pi/2}\left[M_1 - \frac{PR}{2}(1-\cos\phi)\right]\delta M d\phi = 0$$

which reduces to

$$\frac{4R}{EI_z}\left[\frac{\pi}{2}M_1 - \frac{\pi}{4}PR + \frac{1}{2}PR\right] = 0$$

so that

$$M_1 = \frac{\pi - 2}{2\pi}PR = 0.1817PR \tag{c}$$

7.8 Statically Indeterminate Beams and Frames

Substituting M_1 into Equation a yields the bending moment in terms of ϕ and the applied load P. This is plotted in Figure 7.8.24, which shows that the maximum bending moment occurs at point 3 (and, by symmetry, point 4).

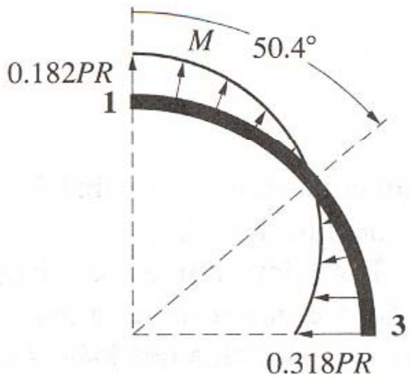


Figure 7.8.24 Symmetrical bending moment distribution over a quadrant of the circular frame of Figure 7.8.21.

The bending moment is plotted on the side of the frame in compression.

7.8 Statically Indeterminate Beams and Frames

Example 7.8.9 A frame composed of a semicircular beam and a horizontal floor
beam has a point load *P* applied to the midspan of the floor beam, as
shown in Figure 7.8.25a. This load is equilibrated by a uniform shear
flow *P/2R* acting around the periphery of the circular beam. Using the
l work, find the
maximum bendir
he point where the
moment is zero.
me that all material
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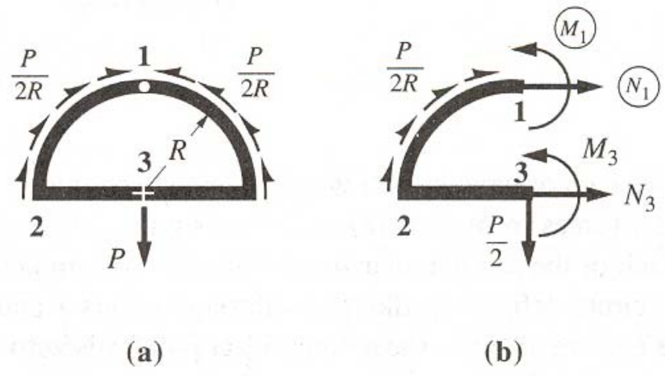

Figure 7.8.25 (a) Statically indeterminate frame. (b) Free-body diagram showing internal loads at 1 and 3.

7.8 Statically Indeterminate Beams and Frames

To find the true and virtual bending moment distribution, we start with the free-body diagrams in Figure 7.8.26. From part (a) of the figure, using Equation 2.5.11 to compute the moment of the shear flow, summing the moments about the cut at an angle ϕ from 1, and equating this to zero yields

$$-M + M_1 - N_1 R\,(1 - \cos\phi) - 2A_1\left(\frac{P}{2R}\right) = 0 \qquad \text{[a]}$$

where A_1, the area of the shaded segment of the circle, is the area $R^2\phi/2$ of the pie-shaped sector subtended by the angle ϕ minus the area $R^2 \sin\phi/2$ of the isosceles triangle with legs R and vertex angle ϕ. Thus,

$$A_1 = \frac{R^2}{2}(\phi - \sin\phi) \qquad \text{[b]}$$

Substituting this into Equation a and solving for the true bending moment M, we get

$$M = M_1 - N_1 R\,(1 - \cos\phi) - \frac{PR}{2}(\phi - \sin\phi) \qquad \text{[c]}$$

Similarly, an analysis of Figure 7.8.26b yields the following for the virtual bending moment:

$$\delta M = \delta M_1 - \delta N_1 R\,(1 - \cos\phi) \qquad \text{[d]}$$

Moving into the horizontal beam, we have the free-body diagrams shown in Figure 7.8.27. From part (a) of the figure, summing the moments about the bottom cut, which is a distance s from point 2, we obtain

$$M + M_1 - N_1 R - 2(A_1 + A_2)\left(\frac{P}{2R}\right) = 0 \qquad \text{[e]}$$

7.8 Statically Indeterminate Beams and Frames

$$A_1 = \frac{R^2}{2}\left(\frac{\pi}{2} - 1\right)$$

It can be seen from the figure that A_2 is the area of the triangle with base s and height R, or $A_2 = Rs/2$. We substitute A_1 and A_2 into Equation e and solve for the true bending moment in this portion of the frame, as follows:

$$M = -M_1 + N_1 R + \frac{Ps}{2} + \frac{PR}{2}\left(\frac{\pi}{2} - 1\right) \qquad \text{[f]}$$

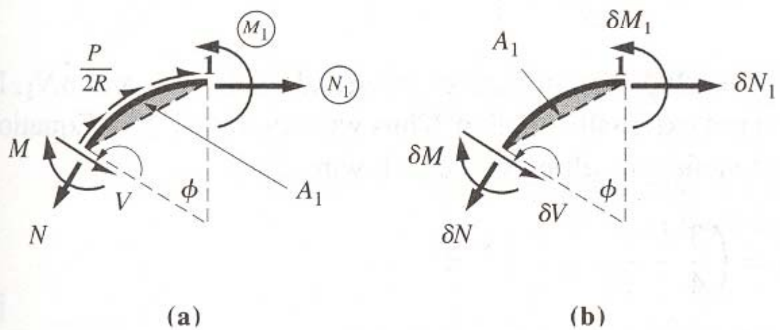

Figure 7.8.26 (a) Free-body diagram of the semicircular portion of the statically determinate base frame acted on by the true loads. (b) The same free-body diagram acted on by the virtual loads applied at 1.

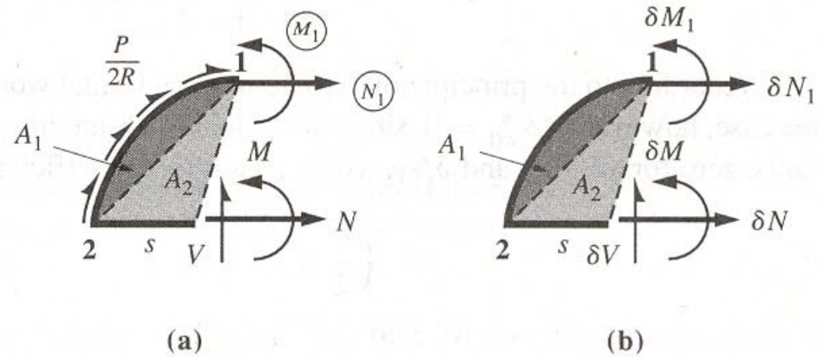

Figure 7.8.27 (a) Free-body diagram of the frame for a cut through the floor beam, showing the actual applied loads. (b) The same free-body diagram with only the virtual loads applied.

7.8 Statically Indeterminate Beams and Frames

A similar analysis of the free-body diagram in Figure 7.8.27b yields the virtual bending moment,

$$\delta M = -\delta M_1 + \delta N_1 R \tag{g}$$

We are now in a position to determine the internal complementary virtual work for the structure in Figure 7.8.25a. Considering only bending and using symmetry, we have

$$\delta W^*_{\text{int}} = 2\left(\frac{1}{EI_z}\int_1^2 M\delta M ds + \frac{1}{EI_z}\int_2^3 M\delta M ds\right)$$

where $ds = Rd\phi$ in the first integral. Substituting Equations c, d, f, and g into the integrands yields

$$\begin{aligned}\delta W^*_{\text{int}} = {} & \frac{2}{EI_z}\int_0^{\pi/2}\left[M_1 - N_1 R(1-\cos\phi) - \frac{PR}{2}(\phi - \sin\phi)\right][\delta M_1 - \delta N_1 (1-\cos\phi)]\,Rd\phi \\ & + \frac{2}{EI_z}\int_0^R\left[-M_1 + N_1 R + \frac{Ps}{2} + \frac{PR}{2}\left(\frac{\pi}{2}-1\right)\right][-\delta M_1 + \delta N_1 R]\,ds\end{aligned} \tag{h}$$

7.8 Statically Indeterminate Beams and Frames

Expanding the integrands and collecting coefficients of the virtual loads δM_1 and δN_1, we get the following lengthy expression:

$$\delta W^*_{\text{int}} = \frac{2\delta M_1}{EI_z}\left\{\int_0^{\pi/2}\left[M_1 - N_1R\,(1-\cos\phi) - \frac{PR}{2}(\phi - \sin\phi)\right]Rd\phi + \int_0^R\left[M_1 - N_1R - \frac{Ps}{2} - \frac{PR}{2}\left(\frac{\pi}{2}-1\right)\right]ds\right\}$$

$$+\frac{2\delta N_1}{EI_z}\left\{\int_0^{\pi/2}\left[M_1R\,(\cos\phi-1) + N_1R^2\left(1-2\cos\phi+\cos^2\phi\right) + \frac{PR^2}{2}(\phi - \sin\phi - \phi\cos\phi + \sin\phi\cos\phi)\right]Rd\phi\right.$$

$$\left. + \int_0^R\left[-RM_1 + N_1R^2 + \frac{RPs}{2} + \frac{PR^2}{2}\left(\frac{\pi}{2}-1\right)\right]ds\right\}$$

Carrying out the simple integrations leads ultimately to

$$\delta W^*_{\text{int}} = \frac{2\delta M_1}{EI_z}\left[\left(\frac{\pi}{2}+1\right)RM_1 - \frac{\pi}{2}R^2N_1 + \left(\frac{3}{4} - \frac{\pi^2}{16} - \frac{\pi}{4}\right)PR^2\right]$$

$$+\frac{2\delta N_1}{EI_z}\left[-\frac{\pi}{2}R^2M_1 + \left(\frac{3\pi}{4}-1\right)R^3N_1 + \frac{\pi^2}{16}PR^3\right] \qquad \text{[i]}$$

7.8 Statically Indeterminate Beams and Frames

According to the principle of complementary virtual work, $\delta W^*_{int} = \delta W^*_{ext}$ for all choices of virtual loads δM_1 and δN_1. In this case, however, $\delta W^*_{ext} = 0$, since the virtual loads are internal and not externally applied. Thus we require δW^*_{int} in Equation i to be zero for all δM_1 and δN_1, which means their coefficients must vanish, resulting in the following two equations:

$$\begin{aligned} \left(\frac{\pi}{2}+1\right) R M_1 \qquad - \frac{\pi}{2} R^2 N_1 &= \left(\frac{3}{4} - \frac{\pi^2}{16} - \frac{\pi}{4}\right) P R^2 \\ -\frac{\pi}{2} R^2 M_1 + \left(\frac{3\pi}{4} - 1\right) R^3 N_1 &= -\frac{\pi^2}{16} P R^3 \end{aligned} \qquad \text{[j]}$$

Solving for M_1 and N_1, we find that

$$M_1 = -0.08279PR \qquad N_1 = -0.5507P \qquad \text{[k]}$$

7.8 Statically Indeterminate Beams and Frames

Substituting these values into Equations c and f yields the bending moment equations, which can be plotted on the frame, as shown in Figure 7.8.28. As usual, the bending moment is plotted on the side of the frame in compression. As we can see, the maximum bending moment occurs under the point load P at the middle of the floor beam.

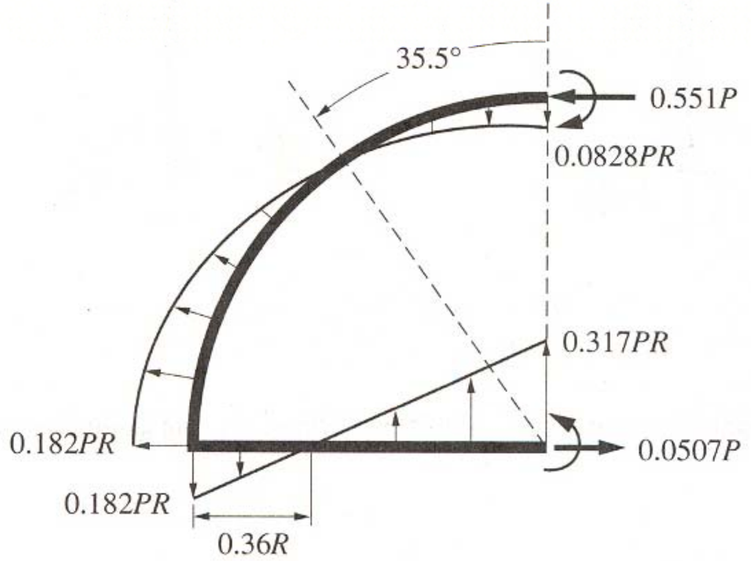

Figure 7.8.28 Symmetric bending moment distribution around the frame of Figure 7.8.25.

7.8 Statically Indeterminate Beams and Frames

Example 7.8.10 Using the principle of complementary virtual work, find the bending moment distribution in the portal frame of Figure 7.8.29a.

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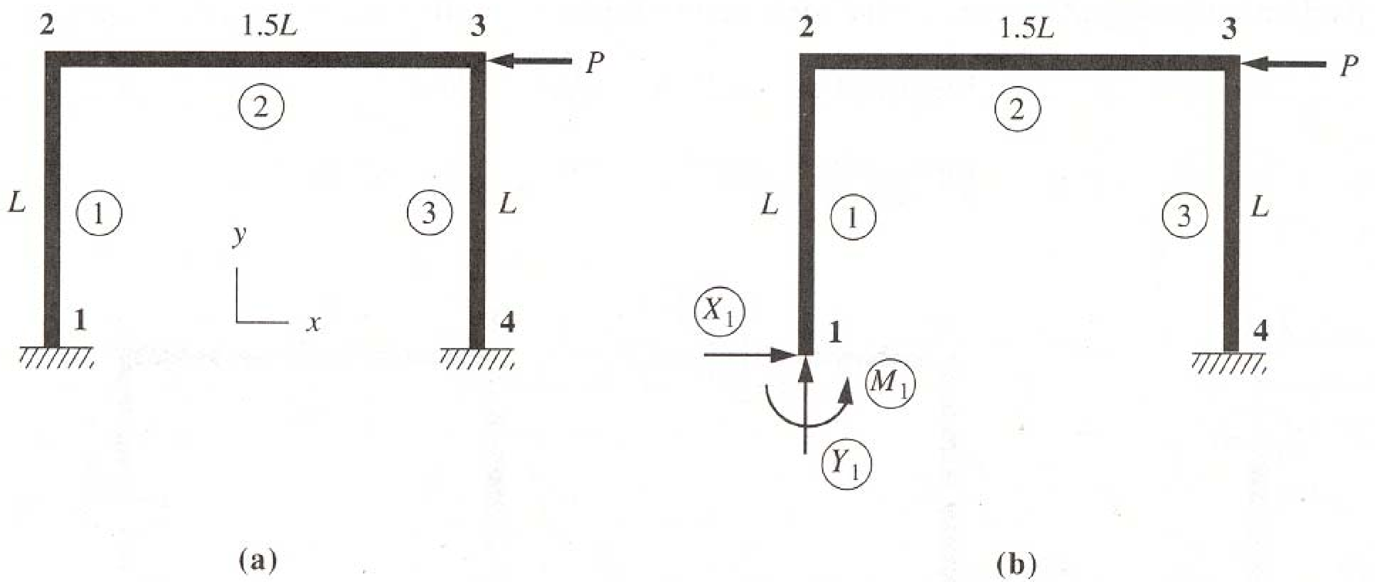


Figure 7.8.29 (a) Portal frame built in at each support. (b) Free-body diagram of the base structure, showing the three support reactions chosen as the redundants.

7.8 Statically Indeterminate Beams and Frames

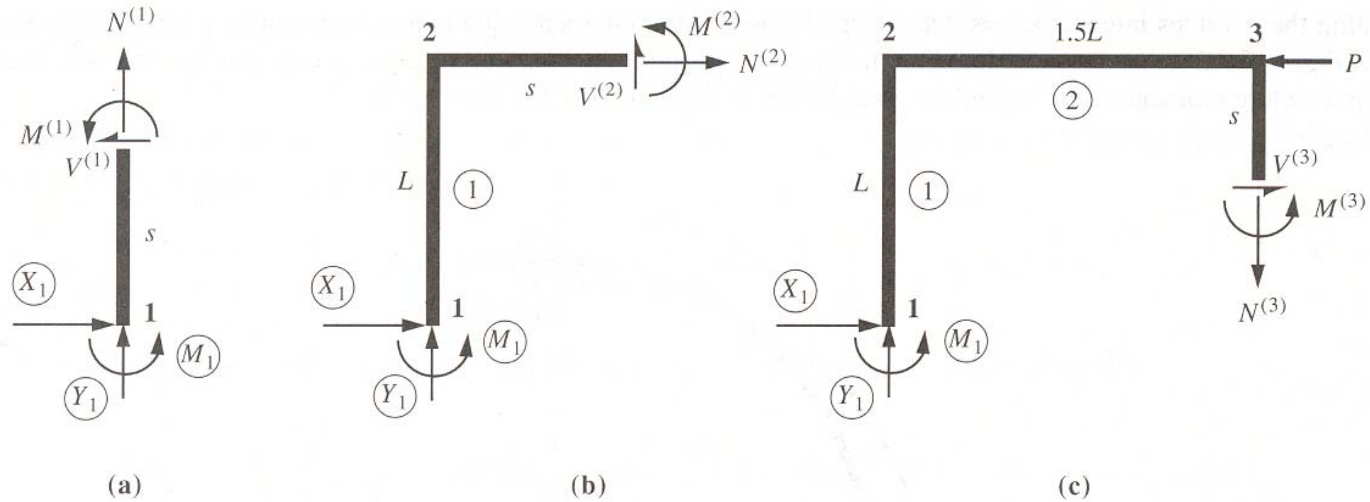


Figure 7.8.30 Free-body diagrams of portions of the base frame, revealing the true section loads in each beam element.

$$\text{Element 1}: \quad M^{(1)} = -X_1 s - M_1 \qquad \text{[a]}$$

$$\text{Element 2}: \quad M^{(2)} = -X_1 L + Y_1 s - M_1 \qquad \text{[b]}$$

$$\text{Element 3}: \quad M^{(3)} = -X_1 (L - s) + 1.5LY_1 - M_1 - Ps \qquad \text{[c]}$$

Likewise, from the free-body diagrams of the virtual load, Figure 7.8.31, we find

$$\text{Element 1}: \quad \delta M^{(1)} = -\delta X_1 s - \delta M_1 \qquad \text{[d]}$$

$$\text{Element 2}: \quad \delta M^{(2)} = -\delta X_1 L + \delta Y_1 s - \delta M_1 \qquad \text{[e]}$$

$$\text{Element 3}: \quad \delta M^{(3)} = -\delta X_1 (L - s) + 1.5L\delta Y_1 - \delta M_1$$

7.8 Statically Indeterminate Beams and Frames

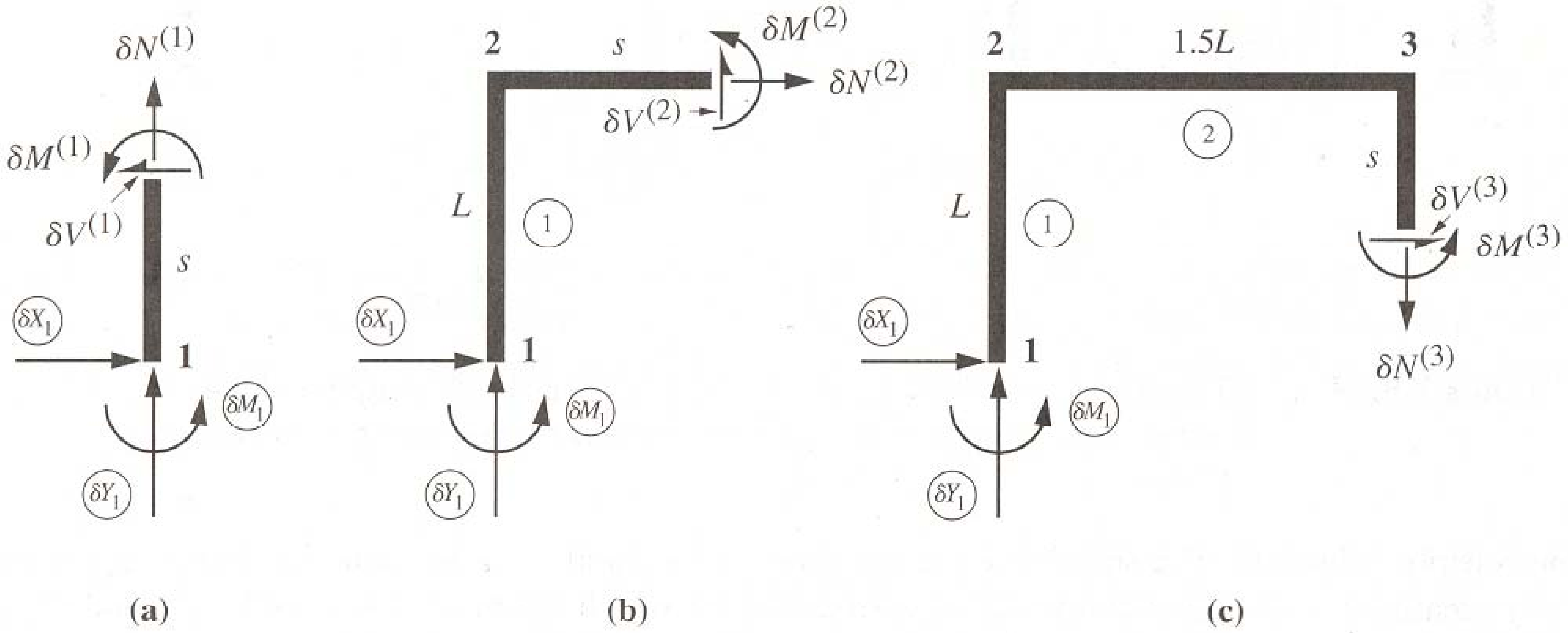


Figure 7.8.31 Free-body diagrams of portions of the base frame, revealing the virtual section loads in each beam element.

7.8 Statically Indeterminate Beams and Frames

The internal complementary virtual work for each element can be written in turn using Equations a through f, as follows:

$$\delta W_{\text{int}}^{*(1)} = \frac{1}{EI_z}\int_0^{L^{(1)}} M^{(1)}\delta M^{(1)} ds = \frac{1}{EI_z}\int_0^{L} (-X_1 s - M_1)(-\delta X_1 s - \delta M_1)\, ds$$

$$= \frac{1}{EI_z}\left[\left(\tfrac{1}{3}L^3X_1 + \tfrac{1}{2}L^2M_1\right)\delta X_1 + \left(\tfrac{1}{2}L^2X_1 + LM_1\right)\delta M_1\right] \qquad \text{[g]}$$

$$\delta W_{\text{int}}^{*(2)} = \frac{1}{EI_z}\int_0^{L^{(2)}} M^{(2)}\delta M^{(2)} ds = \frac{1}{EI_z}\int_0^{1.5L} (-X_1 L + Y_1 s - M_1)(-\delta X_1 L + \delta Y_1 s - \delta M_1)\, ds$$

$$= \frac{1}{EI_z}\left[\left(\tfrac{3}{2}L^3X_1 - \tfrac{9}{8}L^3Y_1 + \tfrac{3}{2}L^2M_1\right)\delta X_1 + \left(-\tfrac{9}{8}L^3X_1 + \tfrac{9}{8}L^3Y_1 - \tfrac{9}{8}L^2M_1\right)\delta Y_1\right.$$
$$\left. + \left(\tfrac{3}{2}L^2X_1 - \tfrac{9}{8}L^2Y_1 + \tfrac{3}{2}LM_1\right)\delta M_1\right] \qquad \text{[h]}$$

$$\delta W_{\text{int}}^{*(3)} = \frac{1}{EI_z}\int_0^{L^{(3)}} M^{(3)}\delta M^{(3)} ds = \frac{1}{EI_z}\int_0^{L} [-X_1(L-s) + 1.5LY_1 - M_1 - Ps][-\delta X_1(L-s) + 1.5L\delta Y_1 - \delta M_1]\, ds$$

$$= \frac{1}{EI_z}\left[\left(\tfrac{1}{3}L^3X_1 - \tfrac{3}{4}L^3Y_1 + \tfrac{1}{2}L^2M_1 + \tfrac{1}{6}L^3P\right)\delta X_1 + \left(-\tfrac{3}{4}L^3X_1 + \tfrac{9}{4}L^3Y_1 - \tfrac{3}{2}L^2M_1 - \tfrac{3}{4}L^3P\right)\delta Y_1\right.$$
$$\left. + \left(\tfrac{1}{2}L^2X_1 - \tfrac{3}{2}L^2Y_1 + LM_1 + \tfrac{1}{2}L^2P\right)\delta M_1\right] \qquad \text{[i]}$$

7.8 Statically Indeterminate Beams and Frames

The total internal complementary virtual work of the frame is the sum of the values for the three beam elements, or

$$\delta W^*_{\text{int}} = \delta W^{*\,(1)}_{\text{int}} + \delta W^{*\,(2)}_{\text{int}} + \delta W^{*\,(3)}_{\text{int}}$$

$$= \frac{1}{EI_z}\left[\left(\tfrac{13}{6}L^3X_1 - \tfrac{15}{8}L^3Y_1 + \tfrac{5}{2}L^2M_1 + \tfrac{1}{6}L^3P\right)\delta X_1 + \left(-\tfrac{15}{8}L^3X_1 + \tfrac{27}{8}L^3Y_1 - \tfrac{21}{8}L^2M_1 - \tfrac{3}{4}L^3P\right)\delta Y_1\right. \quad \text{[j]}$$
$$\left. + \left(\tfrac{5}{2}L^2X_1 - \tfrac{21}{8}L^2Y_1 + \tfrac{7}{2}LM_1 + \tfrac{1}{2}L^2P\right)\delta M_1\right]$$

The external complementary virtual work is zero, since the true displacement in the direction of each of the three redundants is zero. It follows then that $\delta W^*_{\text{int}} = 0$ for an arbitrary choice of δX_1, δY_1, and δM_1. Therefore, their coefficients in Equation j must each be zero, yielding the following three equations,

$$\begin{aligned}
\tfrac{13}{6}L^3X_1 - \tfrac{15}{8}L^3Y_1 + \tfrac{5}{2}L^2M_1 &= -\tfrac{1}{6}PL \\
-\tfrac{15}{8}L^3X_1 + \tfrac{27}{8}L^3Y_1 - \tfrac{21}{8}L^2M_1 &= \tfrac{3}{4}PL^3 \\
\tfrac{5}{2}L^2X_1 - \tfrac{21}{8}L^2Y_1 + \tfrac{7}{2}LM_1 &= -\tfrac{1}{2}PL^2
\end{aligned} \quad \text{[k]}$$

The solution to this system is

$$X_1 = \tfrac{1}{2}P \qquad Y_1 = \tfrac{4}{15}P \qquad M_1 = -\tfrac{3}{10}PL \quad \text{[l]}$$

We can now substitute these values for the redundant loads into Equations a, b, and c to obtain the bending moments throughout the frame in terms of the applied load P. These are plotted in Figure 7.8.32.

7.8 Statically Indeterminate Beams and Frames

Circular frames are a common component of aircraft structures. Normally, many stringers are distributed around the periphery of such frames. The shear flow applied to a frame between each pair of stringers varies with position, as illustrated in Example 5.2.3. To determine the contribution of a given portion of the shear flow to the bending moment at another point of the frame, a formula is useful. First, let A be the shaded area abc subtended by the uniform shear flow q in Figure 7.8.33. Then, the moment of that shear flow about point c is

$$M_c = 2Aq$$

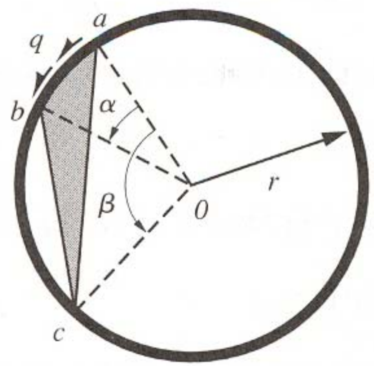


Figure 7.8.33 Shaded area subtended by the constant shear flow q.

7.8 Statically Indeterminate Beams and Frames

However, A is the area of segment abc of the circle minus the area of segment bc (see Equation b of Example 7.8.9). Using that equation to calculate the areas of the two segments in question and then subtracting the larger from the smaller, we get

$$A = \frac{r^2}{2}(\beta - \sin\beta) - \frac{r^2}{2}[(\beta - \alpha) - \sin(\beta - a)] = \frac{r^2}{2}[\alpha - \sin\beta + \sin(\beta - \alpha)]$$

The moment of the shear flow q about point c is therefore

$$M_c = qr^2[\alpha - \sin\beta + \sin(\beta - \alpha)] = qr^2[\alpha - (1 - \cos\alpha)\sin\beta - \sin\alpha\cos\beta] \qquad \text{[7.8.1]}$$

7.8 Statically Indeterminate Beams and Frames

Example 7.8.11 Use the principle of complementary virtual work to find the bending moment distribution in the circular fuselage of Figure 7.8.34. Assume that the flexural rigidity EI_z is uniform through out both the ring and ... stretching. The area of all the

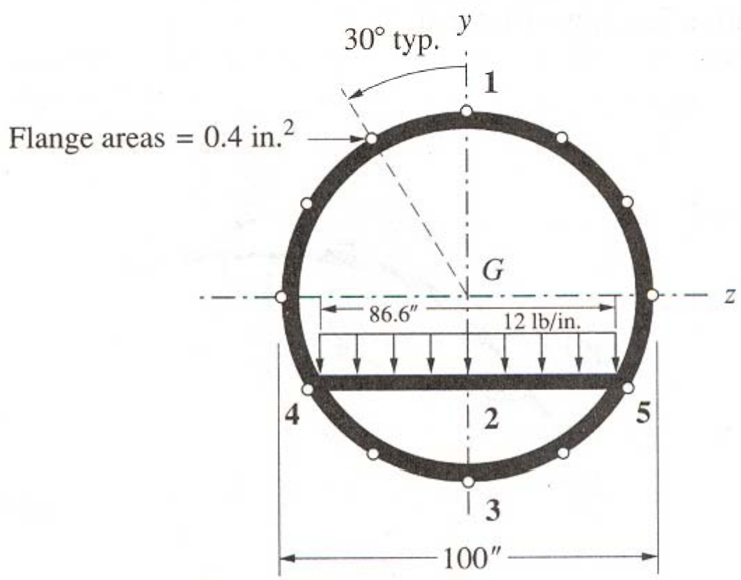


Figure 7.8.34 Circular frame and floor beam combination. The floor beam carries a uniformly distributed line load.

7.8 Statically Indeterminate Beams and Frames

The load on the floor beam is equilibrated by shear flows applied by the skin to the periphery of the frame. The method of determining these shear flows is described in Chapter 5. Following a procedure similar to that of Example 5.2.3, we obtain the shear flows shown in Figure 7.8.35.

To reveal the bending moments within the frame, we must section the structure appropriately. The vertical diameter through points 1 and 3 is a symmetry axis, so we will cut through the frame along that line, dividing it into two free bodies that are mirror images. The left one is shown in Figure 7.8.36. There is no shear force on the sections at 1, 2, and 3, since they lie on the axis of symmetry. Two of the three pairs of normal force and bending moments in Figure 7.8.36 are redundant. Our choice will be those at 1 and 2, which are circled for emphasis.

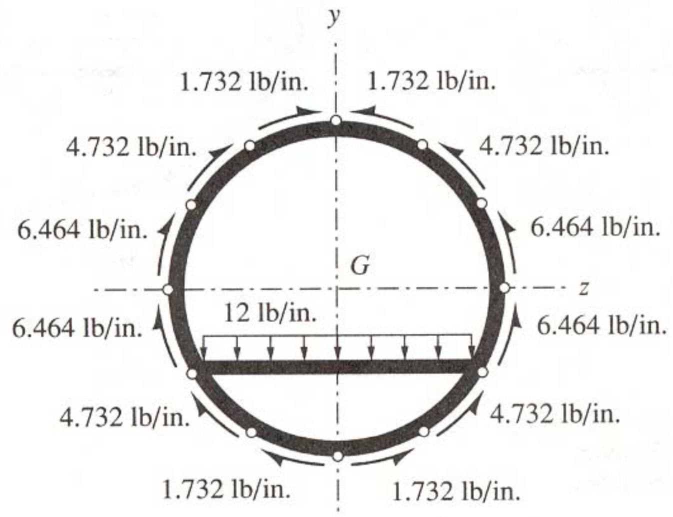


Figure 7.8.35 Shear flows acting around the periphery of the circular fuselage frame in response to the load on the floor beam.

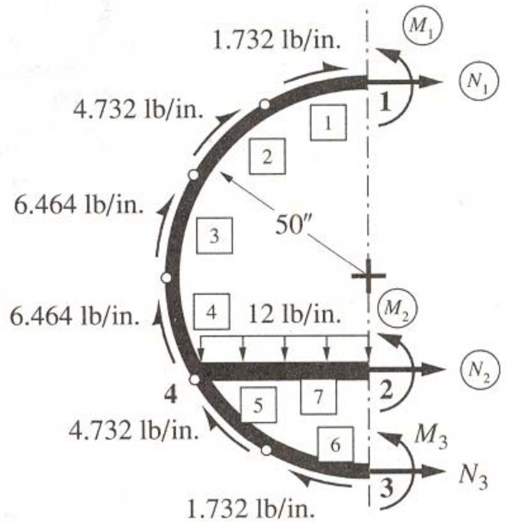
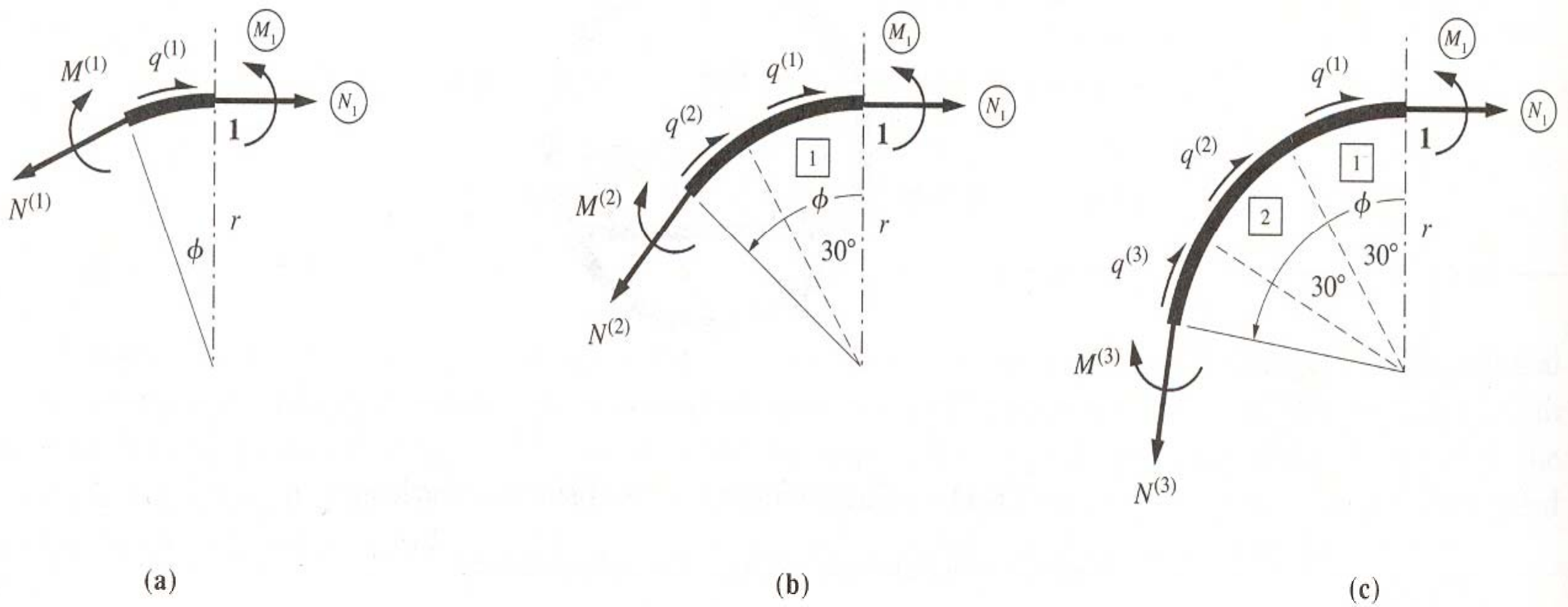


Figure 7.8.36 Free-body diagram of one-half of the symmetric frame, with the chosen redundant loads circled.

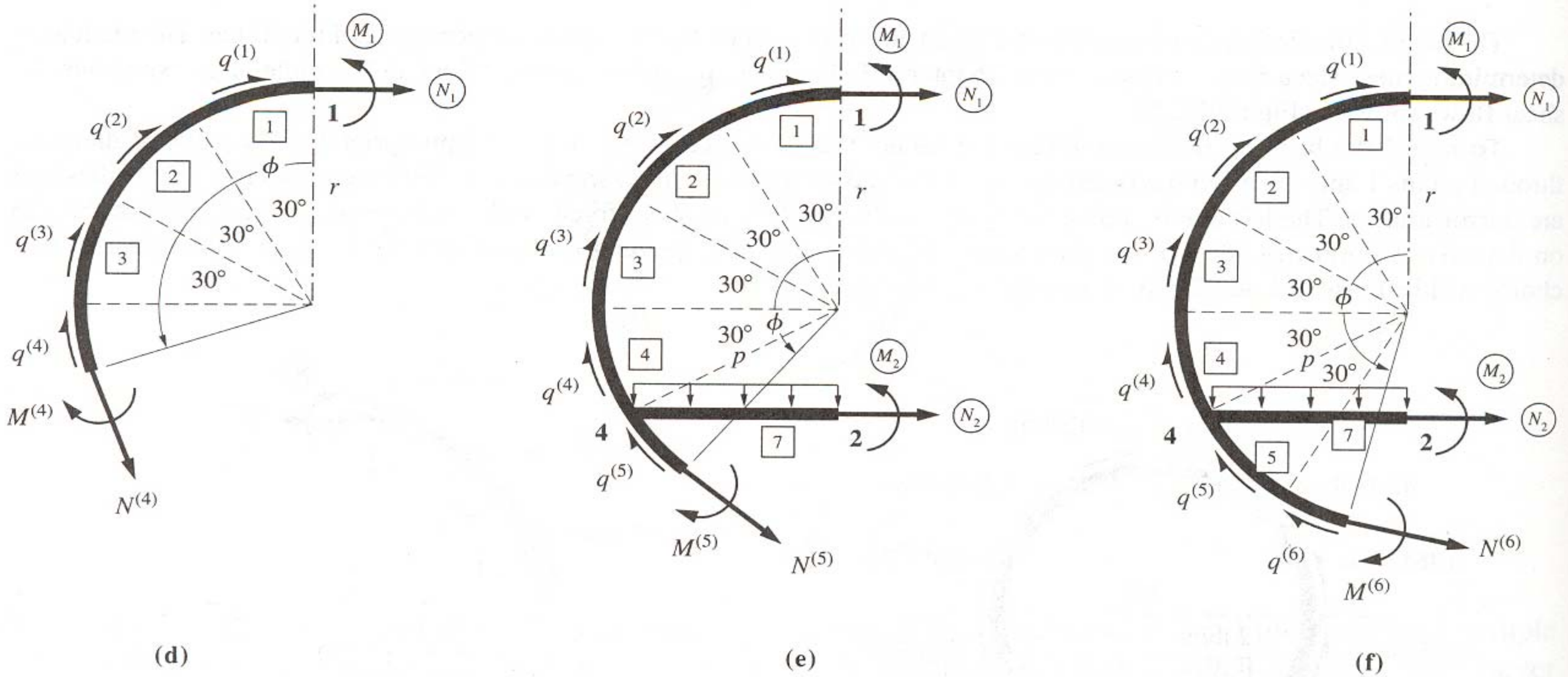
7.8 Statically Indeterminate Beams and Frames

It is convenient to use a number to identify the portions of the circular frame lying between two stringers. Therefore, including the half-span of floor beam, we can consider the symmetric half of the structure in Figure 7.8.36 to be composed of seven elements. Starting with point 1, at the top of the structure, we work our way around the frame counterclockwise, sectioning each of the six curved elements in turn and drawing the free-body diagrams shown in Figure 7.8.37. Observe that we have identified the shear flows symbolically ($q^{(1)}$, $q^{(2)}$, etc.) and denoted the 12 lb/in. load on the floor beam as p.

From Figure 7.8.37a, using Equation 7.8.1, we find that



7.8 Statically Indeterminate Beams and Frames



$q^{(1)} = q^{(6)} = 1.732$ lb/in.
$q^{(2)} = q^{(5)} = 4.732$ lb/in.
$q^{(3)} = q^{(4)} = 6.464$ lb/in.
$p = 12$ lb/in.
$r = 50$ in.

Figure 7.8.37 Free-body diagrams required to find the bending moment in each of the six elements of the frame in terms of the redundants.

7.8 Statically Indeterminate Beams and Frames

$$M^{(1)} = M_1 - N_1 r(1-\cos\phi) - q^{(1)} r^2(\phi - \sin\phi) \qquad \text{[a1]}$$

The virtual bending moment is found by removing the shear flow and replacing M_1 and N_1 by δM_1 and δN_1, respectively, so that

$$\delta M^{(1)} = \delta M_1 - \delta N_1 r(1-\cos\phi) \qquad \text{[a2]}$$

The true and virtual bending moments in elements 2 through 4 are found in a similar fashion by using the free-body diagrams in parts (b) through (d) of Figure 7.8.37. The number of terms in the equations increases as we move from element to element, picking up more and more of the applied loads.

$$M^{(2)} = M_1 - N_1 r\,(1-\cos\phi) - q^{(1)} r^2 \left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\phi - \sin\tfrac{\pi}{6}\cos\phi\right] - q^{(2)} r^2 \left[\phi - \tfrac{\pi}{6} - \sin\left(\phi - \tfrac{\pi}{6}\right)\right] \qquad \text{[b1]}$$

$$\delta M^{(2)} = \delta M_1 - \delta N_1 r(1-\cos\phi) \qquad \text{[b2]}$$

$$\begin{aligned} M^{(3)} = {} & M_1 - N_1 r\,(1-\cos\phi) - q^{(1)} r^2 \left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\phi - \sin\tfrac{\pi}{6}\cos\phi\right] \\ & - q^{(2)} r^2 \left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\left(\phi - \tfrac{\pi}{6}\right) - \sin\tfrac{\pi}{6}\cos\left(\phi - \tfrac{\pi}{6}\right)\right] - q^{(3)} r^2 \left[\phi - \tfrac{\pi}{3} - \sin\left(\phi - \tfrac{\pi}{3}\right)\right] \end{aligned} \qquad \text{[c1]}$$

$$\delta M^{(3)} = \delta M_1 - \delta N_1 r(1-\cos\phi) \qquad \text{[c2]}$$

$$\begin{aligned} M^{(4)} = {} & M_1 - N_1 r\,(1-\cos\phi) - q^{(1)} r^2 \left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\phi - \sin\tfrac{\pi}{6}\cos\phi\right] \\ & - q^{(2)} r^2 \left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\left(\phi - \tfrac{\pi}{6}\right) - \sin\tfrac{\pi}{6}\cos\left(\phi - \tfrac{\pi}{6}\right)\right] \\ & - q^{(3)} r^2 \left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\left(\phi - \tfrac{\pi}{3}\right) - \sin\tfrac{\pi}{6}\cos\left(\phi - \tfrac{\pi}{3}\right)\right] - q^{(4)} r^2 \left[\phi - \tfrac{\pi}{2} - \sin\left(\phi - \tfrac{\pi}{2}\right)\right] \end{aligned} \qquad \text{[d1]}$$

$$\delta M^{(4)} = \delta M_1 - \delta N_1 r(1-\cos\phi) \qquad \text{[d2]}$$

7.8 Statically Indeterminate Beams and Frames

The free-body diagrams in parts (e) and (f) of Figure 7.8.37 reveal the distributed load p on the floor beam and the redundant loads M_2 and N_2 acting on the cut at point 2. The corresponding free-body diagrams for the virtual load would show all of the shear flows removed and the normal load and bending moment at points 1 and 2 replaced, as usual, by their virtual counterparts. Therefore, for elements 5 and 6, moment equilibrium implies

$$\begin{aligned} M^{(5)} = M_1 - N_1 r\,(1-\cos\phi) - q^{(1)}r^2\left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\phi - \sin\tfrac{\pi}{6}\cos\phi\right] \\ - q^{(2)}r^2\left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\left(\phi - \tfrac{\pi}{6}\right) - \sin\tfrac{\pi}{6}\cos\left(\phi - \tfrac{\pi}{6}\right)\right] \\ - q^{(3)}r^2\left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\left(\phi - \tfrac{\pi}{3}\right) - \sin\tfrac{\pi}{6}\cos\left(\phi - \tfrac{\pi}{3}\right)\right] \\ - q^{(4)}r^2\left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\left(\phi - \tfrac{\pi}{2}\right) - \sin\tfrac{\pi}{6}\cos\left(\phi - \tfrac{\pi}{2}\right)\right] \\ - q^{(5)}r^2\left[\phi - \tfrac{2\pi}{3} - \sin\left(\phi - \tfrac{2\pi}{3}\right)\right] + M_2 + N_2 r\left(\cos\tfrac{\pi}{3} + \cos\phi\right) + pr^2\sin\tfrac{\pi}{3}\left(\tfrac{1}{2}\sin\tfrac{\pi}{3} - \sin\phi\right) \end{aligned} \qquad \text{[e1]}$$

$$\delta M^{(5)} = \delta M_1 - \delta N_1 r(1-\cos\phi) + \delta M_2 + \delta N_2 r(\cos\tfrac{\pi}{3} + \cos\phi) \qquad \text{[e2]}$$

$$\begin{aligned} M^{(6)} = M_1 - N_1 r\,(1-\cos\phi) - q^{(1)}r^2\left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\phi - \sin\tfrac{\pi}{6}\cos\phi\right] \\ - q^{(2)}r^2\left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\left(\phi - \tfrac{\pi}{6}\right) - \sin\tfrac{\pi}{6}\cos\left(\phi - \tfrac{\pi}{6}\right)\right] \\ - q^{(3)}r^2\left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\left(\phi - \tfrac{\pi}{3}\right) - \sin\tfrac{\pi}{6}\cos\left(\phi - \tfrac{\pi}{3}\right)\right] \\ - q^{(4)}r^2\left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\left(\phi - \tfrac{\pi}{2}\right) - \sin\tfrac{\pi}{6}\cos\left(\phi - \tfrac{\pi}{2}\right)\right] \\ - q^{(5)}r^2\left[\tfrac{\pi}{6} + \left(\cos\tfrac{\pi}{6} - 1\right)\sin\left(\phi - \tfrac{2\pi}{3}\right) - \sin\tfrac{\pi}{6}\cos\left(\phi - \tfrac{2\pi}{3}\right)\right] \\ - q^{(6)}r^2\left[\phi - \tfrac{5\pi}{6} - \sin\left(\phi - \tfrac{5\pi}{6}\right)\right] + M_2 + N_2 r\left(\cos\tfrac{\pi}{3} + \cos\phi\right) + pr^2\sin\tfrac{\pi}{3}\left(\tfrac{1}{2}\sin\tfrac{\pi}{3} - \sin\phi\right) \end{aligned} \qquad \text{[f1]}$$

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$$\delta M^{(6)} = \delta M_1 - \delta N_1 r\,(1-\cos\phi) + \delta M_2 + \delta N_2 r\left(\cos\tfrac{\pi}{3} + \cos\phi\right) \qquad \text{[f2]}$$

For the floor beam between points 2 and 4, the true and virtual bending moments are simply

$$M^{(7)} = M_2 - \frac{ps^2}{2} \qquad \text{[g1]}$$

$$\delta M^{(7)} = \delta M_2 \qquad \text{[g2]}$$

The internal complementary virtual work for the frame is the sum of the individual contributions from each of the seven elements (multiplied by 2 to account for both symmetric halves of the complete frame):

$$\begin{aligned}\delta W^*_{\text{int}} = \frac{2}{EI_z}\Bigg(&\int_0^{\pi/6} M^{(1)}\delta M^{(1)} r\,d\phi + \int_{\pi/6}^{\pi/3} M^{(2)}\delta M^{(2)} r\,d\phi + \int_{\pi/3}^{\pi/2} M^{(3)}\delta M^{(3)} r\,d\phi + \int_{\pi/2}^{2\pi/3} M^{(4)}\delta M^{(4)} r\,d\phi \\ &+ \int_{2\pi/3}^{5\pi/6} M^{(5)}\delta M^{(5)} r\,d\phi + \int_{5\pi/6}^{\pi} M^{(6)}\delta M^{(6)} r\,d\phi + \int_0^{r\sin\frac{\pi}{3}} M^{(7)}\delta M^{(7)}\,ds\Bigg)\end{aligned} \qquad \text{[h]}$$

Substituting Equations a through g into this equation, inserting the numerical values for the shear flows, distributed load, and frame radius, collecting terms, simplifying, and doing the integrals finally yields

$$\begin{aligned}\delta W^*_{\text{int}} = \frac{1}{EI_z}\Big[&\left(157.1M_1 - 7854N_1 + 52.36M_2 - 856.1N_2 - 1.420\times10^6\right)\delta M_1 \\ &+ \left(-7854M_1 + 589.0\times10^3 N_1 - 4783M_2 + 81190N_2 + 121.2\times10^6\right)\delta N_1 \\ &+ \left(52.36M_1 - 4783N_1 + 95.66M_2 - 856.1N_2 - 1.334\times10^6\right)\delta M_2 \\ &+ \left(-856.1M_1 + 81190N_1 - 856.1M_2 + 16990N_2 + 19.25\times10^6\right)\delta N_2\Big]\end{aligned} \qquad \text{[i]}$$

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All of the redundants are internal loads, so $\delta W^*_{ext} = 0$. Therefore, $\delta W^*_{int} = 0$ and, by the usual argument, the principle of complementary virtual work requires the four coefficients of the virtual loads in Equation i to vanish. The result is the four equations in the four unknowns M_1, N_1, M_2, and N_2, as follows:

$$
\begin{aligned}
157.1M_1 - 7854N_1 + 52.36M_2 - 856.1N_2 &= 1.420 \times 10^6 EI_z \\
-7854M_1 + 589.0 \times 10^3 N_1 - 4783M_2 + 81190N_2 &= -121.2 \times 10^6 EI_z \\
52.36M_1 - 4783N_1 + 95.66M_2 - 856.1N_2 &= 1.334 \times 10^6 EI_z \\
-856.1M_1 + 81190N_1 - 856.1M_2 + 16990N_2 &= -19.25 \times 10^6 EI_z
\end{aligned}
\qquad [j]
$$

The solution of this system is

$$M_1 = -1788 \text{ in.-lb} \qquad N_1 = -161 \text{ lb} \qquad M_2 = 5115 \text{ in.-lb} \qquad N_2 = -196 \text{ in.-lb}$$

The bending moment distribution around the frame is found by substituting these redundant load values into Equations a1 through g1. The bending moment is plotted in Figure 7.8.38. We can see that the maximum value occurs at the junction of the circular frame and the floor beam, and the next largest value occurs at the midpoint of the floor beam.

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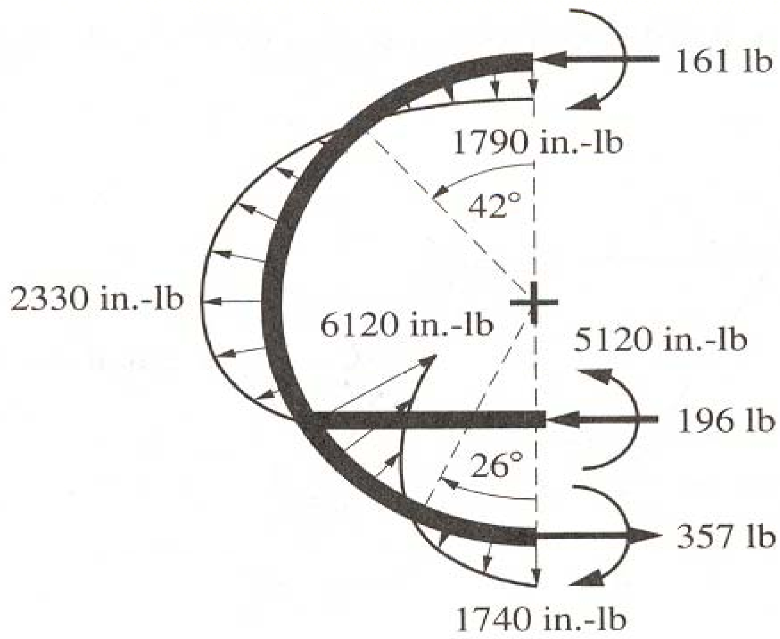

Figure 7.8.38 Bending moment distribution around the circular portion of the frame.

The moment is plotted on the side of the frame in compression.