Aircraft Structural Analysis

Chapter 8

Force Method: Idealized Thin-Walled Structures

8.1 Introduction

This chapter applies the force method based on the complementary virtual work principle to the analysis of

- assembles of thin shear panels and stiffeners
- deflections of box beams, with and without taper
- shear flows in multicell box beams
- the unrestrained warping of beam cross sections due to the torsional component of loading
- the effects on shear flows of warping restraints, as occurs near supports.

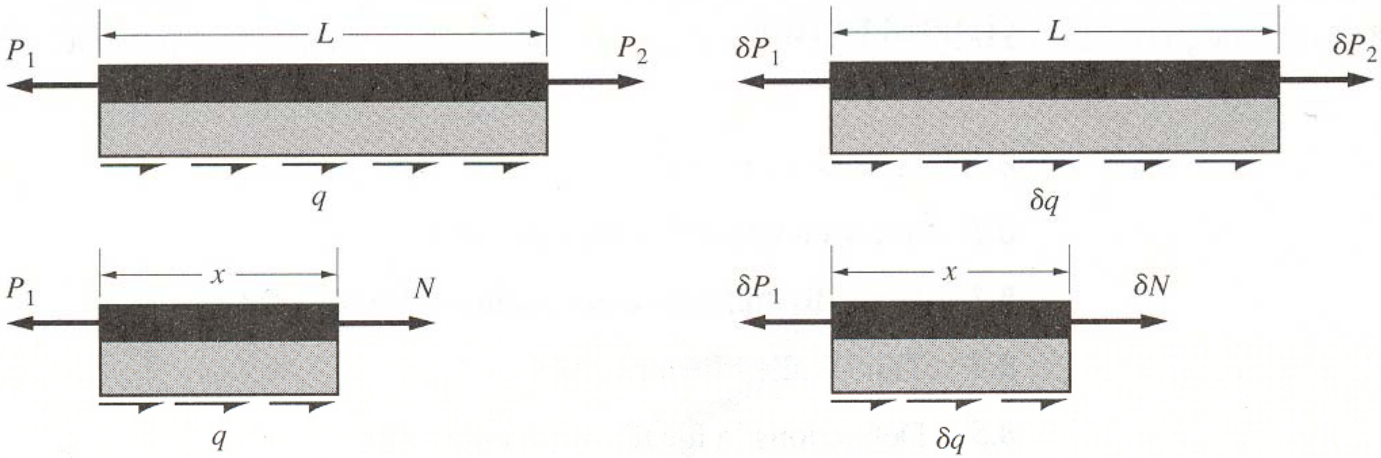
The chapter concludes with a discussion of shear lag, which is not so much an aspect of the force method as it is a means of assessing the influence of deformation restraints on shear flow distribution.

8.2 Shear planes and stiffeners

The rod element discussed in section 7.2 commonly plays the role of a constant-area stiffener attached to one or more constant shear flow panels as in a box beam. This situation is depicted in Figure 8.2.1.

$$P_2 + qL - P_1 = 0$$

$$\text{or} \qquad q = \frac{P_1 - P_2}{L} \qquad \textbf{[8.2.1]}$$



(a) Actual load

(b) Virtual load

Figure 8.2.1 Rod and adjacent shear web.

8.2 Shear planes and stiffeners

$$N = P_1\left(1-\frac{x}{L}\right) + P_2\left(\frac{x}{L}\right) \qquad [8.2.2]$$

$$\delta N = \delta P_1\left(1-\frac{x}{L}\right) + \delta P_2\left(\frac{x}{L}\right) \qquad [8.2.3]$$

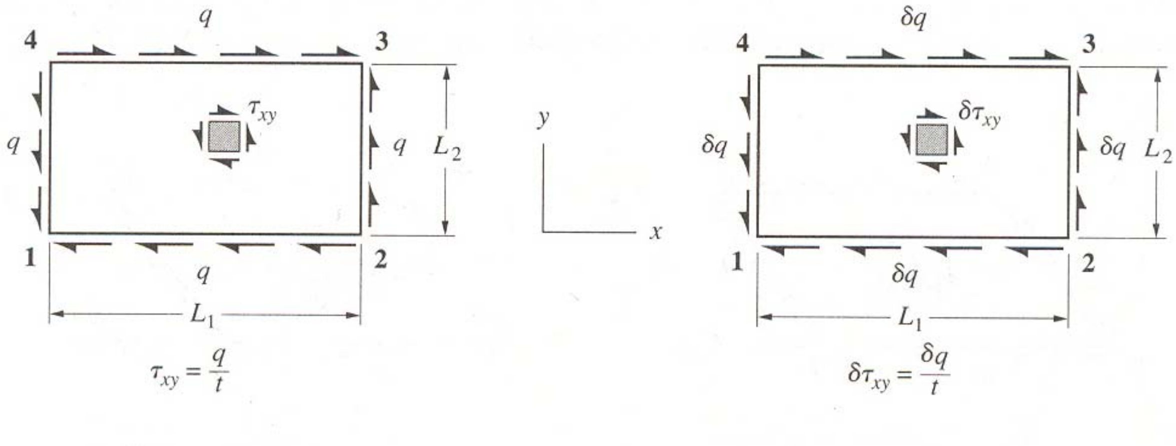
$$N\delta N = P_1\delta P_1\left(1-\frac{x}{L}\right)^2 + P_2\delta P_2\left(\frac{x}{L}\right)^2 + (P_1\delta P_2 + P_2\delta P_1)\left(1-\frac{x}{L}\right)\left(\frac{x}{L}\right) \qquad [8.2.4]$$

After substituting this expression into Equation 7.2.1, and doing the first integral,

$$\delta W^*_{\text{int}} = \int_0^L \frac{N\delta N}{AE}\,dx + \int_0^L (\alpha T)\,\delta N\,dx \qquad [7.2.1]$$

$$\boxed{\delta W^*_{\text{int}} = \frac{L}{3AE}\left[\left(P_1 + \tfrac{1}{2}P_2\right)\delta P_1 + \left(P_2 + \tfrac{1}{2}P_1\right)\delta P_2\right] + \int_0^L (\alpha T)\left[\delta P_1\left(1-\frac{x}{L}\right) + \delta P_2\left(\frac{x}{L}\right)\right]dx} \quad \text{web stiffener} \qquad [8.2.5]$$

8.2 Shear planes and stiffeners



(a) Actual load (b) Virtual load

Figure 8.2.2 Rectangular, constant shear flow panel.

$$\delta W^*_{\text{int}} = \iiint_V \delta\tau_{xy}\gamma_{xy} dV$$
$$= \iint_A \delta\tau_{xy}\left(\frac{\tau_{xy}}{G}\right) t dA$$
$$= \iint_A \left(\frac{\delta q}{t}\right)\left(\frac{q}{Gt}\right) t dA$$
$$= \frac{q\delta q}{Gt}\iint_A dA$$

For a rectangular shear panel, $$\delta W^*_{\text{int}} = \frac{A}{Gt} q\delta q \qquad [8.2.6]$$

where $A = L_1 * L_2$ is the area of the rectangle.

8.2 Shear planes and stiffeners

For a curved cylindrical panels,

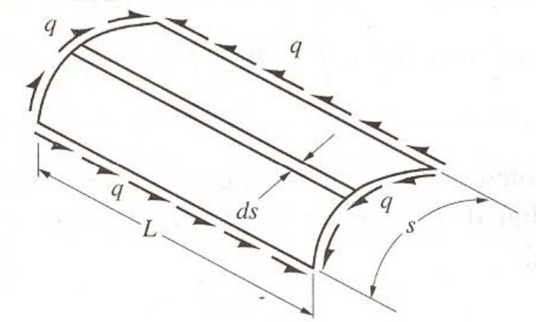
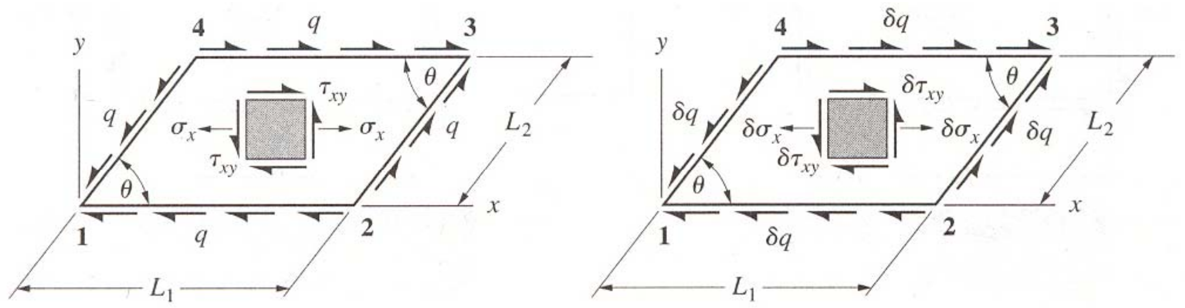


Figure 8.2.3 Cylindrical shear panel.



$\sigma_x = 2\frac{q}{t}\cot\theta \qquad \tau_{xy} = \frac{q}{t}$

(a) Actual load

$\delta\sigma_x = 2\frac{\delta q}{t}\cot\theta \qquad \delta\tau_{xy} = \frac{\delta q}{t}$

(b) Virtual load

Figure 8.2.4 Parallelogram shear panel.

The only nonzero stress components are

$$\sigma_x = 2\frac{q}{t}\cot\theta \qquad \tau_{xy} = \frac{q}{t}$$

Where t is the panel thickness and θ is the acute included angle of the parallelogram

8.2 Shear planes and stiffeners

Substituting these stresses into Equation 6.6.4,

$$\delta W_{\text{int}}^{*} = \iiint_V \delta\sigma_x \varepsilon_x dV + \iiint_V \delta\tau_{xy}\gamma_{xy} dV$$

$$= \iint_A \delta\sigma_x \frac{\sigma_x}{E}\, t dA + \iint_A \delta\tau_{xy} \frac{\tau_{xy}}{G}\, t dA$$

$$= \iint_A \left(2\frac{\delta q}{t}\cot\theta\right)\left[\frac{2(q/t)\cot\theta}{2\,(1+\nu)\,G}\right] t dA + \iint_A \left(\frac{\delta q}{t}\right)\left(\frac{q/t}{G}\right) t dA$$

$$\delta W_{\text{int}}^{*} = \left[1 + \frac{2\cot^2\theta}{1+\nu}\right]\frac{A}{Gt} q\delta q \qquad \text{parallelogram shear panel} \qquad \textbf{[8.2.8]}$$

Where θ is 90˚, this expression reduces to that for a rectangular panel, Equation 8.2.6

8.2 Shear planes and stiffeners

Figure 8.2.5 shows a flat trapezoidal panel, two edges of which are parallel while the other two (extended) intersect at the vertex P through which a baseline parallel to the parallel edges is drawn.

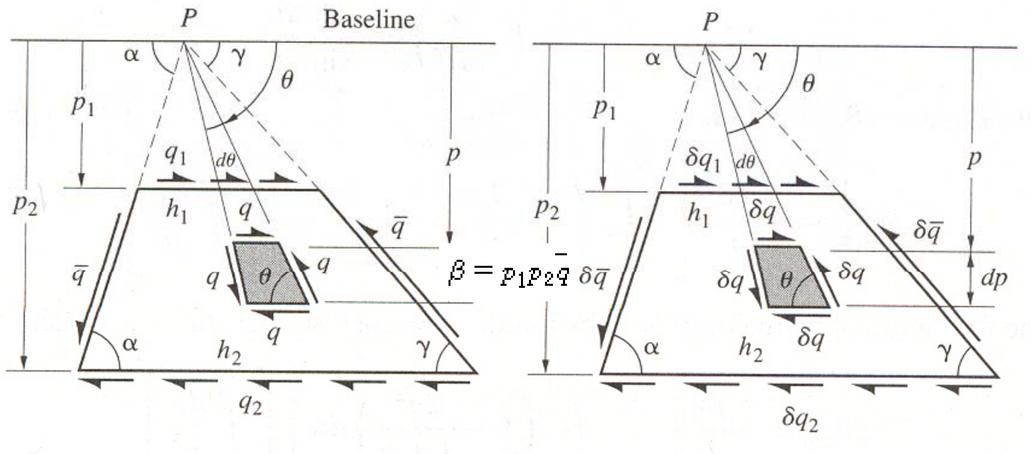


Figure 8.2.5 Trapezoidal shear panel, where h_1 and h_2 are the lengths of the parallel sides.

$$q = \frac{\beta}{p^2} \qquad [8.2.9]$$

P : the perpendicular distance from the vertex to the differential element

β : a constant related to the average shear flow in the panel ($\beta = p_1 p_2 \bar{q}$)

8.2 Shear planes and stiffeners

$$\delta W^*_{\text{int}} = \int_{\text{trapezoid}} \delta W^*_{\text{int,differential parallelogram}}$$

$$\delta W^*_{\text{int}} = \iint_A \frac{1}{Gt}\left[1 + \frac{2\cot^2\theta}{1+\nu}\right]\frac{\beta\delta\beta}{p^4}dA \qquad \text{[8.2.10]}$$

$$dA = \left(\frac{p}{\sin\theta}d\theta\right)\left(\frac{dp}{\sin\theta}\right) = \frac{p\,dp\,d\theta}{\sin^2\theta} \qquad \text{[8.2.11]}$$

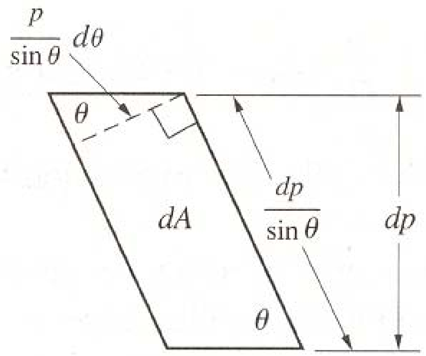


Figure 8.2.6 Detail of the shaded differential parallelogram in Figure 8.2.5.

$$\delta W^*_{\text{int}} = \frac{\beta\,\delta\beta}{Gt}\left[\int_{\gamma}^{180-\alpha}\left(1 + \frac{2\cot^2\theta}{1+\nu}\right)\frac{d\theta}{\sin^2\theta}\right]\left[\int_{p_1}^{p_2}\frac{dp}{p^3}\right] \qquad \text{[8.2.12]}$$

$$u = \cot\theta,\ du = -d\theta/\sin^2\theta$$

$$\delta W^*_{\text{int}} = -\frac{\beta\delta\beta}{Gt}\left[\int_{\cot\gamma}^{-\cot\alpha}\left(1 + \frac{2u^2}{1+\nu}\right)du\right]\left[\int_{p_1}^{p_2}\frac{dp}{p^3}\right]$$

8.2 Shear planes and stiffeners

For a trapezoidal shear panel, $\beta = p_1 p_2 \bar{q}$ and $\delta\beta = p_1 p_2 \delta\bar{q}$

$$\delta W^*_{\text{int}} = \left[1 + \frac{2}{3(1+\nu)}\left(\cot^2\gamma - \cot\alpha \; \cot\gamma + \cot^2\alpha\right)\right] \frac{A}{Gt} \bar{q}\delta\bar{q} \qquad [8.2.13]$$

which is $A = \frac{1}{2}(h_1 + h_2)(p_2 - p_1)$

For a quadrilateral shear panel no two sides of which are parallel, the expression for is even more complicated will not be given here.

8.3 Statically Indeterminate Stiffened Webs

According to Equation 2.5.7, a plane stiffened panel is statically determinate if

$$(\text{no. rods}) + (\text{no. panels}) + (\text{no. reactions}) = 2 \times (\text{no. nodes})$$

$$(\text{no. rods}) + (\text{no. panels}) + (\text{no. reactions}) - 2 \times (\text{no. nodes}) = \text{degree of static indeterminancy} \quad [8.3.1]$$

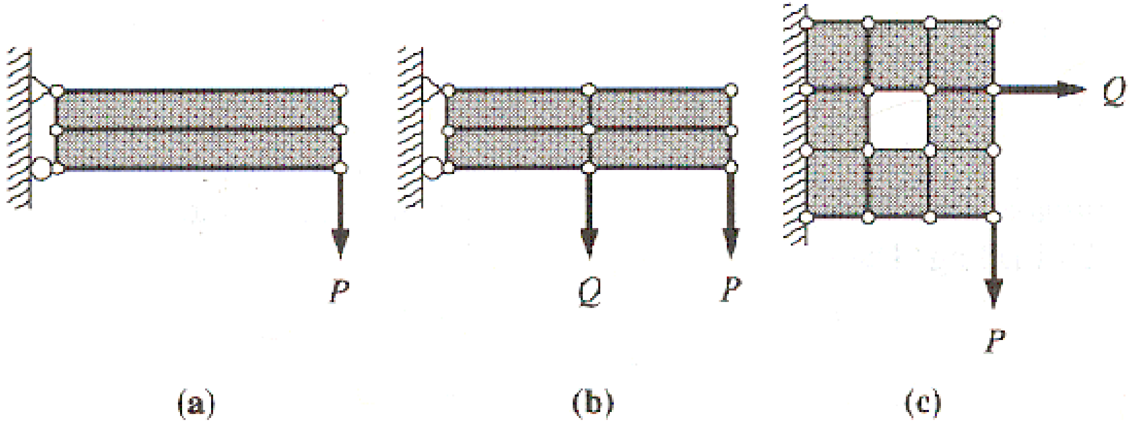


Figure 8.3.1 (a) Statically determinate, stiffened panel assembly. (b) and (c) Statically indeterminate structures.

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.1

Use the principle of complementary virtual work to calculate the shear flows in the stiffened web structure in Figure 8.3.2. All of the stiffeners have the same cross-sectional area, all of the panels have the same thickness t. The material properties are uniform throughout.

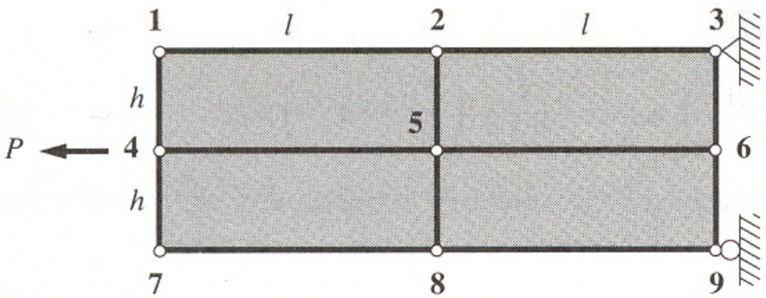


Figure 8.3.2 Singly-redundant stiffened panel structure.

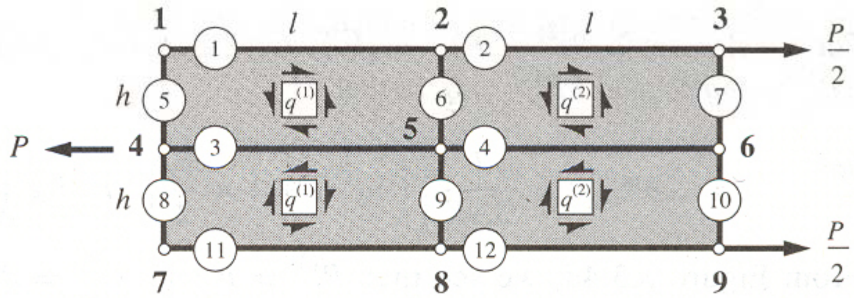


Figure 8.3.3 Free-body diagram of the symmetric structure of Figure 8.3.2, showing the shear flows and rod element numbering scheme.

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.1

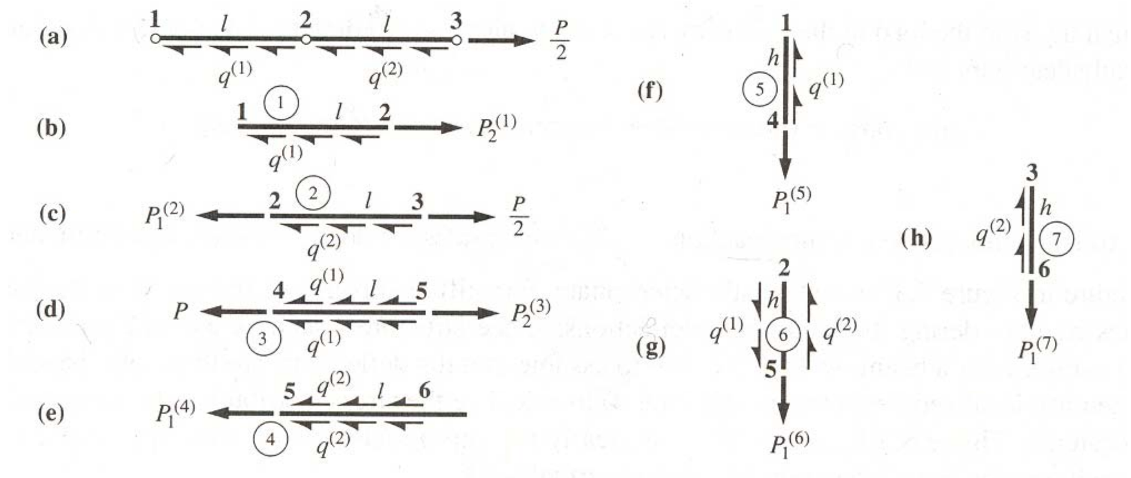


Figure 8.3.4 Free-body diagrams of individual stiffener elements of the structure in Figure 8.3.2.

Let us choose $q^{(1)}$ as the redundant shear flow,

$$q^{(1)}l + q^{(2)}l = \frac{P}{2} \quad \Rightarrow \quad \delta q^{(2)} = -\delta q^{(1)}$$

For rod element 1, $P_2^{(1)} = q^{(1)}l$ and $\delta P_2^{(1)} = \delta q^{(1)}l$

$$\begin{aligned} \delta W_{\text{int}}^{*(1)} &= \frac{l}{3AE}\left[\left(P_1^{(1)} + \tfrac{1}{2}P_2^{(1)}\right)\delta P_1^{(1)} + \left(P_2^{(1)} + \tfrac{1}{2}P_1^{(1)}\right)\delta P_2^{(1)}\right] \\ &= \frac{l}{3AE}\left[\left(0 + \tfrac{1}{2}q^{(1)}l\right)0 + \left(q^{(1)}l + \tfrac{1}{2}\times 0\right)\delta q^{(1)}l\right] \end{aligned} \quad \Rightarrow \quad \delta W_{\text{int}}^{*(1)} = \frac{l^3}{3AE}q^{(1)}\delta q^{(1)}$$

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.1

For rod element 2, $P_1^{(2)} = P_2^{(1)} = q^{(1)}l$, $P_2^{(2)} = P/2$ and $\delta P_1^{(2)} = \delta q^{(1)}l$, $\delta P_2^{(2)} = 0$

$$\delta W_{\text{int}}^{*(2)} = \frac{l}{3AE}\left[\left(\delta q^{(1)}l + \tfrac{1}{2}\times\frac{P}{2}\right)\delta q^{(1)}l + \left(\frac{P}{2} + \tfrac{1}{2}\delta q^{(1)}l\right)\times 0\right] = \frac{l^2}{3AE}\left(\frac{P}{4} + lq^{(1)}\right)\delta q^{(1)}$$

For rod element 3, $P_1^{(3)} = P$, $P_2^{(3)} = P - 2q^{(1)}l$ and $\delta P_1^{(3)} = 0$, $\delta P_2^{(3)} = -2\delta q^{(1)}l$

$$\delta W_{\text{int}}^{*(3)} = \frac{l}{3AE}\left\{\left[P + \tfrac{1}{2}\left(P - 2q^{(1)}l\right)\right]\times 0 + \left[\left(P - 2q^{(1)}l\right) + \tfrac{1}{2}P\right]\left(-2\delta q^{(1)}l\right)\right\} = \frac{2l^2}{3AE}\left(\tfrac{3}{2}P - 2q^{(1)}l\right)\delta q^{(1)}$$

Using the remaining free-body diagrams in Figure 8.3.4 and processing as before leads to the following complementary virtual work expressions for the remaining rods:

$$\delta W_{\text{int}}^{*(4)} = \frac{2l^2}{3AE}\left(P - 2q^{(1)}l\right)\delta q^{(1)}$$

$$\delta W_{\text{int}}^{*(5)} = \frac{h^3}{3AE}q^{(1)}\delta q^{(1)}$$

$$\delta W_{\text{int}}^{*(6)} = \frac{h^3}{3AE}\left(4q^{(1)} - \frac{P}{l}\right)\delta q^{(1)}$$

$$\delta W_{\text{int}}^{*(7)} = \frac{h^3}{3AE}\left(q^{(1)} - \frac{P}{2l}\right)\delta q^{(1)}$$

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.1

The total internal complementary virtual work for the structure, including all 12 rods, is

$$\delta W^*_{\text{int,stiffeners}} = 2\delta W^{*(1)}_{\text{int}} + 2\delta W^{*(2)}_{\text{int}} + \delta W^{*(3)}_{\text{int}} + \delta W^{*(4)}_{\text{int}} + 2\delta W^{*(5)}_{\text{int}} + 2\delta W^{*(6)}_{\text{int}} + 2\delta W^{*(7)}_{\text{int}}$$
$$= \frac{l^2}{AE}\left\{4q^{(1)}l\left[1+\left(\frac{h}{l}\right)^3\right] - P\left[\frac{3}{2}+\left(\frac{h}{l}\right)^3\right]\right\}\delta q^{(1)}$$

For the complementary internal virtual work of the webs,

$$\delta W^*_{\text{int,panels}} = 2\times\left(\frac{hl}{Gt}q^{(1)}\delta q^{(1)} + \frac{hl}{Gt}q^{(2)}\delta q^{(2)}\right)$$
$$= \frac{h}{Gt}\left(4q^{(1)}l - P\right)\delta q^{(1)}$$

The total complementary internal virtual work for the structure is

$$\delta W^*_{\text{int}} = \delta W^*_{\text{int,stiffeners}} + \delta W^*_{\text{int,panels}}$$
$$= \frac{1}{2AEGlt}\left\{8\left[AEhl^2 + Glt\left(l^3+h^3\right)\right]q^{(1)} - \left[2AEl + Gt\left(2h^3+3l^3\right)\right]P\right\}\delta q^{(1)}$$

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.1

Since the redundant shear flow, $q^{(1)}$, is an internal load, the external CVW is zero.

The principle of complementary virtual work therefore requires $\delta W^*_{\text{int}} = 0.$

$$q^{(1)} = \frac{\frac{3}{2} + \frac{AE}{Glt}\frac{h}{l} + \frac{h^3}{l^3}}{4\left(1 + \frac{AE}{Glt}\frac{h}{l} + \frac{h^3}{l^3}\right)}\frac{P}{l} \qquad q^{(2)} = \frac{\frac{1}{2} + \frac{AE}{Glt}\frac{h}{l} + \frac{h^3}{l^3}}{4\left(1 + \frac{AE}{Glt}\frac{h}{l} + \frac{h^3}{l^3}\right)}\frac{P}{l}$$

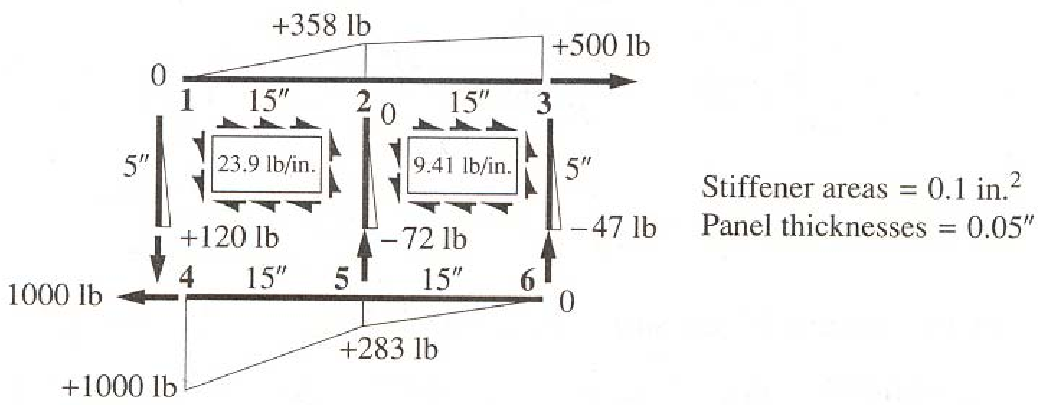


Figure 8.3.5 Panel shear flows and stiffener axial load distributions in the top half of the symmetric structure of Figure 8.3.2 corresponding to the indicated numerical data.

8.3 Statically Indeterminate Stiffened Webs

Using the same numerical data as Example 8.3.1, a shear lag analysis (in the section 8.10) yields the shear flow distribution shown in Figure 8.3.6. The Associated average shear flows computed in the example are 23.9 lb/in. and 9.41 lb/in.

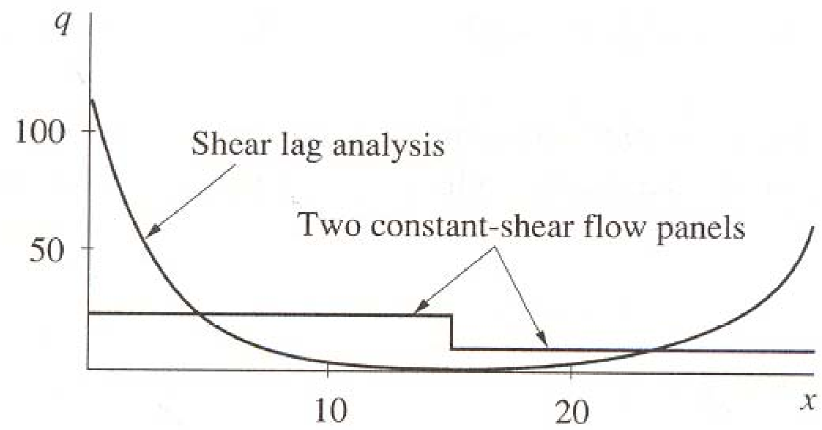


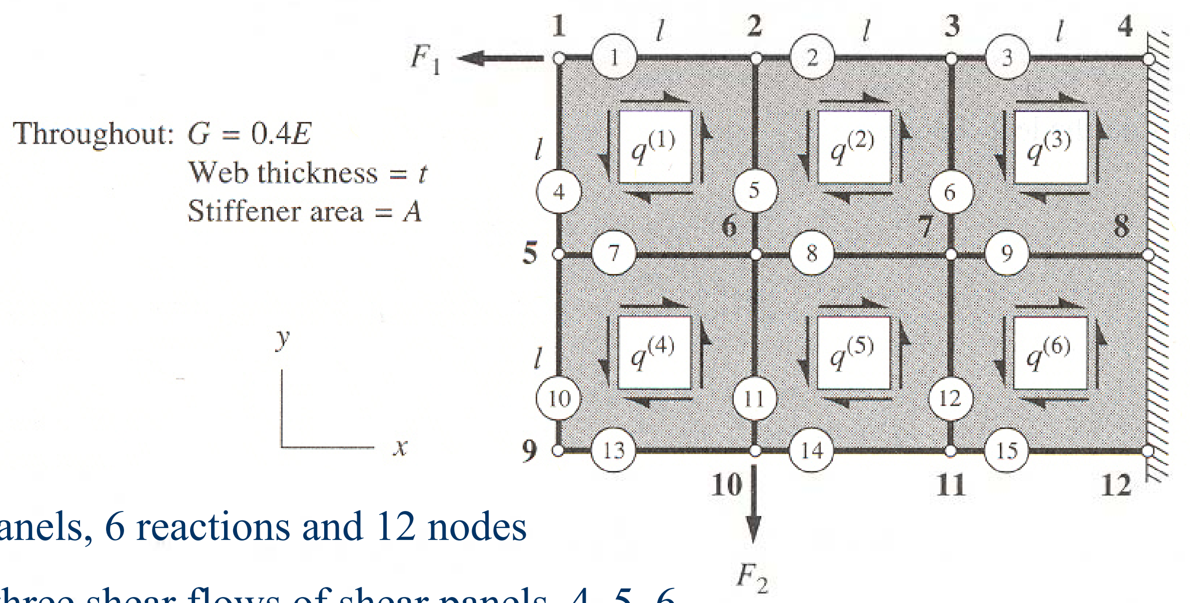
Figure 8.3.6 Shear flow q (lb/in.) vs. station x (in.) for the stiffened web structure of Figure 8.3.2.

Shear lag results are compared to those in the previous example, using the same numerical data.

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.2

Using the principle of complementary virtual work to find the shear flows in each of six panels of the stiffened web structure in Figure 8.3.7.



15 rods, 6 panels, 6 reactions and 12 nodes

(15+6+6)-2x2=3

We choose three shear flows of shear panels, 4, 5, 6

3 redundants

Figure 8.3.7 Stiffened web structure with three degrees of static indeterminacy.

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.2

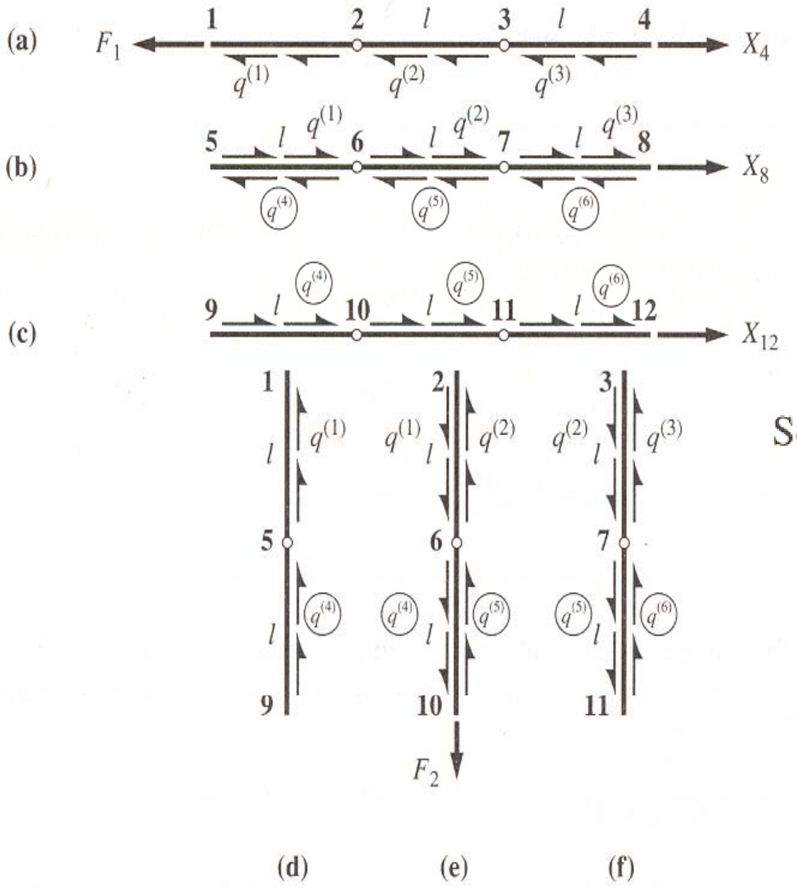


Figure 8.3.8 Free-body diagrams of the six stiffeners in Figure 8.3.7.

from parts (a) through (f) of the figure, we obtain,

$$-lq^{(1)} - lq^{(2)} - lq^{(3)} + X_4 = F_1$$

$$lq^{(1)} + lq^{(2)} + lq^{(3)} + X_8 = lq^{(4)} + lq^{(5)} + lq^{(6)}$$

$$X_{12} = -lq^{(4)} - lq^{(5)} - lq^{(6)}$$

$$lq^{(1)} = -lq^{(4)}$$

$$-lq^{(1)} + lq^{(2)} = lq^{(4)} - lq^{(5)} + F_2$$

$$-lq^{(2)} + lq^{(3)} = lq^{(5)} - lq^{(6)}$$

Solving for the unknowns on the left, we get

$$q^{(1)} = -q^{(4)} \qquad [g1]$$

$$q^{(2)} = -q^{(5)} + \frac{F_2}{l} \qquad [h1]$$

$$q^{(3)} = -q^{(6)} + \frac{F_2}{l} \qquad [i1]$$

$$X_4 = -lq^{(4)} - lq^{(5)} - lq^{(6)} + F_1 + 2F_2 \qquad [j1]$$

$$X_8 = 2lq^{(4)} + 2lq^{(5)} + 2lq^{(6)} - 2F_2 \qquad [k1]$$

$$X_{12} = -lq^{(4)} - lq^{(5)} - lq^{(6)} \qquad [l1]$$

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.2

To obtain the virtual load counterparts of these true loads,

$$\delta q^{(1)} = -\delta q^{(4)} \qquad [g2]$$

$$\delta q^{(2)} = -\delta q^{(5)} \qquad [h2]$$

$$\delta q^{(3)} = -\delta q^{(6)} \qquad [i2]$$

$$\delta X_4 = -l\delta q^{(4)} - l\delta q^{(5)} - l\delta q^{(6)} \qquad [j2]$$

$$\delta X_8 = 2l\delta q^{(4)} + 2l\delta q^{(5)} + 2l\delta q^{(6)} \qquad [k2]$$

$$\delta X_{12} = -l\delta q^{(4)} - l\delta q^{(5)} - l\delta q^{(6)} \qquad [l2]$$

The complementary virtual work of a stiffener element is found in Equation 8.2.5,

$$\delta W_{\text{int}}^{*(e)} = \frac{L^{(e)}}{3A^{(e)}E^{(e)}}\left[\left(P_1^{(e)} + \frac{1}{2}P_2^{(e)}\right)\delta P_1^{(e)} + \left(P_2^{(e)} + \frac{1}{2}P_1^{(e)}\right)\delta P_2^{(e)}\right] \qquad [m]$$

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.2

for stiffener element 1,

$$P_2^{(1)} = F_1 + q^{(1)} l$$

according to Equation (g1), $q^{(1)} = -q^{(4)}$.

$$P_2^{(1)} = F_1 - q^{(4)} l$$

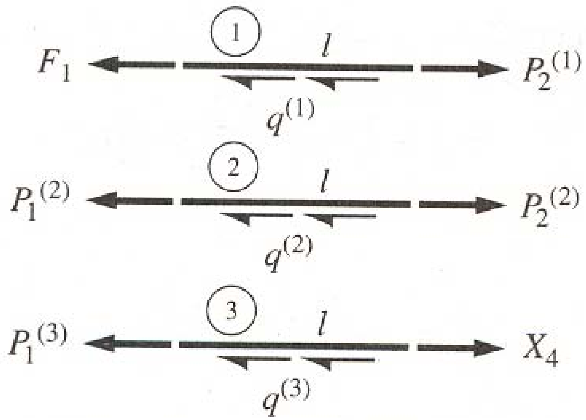


Figure 8.3.9 Free-body diagrams of the individual elements comprising the topmost stiffener.

The virtual end loads are then

$$\delta P_1^{(1)} = \delta F_1 = 0 \quad \text{and} \quad \delta P_2^{(1)} = \delta F_1 - \delta q^{(4)} l = -\delta q^{(4)} l$$

Thus, for element 1,

$$\delta W_{\text{int}}^{*(1)} = \frac{l}{3AE} \left\{ \left[F_1 + \tfrac{1}{2} \left(F_1 - q^{(4)} l \right) \right] (0) + \left(F_1 - q^{(4)} l + \tfrac{1}{2} F_1 \right) \left(-\delta q^{(4)} l \right) \right\} = \frac{l}{3AE} \left(q^{(4)} l^2 - \frac{3}{2} F_1 l \right)$$

Example 8.3.2

Moving on to stiffener element 2.

$$P_1^{(2)} = P_2^{(1)} = F_1 - q^{(4)} l$$

For equilibrium,

$$P_2^{(2)} = P_2^{(1)} + q^{(2)} l = \left(F_1 - q^{(4)} l\right) + q^{(2)} l$$

Substituting Equation h1,

$$P_2^{(2)} = -q^{(4)} l - q^{(5)} l + F_1 + F_2$$

From these relationshps,

$$\delta P_2^{(1)} = -\delta q^{(4)} l \qquad \text{and} \qquad \delta P_2^{(2)} = -\delta q^{(4)} l - \delta q^{(5)} l$$

From Equation m, we therefore obtain, :

$$\delta W_{\text{int}}^{*(2)} = \frac{l}{3AE} \left[\left(3q^{(4)} l^2 + \tfrac{3}{2} q^{(5)} l^2 - 3F_1 l - \tfrac{3}{2} F_2 l\right) \delta q^{(4)} + \left(\tfrac{3}{2} q^{(4)} l^2 + q^{(5)} l^2 - \tfrac{3}{2} F_1 l - F_2 l\right) \delta q^{(5)} \right]$$

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.2

For stiffener element 3,

$$P_1^{(3)} = P_2^{(2)} = -q^{(4)}l - q^{(5)}l + F_1 + F_2$$

so that $$\delta P_1^{(3)} = -\delta q^{(4)}l - \delta q^{(5)}l_2$$

At the other end, we see that $P_2^{(3)} = X_4$. X_4, and δX_4, in terms of the redundant and applied loads, were found in Equations j1 and j2. Therefore, the complementary virtual work of element 3, from Equation m, is

$$\delta W_{\text{int}}^{*(3)} = \frac{l}{3AE}\left[\left(3q^{(4)}l^2 + 3q^{(5)}l^2 + \tfrac{3}{2}q^{(6)}l^2 - 3F_1 l - \tfrac{9}{2}F_2 l\right)\delta q^{(4)} + \left(3q^{(4)}l^2 + 3q^{(5)}l^2 + \tfrac{3}{2}q^{(6)}l^2 - 3F_1 l - \tfrac{9}{2}F_2 l\right)\delta q^{(5)} + \left(\tfrac{3}{2}q^{(4)}l^2 + \tfrac{3}{2}q^{(5)}l^2 + q^{(6)}l^2 - \tfrac{3}{2}F_1 l - \tfrac{5}{2}F_2 l\right)\delta q^{(6)}\right]$$

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.2

The total complementary virtual work of all the stiffeners

$$\begin{aligned}\delta W^*_{\text{int, stiffeners}} &= \sum_{e=1}^{15} \delta W^{*(e)}_{\text{int}} \\ &= \frac{l^2}{6AE}[(92q^{(4)}l + 50q^{(5)}l + 18q^{(6)}l - 15F_1 - 31F_2)\delta q^{(4)} + \\ &+ (50q^{(4)}l + 56q^{(5)}l + 14q^{(6)}l - 15F_1 - 31F_2)\delta q^{(5)} \\ &+ .(18q^{(4)}l + 14q^{(5)}l + 16q^{(6)}l - 3F_1 - 15F_2)\delta q^{(6)}]\end{aligned} \qquad \text{[n]}$$

To calculate the complementary virtual work of the shear panels, Equation 8.2.6 applies.

$$\begin{aligned}\delta W^*_{\text{int, panels}} &= \frac{l^2}{Gt}\sum_{e=1}^{6} q^{(e)}\delta q^{(e)} \\ &= \frac{l^2}{Gt}\left[2q^{(4)}\delta q^{(4)} + \left(2q^{(5)} - \frac{F_2}{l}\right)\delta q^{(5)} + \left(2q^{(6)} - \frac{F_2}{l}\right)\delta q^{(6)}\right]\end{aligned} \qquad \text{[o]}$$

The total internal complementary virtual work for the structure is that of the stiffeners, Equation n, plus that of the panels, Equations o,

$$\begin{aligned}\delta W^*_{\text{int}} = &\frac{1}{6AEGt}\left[\left(12AEl^2 + 92Gl^3t\right)q^{(4)} + 50Gl^3tq^{(5)} + 18Gl^3tq^{(6)} - 15Gl^2tF_1 - 31Gl^2tF_2\right]\delta q^{(4)} \\ &+ \frac{1}{6AEGt}\left[50Gl^3tq^{(4)} + \left(12AEl^2 + 56Gl^3t\right)q^{(5)} + 14Gl^3tq^{(6)} - 9Gl^2tF_1 - \left(6AEl + 38Gl^2t\right)F_2\right]\delta q^{(5)} \\ &+ \frac{1}{6AEGt}\left[18Gl^3tq^{(4)} + 14Gl^3tq^{(5)} + \left(12AEl^2 + 16Gl^3t\right)q^{(6)} - 3Gl^2tF_1 - \left(6AEl + 15Gl^2t\right)F_2\right]\delta q^{(6)}\end{aligned}$$

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.2

Three equations for the true redundants,

$$\begin{aligned}
&\left(12AEl^2+92Gl^3t\right)q^{(4)} &&+50Gl^3tq^{(5)} &&+18Gl^3q^{(6)} = 15Gl^2tF_1+31Gl^2tF_2\\
&50Gl^3tq^{(4)} &&+\left(12AEl^2+56Gl^3t\right)q^{(5)} &&+14Gl^3tq^{(6)} = 9Gl^2tF_1+\left(6AEl+38Gl^2t\right)F_2\\
&18Gl^3q^{(4)} &&+14Gl^3tq^{(5)} &&+\left(12AEl^2+16Gl^3t\right)q^{(6)} = 3Gl^2tF_1+\left(6AEl+15Gl^2t\right)F_2
\end{aligned}$$

$$q^{(1)}=\frac{1}{\Delta}\left\{-\left[2160\left(\frac{AE}{Glt}\right)^2+6912\frac{AE}{Glt}+4644\right]\frac{F_1}{l}+\left[432\left(\frac{AE}{Glt}\right)^2+4392\frac{AE}{Glt}+3744\right]\frac{F_2}{l}\right\} \quad [p]$$

$$q^{(2)}=\frac{1}{\Delta}\left\{-\left[1296\left(\frac{AE}{Glt}\right)^2+2160\frac{AE}{Glt}+948\right]\frac{F_1}{l}+\left[864\left(\frac{AE}{Glt}\right)^3+11,376\left(\frac{AE}{Glt}\right)^2+21,312\frac{AE}{Glt}+10,640\right]\frac{F_2}{l}\right\} \quad [q]$$

$$q^{(3)}=\frac{1}{\Delta}\left\{\left[-432\left(\frac{AE}{Glt}\right)^2-576\frac{AE}{Glt}+156\right]\frac{F_1}{l}+\left[864\left(\frac{AE}{Glt}\right)^3+11,808\left(\frac{AE}{Glt}\right)^2+26,856\frac{AE}{Glt}+15,968\right]\frac{F_2}{l}\right\} \quad [r]$$

$$q^{(4)}=\frac{1}{\Delta}\left\{\left[2160\left(\frac{AE}{Glt}\right)^2+6912\frac{AE}{Glt}+4644\right]\frac{F_1}{l}-\left[432\left(\frac{AE}{Glt}\right)^2+4392\frac{AE}{Glt}+3744\right]\frac{F_2}{l}\right\} \quad [s]$$

$$q^{(5)}=\frac{1}{\Delta}\left\{\left[1296\left(\frac{AE}{Glt}\right)^2+2160\frac{AE}{Glt}+948\right]\frac{F_1}{l}+\left[864\left(\frac{AE}{Glt}\right)^3+12,240\left(\frac{AE}{Glt}\right)^2+32,688\frac{AE}{Glt}+20,816\right]\frac{F_2}{l}\right\} \quad [t]$$

$$q^{(6)}=\frac{1}{\Delta}\left\{\left[432\left(\frac{AE}{Glt}\right)^2+576\frac{AE}{Glt}-156\right]\frac{F_1}{l}+\left[864\left(\frac{AE}{Glt}\right)^3+11,808\left(\frac{AE}{Glt}\right)^2+27,144\frac{AE}{Glt}+15,488\right]\frac{F_2}{l}\right\} \quad [u]$$

where $\Delta = 1728\left(\frac{AE}{Glt}\right)^3+23,616\left(\frac{AE}{Glt}\right)^2+54,000\frac{AE}{Glt}+31,456$ [v]

8.3 Statically Indeterminate Stiffened Webs

Example 8.3.2

At one extreme, in which the panels are very rigid compared to the relatively flexible stiffeners, we have $AE/Glt \to 0.$ In this case Equations p through v imply in the limit that

$$q^{(1)} = -0.1476\frac{F_1}{l} + 0.1190\frac{F_2}{l}$$

$$q^{(2)} = -0.03014\frac{F_1}{l} + 0.3382\frac{F_2}{l}$$

$$q^{(3)} = 0.004959\frac{F_1}{l} + 0.5076\frac{F_2}{l}$$

$$q^{(4)} = 0.1476\frac{F_1}{l} - 0.1190\frac{F_2}{l}$$

$$q^{(5)} = 0.03014\frac{F_1}{l} + 0.6617\frac{F_2}{l}$$

$$q^{(6)} = -0.004959\frac{F_1}{l} + 0.4924\frac{F_2}{l}$$

At the other extreme, in which the stiffeners are much more rigid than the panels, the shear flows are

$$q^{(1)} = q^{(4)} = 0$$

$$q^{(2)} = q^{(3)} = q^{(5)} = q^{(6)} = \frac{1}{2}\frac{F_2}{l}$$

Example 8.3.2

For a more typical situation, set $A = 0.5$ in.^{2}, $t = 0.05$ in., $E = 10^7$ lb/in.^{2} and $G = 0.4E$. If $l = 15$ in.

then $AE/Glt = 1.666$, so that

$$q^{(1)} = -0.1136\frac{F_1}{l} + 0.06287\frac{F_2}{l}$$

$$q^{(2)} = -0.04177\frac{F_1}{l} + 0.4192\frac{F_2}{l}$$

$$q^{(3)} = 0.01027\frac{F_1}{l} + 0.5000\frac{F_2}{l}$$

$$q^{(4)} = 0.1136\frac{F_1}{l} - 0.06287\frac{F_2}{l}$$

$$q^{(5)} = 0.04177\frac{F_1}{l} + 0.5808\frac{F_2}{l}$$

$$q^{(6)} = -0.01027\frac{F_1}{l} + 0.5000\frac{F_2}{l}$$

8.4 Thin-Walled Beams

In three-dimensional idealized beams of arbitrary cross section and lateral dimensions, the internal virtual work is divided between bending and shear, as follows:

$$\delta W^*_{\text{int}} = \delta W^*_{\text{int, bending}} + \delta W^*_{\text{int, shear}} \qquad [8.4.1]$$

where

$$\delta W^*_{\text{int, bending}} = \iiint_V \delta\sigma_x \varepsilon_x dV = \iiint_V \frac{\sigma_x \delta\sigma_x}{E} dAdx \qquad [8.4.2]$$

and

$$\delta W^*_{\text{int, shear}} = \iiint_V \delta\tau_{xs} \gamma_{xs} dV = \iiint_V \frac{\tau_{xs}\delta\tau_{xs}}{G} dAdx \qquad [8.4.3]$$

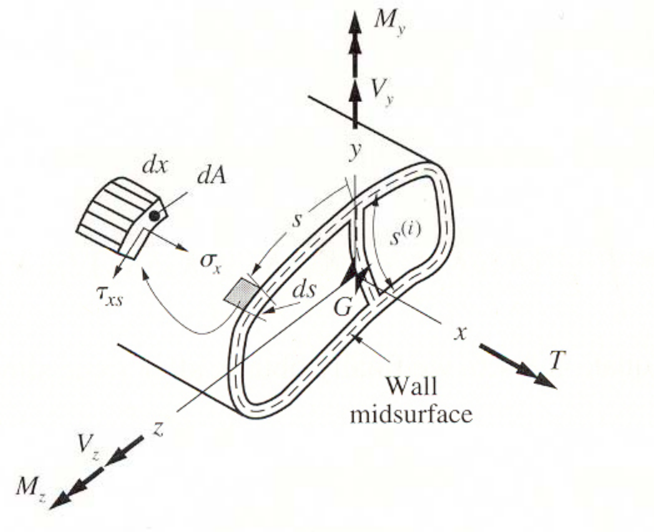


Figure 8.4.1 Section of a thin-walled box beam.

Where τ_{xs} is the shear stress, directed along the tangent to the middle surface of the wall

Since $\tau_{xs} = q/t$,

$$\delta W^*_{\text{int, shear}} = \iiint_V \frac{q\delta q}{Gt^2} dAdx = \int_0^L [\sum_{i=1}^{\text{No. walls}} \int_0^{s^{(i)}} \frac{q^{(i)}\delta q^{(i)}}{G^{(i)}t^{(i)^2}} (t^{(i)}ds)]dx = L \sum_{i=1}^{\text{No. walls}} \int_0^{s^{(i)}} \frac{q^{(i)}\delta q^{(i)}}{G^{(i)}t^{(i)}} ds$$

8.4 Thin-Walled Beams

If the thickness and shear modulus of each wall are over the length of the wall, then

$$\delta W^*_{\text{int, shear}} = L \sum_{i=1}^{\text{No. walls}} \frac{1}{G^{(i)} t^{(i)}} \int_0^{s^{(i)}} q^{(i)} \delta q^{(i)} ds \qquad [8.4.4]$$

according to Equation 4.6.6,

$$\sigma_x = K_y y + K_z z$$

where y and z are the coordinates of the point in the cross section relative to the centroid,

$$K_y = -\frac{M_z I_y + M_y I_{yz}}{I_y I_z - I_{yz}^2} \qquad K_z = \frac{M_y I_z + M_z I_{yz}}{I_y I_z - I_{yz}^2} \qquad [8.4.5]$$

Thus,

$$\delta W^*_{\text{int, bending}} = \iiint_V \frac{1}{E} \left(K_y y + K_z z\right) \left(\delta K_y y + \delta K_z z\right) dV$$

8.4 Thin-Walled Beams

Expanding the integrand,

$$\delta W^{*}_{\text{int, bending}} = \int_0^L \left\{ \iint_A \frac{1}{E} \left[K_y \delta K_y y^2 + \left(K_y \delta K_z + K_z \delta K_y \right) yz + K_z \delta K_z z^2 \right] dA \right\} dx$$

$$= \int_0^L \frac{1}{E} \left[K_y \delta K_y \left(\iint_A y^2 dA \right) + \left(K_y \delta K_z + K_z \delta K_y \right) \left(\iint_A yz dA \right) + K_z \delta K_z \left(\iint_A z^2 dA \right) \right] dx$$

The three integrals are the area moments and the product of inertia. Therefore,

$$\delta W^{*}_{\text{int, bending}} = \int_0^L \frac{1}{E} \left[K_y \delta K_y I_z + \left(K_y \delta K_z + K_z \delta K_y \right) I_{yz} + K_z \delta K_z I_y \right] dx \qquad \textbf{[8.4.6]}$$

Substituting Equation 8.4.5 into Equation 8.4.6,

$$\delta W^{*}_{\text{int, bending}} = \int_0^L \frac{1}{E \left(I_y I_z - {I_{yz}}^2 \right)} \left\{ \left[M_y I_z + M_z I_{yz} \right] \delta M_y + \left[M_z I_y + M_y I_{yz} \right] \delta M_z \right\} dx \qquad \textbf{[8.4.7]}$$

8.4 Thin-Walled Beams

Example 8.4.1

find the displacement v_P in the direction of the 1000 lb load for the box beam in figure 8.4.2. The elastic moduli are E = 10 * 10^6 psi and G = 4 * 10^6 psi.

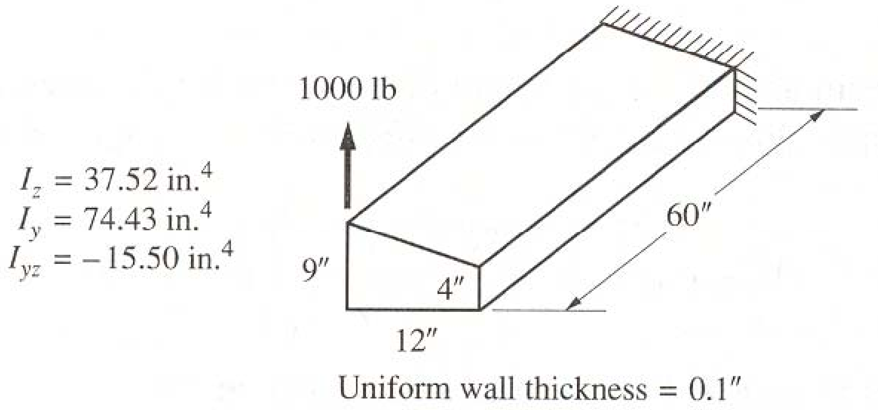


Figure 8.4.2 Cantilever box beam. The section properties were calculated in Example 4.7.3.

8.4 Thin-Walled Beams

Example 8.4.1

Using Equation 8.4.7 to obtain the internal complementary virtual work due to bending in $M_y = 0$ and $\delta M_y = 0$ for this problem,

$$\delta W^*_{\text{int, bending}} = \int_0^L \frac{I_y}{E\left(I_y I_z - I_{yz}{}^2\right)} M_z \delta M_z dx$$

Substituting the material and section properties and the true and virtual bending moments $M_z = 1000x$, $\delta M_z = \delta P x$

$$\delta M_z = 9.720 \frac{L^3}{E} \delta P = 0.210 \delta P$$

Since the four walls of the cross section have a common thickness and shear modulus,

$$\delta W^*_{\text{int, shear}} = \frac{L}{Gt}\left(\int_0^4 q^{(1)} \delta q^{(1)} ds + \int_0^{13} q^{(2)} \delta q^{(2)} ds + \int_0^9 q^{(3)} \delta q^{(3)} ds + \int_0^{12} q^{(4)} \delta q^{(4)} ds \right)$$

8.4 Thin-Walled Beams

Example 8.4.1

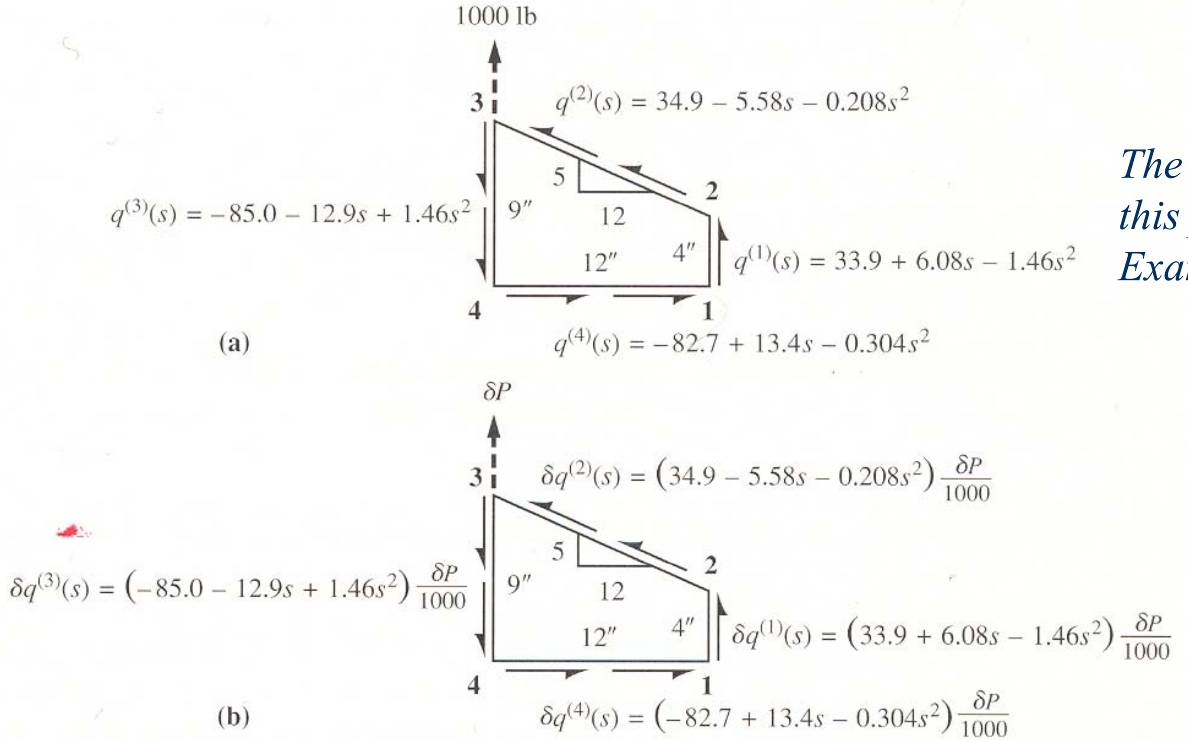


Figure 8.4.3 Wall shear flows accompanying (a) the actual load, and (b) the virtual load.

The variable shear flows for this problem were calculated in Example 4.7.3

Substituting the true and virtual shear flows shown in Figure 8.4.3

$$\delta W^*_{\text{int, shear}} = 135\frac{L}{Gt}\delta P = 0.0203\delta P$$

8.4 Thin-Walled Beams

Example 8.4.1

The total internal complementary virtual work is therefore

$$\delta W^*_{\text{int}} = \delta W^*_{\text{int, bending}} + \delta W^*_{\text{int, shear}} = 0.230\delta P$$

Setting this equal to the external complementary virtual work, we get $v_P \delta P = 0.230\delta P$, or

$$v_P = 0.230 \text{ in.}$$

Observe that bending accounts for 91 percent of the displacement.

Example 8.4.2

Calculate the angle of twist per unit length for the beam in Figure 8.4.2.

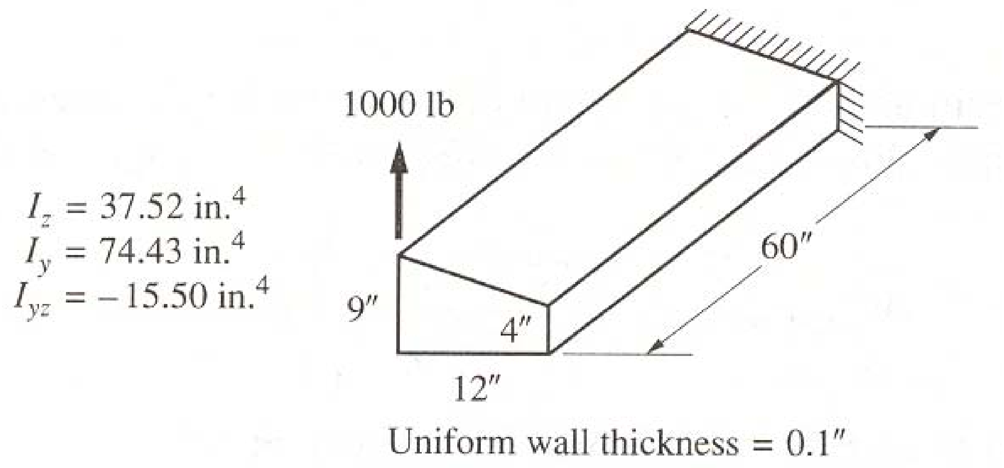


Figure 8.4.2 Cantilever box beam. The section properties were calculated in Example 4.7.3.

Example 8.4.2

Applying a virtual torque δT to the section produces the uniform shear flow

$$\delta q = \frac{\delta T}{2A} = \frac{\delta T}{2 \times \frac{1}{2}(9+4)\cdot 12} = \frac{\delta T}{156}$$

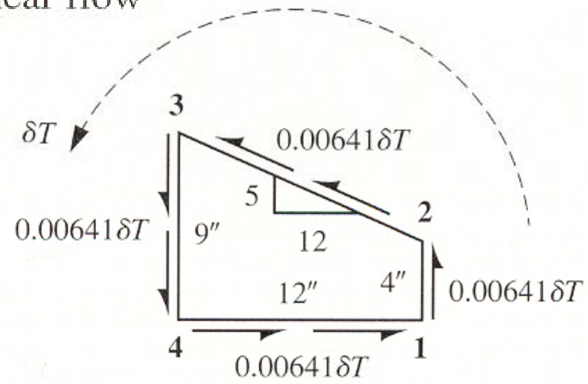


Figure 8.4.4 Constant shear flow due to a virtual torque δT.

Since the virtual bending moments are zero,

$$\delta W^*_{\text{int}} = \delta W^*_{\text{int, shear}} = \frac{L}{Gt}\left(\int_0^4 q^{(1)}ds + \int_0^{13} q^{(2)}ds + \int_0^{9} q^{(3)}ds + \int_0^{12} q^{(4)}ds\right) \times \frac{\delta T}{156}$$

Substituting the shear flows from Figure 8.4.3a and integrating yields,

$$\delta W^*_{int} = \frac{L}{Gt}(-7.39)\delta T = -(18.5\times 10^{-6})L\delta T$$

Since $\delta W^*_{\text{ext}} = \theta_x \delta T$, $\quad \dfrac{\theta_x}{L} = -18.5 \times 10^{-6}$ rad/in. $= -0.00106$ degree/in.

The negative sign means that the angle of twist due to the actual load is clockwise, in the direction opposite to the virtual torque in Figure 8.4.4.

8.5 Deflections in Idealized Beams

The complementary internal virtual work formula for idealized beams built up of stringers and shear panels combines Equation 8.4.7 for the stringers with the expressions obtained in section 8.2 for shear panels as follows:

$$\delta W^*_{\text{int}} = \int_0^L \frac{1}{E(I_y I_z - I_{yz}^2)} \{[M_y I_z + M_z I_{yz}]\delta M_y + [M_z I_y + M_y I_{yz}]\delta M_z\} dx + \sum_{i=1}^{\text{No. panels}} \delta W^{*(i)}_{\text{int}} \qquad \text{[8.5.1]}$$

For each panel we substitute the appropriate virtual work expression, depending on whether it is a rectangle(Equation 8.2.6), parallelogram (Equation 8.2.8), or trapezoid (Equation 8.2.13).

$$\delta W^*_{\text{int}} = \frac{A}{Gt} q \delta q \qquad [8.2.6]$$

$$\delta W^*_{\text{int}} = \left[1 + \frac{2\cot^2\theta}{1+\nu}\right] \frac{A}{Gt} q \delta q \qquad [8.2.8]$$

$$\delta W^*_{\text{int}} = \left[1 + \frac{2}{3(1+\nu)} \left(\cot^2\gamma - \cot\alpha \ \cot\gamma + \cot^2\alpha\right)\right] \frac{A}{Gt} \bar{q} \delta \bar{q} \qquad [8.2.13]$$

8.5 Deflections in Idealized Beams

Example 8.5.1

calculate the horizontal (z) displacement of the centroid at the free end of the idealized, single-cell box beam depicted in Figure 8.5.1, given that a vertical shear of 1000 lb acts through the centroid. The location of the centroid and the values of the centroidal moments of inertia are shown.

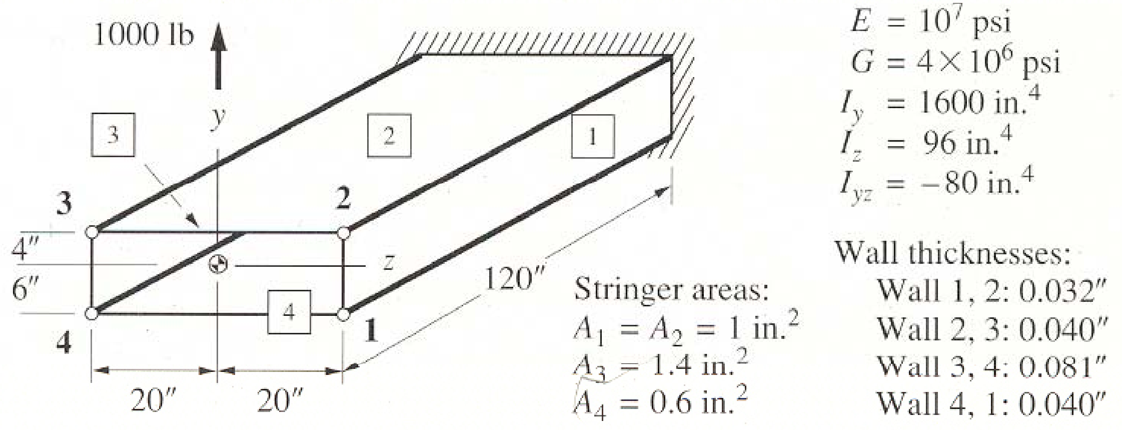


Figure 8.5.1 Single-cell box beam subjected to a shear load. Material and geometric properties are given.

8.5 Deflections in Idealized Beams

Example 8.5.1

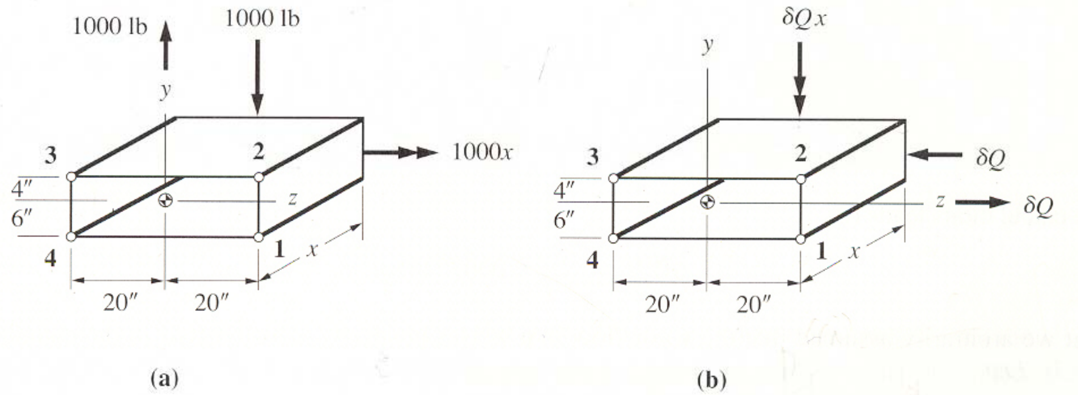


Figure 8.5.2 (a) Actual load and the internal reactions at station x. (b) Virtual load and the internal reactions at station x.

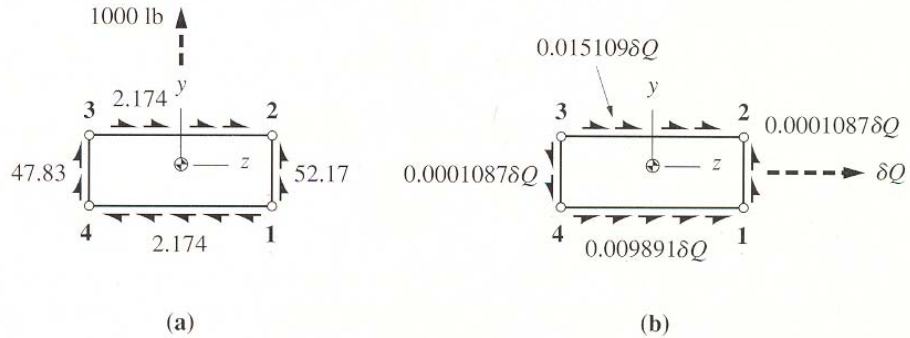


Figure 8.5.3 (a) Shear flows (lb/in.) accompanying the true 10,000 lb load. (b) Virtual shear flows due to the virtual load.

8.5 Deflections in Idealized Beams

Example 8.5.1

the internal complementary virtual work of the stringers,

$$\delta W^*_{\text{int, stringers}} = \int_0^{120} \frac{1}{10^7\left[1600 \cdot 96 - (-80)^2\right]}\left\{\left[0 \cdot 96 + (1000x)(-80)\right](-\delta Q\, x) + \left[1000x \cdot 1600 + 0 \cdot (-80)\right](0)\right\} dx$$

$$= 54.35 \times 10^{-9}\delta Q \int_0^{120} x^2 dx \qquad = 0.03130\delta Q$$

For the shear panels,

$$\delta W^*_{\text{int, panels}} = \sum_{\text{panels}} \frac{A}{Gt} q\delta q$$

$$= \frac{120 \cdot 10}{4(10^6)(0.032)}(52.17)(0.0001087\delta Q) + \frac{120 \cdot 40}{4(10^6)(0.04)}(-2.174)(-0.015109\delta Q)$$

$$+ \frac{120 \cdot 10}{4(10^6)(0.081)}(-47.83)(0.0001087\delta Q) + \frac{120 \cdot 40}{4(10^6)(0.04)}(-2.174)(0.009891\delta Q)$$

$$= 0.0003743\delta Q$$

The total complementary virtual work for the beam is

$$\delta W^*_{\text{int}} = 0.03130\delta Q + 0.0003743\delta Q = 0.03168\delta Q$$

The complementary external virtual work is

$$\delta W^*_{\text{ext}} = w_G \times \delta Q$$

(Where w_G is the displacement of the centroid in the direction of the virtual load δQ)

$$w_G = 0.03168 \text{ in.}$$

8.5 Deflections in Idealized Beams

The angle of twist of a box beam with constant cross section

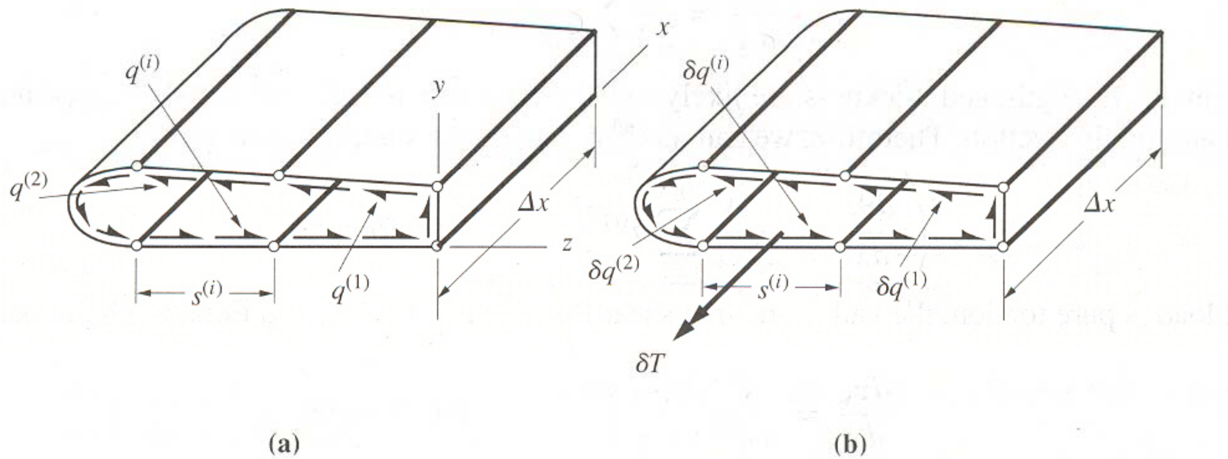


Figure 8.5.4 (a) Shear flows due to an arbitrary load on the box beam. (b) Virtual shear flows due to a virtual pure torque δT.

The virtual load is pure torsion. This means that the virtual bending moments are zero, so the stringers are not involved in the virtual work calculations.

For such panels, the internal complementary virtual work for panel i is

$$\delta W_{\text{int}}^{*(i)} = \frac{A^{(i)}}{G^{(i)} t^{(i)}} q^{(i)} \delta q^{(i)} = \frac{s^{(i)} \Delta x}{G^{(i)} t^{(i)}} q^{(i)} \delta q^{(i)}$$

8.5 Deflections in Idealized Beams

Since the virtual load is pure torsion, the virtual shear flow is the same in every web and is

$$\delta q^{(1)} = \delta q^{(2)} = \ldots = \delta q^{(i)} = \ldots = \frac{\delta T}{2A}$$

For each panel,

$$\delta W^{*(i)}_{\text{int}} = \frac{\delta T \, \Delta x}{2A} \frac{s^{(i)}}{G^{(i)} t^{(i)}} q^{(i)}$$

The external complementary virtual work is $\Delta\theta \, \delta T$, where $\Delta\theta$ is the true rotation of the cross section at $x + \Delta x$ relative to that at x.

Setting the external complementary virtual work equal to the total internal complementary virtual work of all n panels comparing the cross section yields,

$$\Delta\theta_x \, \delta T = \frac{\delta T \, \Delta x}{2A} \sum_{i=1}^{n} \frac{s^{(i)}}{G^{(i)} t^{(i)}} q^{(i)}$$

or

$$\frac{\Delta\theta_x}{\Delta x} = \frac{1}{2A} \sum_{i=1}^{n} \frac{s^{(i)}}{G^{(i)} t^{(i)}} q^{(i)}$$

8.5 Deflections in Idealized Beams

In the limit as Δx approaches zero,

$$\frac{d\theta_x}{dx} = \frac{1}{2A}\sum_{i=1}^{n}\frac{s^{(i)}}{G^{(i)}t^{(i)}}q^{(i)} \qquad \textbf{[8.5.2]}$$

$$\frac{d\theta_x}{dx} = \frac{1}{2AG}\sum_{i=1}^{n}q^{(i)}\frac{s^{(i)}}{t^{(i)}} \qquad \text{uniform G} \qquad \textbf{[8.5.3]}$$

$$\frac{d\theta_x}{dx} = \frac{T}{4A^2G}\sum_{i=1}^{n}\frac{s^{(i)}}{t^{(i)}} \qquad \text{pure torsion} \qquad \textbf{[8.5.4]}$$

According to Equation 4.4.14, the torsion constant J is given by $J = T/G\phi$, where $\phi = d\theta_x/dx$. For an idealized box beam, Equation 8.5.4 therefore yields

$$J = \frac{4A^2}{\sum_{i=1}^{n}\frac{s^{(i)}}{t^{(i)}}} \qquad [8.5.5]$$

Remember that Equation 8.5.2 through 8.5.5 are valid *only for beams of constant cross section.*

8.5 Deflections in Idealized Beams

Example 8.5.2

Calculate the angle of twist of the free end of the cantilevered idealized box beam in Figure 8.5.5. The location of the centroid, as well as the values of the centroidal moments of inertia, are shown in the figure.

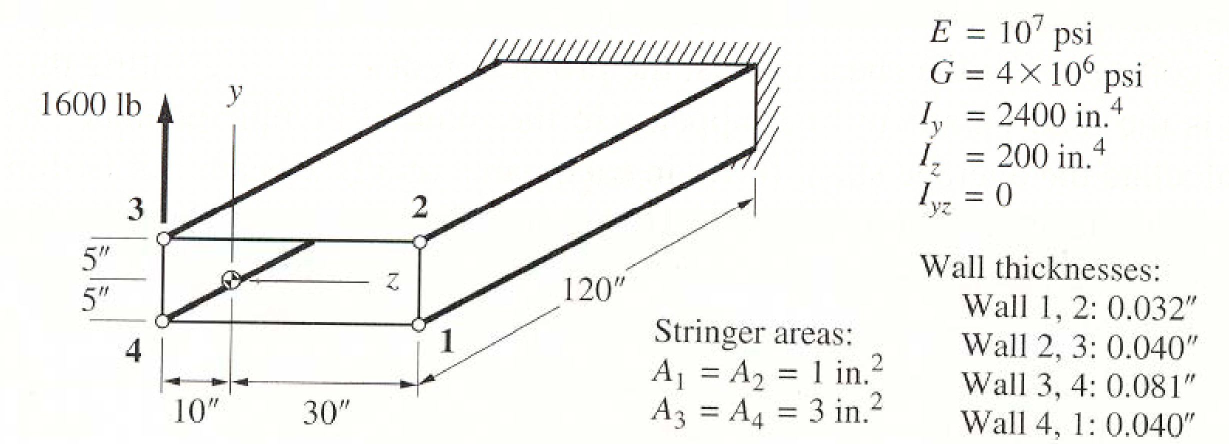


Figure 8.5.5 Single-cell box beam subjected to a shear load. The material and geometric properties are shown.

8.5 Deflections in Idealized Beams

Example 8.5.2

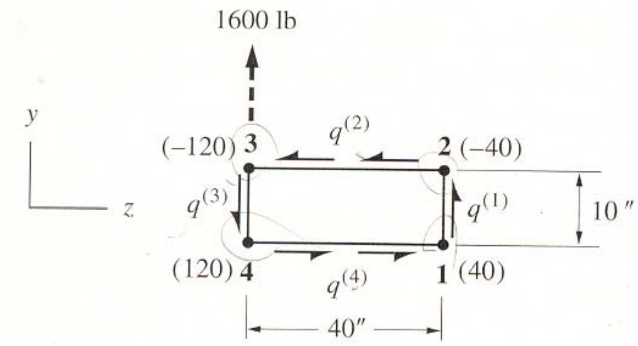


Figure 8.5.6 Flange load gradients (lb/in.) and the assumed shear flow directions.

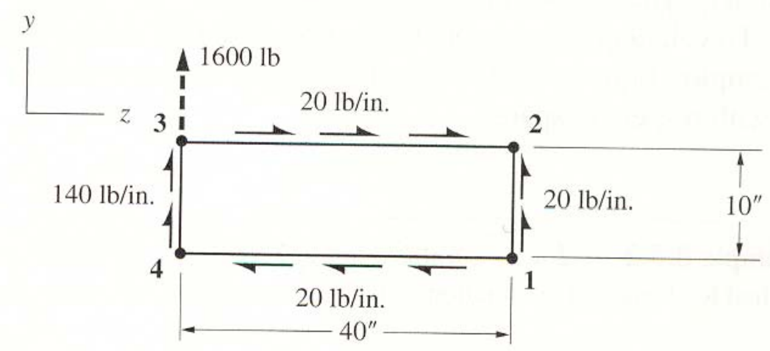


Figure 8.5.7 Computed shear flows in the box beam.

From Equation 4.8.2,

$$\begin{aligned} q^{(1)} &= q^{(4)} + 40 \\ q^{(2)} &= q^{(1)} - 40 = \left(q^{(4)} + 40\right) - 40 = q^{(4)} \\ q^{(3)} &= q^{(2)} - 120 = q^{(4)} - 120 \end{aligned} \qquad [a]$$

Moment equivalence about flange 4 requires that

$$\left(10q^{(1)}\right) \times 40 + \left(40q^{(2)}\right) \times 10 = 0$$

using Equation a,

$$400\left[\left(q^{(4)} + 40\right) + q^{(4)}\right] = 0$$

8.5 Deflections in Idealized Beams

Example 8.5.2

Solving this for $q^{(4)}$ and substituting the result into Equation a yields the shear flows shown in Figure 8.5.7

Using Equation 8.5.3,

$$\frac{d\theta_x}{dx} = \frac{1}{2(40 \times 10)(4 \times 10^6)}\left[(20)\frac{10}{0.032} + (-20)\frac{40}{0.04} + (-140)\frac{10}{0.081} + (-20)\frac{40}{0.04}\right]$$

$$= -15.9 \times 10^{-6} \text{ radians/in.}$$

Therefore,

$$\theta_x = \theta_x)_{x=0} - 15.9 \times 10^{-6} x$$

Since $\theta_x = 0$ at $x = 120$ in.,

$$0 = \theta_x)_{x=0} - 15.9 \times 10^{-6}(120)$$

so that

$$\theta_x)_{x=0} = 0.00191 \text{ rad} = 0.109^\circ$$

The positive sign means that the rotation of the section is clockwise.

8.5 Deflections in Idealized Beams

Example 8.5.3

for the idealized tapered box beam pictured in Figure 8.5.8, calculate: (a) the deflection in the direction of the applied load, and (b) the rotation of the free end. The figure shows the geometric and material property data for the structure.

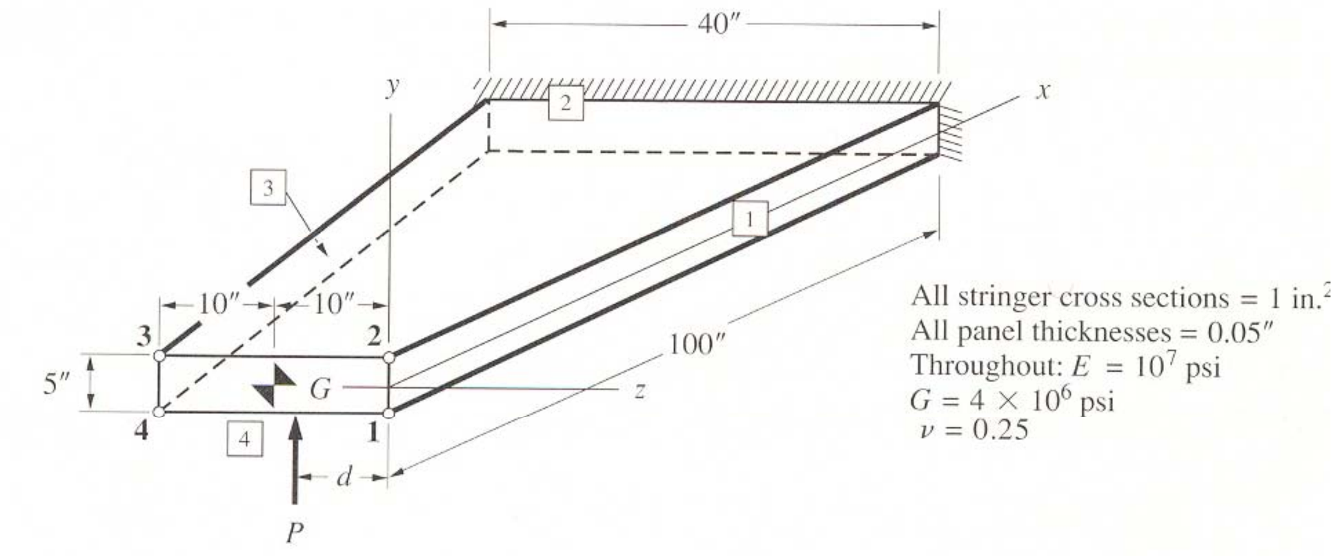


Figure 8.5.8 Cantilevered, tapered, idealized box beam with a point load at the free end.

8.5 Deflections in Idealized Beams

Example 8.5.3

(a)

First, we note that the centroidal area moments of inertia are

$$I_{yz} = 0 \text{ and } I_z = 4 \times \left[1 \times (5/2)^2\right] = 25 \text{ in.}^4$$

$$M_y = 0$$

$$M_z = Px$$

the virtual bending moment is $\delta M_z = \delta P\, x$

the internal complementary virtual work for the stringers is

$$\delta W^*_{\text{int,stringers}} = \int_0^{100} \frac{1}{EI_z} M_z \delta M_z dx = \int_0^{100} \frac{1}{10^7 \times 25} (Px)(\delta P x)\, dx = 0.001333 P \delta P$$

we first establish spanwise equilibrium of each of the stringers by means of Equation 4.9.9,

$$\sum_{\text{adjoining webs}} \bar{q}_{\text{out}} = \overline{\frac{dP_x^{(i)}}{dx}}$$

8.5 Deflections in Idealized Beams

Example 8.5.3

The average flange load gradient $\overline{dP_x^{(i)}/dx}$. is found by computing the flange load at each end of the beam and using Equation 4.9.8,

$$\overline{\frac{dP_x^{(i)}}{dx}} = \frac{P_x^{(i)}(L) - P_x^{(i)}(0)}{L} \qquad \text{[e]}$$

The flange loads in this expression are obtained from Equation 4.8.1,

$$P_x^{(i)} = -\frac{P x_i y_i}{25} A_i$$

All of the flange loads are zero at the free end of the beam ($x = 0$).

at $x = 100$ in., $P_x^{(1)}(100) = 10P \quad P_x^{(2)}(100) = -10P \quad P_x^{(3)}(100) = -10P \quad P_x^{(4)}(100) = 10P$

Substituting these into Equation e,

$$\overline{\frac{dP_x^{(1)}}{dx}} = \frac{P}{10} \qquad \overline{\frac{dP_x^{(2)}}{dx}} = -\frac{P}{10} \qquad \overline{\frac{dP_x^{(3)}}{dx}} = \frac{P}{10} \qquad \overline{\frac{dP_x^{(4)}}{dx}} = -\frac{P}{10}$$

we can obtain the average shear flows

$$\bar{q}^{(2)} = \bar{q}^{(1)} - \frac{P}{10} \qquad \bar{q}^{(3)} = \bar{q}^{(1)} - \frac{P}{5} \qquad \bar{q}^{(4)} = \bar{q}^{(1)} - \frac{P}{10} \qquad \text{[i]}$$

8.5 Deflections in Idealized Beams

Example 8.5.3

Setting the moments of the shear flows about flange 1 equal to the moment of the load P,

$$20q^{(2)}(0) \times 5 + 5q^{(3)}(0) \times 20 = -P \times d \qquad \text{[j]}$$

In the top and bottom panels.

$$q(x) = \bar{q}\frac{h(0)h(100)}{h(x)^2}$$

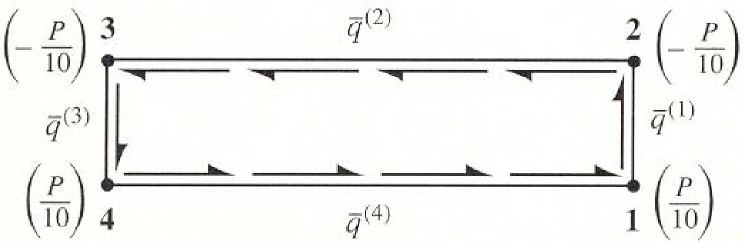


Figure 8.5.9 A generic cross section of the tapered beam, showing the average flange load gradients (in parentheses) and the assumed directions of the average shear flows.

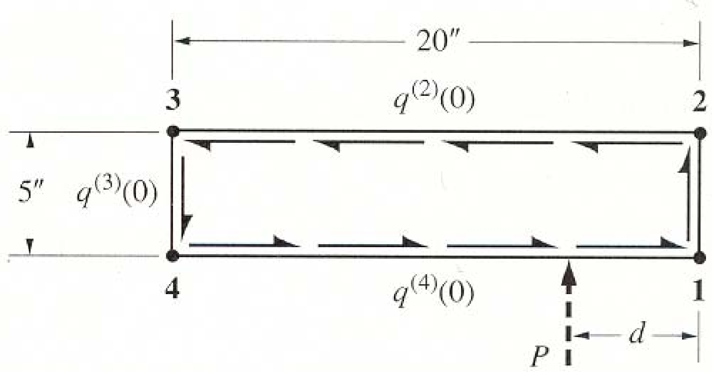


Figure 8.5.10 Shear flows at the free end of the tapered beam.

8.5 Deflections in Idealized Beams

Example 8.5.3

where the variable width of these panels is given by

$$h(x) = 20(1 + 0.01x)$$

Thus, $q^{(2)}(0) = 2\bar{q}^{(2)}$. Equation j therefore becomes

$$200\bar{q}^{(2)} + 100\bar{q}^{(3)} = -P \times d$$

Substituting for $\bar{q}^{(2)}$ and $\bar{q}^{(3)}$ in terms of $\bar{q}^{(1)}$, using Equation i, yields

$$200\left(\bar{q}^{(1)} - \frac{P}{10}\right) + 100\left(\bar{q}^{(1)} - \frac{P}{5}\right) = -P \times d$$

$$\bar{q}^{(1)} = \frac{(40-d)P}{300} \qquad \bar{q}^{(2)} = \frac{(10-d)P}{300} \qquad \bar{q}^{(3)} = -\frac{(20+d)P}{300} \qquad \bar{q}^{(4)} = \frac{(10-d)P}{300} \qquad \text{[k]}$$

we use Equation k to find the virtual average shear flows:

$$\delta\bar{q}^{(1)} = \frac{(40-d)\delta P}{300} \qquad \delta\bar{q}^{(2)} = \frac{(10-d)\delta P}{300} \qquad \delta\bar{q}^{(3)} = -\frac{(20+d)\delta P}{300} \qquad \delta\bar{q}^{(4)} = \frac{(10-d)\delta P}{300}$$

8.5 Deflections in Idealized Beams

Example 8.5.3

Equation 8.2.6 for the rectangles

$$\delta W^*_{\text{int}} = \frac{A}{Gt} q \delta q$$

Using Equation 8.2.13 for the trapezoids,

$$\delta W^*_{\text{int}} = \left[1 + \frac{2}{3\,(1+\nu)}\left(\cot^2\gamma \text{ or } \cot\alpha \ \cot\gamma + \cot^2\alpha\right)\right] \frac{A}{Gt} \bar{q} \delta \bar{q}$$

$$\delta W^*_{\text{int,panels}} = \overbrace{\frac{5 \cdot 100}{4(10^6) \cdot 0.05}\left[\frac{P\,(40-d)}{300}\right]\left[\frac{\delta P\,(40-d)}{300}\right]}^{\text{panel 1}} + \overbrace{\frac{5 \cdot \sqrt{100^2 + 20^2}}{4(10^6) \cdot 0.05}\left[-\frac{P\,(20+d)}{300}\right]\left[-\frac{\delta P\,(20+d)}{300}\right]}^{\text{panel 3}}$$

$$+ \overbrace{2 \times \left[1 + \frac{2}{3(1+0.25)} \cot^2 78.69^\circ\right] \frac{\frac{1}{2}\,(20+40) \cdot 100}{4(10^6) \cdot 0.05}\left[\frac{P\,(10-d)}{300}\right]\left[\frac{\delta P\,(10-d)}{300}\right]}^{\text{panels 2 and 4}}$$

or $$\delta W^*_{\text{int,panels}} = (3.966 \times 10^{-7} d^2 - 7.898 \times 10^{-6} d + 8.982 \times 10^{-5})\, P \delta P$$

8.5 Deflections in Idealized Beams

Example 8.5.3

The total internal complementary virtual work, $\delta W^*_{\text{int,stringers}} + \delta W^*_{\text{int,panels}}$,

$$\delta W^*_{\text{int}} = (3.966 \times 10^{-7} d^2 - 7.898 \times 10^{-6} d + 142.3 \times 10^{-5}) P \delta P$$

The external complementary virtual work is $v\delta P$

the location of the applied load,

$$v = (3.966 \times 10^{-7} d^2 - 7.898 \times 10^{-6} d + 142.3 \times 10^{-5}) P$$

For example, if we select $d = 5$ in. for the load application point, the vertical displacement of that point is

$$v = 0.001393 P \text{ in.} \qquad \text{(where } P \text{ is measured in pounds)}$$

(b)

Just as for Equation j, we sum moments about flange 1:

$$\delta q^{(2)}(0) \times 20 \times 5 + \delta q^{(3)}(0) \times 5 \times 20 = \delta T$$

$$\boxed{\begin{aligned} \delta q^{(3)}(0) &= \delta \bar{q}_T; \\ \delta q^{(2)}(0) &= 2\delta \bar{q}_T \end{aligned}}$$

$$2\delta \bar{q}_T \times 20 \times 5 + \delta \bar{q}_T \times 5 \times 20 = \delta T$$

8.5 Deflections in Idealized Beams

Example 8.5.3

so that

$$\delta \bar{q}_T = \frac{\delta T}{300}$$

Since there are no virtual bending moments associated with the applied virtual torque δT. That is, $\delta W^*_{\text{int}} = \delta W^*_{\text{int,panels}}$,

so that

$$\delta W^*_{\text{int}} = \overbrace{\frac{5 \cdot 100}{4(10^6) \cdot 0.05} \left[\frac{P\,(40-d)}{300}\right]\left[\frac{\delta T}{300}\right]}^{\text{panel 1}} + \overbrace{\frac{5 \cdot \sqrt{100^2+20^2}}{4(10^6) \cdot 0.05}\left[-\frac{P\,(20+d)}{300}\right]\left[\frac{\delta T}{300}\right]}^{\text{panel 3}}$$

$$+ \overbrace{2 \times \left[1 + \frac{2}{3(1+0.25)} \cot^2 78.69^\circ\right] \frac{\frac{1}{2}\,(20+40) \cdot 100}{4(10^6) \cdot 0.05}\left[\frac{P\,(10-d)}{300}\right]\left[\frac{\delta T}{300}\right]}^{\text{panels 2 and 4}}$$

Upon simplification, this becomes

$$\delta W^*_{\text{int}} = (3.949 \times 10^{-6} – 3.966 \times 10^{-7} d)\, P\, \delta T$$

$$\therefore \quad \theta = (3.949 \times 10^{-6} – 3.966 \times 10^{-7} d)\, P \quad \text{radians} \quad (\text{where } P \text{ is measured in pounds})$$

As expected, the twist angle depends on the location of the applied load P. That angle is zero if the load passes through the point $d = 9.96$ inches.

8.6 Shear Center of Closed Sections

The shear center of constant-cross-section beam is the point through which the shear load borne by a section must pass if the accompanying stresses in the cross section are those dictated by beam theory. For thin-walled sections, this means Equation 4.7.3 alone governs the shear stress distribution.

By definition, there are no torsional shear stresses if the shear load acts through the shear center.

8.6 Shear Center of Closed Sections

Example 8.6.1

Find the shear center of the section shown in Figure 8.6.1

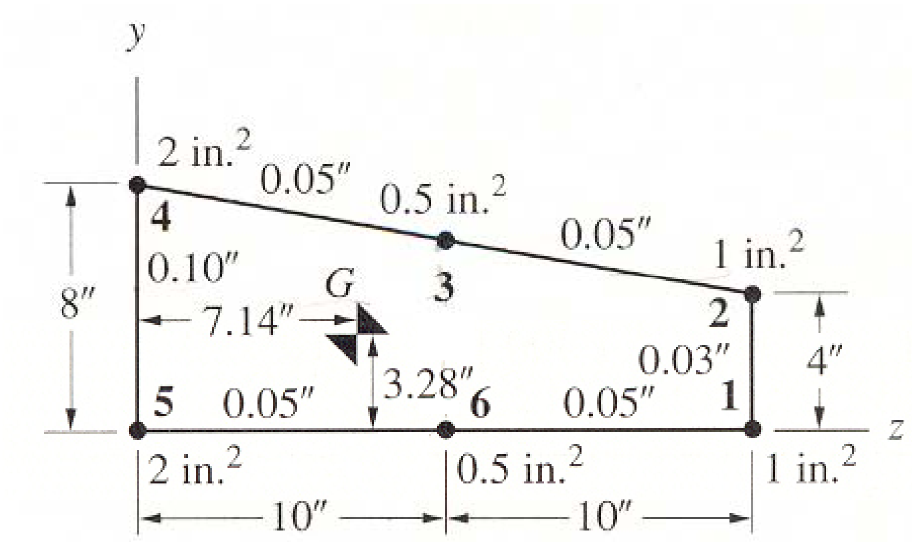


Figure 8.6.1 Idealized beam section.

8.6 Shear Center of Closed Sections

Example 8.6.1

First we calculate the location of the centroid G.

$$y_G = \frac{(8 \times 2) + (6 \times 0.5) + (4 \times 1)}{2 \times (2 + 0.5 + 1)} = 3.286 \text{ in.}$$

$$z_G = \frac{2 \times (10 \times 0.5) + 2 \times (20 \times 1)}{2 \times (2 + 0.5 + 1)} = 7.143 \text{ in.}$$

The centroidal area moments of inertia are

$$I_{Gy} = 2 \times \left[2 \times (7.143)^2\right] + 2 \times \left[0.5 \times (10 - 7.143)^2\right] + 2 \times \left[1 \times (20 - 7.143)^2\right] = 542.9 \text{ in.}^4$$

$$I_{Gz} = 2 \times \left[(8 - 3.286)^2 + (3.286)^2\right] + 0.5 \times \left[(6 - 3.286)^2 + (3.286)^2\right] + 1 \times \left[(4 - 3.286)^2 + (3.286)^2\right] = 86.43 \text{ in.}^4$$

$$I_{Gyz} = 2 \times [(8 - 3.286)(-7.143) + (-3.286)(-7.143)] + 0.5 \times [(6 - 3.286)(10 - 7.143) + (-3.286)(10 - 7.143)] + 1 \times [(4 - 3.286)(20 - 7.143) + (-3.286)(20 - 7.143)] = -54.29 \text{ in.}^4$$

From Equation 4.8.2, the flange load gradient at the ith flange is,

$$P_x'^{(i)} = 2.274 \times 10^{-5} \left\{ V_y [542.9 (y_i - y_G) + 54.29 (z_i - z_G)] + V_z [54.29 (y_i - y_G) + 86.43 (z_i - z_G)] \right\} A_i$$

8.6 Shear Center of Closed Sections

Example 8.6.1

$$\theta' = \frac{1}{2AG} \sum_{i=1}^{\text{No. walls}} q^{(i)} \frac{s^{(i)}}{t^{(i)}} = 0$$

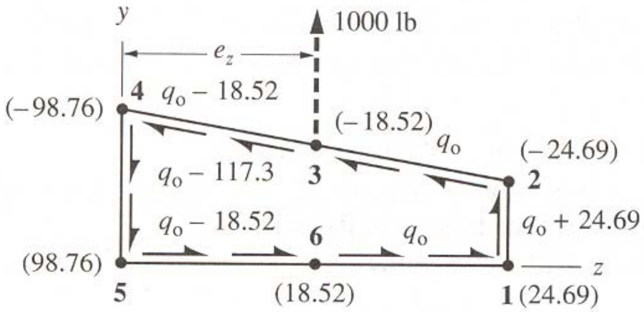


Figure 8.6.2 Flange load gradients (lb/in.) and wall shear flows.

Starting with the wall joining flanges 1 and 2, and moving counterclockwise around the cell,

$$\frac{1}{2AG}\left[(q_o + 24.69)\left(\frac{4}{0.03}\right) + q_o\left(\frac{10.2}{0.05}\right) + (q_o - 18.52)\left(\frac{10.2}{0.05}\right)\right.$$
$$\left. + (q_o - 117.3)\left(\frac{8}{0.1}\right) + (q_o - 18.52)\left(\frac{10}{0.05}\right) + q_o\left(\frac{10}{0.05}\right)\right] = 0$$

$$\Rightarrow \quad q_o = 13.29 \text{ lb/ in.}$$

8.6 Shear Center of Closed Sections

Example 8.6.1

Finally, we locate the shear center by requiring that the moments of the shear flows corresponding to zero twist angle equal that of the 1000 lb load acting through the shear center. Summing the moments about the lower left corner of the section, $(2 \times 40 \times 37.98) + (2 \times 40 \times 13.29) - (2 \times 40 \times 5.229) = 1000 \times e_z$

$$\Rightarrow \quad e_z = 3.683 \text{ in.}$$

As before, we locate the 1000 lb force by summing the moments around flange 5,

$$-(2 \times 40 \times 26.64) - (2 \times 40 \times 22.16) + (2 \times 40 \times 3.992) = -1000 \times e_y$$

$$\Rightarrow \quad e_y = 3.585 \text{ in.}$$

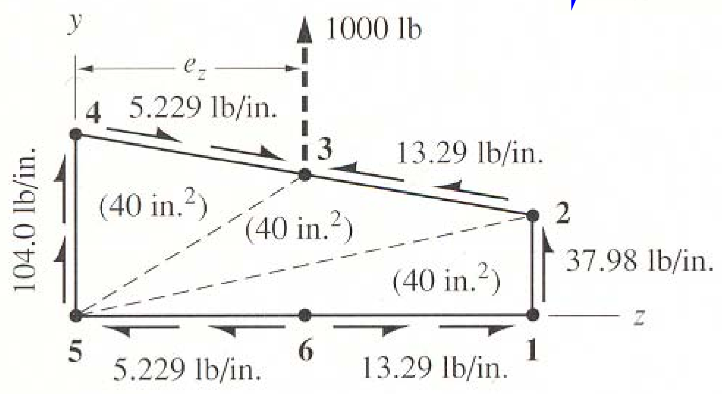


Figure 8.6.3 Shear flows required to prevent the section from rotating.

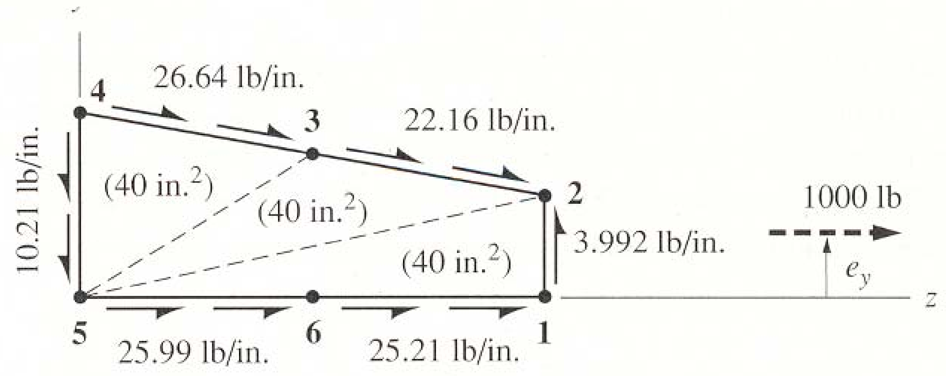


Figure 8.6.4 Shear flows for a 1000 lb load to the right and zero twist.

8.7 Warping Deflections

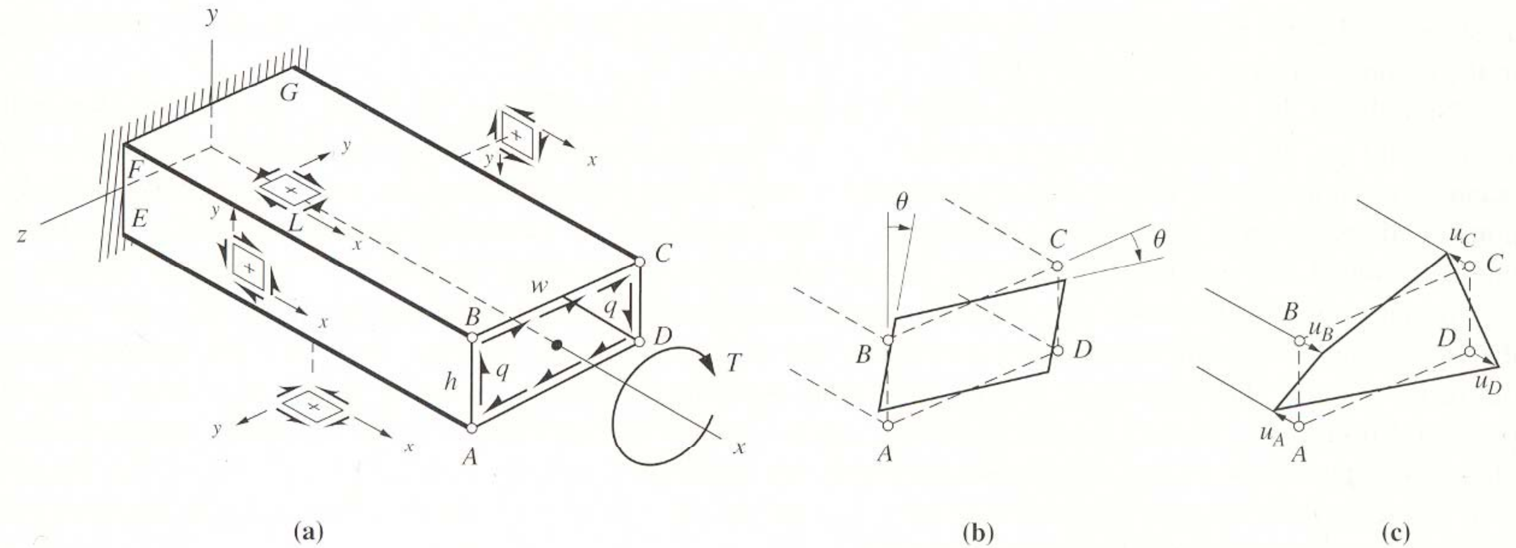


Figure 8.7.1 (a) Thin-walled box beam in pure torsion. Each panel shows a "local" *xy* axis relative to which the sign of stress and strain in that panel may be established. (b) Rigid-body rotation of the cross section. (c) Cross sectional warping, which is superimposed on the rigid rotation.

Figure 8.7.1a shows a thin-walled, idealized box beam in pure torsion. The dimensions of the rectangular cross section are h and w, where w> h.

The deformation at a given section of the torque box consists of the modes, shown in Figure 8.7.1b and c, respectively. The first is a rigid-body rotation of the cross section around the twist axis.

8.7 Warping Deflections

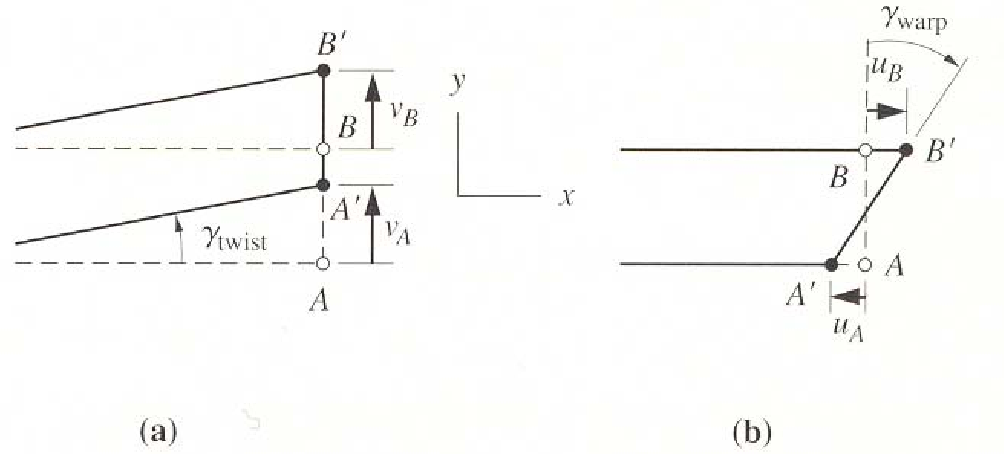


Figure 8.7.2 Portion of the side panel *ABCD* of Figure 8.7.1. Shear strain due to (a) twist and (b) warping.

Warping occurs because of the torque-induced shear strain in the walls of the box beam.

Suppose for simplicity that all of the walls have the same thickness t, which means that the shear stress, and therefore the net s γ_{xy}, **strain is the same in every pan** $\gamma_{xy,\text{twist}}$ **e varies from panel to panel, there must be another component of the** $\gamma_{xy,\text{warp}}$, **ain, such that**

$$\gamma_{xy} = \gamma_{xy,\text{twist}} + \gamma_{xy,\text{warp}}$$

8.7 Warping Deflections

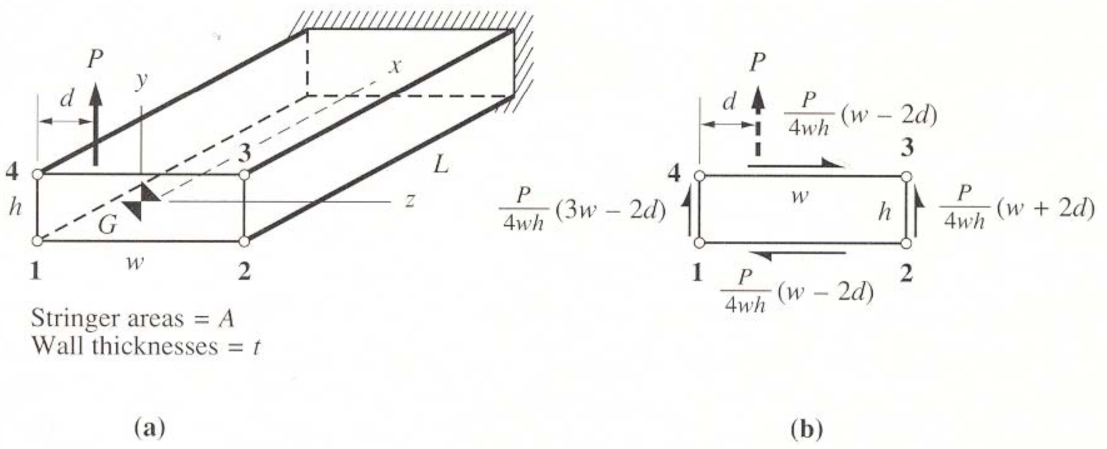


Figure 8.7.3 (a) Idealized box beam with a vertical shear load P. (b) The corresponding section shear flows.

Consider the idealized box beam shown in Figure 8.7.3a. For purposes of illustration, let the four stringers all have the same area A and the panels have a common thickness t and a common shear modulus.

$$\delta W^*_{\text{ext}} = \delta P u_1 - \delta P u_2 + \delta P u_3 - \delta P u_4$$

The minus signs account for the opposite directions of virtual load and displacement at flanges 2 and 4. This expression can also be written as

$$\delta W^*_{\text{ext}} = \delta P\left[(u_1 - u_4) - (u_2 - u_3)\right]$$

8.7 Warping Deflections

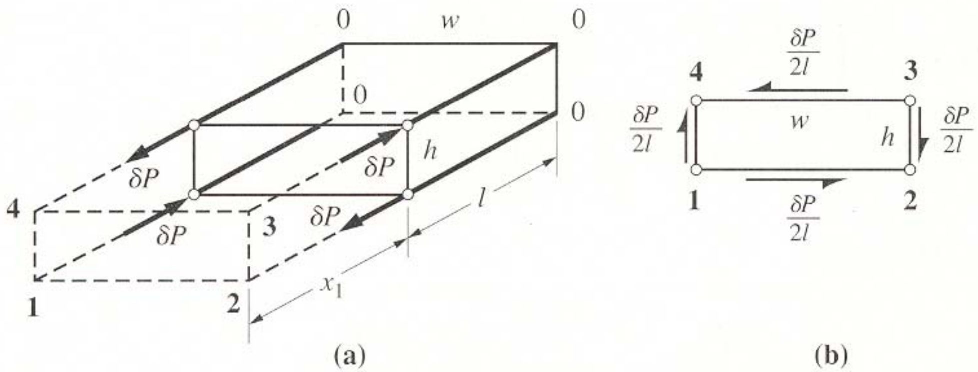


Figure 8.7.4 (a) Self-equilibrating set of virtual stringer loads applied at a given section of the box beam of the previous figure. (b) The corresponding virtual shear flows.

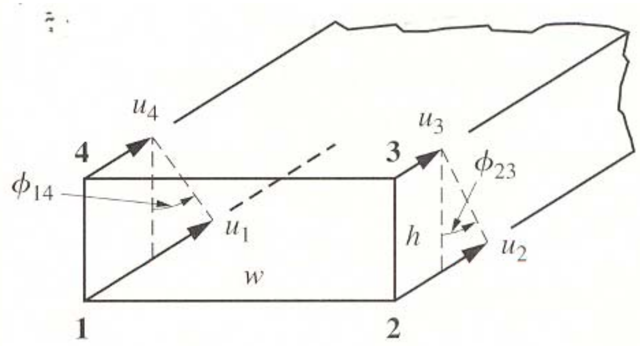


Figure 8.7.5 Relation between flange displacements and the rotations of the vertical sides of the cell.

$$\delta W^*_{\text{ext}} = \delta P[\phi_{14}h - \phi_{23}h] = h\phi\delta P \qquad \textbf{[8.7.1]}$$

$$\delta W^*_{\text{int}} = \sum_{\text{panels}} \frac{A}{Gt} q\delta q = \frac{1}{Gt}\left[wlq^{(1\text{–}2)}\delta q^{(1\text{–}2)} + hlq^{(2\text{–}3)}\delta q^{(2\text{–}3)} + wlq^{(3\text{–}4)}\delta q^{(3\text{–}4)} + hlq^{(4\text{–}1)}\delta q^{(4\text{–}1)}\right] \qquad \textbf{[8.7.2]}$$

$$\delta W^*_{\text{int}} = \frac{1}{Gt}\left\{wl\left[-\frac{P}{4wh}(w\text{–}2d)\right]\left(\frac{\delta P}{2l}\right) + hl\left[\frac{P}{4wh}(w+2d)\right]\left(-\frac{\delta P}{2l}\right)\right.$$
$$\left. + wl\left[-\frac{P}{4wh}(w\text{–}2d)\right]\left(\frac{\delta P}{2l}\right) + hl\left[-\frac{P}{4wh}(3w\text{–}2d)\right]\left(-\frac{\delta P}{2l}\right)\right\}$$

8.7 Warping Deflections

Upon simplification,

$$\delta W_{\text{int}}^{*} = -\frac{P\,\delta P}{2Gtwh}(w-h)\left(\frac{w}{2}-d\right) \qquad [8.7.3]$$

the warp angle for a section

$$\phi = -\frac{P}{2Gtwh^{2}}(w-h)\left(\frac{w}{2}-d\right) \qquad [8.7.4]$$

pure torque with the same moment,

$$T = -P\left(\frac{w}{2}-d\right)$$

according to Equation 8.7.4, if the beam is in pure torsion, the warp angle is

$$\phi = \frac{T}{2Gtwh^{2}}(w-h) \qquad [8.7.5]$$

8.7 Warping Deflections

Example 8.7.1

Calculate the warping angle for an idealized beam with constant cross section loaded in shear, as illustrated in Figure 8.7.6.

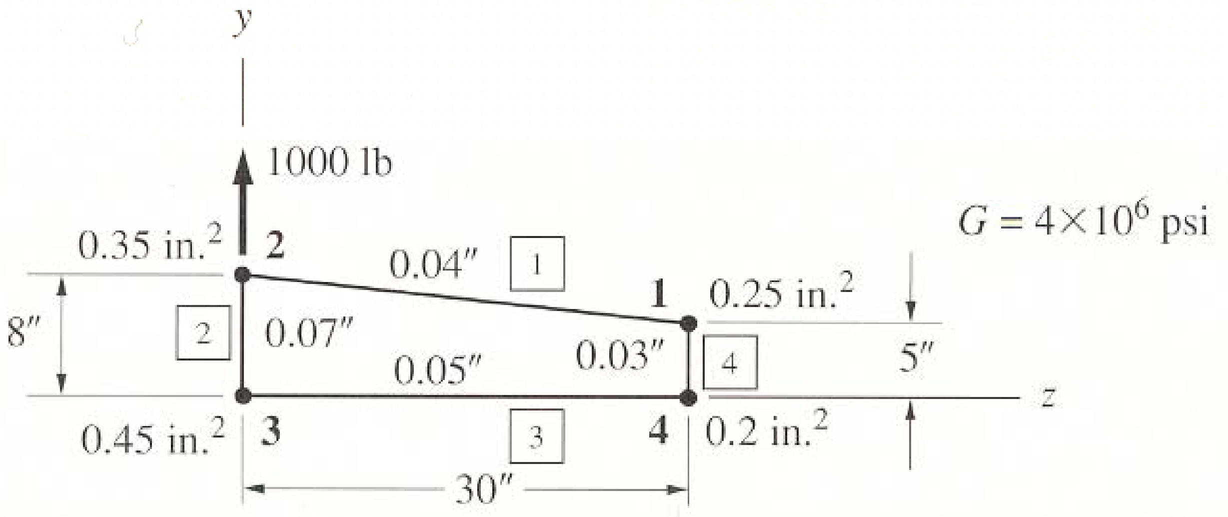


Figure 8.7.6 Section of a constant cross section box beam.

8.7 Warping Deflections

Example 8.7.1

$$y_G = \frac{\sum_{i=1}^{4} y_i A_i}{\sum_{i=1}^{4} A_i} = \frac{4.050}{1.250} = 3.240 \text{ in.} \qquad z_G = \frac{\sum_{i=1}^{4} z_i A_i}{\sum_{i=1}^{4} A_i} = \frac{13.50}{1.250} = 10.80 \text{ in.}$$

$$I_{G_y} = \sum_{i=1}^{4} (z_i - z_G)^2 A_i = 259.2 \text{ in.}^4$$

$$I_{G_z} = \sum_{i=1}^{4} (y_i - y_G)^2 A_i = 15.53 \text{ in.}^4$$

$$I_{G_{yz}} = \sum_{i=1}^{4} (y_i - y_G)(z_i - z_G) A_i = -6.240 \text{ in.}^4$$

Substituting $V_y = -1000$ lb and $V_z = 0$ into Equation 4.8.2,

$$P_x'^{(i)} = \frac{1}{I_{G_y} I_{G_z} - I_{G_{yz}}^2} [(I_{G_y} V_y - I_{G_{yz}} V_z)(y_i - y_G) + (I_{G_z} V_z - I_{G_{yz}} V_y)(z_i - z_G)] A_i$$

8.7 Warping Deflections

Example 8.7.1

$$q^{(1)} = q^{(4)} + P_x'^{(1)} = q^{(4)} - 36.13$$

$$q^{(2)} = q^{(4)} + P_x'^{(2)} = (q^{(4)} + P_x'^{(1)}) + P_x'^{(2)} = q^{(4)} - 138.6$$

$$q^{(3)} = q^{(2)} + P_x'^{(3)} = (q^{(4)} + P_x'^{(1)} + P_x'^{(2)}) + P_x'^{(3)} = q^{(4)} - 36.13$$

Invoking moment equivalence about flange 2,

$$(30q^{(3)}) \times 8 + (5q^{(4)}) \times 30 = 0$$

This can be written in terms of $q^{(4)}$ alone by substituting the third of Equation c:

$$240\left(q^{(4)} - 36.13\right) + 150q^{(4)} = 0$$

Solving this for $q^{(4)}$ and then substituting into Equations c,

$$q^{(4)} = 22.23 \text{ lb/in.}$$

$$q^{(3)} = -13.90 \text{ lb/in.}$$

$$q^{(2)} = -116.3 \text{ lb/in.}$$

$$q^{(1)} = -13.90 \text{ lb/in.}$$

8.7 Warping Deflections

Example 8.7.1

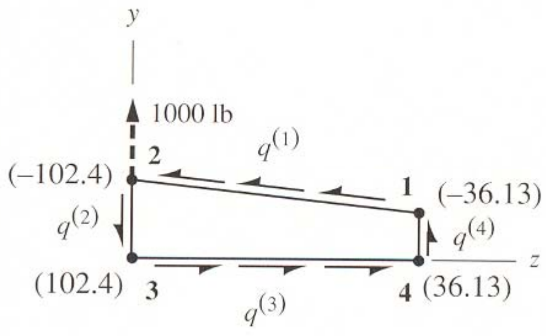


Figure 8.7.7 Flange load gradients (lb/in.) and assumed directions of shear flow.

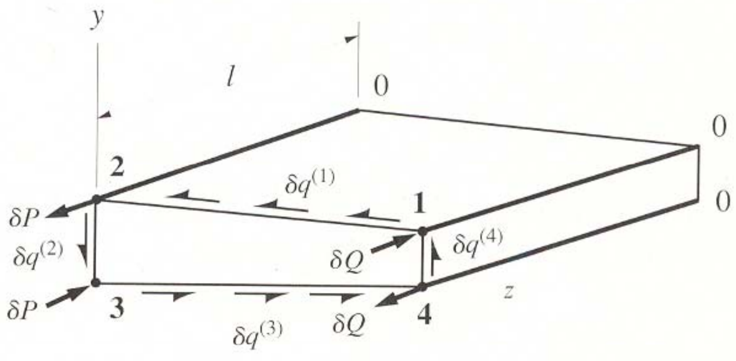


Figure 8.7.8 Self-equilibrating set of virtual loads applied to the stringers of an isolated free body of a portion of the box beam, together with the corresponding set of virtual shear flows.

$$\delta Q = \frac{h_2}{h_4}\delta P = 1.6\delta P$$

the equilibrium of stringers 2, 3 and 4,

$$\delta q^{(2)} - \delta q^{(1)} = -\frac{\delta P}{l} \qquad or \qquad \delta q^{(2)} = \delta q^{(1)} - \frac{\delta P}{l}$$

$$\delta q^{(3)} - \delta q^{(2)} = \frac{\delta P}{l} \qquad or \qquad \delta q^{(3)} = \delta q^{(2)} + \frac{\delta P}{l}$$

$$\delta q^{(4)} - \delta q^{(3)} = -\frac{\delta Q}{l} \qquad or \qquad \delta q^{(4)} = \delta q^{(3)} - \frac{\delta Q}{l}$$

8.7 Warping Deflections

Example 8.7.1

Using flange 2 as the moment summation point,

$$(30\delta q^{(3)}) \times 8 + (5\delta q^{(4)}) \times 30 = 0$$

Substituting the second and third of Equation g into h yields

$$240\delta q^{(1)} + 150(\delta q^{(1)} - 1.6\frac{\delta P}{l}) = 0$$

From this and Equations g, we find all of the virtual shear flows in terms of the virtual load δP, as follows:

$$\delta q^{(1)} = 0.6154\frac{\delta P}{l} \qquad \delta q^{(2)} = -0.3846\frac{\delta P}{l} \qquad \delta q^{(3)} = 0.6154\frac{\delta P}{l} \qquad \delta q^{(4)} = -0.9846\frac{\delta P}{l}$$

With the true and virtual shear flows in hand,

$$\delta W^*_{\text{int}} = \sum_{\text{panels}} \frac{A}{Gt} q\delta q = \frac{l}{G}\left[\frac{h^{(1)}}{t^{(1)}} q^{(1)}\delta q^{(1)} + \frac{h^{(2)}}{t^{(2)}} q^{(2)}\delta q^{(2)} + \frac{h^{(3)}}{t^{(3)}} q^{(3)}\delta q^{(3)} + \frac{h^{(4)}}{t^{(4)}} q^{(4)}\delta q^{(4)}\right]$$

so that

$$\delta W^*_{\text{int}} = \frac{l}{4(10^6)}\left[\frac{\sqrt{30^2+3^2}}{0.04}(-13.90) \times 0.6154\frac{\delta P}{l} + \frac{8}{0.07}(-116.3) \times \left(-0.3846\frac{\delta P}{l}\right)\right.$$
$$\left. + \frac{30}{0.05}(-13.90) \times 0.6154\frac{\delta P}{l} + \frac{5}{0.03}22.23 \times \left(-0.9846\frac{\delta P}{l}\right)\right]$$

or

$$\delta W^*_{\text{int}} = -0.002528\delta P$$

8.7 Warping Deflections

Example 8.7.1

the external complementary virtual work is

$$\delta W^*_{\text{ext}} = -\delta P u_2 + \delta P u_3 - \delta Q u_4 + \delta Q u_1 = (u_3 - u_2)\delta P - (u_4 - u_1)\delta Q$$

The counterclockwise rotation ϕ_2 about the z axis of web 2-3 in terms of the axial displacements of stringers 2 and 3,

$$\phi_2 = \frac{u_3 - u_2}{h_2}$$

Likewise, for web 1-4,

$$\phi_4 = \frac{u_4 - u_1}{h_1}$$

Therefore,

$$\delta W^*_{\text{ext}} = \phi_2 h_2 \delta P - \phi_4 h_4 \delta Q$$

we obtain the external complementary virtual work in terms of the warp angle $\phi = \phi_3 - \phi_1$,

$$\delta W^*_{\text{ext}} = \phi h_2 \delta P = 8\phi_2 \delta P$$

$$\Rightarrow \quad \phi = -0.000316 \text{ radians} = -0.0181 \text{ degrees}$$

8.8 Multicell Idealized Box Beams

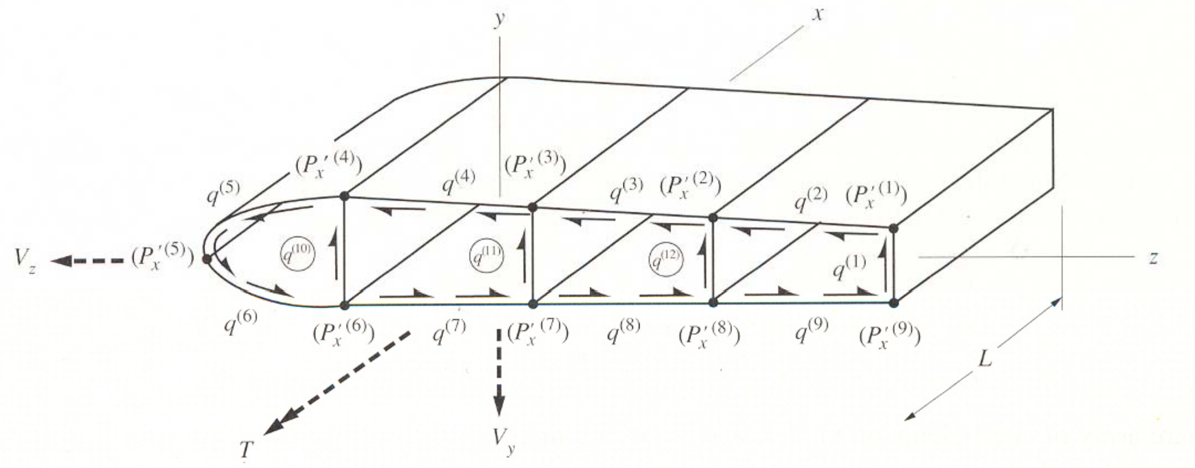


Figure 8.8.1 Multicell idealized box beam section.

We have the nine dependent shear flows as linear functions of the redundants and the applied loads, as follows:

$$q^{(i)} = a_i q^{(10)} + b_i q^{(11)} + c_i q^{(12)} + d_i V_z + e_i V_y + f_i T \qquad i = 1, \ldots, 9 \qquad [8.8.1]$$

Virtual shear flows

$$\delta q^{(i)} = a_i \delta q^{(10)} + b_i \delta q^{(11)} + c_i \delta q^{(12)} \qquad i = 1, \ldots, 9 \qquad [8.8.2]$$

The internal complementary virtual work is given by the familiar expression

$$\delta W_{\text{int}}^* = \sum_{i=1}^{12} \frac{k^{(i)} A^{(i)}}{G^{(i)} t^{(i)}} q^{(i)} \delta q^{(i)} \qquad [8.8.3]$$

8.8 Multicell Idealized Box Beams

The principle of complementary virtual work requires that the internal complementary work also vanish. Equation 8.8.3 then implies that

$$\frac{L}{G}\sum_{i=1}^{12}\frac{s^{(i)}}{t^{(i)}}q^{(i)}\delta q^{(i)} = 0$$

or

$$\frac{s^{(i)}}{t^{(i)}}q^{(i)}\delta q^{(i)} + \frac{s^{(10)}}{t^{(10)}}q^{(10)}\delta q^{(10)} + \frac{s^{(11)}}{t^{(11)}}q^{(11)}\delta q^{(11)} + \frac{s^{(12)}}{t^{(12)}}q^{(12)}\delta q^{(12)} = 0 \qquad [8.8.4]$$

Substituting Equations 8.8.1 and 8.8.2 into the first terms of this equation and factoring out the independent virtual $\delta q^{(10)}$, $\delta q^{(11)}$ and $\delta q^{(12)}$, we obtain

$$\begin{aligned}&\left(c_{1,1}q^{(10)} + c_{1,2}q^{(11)} + c_{1,3}q^{(12)} + b_{1,1}V_z + b_{1,2}V_y + b_{1,3}T\right)\delta q^{(10)}\\ &\quad + \left(c_{2,1}q^{(10)} + c_{2,2}q^{(11)} + c_{2,3}q^{(12)} + b_{2,1}V_z + b_{2,2}V_y + b_{2,3}T\right)\delta q^{(11)}\\ &\qquad + \left(c_{3,1}q^{(10)} + c_{3,2}q^{(11)} + c_{3,3}q^{(12)} + b_{3,1}V_z + b_{3,2}V_y + b_{3,3}T\right)\delta q^{(12)} = 0\end{aligned}$$

we thereby obtain three equations for the three redundant shear flows as follows:

$$\begin{aligned}c_{1,1}q^{(10)} + c_{1,2}q^{(11)} + c_{1,3}q^{(12)} &= -\left(b_{1,1}V_z + b_{1,2}V_y + b_{1,3}T\right)\\ c_{2,1}q^{(10)} + c_{2,2}q^{(11)} + c_{2,3}q^{(12)} &= -\left(b_{2,1}V_z + b_{2,2}V_y + b_{2,3}T\right) \qquad [8.8.5]\\ c_{3,1}q^{(10)} + c_{3,2}q^{(11)} + c_{3,3}q^{(12)} &= -\left(b_{3,1}V_z + b_{3,2}V_y + b_{3,3}T\right)\end{aligned}$$

8.8 Multicell Idealized Box Beams

Example 8.8.1

The sections of a constant cross-section beam in Figure 8.8.2 carries a pure torque of 50,000 in-lb counterclockwise. Find the shear flow and rate of twist.

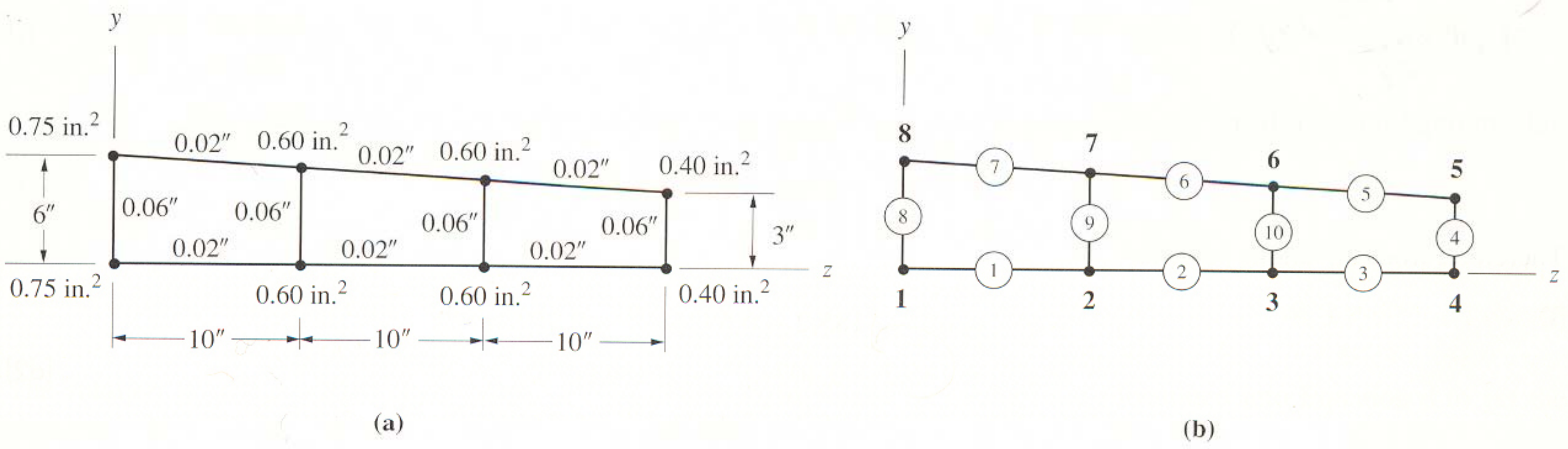


Figure 8.8.2 (a) Section size and properties. (b) Flange and wall numbering.

8.8 Multicell Idealized Box Beams

Example 8.8.1

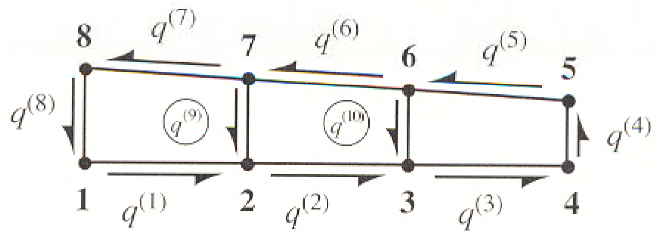


Figure 8.8.3 Assumed directions for the shear flows.

The circled shear flows are the chosen redundants.

$$
\begin{aligned}
&1 \quad q^{(1)} - q^{(8)} = 0 \\
&2 \quad q^{(2)} - q^{(1)} - q^{(9)} = 0 \\
&3 \quad q^{(3)} - q^{(2)} - q^{(10)} = 0 \\
&4 \quad q^{(4)} - q^{(3)} = 0 \\
&5 \quad q^{(5)} - q^{(4)} = 0 \\
&6 \quad -q^{(5)} + q^{(6)} + q^{(10)} = 0 \\
&7 \quad -q^{(6)} + q^{(7)} + q^{(9)} = 0
\end{aligned}
$$

$$
\begin{aligned}
q^{(1)} &= q^{(8)} \\
q^{(2)} &= q^{(8)} + q^{(9)} \\
q^{(3)} &= q^{(8)} + q^{(9)} + q^{(10)} \\
q^{(4)} &= q^{(8)} + q^{(9)} + q^{(10)} \\
q^{(5)} &= q^{(8)} + q^{(9)} + q^{(10)} \\
q^{(6)} &= q^{(8)} + q^{(9)} \\
q^{(7)} &= q^{(8)}
\end{aligned}
$$

For moment equivalence, about flange 8

$$2\left(\frac{1}{2}10\times 6\right)q^{(1)} + 2\left(\frac{1}{2}10\times 6\right)q^{(2)} + 2\left(\frac{1}{2}10\times 6\right)q^{(3)} + 2\left(\frac{1}{2}3\times 30\right)q^{(4)} - 2\left(\frac{1}{2}5\times 10\right)q^{(9)} - 2\left(\frac{1}{2}4\times 20\right)q^{(10)} = 50,000$$

8.8 Multicell Idealized Box Beams

Example 8.8.1

$$q^{(8)} = 185.2 - 0.5926q^{(9)} - 0.2593q^{(10)}$$

$$q^{(1)} = q^{(7)} = q^{(8)} = 185.2 - 0.5926q^{(9)} - 0.2593q^{(10)}$$

$$q^{(2)} = q^{(6)} = 185.2 + 0.4074q^{(9)} - 0.2593q^{(10)}$$

$$q^{(3)} = q^{(4)} = q^{(5)} = 185.2 + 0.4074q^{(9)} + 0.7407q^{(10)}$$

Virtual shear flows

$$\delta q^{(1)} = \delta q^{(7)} = \delta q^{(8)} = -0.5926\delta q^{(9)} - 0.2593\delta q^{(10)}$$

$$\delta q^{(2)} = \delta q^{(6)} = 0.4074\delta q^{(9)} - 0.2593\delta q^{(10)}$$

$$\delta q^{(3)} = \delta q^{(4)} = \delta q^{(5)} = 0.4074\delta q^{(9)} + 0.7407\delta q^{(10)}$$

the internal complementary virtual work is just that of the panels,

$$\delta W^*_{\text{int}} = \sum_{i=1}^{10} \frac{A^{(i)}}{G^{(i)}t^{(i)}} q^{(i)} \delta q^{(i)} = \frac{L}{G} \sum_{i=1}^{10} \frac{s^{(i)}}{t^{(i)}} q^{(i)} \delta q^{(i)}$$

8.8 Multicell Idealized Box Beams

Example 8.8.1

$$\delta W^*_{\text{int}} = \frac{L}{G}\Bigg\{\overbrace{\left(\frac{10}{0.02}+\frac{10.05}{0.02}+\frac{6}{0.06}\right)\left(185.2-0.5926q^{(9)}-0.2593q^{(10)}\right)\left(-0.5926\delta q^{(9)}-0.2593\delta q^{(10)}\right)}^{\text{webs 1, 7, and 8}}+$$

$$+\overbrace{\left(\frac{10}{0.02}+\frac{10.05}{0.02}\right)\left(185.2+0.4074q^{(9)}-0.2593q^{(10)}\right)\left(0.4074\delta q^{(9)}-0.2593\delta q^{(10)}\right)}^{\text{webs 2 and 6}}$$

$$+\overbrace{\left(\frac{10}{0.02}+\frac{3}{0.06}+\frac{10.05}{0.02}\right)\left(185.2+0.4074q^{(9)}+0.7407q^{(10)}\right)\left(0.4074\delta q^{(9)}+0.7407\delta q^{(10)}\right)}^{\text{webs 3, 4, and 5}}$$

$$+\overbrace{\frac{5}{0.06}q^{(9)}\delta q^{(9)}}^{\text{web 9}}+\overbrace{\frac{4}{0.06}q^{(10)}\delta q^{(10)}}^{\text{web 10}}\Bigg\}$$

Simplifying and combining terms leads to

$$\delta W^*_{\text{int}} = \frac{L}{G}\left[\left(811.6q^{(9)}+381.1q^{(10)}+34,050\right)\delta q^{(9)}+\left(381.1q^{(9)}+785.6q^{(10)}+43,310\right)\delta q^{(10)}\right]$$

8.8 Multicell Idealized Box Beams

Example 8.8.1

According to the principle of complementary virtual work,

$$\frac{L}{G}\left[\left(811.6q^{(9)} + 381.1q^{(10)} + 34,050\right)\delta q^{(9)} + \left(381.1q^{(9)} + 785.6q^{(10)} + 43,310\right)\delta q^{(10)}\right] = 0$$

This results in a system of two equation for the two redundants,

$$811.6q^{(9)} + 381.1q^{(10)} = -34,050$$

$$381.1q^{(9)} + 785.6q^{(10)} = -43,313$$

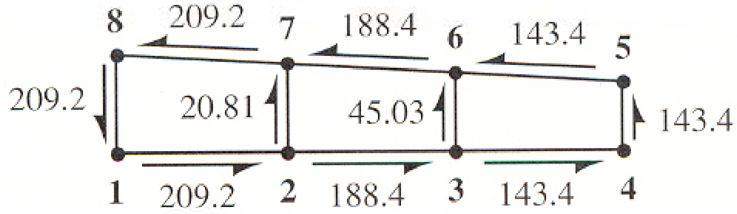


Figure 8.8.4 Shear flows (lb/in.) statically equivalent to a 50,000 in.-lb counter clockwise torque.

The internal complementary virtual work is

$$\delta W^*_{\text{int}} = \frac{L}{G}\sum_{i=1}^{10}\frac{s^{(i)}}{t^{(i)}}q^{(i)}\frac{q^{(i)}}{50,000}\delta T = \frac{L\delta T}{50,000G}\sum_{i=1}^{10}\frac{s^{(i)}}{t^{(i)}}q^{(i)2}$$

$$= \frac{L\delta T}{50,000G}\left[\frac{10}{0.02}(209.2)^2 + \frac{10}{0.02}(188.4)^2 + \frac{10}{0.02}(143.4)^2 + \frac{3}{0.06}(143.4)^2 + \frac{10.05}{0.02}(143.4)^2\right.$$

$$\left. + \frac{10.05}{0.02}(188.4)^2 + \frac{10.05}{0.02}(209.2)^2 + \frac{6}{0.06}(209.2)^2 + \frac{5}{0.06}(-20.81)^2 + \frac{4}{0.06}(-45.03)^2\right]$$

or

$$\delta W^*_{\text{int}} = 2112.4\frac{L}{G}\delta T$$

8.8 Multicell Idealized Box Beams

Example 8.8.1

Equating this to the complementary virtual work of the torque $\theta\delta T$,

$$\theta = 2112.4\frac{L}{G}$$

virtual shear flows acting around the closed cell must all have the same value, denoted δq_T,

$$\delta q^{(10)} = \delta q^{(5)} = \delta q^{(4)} = \delta q^{(3)} = \delta q_T$$

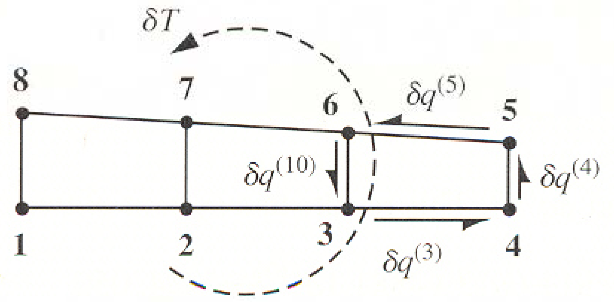


Figure 8.8.5 One alternative virtual shear load distribution representing the virtual torque.

For static equivalence, the moment of the constant shear flow δq_T must equal the virtual torque δT.

$$\delta q_T = \frac{\delta T}{2A_{\text{cell}}} = \frac{\delta T}{2 \times \left[\frac{1}{2}(3+4)(10)\right]} = 0.01428\delta T$$

The internal complementary virtual work is

$$\delta W^*_{\text{int}} = \frac{L}{G}\left(\frac{s^{(3)}}{t^{(3)}}q^{(3)}\delta q^{(3)} + \frac{s^{(4)}}{t^{(4)}}q^{(4)}\delta q^{(4)} + \frac{s^{(5)}}{t^{(5)}}q^{(5)}\delta q^{(5)} + \frac{s^{(10)}}{t^{(10)}}q^{(10)}\delta q^{(10)}\right)$$

$$= 0.01428\left(\frac{s^{(3)}}{t^{(3)}}q_3 + \frac{s^{(4)}}{t^{(4)}}q^{(4)} + \frac{s^{(5)}}{t^{(5)}}q^{(5)} + \frac{s^{(10)}}{t^{(10)}}q^{(10)}\right)\frac{L}{G}\delta T$$

Substituting the web dimensions and the true shear flows previously computed

$$\delta W^*_{\text{int}} = 0.01428\left[\frac{10}{0.02}143.4 + \frac{3}{0.06}143.4 + \frac{\sqrt{10^2+1^2}}{0.02}143.4 + \frac{4}{0.06}(-45.03)\right]\frac{L}{G}\delta T = 2112.4\frac{L}{G}\delta T$$

8.8 Multicell Idealized Box Beams

Example 8.8.2

Calculate the shear flows in the webs of the constant-cross-section idealized beam in the previous example if, instead of pure tension, the beam is subjected to the shear shown in Figure 8.8.6b.

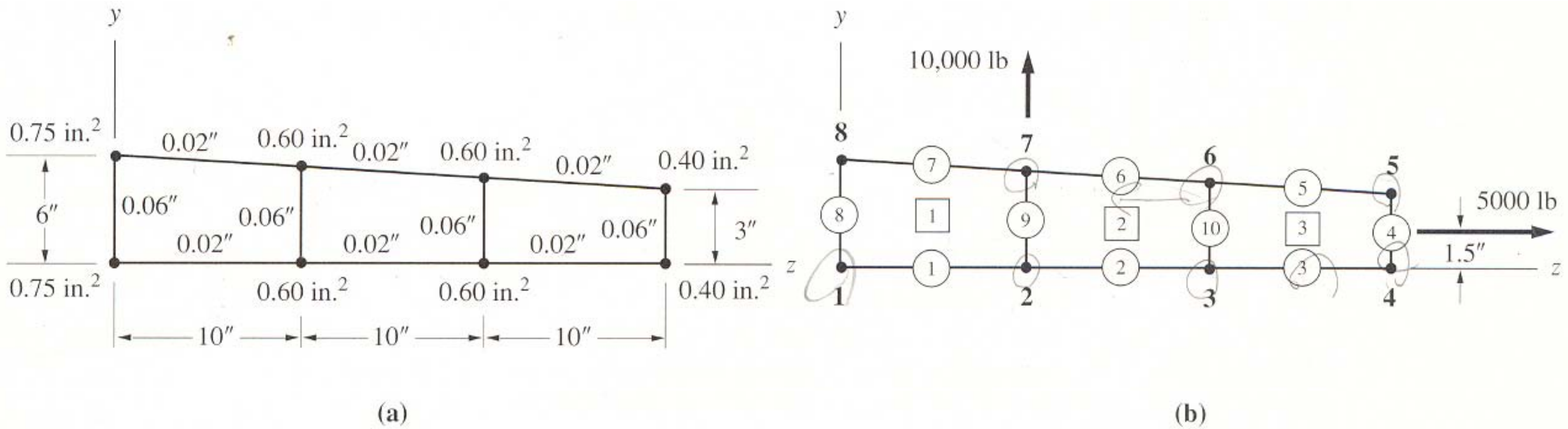


Figure 8.8.6 Three-cell section carrying shear loads.

8.8 Multicell Idealized Box Beams

Example 8.8.2

$$z_G = \frac{\sum_{i=1}^{8} A_i z_i}{\sum_{i=1}^{8} A_i} = \frac{60}{4.7} = 12.76 \text{ in.} \qquad y_G = \frac{\sum_{i=1}^{8} A_i y_i}{\sum_{i=1}^{8} A_i} = \frac{11.10}{4.7} = 2.362 \text{ in.}$$

$$I_{G_y} = \sum_{i=1}^{8} (z_i - z_G)^2 A_i = 554.0 \text{ in.}^4 \qquad I_{G_z} = \sum_{i=1}^{8} (y_i - y_G)^2 A_i = 28.98 \text{ in.}^4 \qquad I_{G_{yz}} = \sum_{i=1}^{8} (y_i - y_G)(z_i - z_G) A_i = -27.70 \text{ in.}^4$$

To compute the flange load gradient at the ith flange,

$$P_x^{\prime(i)} = \frac{1}{I_{G_y} I_{G_z} - I_{G_{yz}}^2} [(I_{G_y} V_y - I_{G_{yz}} V_z)(y_i - y_G) + (I_{G_z} V_z - I_{G_{yz}} V_y)(z_i - z_G)] A_i$$

$$\Rightarrow P_x^{\prime(i)} = [-371.4 y_i - 27.59 z_i + 1230]\, A_i$$

$$q^{(1)} = q^{(8)} + 922.0$$
$$q^{(2)} = q^{(8)} + q^{(9)} + 1494$$
$$q^{(3)} = q^{(8)} + q^{(9)} + q^{(10)} + 1900$$
$$q^{(4)} = q^{(8)} + q^{(9)} + q^{(10)} + 2061$$
$$q^{(5)} = q^{(8)} + q^{(9)} + q^{(10)} + 1776$$
$$q^{(6)} = q^{(8)} + q^{(9)} + 1291$$
$$q^{(7)} = q^{(8)} + 749.2$$

8.8 Multicell Idealized Box Beams

Example 8.8.2

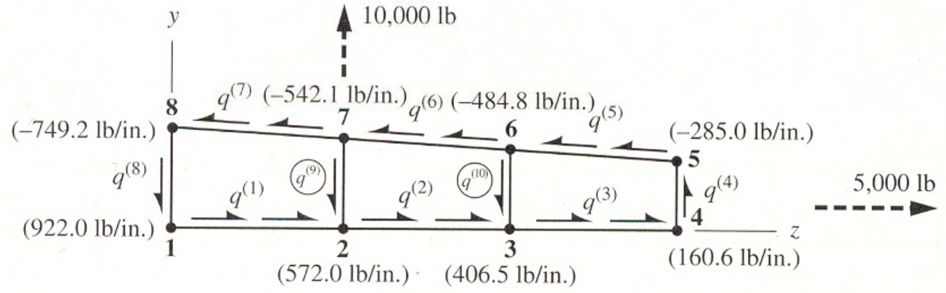


Figure 8.8.7 Flange load gradients (in parentheses) and the assumed directions for the shear flows, with $q^{(9)}$ and $q^{(10)}$ highlighted as the selected redundants.

moment equivalence about flange 8

$$2\left(\frac{1}{2}10\times 6\right)q^{(1)}+2\left(\frac{1}{2}10\times 6\right)q^{(2)}+2\left(\frac{1}{2}10\times 6\right)q^{(3)}+2\left(\frac{1}{2}3\times 30\right)q^{(4)}-2\left(\frac{1}{2}5\times 10\right)q^{(9)}-2\left(\frac{1}{2}4\times 20\right)q^{(10)}=$$

$$10,000\times 10+5000\times 4.5$$

$$\Rightarrow\quad 270q^{(8)}+160q^{(9)}+70q^{(10)}+4.445\times 10^{6}=122,500$$

from which we obtain

$$q^{(8)}=-0.5926q^{(9)}-0.2593q^{(10)}-1,193$$

$$q^{(7)}=-0.5926q^{(9)}-0.2593q^{(10)}-1193$$
$$q^{(6)}=0.4074q^{(9)}-0.2593q^{(10)}+98.70$$
$$q^{(5)}=0.4074q^{(9)}+0.7407q^{(10)}+583.5$$
$$q^{(4)}=0.4074q^{(9)}+0.7407q^{(10)}+868.6$$
$$q^{(3)}=0.4074q^{(9)}+0.7407q^{(10)}+707.9$$
$$q^{(2)}=0.4074q^{(9)}-0.2593q^{(10)}+301.5$$
$$q^{(1)}=-0.5926q^{(9)}-0.2593q^{(10)}-270.6$$

8.8 Multicell Idealized Box Beams

Example 8.8.2

These virtual shear flows

$$
\begin{aligned}
\delta q^{(1)} &= -0.5926\delta q^{(9)} - 0.2593\delta q^{(10)} \\
\delta q^{(2)} &= 0.4074\delta q^{(9)} - 0.2593\delta q^{(10)} \\
\delta q^{(3)} &= 0.4074\delta q^{(9)} + 0.7407\delta q^{(10)} \\
\delta q^{(4)} &= 0.4074\delta q^{(9)} + 0.7407\delta q^{(10)} \\
\delta q^{(5)} &= 0.4074\delta q^{(9)} + 0.7407\delta q^{(10)} \\
\delta q^{(6)} &= 0.4074\delta q^{(9)} - 0.2593\delta q^{(10)} \\
\delta q^{(7)} &= -0.5926\delta q^{(9)} - 0.2593\delta q^{(10)} \\
\delta q^{(8)} &= -0.5926\delta q^{(9)} - 0.2593\delta q^{(10)}
\end{aligned}
$$

the internal complementary virtual work expression,

$$\delta W^*_{\text{int}} = \sum_{i=1}^{10} \frac{Ls^{(i)}}{Gt^{(i)}} q^{(i)} \delta q^{(i)}$$

Setting the result equal to zero ($\delta W^*_{\text{ext}} = 0$ because the redundants are internal loads) leads to

$$\left[\left(811.6q^{(9)} + 381.1q^{(10)}\right) + 6.459 \times 10^5\right]\delta q^{(9)} + \left[\left(381.1q^{(9)} + 785.6q^{(10)}\right) + 5.834 \times 10^5\right]\delta q^{(10)} = 0$$

8.8 Multicell Idealized Box Beams

Example 8.8.2

By the usual argument, this yields the following system of two equations:

$$811.6q^{(9)} + 381.1q^{(10)} = -645,900$$
$$381.1q^{(9)} + 785.6q^{(10)} = -583,400$$

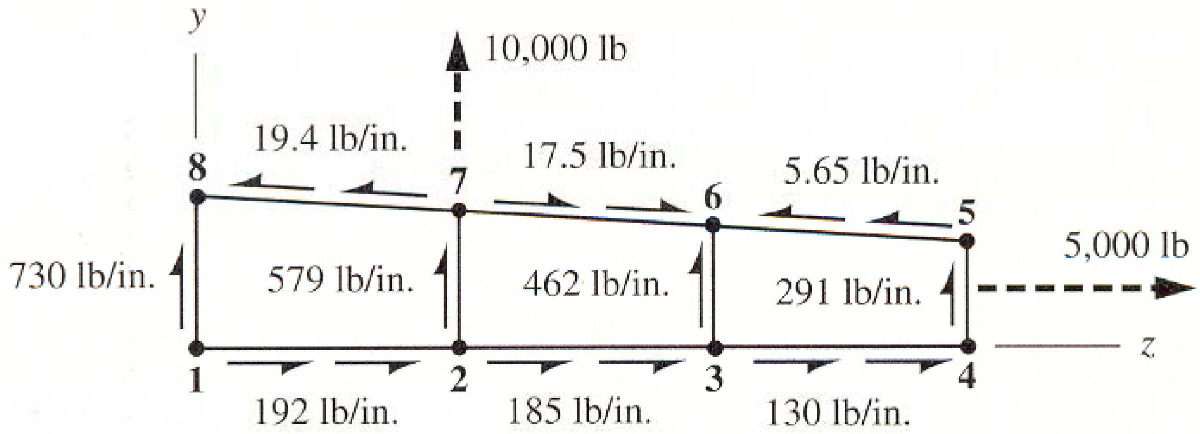


Figure 8.8.8 The shear flows, which are statically equivalent to the combination of an upward-directed 10,000 lb shear force through web 7–2 and a rightward-directed shear force whose line of action bisects web 4–5.

8.8 Multicell Idealized Box Beams

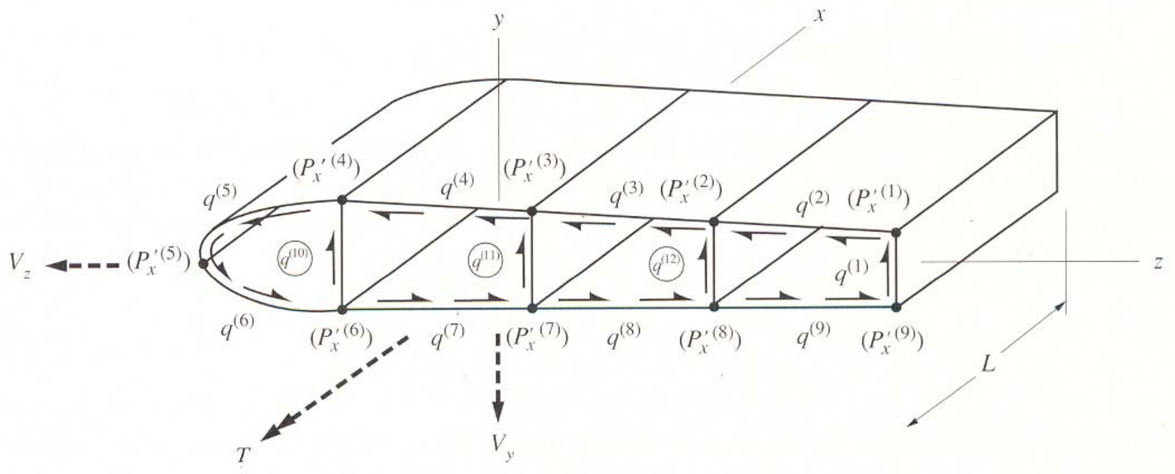


Figure 8.8.1 Multicell idealized box beam section.

$$\delta q^{(i)} = a_i \delta q^{(10)} + b_i \delta q^{(11)} + c_i \delta q^{(12)} \qquad i = 1, \ldots, 9 \qquad [8.8.2]$$

$$q^{(i)} = a_i q^{(10)} + b_i q^{(11)} + c_i q^{(12)} + d_i V_z + e_i V_y + f_i T \qquad i = 1, \ldots, 9 \qquad [8.8.1]$$

$$\delta W_{\text{int}}^* = \sum_{i=1}^{12} \frac{k^{(i)} A^{(i)}}{G^{(i)} t^{(i)}} q^{(i)} \delta q^{(i)} \qquad [8.8.3]$$

8.8 Multicell Idealized Box Beams

$$\frac{L}{G}\sum_{i=1}^{12}\frac{s^{(i)}}{t^{(i)}}q^{(i)}\delta q^{(i)}=0$$

or $$\frac{s^{(i)}}{t^{(i)}}q^{(i)}\delta q^{(i)}+\frac{s^{(10)}}{t^{(10)}}q^{(10)}\delta q^{(10)}+\frac{s^{(11)}}{t^{(11)}}q^{(11)}\delta q^{(11)}+\frac{s^{(12)}}{t^{(12)}}q^{(12)}\delta q^{(12)}=0 \qquad [8.8.4]$$

$$\begin{aligned}&\left(c_{1,1}q^{(10)}+c_{1,2}q^{(11)}+c_{1,3}q^{(12)}+b_{1,1}V_z+b_{1,2}V_y+b_{1,3}T\right)\delta q^{(10)}\\&\quad+\left(c_{2,1}q^{(10)}+c_{2,2}q^{(11)}+c_{2,3}q^{(12)}+b_{2,1}V_z+b_{2,2}V_y+b_{2,3}T\right)\delta q^{(11)}\\&\qquad+\left(c_{3,1}q^{(10)}+c_{3,2}q^{(11)}+c_{3,3}q^{(12)}+b_{3,1}V_z+b_{3,2}V_y+b_{3,3}T\right)\delta q^{(12)}=0\end{aligned}$$

$$\begin{aligned}c_{1,1}q^{(10)}+c_{1,2}q^{(11)}+c_{1,3}q^{(12)}&=-\left(b_{1,1}V_z+b_{1,2}V_y+b_{1,3}T\right)\\c_{2,1}q^{(10)}+c_{2,2}q^{(11)}+c_{2,3}q^{(12)}&=-\left(b_{2,1}V_z+b_{2,2}V_y+b_{2,3}T\right)\\c_{3,1}q^{(10)}+c_{3,2}q^{(11)}+c_{3,3}q^{(12)}&=-\left(b_{3,1}V_z+b_{3,2}V_y+b_{3,3}T\right)\end{aligned} \qquad [8.8.5]$$

8.8 Multicell Idealized Box Beams

Example 8.8.1

The section of a constant-cross-section beam in Figure 8.8.2 carries a pure torque of 50,000 in-lb counterclockwise. Find the shear flows and rate of twist.

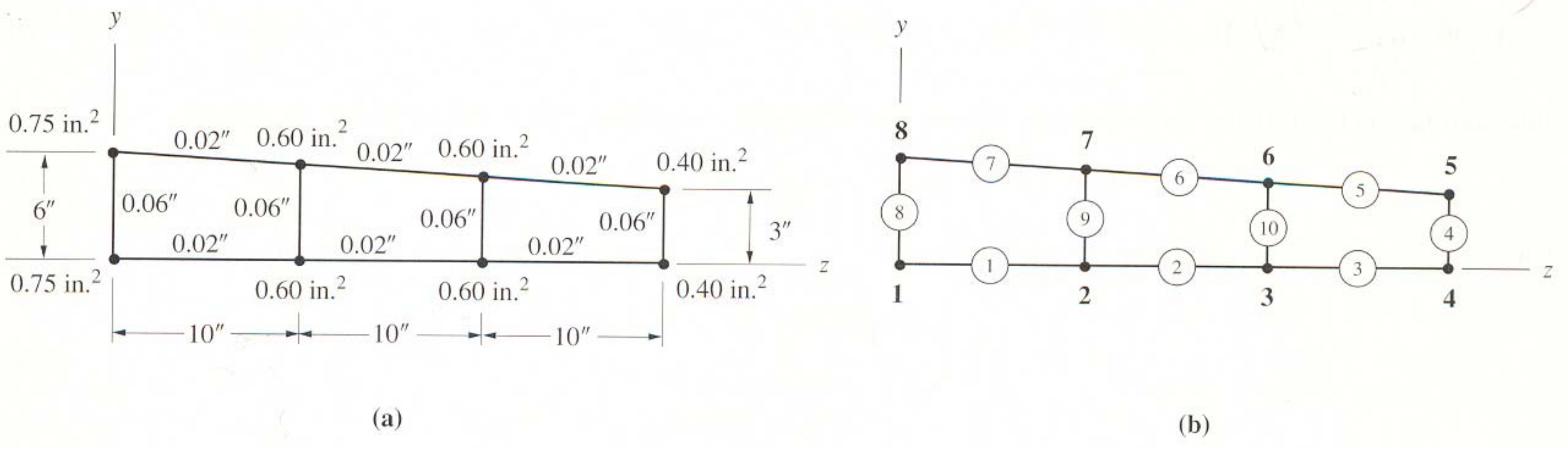


Figure 8.8.2 (a) Section size and properties. (b) Flange and wall numbering.

8.8 Multicell Idealized Box Beams

Example 8.8.1

Solving for shear flows $q^{(1)}$ through $q^{(7)}$ in terms of $q^{(8)}$, $q^{(9)}$, and $q^{(10)}$,

1	$q^{(1)} - q^{(8)} = 0$	$q^{(1)} = q^{(8)}$
2	$q^{(2)} - q^{(1)} - q^{(9)} = 0$	$q^{(2)} = q^{(8)} + q^{(9)}$
3	$q^{(3)} - q^{(2)} - q^{(10)} = 0$	$q^{(3)} = q^{(8)} + q^{(9)} + q^{(10)}$
4	$q^{(4)} - q^{(3)} = 0$	$q^{(4)} = q^{(8)} + q^{(9)} + q^{(10)}$
5	$q^{(5)} - q^{(4)} = 0$	$q^{(5)} = q^{(8)} + q^{(9)} + q^{(10)}$
6	$-q^{(5)} + q^{(6)} + q^{(10)} = 0$	$q^{(6)} = q^{(8)} + q^{(9)}$
7	$-q^{(6)} + q^{(7)} + q^{(9)} = 0$	$q^{(7)} = q^{(8)}$

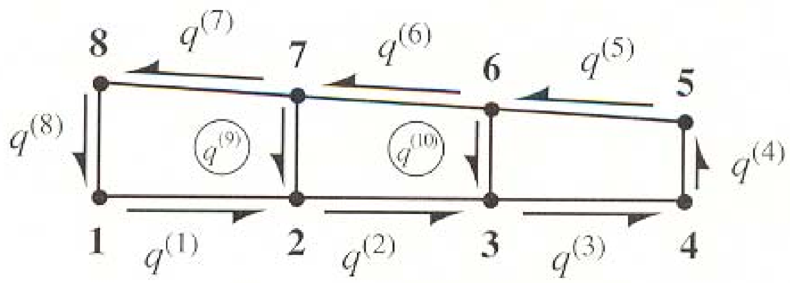


Figure 8.8.3 Assumed directions for the shear flows. The circled shear flows are the chosen redundants.

8.8 Multicell Idealized Box Beams

Example 8.8.1

For moment equivalence,

$$2\left(\frac{1}{2}10\times 6\right)q^{(1)}+2\left(\frac{1}{2}10\times 6\right)q^{(2)}+2\left(\frac{1}{2}10\times 6\right)q^{(3)}+2\left(\frac{1}{2}3\times 30\right)q^{(4)}-2\left(\frac{1}{2}5\times 10\right)q^{(9)}-2\left(\frac{1}{2}4\times 20\right)q^{(10)}=50,000$$

$$q^{(8)}=185.2-0.5926q^{(9)}-0.2593q^{(10)}$$

$$q^{(1)}=q^{(7)}=q^{(8)}=185.2-0.5926q^{(9)}-0.2593q^{(10)}$$

$$q^{(2)}=q^{(6)}=185.2+0.4074q^{(9)}-0.2593q^{(10)}$$

$$q^{(3)}=q^{(4)}=q^{(5)}=185.2+0.4074q^{(9)}+0.7407q^{(10)}$$

virtual shear flows

$$\delta q^{(1)}=\delta q^{(7)}=\delta q^{(8)}=-0.5926\delta q^{(9)}-0.2593\delta q^{(10)}$$

$$\delta q^{(2)}=\delta q^{(6)}=0.4074\delta q^{(9)}-0.2593\delta q^{(10)}$$

$$\delta q^{(3)}=\delta q^{(4)}=\delta q^{(5)}=0.4074\delta q^{(9)}+0.7407\delta q^{(10)}$$

8.8 Multicell Idealized Box Beams

Example 8.8.1

$$\delta W^*_{\text{int}} = \sum_{i=1}^{10} \frac{A^{(i)}}{G^{(i)}t^{(i)}} q^{(i)} \delta q^{(i)} = \frac{L}{G} \sum_{i=1}^{10} \frac{s^{(i)}}{t^{(i)}} q^{(i)} \delta q^{(i)}$$

$$\delta W^*_{\text{int}} = \frac{L}{G} \left\{ \overbrace{\left(\frac{10}{0.02} + \frac{10.05}{0.02} + \frac{6}{0.06}\right) \left(185.2 - 0.5926 q^{(9)} - 0.2593 q^{(10)}\right) \left(-0.5926 \delta q^{(9)} - 0.2593 \delta q^{(10)}\right)}^{\text{webs 1, 7, and 8}} + \right.$$

$$+ \overbrace{\left(\frac{10}{0.02} + \frac{3}{0.06} + \frac{10.05}{0.02}\right) \left(185.2 + 0.4074 q^{(9)} + 0.7407 q^{(10)}\right) \left(0.4074 \delta q^{(9)} + 0.7407 \delta q^{(10)}\right)}^{\substack{\text{webs 2 and 6} \\ \text{webs 3, 4, and 5}}}$$

$$\left. + \overbrace{\frac{5}{0.06} q^{(9)} \delta q^{(9)}}^{\text{web 9}} + \overbrace{\frac{4}{0.06} q^{(10)} \delta q^{(10)}}^{\text{web 10}} \right\}$$

Simplifying and combining terms leads to

$$\delta W^*_{\text{int}} = \frac{L}{G} \left[\left(811.6 q^{(9)} + 381.1 q^{(10)} + 34,050\right) \delta q^{(9)} + \left(381.1 q^{(9)} + 785.6 q^{(10)} + 43,310\right) \delta q^{(10)} \right]$$

The redundant shear flows $q^{(9)}$ and $q^{(10)}$ are internal to the structure; therefore we know $\delta W^*_{\text{ext}} = 0$,

$$\frac{L}{G} \left[\left(811.6 q^{(9)} + 381.1 q^{(10)} + 34,050\right) \delta q^{(9)} + \left(381.1 q^{(9)} + 785.6 q^{(10)} + 43,310\right) \delta q^{(10)} \right] = 0$$

8.8 Multicell Idealized Box Beams

Example 8.8.1

This results in a system of two equations for the two redundants,

$$811.6q^{(9)} + 381.1q^{(10)} = -34,050$$
$$381.1q^{(9)} + 785.6q^{(10)} = -43,313$$

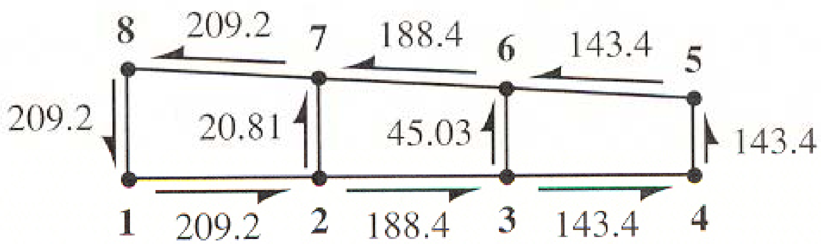


Figure 8.8.4 Shear flows (lb/in.) statically equivalent to a 50,000 in.-lb counter clockwise torque.

Therefore, the internal complementary virtual work is

$$\delta W^*_{\text{int}} = \frac{L}{G}\sum_{i=1}^{10}\frac{s^{(i)}}{t^{(i)}}q^{(i)}\frac{q^{(i)}}{50,000}\delta T = \frac{L\delta T}{50,000G}\sum_{i=1}^{10}\frac{s^{(i)}}{t^{(i)}}q^{(i)2}$$

$$= \frac{L\delta T}{50,000G}\left[\frac{10}{0.02}(209.2)^2 + \frac{10}{0.02}(188.4)^2 + \frac{10}{0.02}(143.4)^2 + \frac{3}{0.06}(143.4)^2 + \frac{10.05}{0.02}(143.4)^2 + \frac{10.05}{0.02}(188.4)^2 + \frac{10.05}{0.02}(209.2)^2 + \frac{6}{0.06}(209.2)^2 + \frac{5}{0.06}(-20.81)^2 + \frac{4}{0.06}(-45.03)^2\right]$$

or $$\delta W^*_{\text{int}} = 2112.4\frac{L}{G}\delta T$$

8.8 Multicell Idealized Box Beams

Example 8.8.1

Equating this to the complementary virtual work of the torque $\theta\delta T$,

$$\theta = 2112.4\frac{L}{G}$$

Since the virtual flange loads are zero,

virtual shear flows acting around the closed cell must all have the same value, denoted δq_T,

$$\delta q^{(10)} = \delta q^{(5)} = \delta q^{(4)} = \delta q^{(3)} = \delta q_T$$

For static equivalence, the moment of the constant shear flow δq_T must equal the virtual torque δT.

$$\delta q_T = \frac{\delta T}{2A_{\text{cell}}} = \frac{\delta T}{2 \times \left[\frac{1}{2}(3+4)(10)\right]} = 0.01428\delta T$$

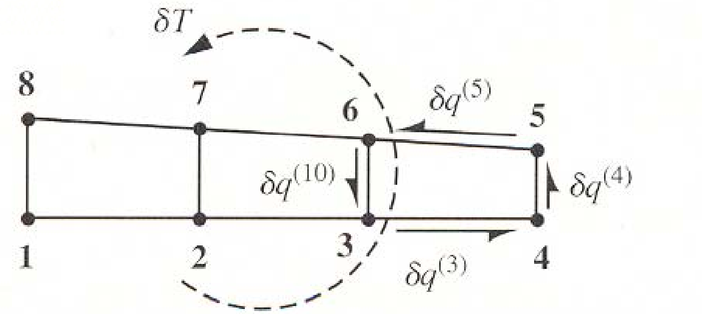


Figure 8.8.5 One alternative virtual shear load distribution representing the virtual torque.

8.8 Multicell Idealized Box Beams

Example 8.8.1

The internal complementary virtual work is

$$\delta W^*_{\rm int} = \frac{L}{G}\left(\frac{s^{(3)}}{t^{(3)}}q^{(3)}\delta q^{(3)} + \frac{s^{(4)}}{t^{(4)}}q^{(4)}\delta q^{(4)} + \frac{s^{(5)}}{t^{(5)}}q^{(5)}\delta q^{(5)} + \frac{s^{(10)}}{t^{(10)}}q^{(10)}\delta q^{(10)}\right)$$

$$= 0.01428\left(\frac{s^{(3)}}{t^{(3)}}q_3 + \frac{s^{(4)}}{t^{(4)}}q^{(4)} + \frac{s^{(5)}}{t^{(5)}}q^{(5)} + \frac{s^{(10)}}{t^{(10)}}q^{(10)}\right)\frac{L}{G}\delta T$$

Substituting the web dimensions and the true shear flows previously computed,

$$\delta W^*_{\rm int} = 0.01428\left[\frac{10}{0.02}143.4 + \frac{3}{0.06}143.4 + \frac{\sqrt{10^2+1^2}}{0.02}143.4 + \frac{4}{0.06}(-45.03)\right]\frac{L}{G}\delta T = 2112.4\frac{L}{G}\delta T$$

8.8 Multicell Idealized Box Beams

Example 8.8.2

Calculate the shear flows in the webs of the constant-cross-section idealized beam in the previous example if, instead of pure torsion, the beam is subjected to the shear loads shown in Figure 8.8.6b.

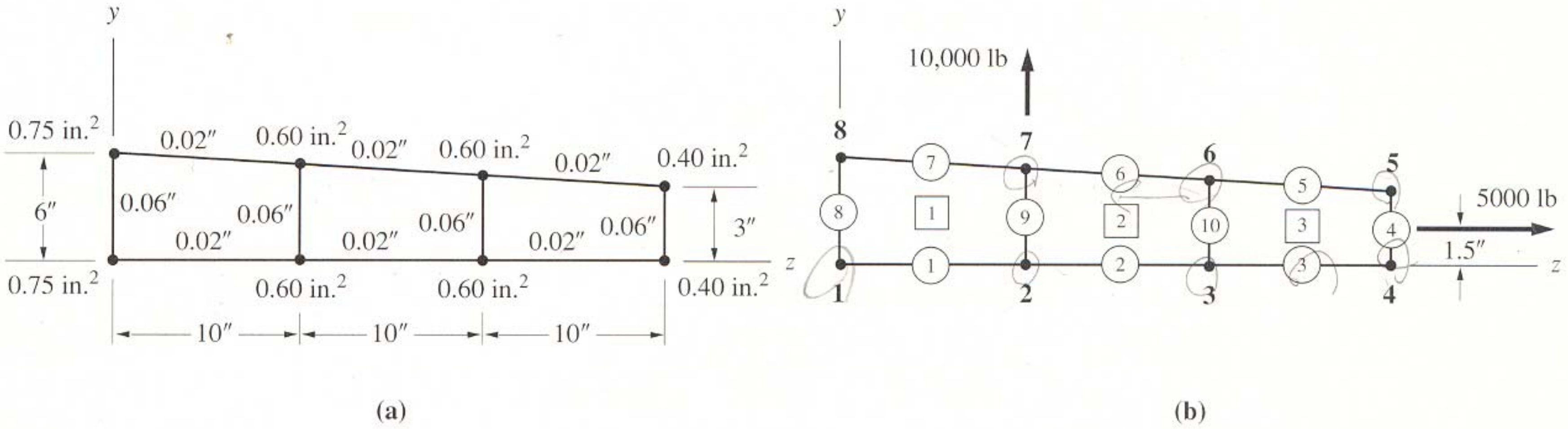


Figure 8.8.6 Three-cell section carrying shear loads.

8.8 Multicell Idealized Box Beams

Example 8.8.2

$$z_G = \frac{\sum_{i=1}^{8} A_i z_i}{\sum_{i=1}^{8} A_i} = \frac{60}{4.7} = 12.76 \text{ in.} \qquad y_G = \frac{\sum_{i=1}^{8} A_i y_i}{\sum_{i=1}^{8} A_i} = \frac{11.10}{4.7} = 2.362 \text{ in.}$$

$$I_{G_y} = \sum_{i=1}^{8} (z_i - z_G)^2 A_i = 554.0 \text{ in.}^4 \qquad I_{G_z} = \sum_{i=1}^{8} (y_i - y_G)^2 A_i = 28.98 \text{ in.}^4 \qquad I_{G_{yz}} = \sum_{i=1}^{8} (y_i - y_G)(z_i - z_G) A_i = -27.70 \text{ in.}^4$$

To compute the flange load gradient at the ith flange, we use Equation 4.8.2, which is

$$P_x'^{(i)} = \frac{1}{I_{G_y} I_{G_z} - I_{G_{yz}}^2} [(I_{G_y} V_y - I_{G_{yz}} V_z)(y_i - y_G) + (I_{G_z} V_z - I_{G_{yz}} V_y)(z_i - z_G)] A_i$$

$$P_x'^{(i)} = [-371.4 y_i - 27.59 z_i + 1230] A_i$$

$$\begin{aligned}
q^{(1)} &= q^{(8)} + 922.0 \\
q^{(2)} &= q^{(8)} + q^{(9)} + 1494 \\
q^{(3)} &= q^{(8)} + q^{(9)} + q^{(10)} + 1900 \\
q^{(4)} &= q^{(8)} + q^{(9)} + q^{(10)} + 2061 \\
q^{(5)} &= q^{(8)} + q^{(9)} + q^{(10)} + 1776 \\
q^{(6)} &= q^{(8)} + q^{(9)} + 1291 \\
q^{(7)} &= q^{(8)} + 749.2
\end{aligned}$$

8.8 Multicell Idealized Box Beams

Example 8.8.2

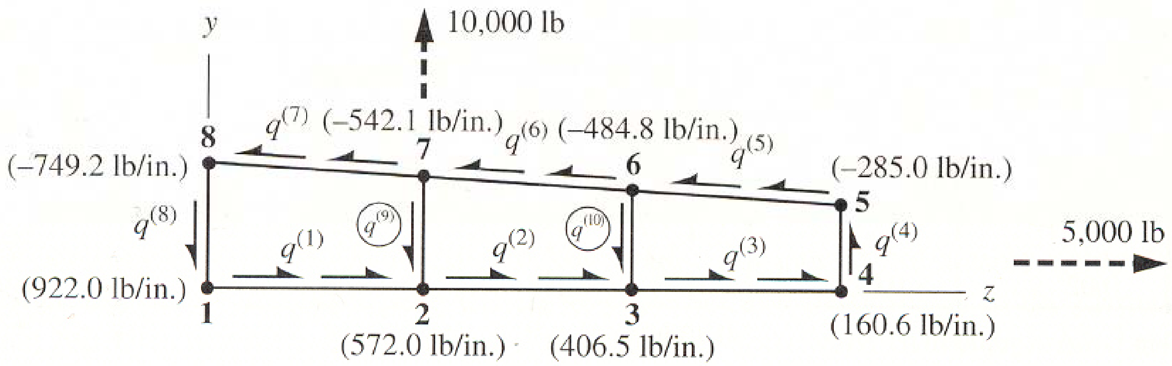


Figure 8.8.7 Flange load gradients (in parentheses) and the assumed directions for the shear flows, with $q^{(9)}$ and $q^{(10)}$ highlighted as the selected redundants.

We now invoke moment equivalence about flange 8 to get $q^{(8)}$ in terms of the redundants $q^{(9)}$ and $q^{(10)}$,

$$2\left(\frac{1}{2}10\times 6\right)q^{(1)}+2\left(\frac{1}{2}10\times 6\right)q^{(2)}+2\left(\frac{1}{2}10\times 6\right)q^{(3)}+2\left(\frac{1}{2}3\times 30\right)q^{(4)}-2\left(\frac{1}{2}5\times 10\right)q^{(9)}-2\left(\frac{1}{2}4\times 20\right)q^{(10)}=$$
$$10,000\times 10+5000\times 4.5$$

we can reduce this to

$$270q^{(8)}+160q^{(9)}+70q^{(10)}+4.445\times 10^6=122,500$$

from which we obtain $$q^{(8)}=-0.5926q^{(9)}-0.2593q^{(10)}-1,193$$

8.8 Multicell Idealized Box Beams

Example 8.8.2

True shear flows

$$q^{(7)} = -0.5926q^{(9)} - 0.2593q^{(10)} - 1193$$
$$q^{(6)} = 0.4074q^{(9)} - 0.2593q^{(10)} + 98.70$$
$$q^{(5)} = 0.4074q^{(9)} + 0.7407q^{(10)} + 583.5$$
$$q^{(4)} = 0.4074q^{(9)} + 0.7407q^{(10)} + 868.6$$
$$q^{(3)} = 0.4074q^{(9)} + 0.7407q^{(10)} + 707.9$$
$$q^{(2)} = 0.4074q^{(9)} - 0.2593q^{(10)} + 301.5$$
$$q^{(1)} = -0.5926q^{(9)} - 0.2593q^{(10)} - 270.6$$

Virtual shear flows

$$\delta q^{(1)} = -0.5926\delta q^{(9)} - 0.2593\delta q^{(10)}$$
$$\delta q^{(2)} = 0.4074\delta q^{(9)} - 0.2593\delta q^{(10)}$$
$$\delta q^{(3)} = 0.4074\delta q^{(9)} + 0.7407\delta q^{(10)}$$
$$\delta q^{(4)} = 0.4074\delta q^{(9)} + 0.7407\delta q^{(10)}$$
$$\delta q^{(5)} = 0.4074\delta q^{(9)} + 0.7407\delta q^{(10)}$$
$$\delta q^{(6)} = 0.4074\delta q^{(9)} - 0.2593\delta q^{(10)}$$
$$\delta q^{(7)} = -0.5926\delta q^{(9)} - 0.2593\delta q^{(10)}$$
$$\delta q^{(8)} = -0.5926\delta q^{(9)} - 0.2593\delta q^{(10)}$$

the internal complementary virtual work expression,

$$\delta W^*_{\text{int}} = \sum_{i=1}^{10} \frac{Ls^{(i)}}{Gt^{(i)}} q^{(i)} \delta q^{(i)}$$

Setting the result equal to zero ($\delta W^*_{\text{ext}} = 0$ because the redundants are internal loads) leads to

$$\left[\left(811.6q^{(9)} + 381.1q^{(10)}\right) + 6.459 \times 10^5\right]\delta q^{(9)} + \left[\left(381.1q^{(9)} + 785.6q^{(10)}\right) + 5.834 \times 10^5\right]\delta q^{(10)} = 0$$

8.8 Multicell Idealized Box Beams

Example 8.8.2

By the usual argument, this yields the following system of two equations:

$$811.6q^{(9)} + 381.1q^{(10)} = -645,900$$
$$381.1q^{(9)} + 785.6q^{(10)} = -583,400$$

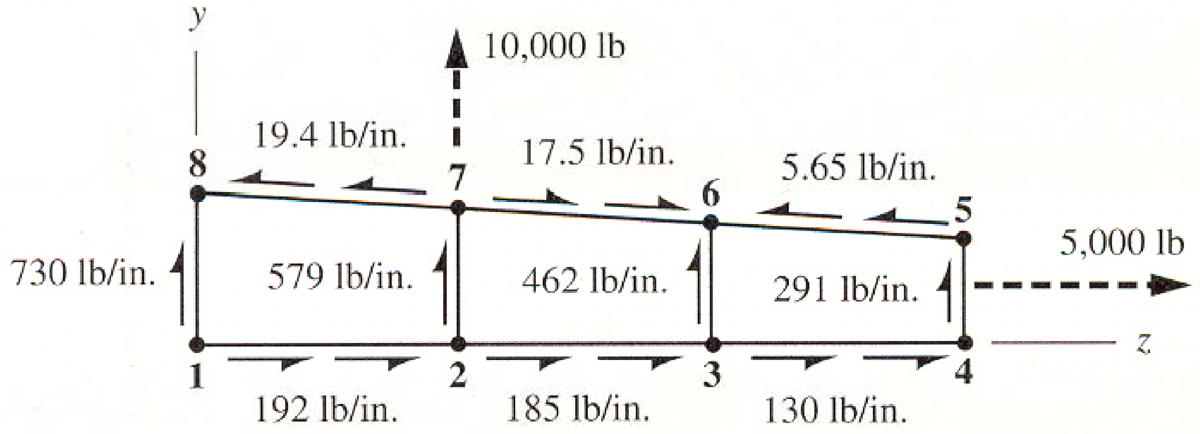


Figure 8.8.8 The shear flows, which are statically equivalent to the combination of an upward-directed 10,000 lb shear force through web 7–2 and a rightward-directed shear force whose line of action bisects web 4–5.

8.8 Multicell Idealized Box Beams

Example 8.8.3

Calculate the angle of twist per unit length for the previous example if G = 4 * 10^6 lb/in.^{2}

$$\delta q_T = \frac{\delta T}{2 \times 55} = \frac{\delta T}{110}$$

Referring to Figures 8.8.6 and 8.8.8, the internal complementary virtual work is

$$\delta W^*_{\text{int}} = \frac{L}{G}\left(\frac{s^{(1)}}{t^{(1)}} q^{(1)} \delta q^{(1)} + \frac{s^{(9)}}{t^{(9)}} q^{(9)} \delta q^{(9)} + \frac{s^{(7)}}{t^{(7)}} q^{(7)} \delta q^{(7)} + \frac{s^{(8)}}{t^{(8)}} q^{(8)} \delta q^{(8)}\right)$$

$$= \frac{1}{110}\left[\frac{10}{0.02} 192 + \frac{5}{0.06} 579 + \frac{\sqrt{10^2 + 1^2}}{0.02} 19.4 + \frac{6}{0.06}(-730)\right]\frac{L}{G}\delta T$$

or

$$\delta W^*_{\text{int}} = 736.3 \frac{L}{G} \delta T$$

Noting that $\delta W^*_{\text{ext}} = \theta \delta T$ and setting $\delta W^*_{\text{ext}} = \delta W^*_{\text{int}}$, we find

$$\frac{\theta}{L} = \frac{736.3}{G} = 0.01055 \ \frac{\text{degrees}}{\text{in.}}$$

8.8 Multicell Idealized Box Beams

Example 8.8.4

use the principle of complementary virtual work to calculate the maximum shear flow in the tapered, redundant cantilevered box beam illustrated in Figure 8.8.9. Loads in both transverse directions are applied to the free end, as shown. Figure 8.8.10 depicts the flange and web numbering and thickness information. Also, E = 10 * 10^6 psi and G = 4* 10^6 psi $(\nu = 0.25)$.

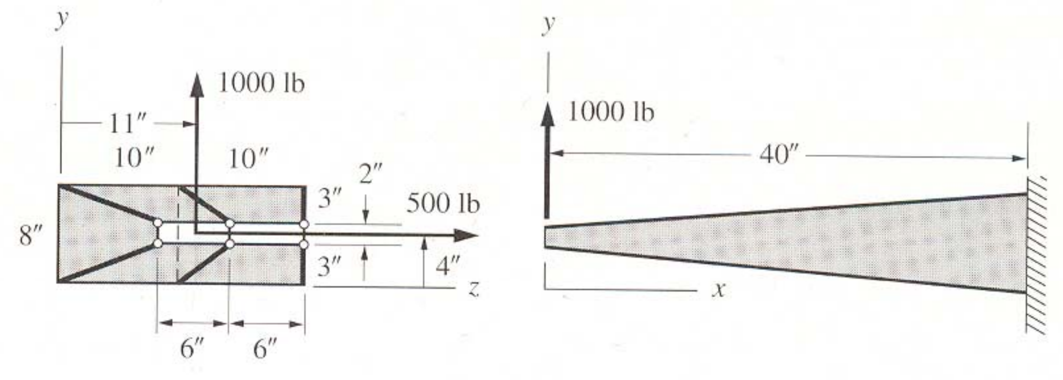


Figure 8.8.9 Tapered box beam with loads applied to the free end at $x=0$.

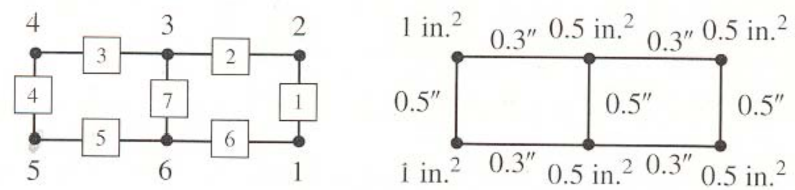


Figure 8.8.10 Flange and web numbering and, on the right, the flange areas and web thicknesses.

8.8 Multicell Idealized Box Beams

Example 8.8.4

$$z_G(40) = \frac{\sum_{i=1}^{6} A_i z_i(40)}{\sum_{i=1}^{6} A_i} = \frac{30}{4} = 7.5 \text{ in.} \qquad y_G(40) = \frac{\sum_{i=1}^{6} A_i y_i(40)}{\sum_{i=1}^{6} A_i} = \frac{16}{4} = 4.0 \text{ in.}$$

$$I_{G_y}(40) = \sum_{i=1}^{6} A_i \left[z_i(40) - z_G(40)\right]^2 = 275 \text{ in.}^4$$

$$I_{G_z}(40) = \sum_{i=1}^{6} A_i \left[y_i(40) - y_G(40)\right]^2 = 64.0 \text{ in.}^4$$

$$I_{G_{yz}}(40) = \sum_{i=1}^{6} A_i \left[y_i(40) - y_G(40)\right]\left[z_i(40) - z_G(40)\right] = 0$$

We then use Equation 4.8.1 to compute flange loads at $x = 40$ in.; that is,

$$P_x^{(i)}(40) = \left\{ \frac{A_i}{I_{G_y} I_{G_z} - I_{G_{yz}}^2} \left[-\left(M_z I_{G_y} + M_y I_{G_{yz}}\right)(y_i - y_G) + \left(M_y I_{G_z} + M_z I_{G_{yz}}\right)(z_i - z_G) \right] \right\}_{x=40''}$$

8.8 Multicell Idealized Box Beams

Example 8.8.4

Therefore, the average flange load gradients are

$$\overline{P_x'^{(1)}} = \frac{P_x^{(1)}(40) - P_x^{(1)}(0)}{40} = \frac{795.4 - 0}{40} = 19.89 \text{ lb/in.}$$

$$\overline{P_x'^{(2)}} = \frac{P_x^{(2)}(40) - P_x^{(2)}(0)}{40} = \frac{-1704 - 0}{40} = -42.61 \text{ lb/in.}$$

$$\overline{P_x'^{(3)}} = \frac{P_x^{(3)}(40) - P_x^{(3)}(0)}{40} = \frac{-1341 - 0}{40} = -33.52 \text{ lb/in.}$$

$$\overline{P_x'^{(4)}} = \frac{P_x^{(4)}(40) - P_x^{(4)}(0)}{40} = \frac{-1954 - 0}{40} = -48.86 \text{ lb/in.}$$

$$\overline{P_x'^{(5)}} = \frac{P_x^{(5)}(40) - P_x^{(5)}(0)}{40} = \frac{3046 - 0}{40} = 76.14 \text{ lb/in.}$$

$$\overline{P_x'^{(6)}} = \frac{P_x^{(6)}(40) - P_x^{(6)}(0)}{40} = \frac{1159 - 0}{40} = 28.98 \text{ lb/in.}$$

$$\bar{q}^{(2)} = \bar{q}^{(1)} + \overline{P_x'^{(2)}} = \bar{q}^{(1)} - 42.61$$

$$\bar{q}^{(3)} = \bar{q}^{(2)} - \bar{q}^{(7)} + \overline{P_x'^{(3)}} = \left(\bar{q}^{(1)} - 42.61\right) - \bar{q}^{(7)} - 33.52 = \bar{q}^{(1)} - \bar{q}^{(7)} - 76.14$$

$$\bar{q}^{(4)} = \bar{q}^{(3)} + \overline{P_x'^{(4)}} = \left(\bar{q}^{(1)} - \bar{q}^{(7)} - 76.14\right) - 48.86 = \bar{q}^{(1)} - \bar{q}^{(7)} - 125$$

$$\bar{q}^{(5)} = \bar{q}^{(4)} + \overline{P_x'^{(5)}} = \left(\bar{q}^{(1)} - \bar{q}^{(7)} - 125\right) + 76.14 = \bar{q}^{(1)} - \bar{q}^{(7)} - 48.86$$

$$\bar{q}^{(6)} = \bar{q}^{(5)} + \bar{q}^{(7)} + \overline{P_x'^{(6)}} = \left(\bar{q}^{(1)} - \bar{q}^{(7)} - 48.86\right) + 28.98 = \bar{q}^{(1)} - \bar{q}^{(7)} - 19.89$$

8.8 Multicell Idealized Box Beams

Example 8.8.4

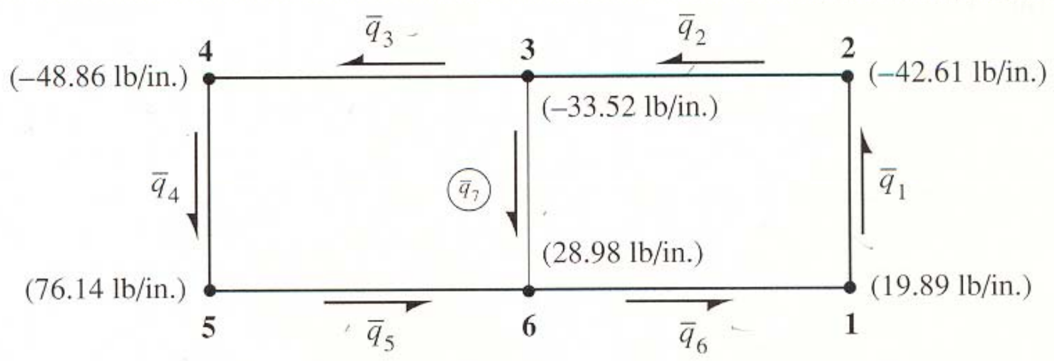


Figure 8.8.11 Computed average flange load gradients and assumed directions of the average shear flows, with $\bar{q}^{(7)}$ highlighted as the chosen redundant shear flow.

moment equivalence at the free end $(x = 0)$ of the beam. Thus, summing moments about flange 5 we get

$$\left[q^{(1)}(0) \times 2\right] \times 12 + \left[q^{(2)}(0) \times 6\right] \times 2 + \left[q^{(3)}(0) \times 6\right] \times 2 - \left[q^{(7)}(0) \times 2\right] \times 6 = 1000 \times 3 - 500 \times 1$$

The shear flows at $x = 0$ are related to the average shear flows by Equation 2.5.4,

$$q^{(i)}(0) = \bar{q}^{(i)} \frac{h^{(i)}(40)}{h^{(i)}(0)} \qquad i = 1, \cdots, 7$$

8.8 Multicell Idealized Box Beams

Example 8.8.4

where $h^{(i)}(x)$ is the width of panel i at station x. We thus have

$$q^{(1)}(0) = \bar{q}^{(1)}\frac{8}{2} = 4\bar{q}^{(1)}$$

$$q^{(2)}(0) = \bar{q}^{(2)}\frac{10}{6} = 1.667\bar{q}^{(2)}$$

$$q^{(3)}(0) = \bar{q}^{(3)}\frac{10}{6} = 1.667\bar{q}^{(3)}$$

$$q^{(4)}(0) = \bar{q}^{(7)}\frac{8}{2} = 4\bar{q}^{(7)}$$

we obtain the relationship between the average shear flows,

$$136.0\bar{q}^{(1)} - 68.0\bar{q}^{(7)} - 2375 = 2500$$

which yields $\bar{q}^{(1)}$ in terms of $\bar{q}^{(7)}$, as follows:

$$\bar{q}^{(1)} = 0.5\bar{q}^{(7)} + 35.85$$

$$\bar{q}^{(2)} = 0.5\bar{q}^{(7)} - 6.768$$

$$\bar{q}^{(3)} = -0.5\bar{q}^{(7)} - 40.29$$

$$\bar{q}^{(4)} = -0.5\bar{q}^{(7)} - 89.15$$

$$\bar{q}^{(5)} = -0.5\bar{q}^{(7)} - 13.02$$

$$\bar{q}^{(6)} = 0.5\bar{q}^{(7)} + 15.96$$

8.8 Multicell Idealized Box Beams

Example 8.8.4

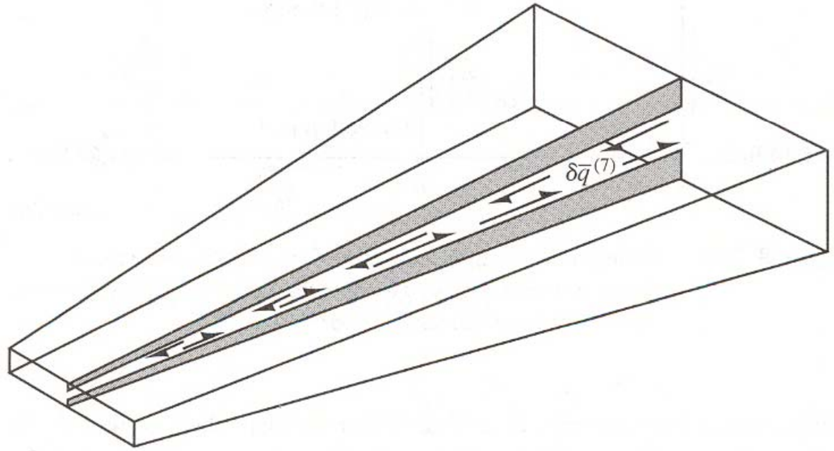


Figure 8.8.12 Virtual shear flow applied to the longitudinal cut in the redundant center spar.

The virtual average shear flows resulting from $\delta\bar{q}^{(7)}$

$$\delta\bar{q}^{(1)} = 0.5\delta\bar{q}^{(7)}$$
$$\delta\bar{q}^{(2)} = 0.5\delta\bar{q}^{(7)}$$
$$\delta\bar{q}^{(3)} = -0.5\delta\bar{q}^{(7)}$$
$$\delta\bar{q}^{(4)} = -0.5\delta\bar{q}^{(7)}$$
$$\delta\bar{q}^{(5)} = -0.5\delta\bar{q}^{(7)}$$
$$\delta\bar{q}^{(6)} = 0.5\delta\bar{q}^{(7)}$$

We are now in a position to calculate the complementary virtual work of each panel,

$$\delta W_{\text{int}}^{*(i)} = \frac{k^{(i)}A^{(i)}}{G^{(i)}t^{(i)}}\bar{q}^{(i)}\delta\bar{q}^{(i)}$$

$$k^{(i)} = 1 + \frac{2}{3\left(1+\nu^{(i)}\right)}\left(\cot^2\alpha^{(i)} - \cot\alpha^{(i)}\cot\gamma^{(i)} + \cot^2\gamma^{(i)}\right)$$

8.8 Multicell Idealized Box Beams

Example 8.8.4

From the geometry of the structure as presented in Figure 8.8.9, we can prepare Table 8.8.1.

Table 8.8.1 Shear panel geometry for the beam of Figure 8.8.9.

Panel	A (in.^{2})	α (degrees)	γ (degrees)	k
1	200.0	85.71	85.71	1.0030
2	320.9	90.00	84.30	1.0053
3	320.9	95.70	78.72	1.037
4	204.0	85.79	85.79	1.0029
5	320.9	78.72	95.70	1.0371
6	320.9	95.70	90.00	1.0053
7	201.0	85.73	85.73	1.0030

we find the total complementary internal virtual work of the beam,

$$\delta W^*_{\text{int}} = \overbrace{0.0005015(0.5\bar{q}^{(7)} + 35.85)\delta\bar{q}^{(7)}}^{\text{panel 1}} + \overbrace{0.001344(0.5\bar{q}^{(7)} - 6.768)\delta\bar{q}^{(7)}}^{\text{panel 2}} + \overbrace{(-0.001387)(-0.5\bar{q}^{(7)} - 40.29)\delta\bar{q}^{(7)}}^{\text{panel 3}}$$
$$+ \overbrace{(-0.0005114)(-0.5\bar{q}^{(7)} - 89.15)\delta\bar{q}^{(7)}}^{\text{panel 4}} + \overbrace{(-0.001387)(-0.5\bar{q}^{(7)} - 13.02)\delta\bar{q}^{(7)}}^{\text{panel 5}} + \overbrace{0.001344(0.5\bar{q}^{(7)} + 15.96)\delta\bar{q}^{(7)}}^{\text{panel 6}}$$
$$+ \overbrace{0.001008\bar{q}^{(7)}\delta\bar{q}^{(7)}}^{\text{panel 7}}$$

8.8 Multicell Idealized Box Beams

Example 8.8.4

Therefore,

$$\delta W^*_{\text{int}} = (0.004245\bar{q}^{(7)} + 0.1498)\,\delta\bar{q}^{(7)}$$

Also, since $\delta\bar{q}^{(7)}$ is an internal force quantity, then $\delta W^*_{\text{ext}} = 0$.

the principle of complementary virtual work that the redundant shear flow is

$$\bar{q}^{(7)} = -35.30 \text{ lb/in.}$$

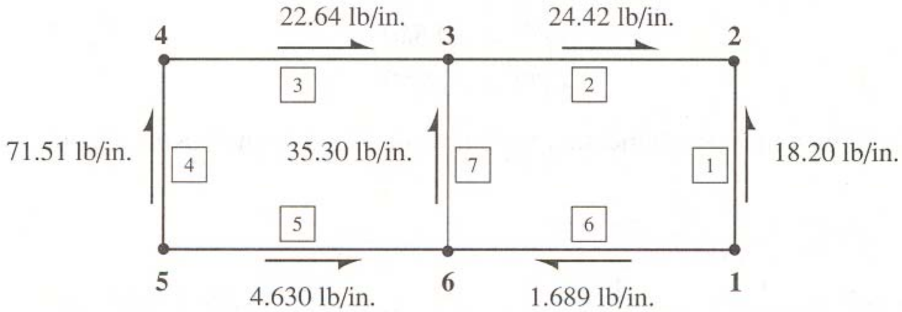


Figure 8.8.13 Average shear flows in the panels, displayed, for simplicity, on a generic cross section of the beam.

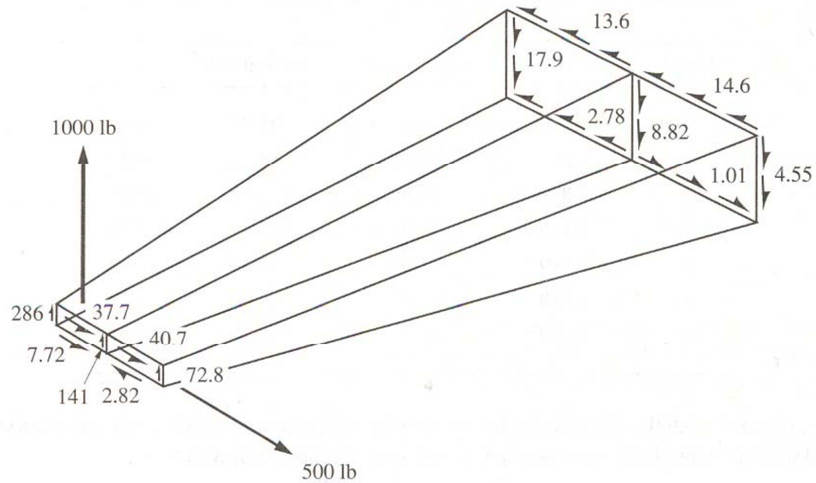


Figure 8.8.14 Shear flows (lb/in.) at each end of the box beam, in response to the loads at the left end.

8.8 Multicell Idealized Box Beams

Example 8.8.5

Calculate the angle of twist θ_x at the free end of the tapered beam in the previous example.

Moment equivalence about flange 5 at the free end of the beam requires that

$$\left[\delta q^{(1)}(0) \times 2\right] \times 12 + \left[\delta q^{(2)}(0) \times 6\right] \times 2 + \left[\delta q^{(3)}(0) \times 6\right] \times 2 = \delta T$$

Using Equation 2.5.4 to relate each of the shear flows in this equation to the average shear flow $\delta \bar{q}_T$ yields

$$[4\delta \bar{q}_T \times 2] \times 12 + \left[\frac{5}{3}\delta \bar{q}_T \times 6\right] \times 2 + \left[\frac{5}{3}\delta \bar{q}_T \times 6\right] \times 2 = \delta T$$

From this we obtain

$$\delta \bar{q}_T = \frac{\delta T}{136}$$

8.8 Multicell Idealized Box Beams

Example 8.8.5

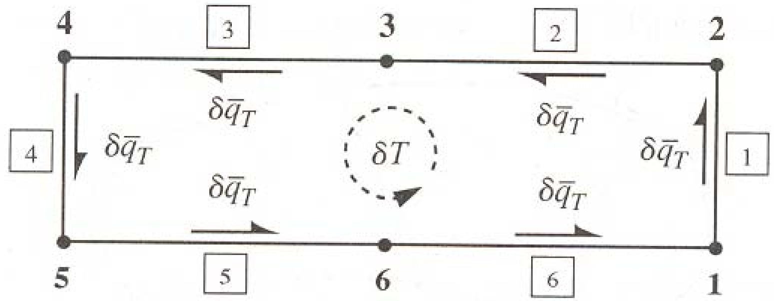


Figure 8.8.15 Generic section of the beam in Figure 8.8.9, without the center web, carrying pure virtual torsion.

the internal complementary virtual work corresponding to the virtual torque is

$$\delta W^*_{\text{int}} = \sum_{i=1}^{6} \frac{k^{(i)} A^{(i)}}{G^{(i)} t^{(i)}} \bar{q}^{(i)} \delta\bar{q}^{(i)} = \frac{\delta\bar{q}_T}{G} \sum_{i=1}^{6} \frac{k^{(i)} A^{(i)}}{t^{(i)}} \bar{q}^{(i)}$$

Substituting Equation a, along with panel data and the true shear flows from Example 8.8.4,

$$\delta W^{*(i)}_{\text{int}} = -0.001287 \delta T$$

Since $\delta W^*_{ext} = \theta_x \delta T$, the principle of complementary virtual work yields the following result for the angle of twist:

$$\theta_x = -0.001287 \text{ radians} = -0.07373 \text{ degrees}$$

The minus sign means that the rotation is clockwise looking inboard, towards the wall.

8.8 Multicell Idealized Box Beams

Consider the two-cell section illustrated in Figure 8.8.16

$$\delta W_{\text{int}}^{*(i)} = \frac{Ls^{(i)}}{Gt^{(i)}} q^{(i)} \delta q^{(i)} \qquad i = 1, 2, 3$$

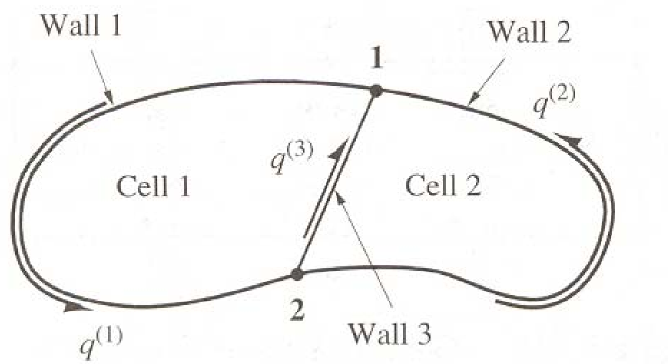


Figure 8.8.16 Two-cell idealized beam section where $q^{(3)}$ is redundant.

Since the redundant is an internal load, the external complementary virtual work is zero; so is the internal virtual work,

$$\sum_{i=1}^{3} \delta W_{\text{int}}^{*(i)} = 0$$

or
$$\frac{L}{G}\left(\frac{s^{(1)}}{t^{(1)}} q^{(1)} \delta q^{(1)} + \frac{s^{(2)}}{t^{(2)}} q^{(2)} \delta q^{(2)} + \frac{s^{(3)}}{t^{(3)}} q^{(3)} \delta q^{(3)}\right) = 0 \qquad [8.8.6]$$

With no virtual shear loads, the virtual flange load gradients vanish.

. For equilibrium,

$$\delta q^{(1)} = \delta q^{(2)} + \delta q^{(3)}$$

8.8 Multicell Idealized Box Beams

Summing the moments about flange 1 we therefore have

$$2A_1\delta q^{(1)} + 2A_2\delta q^{(2)} = 0$$

$$\delta q^{(1)} = \frac{A_2}{A_1 + A_2}\delta q^{(3)} \quad \delta q^{(2)} = -\frac{A_2}{A_1 + A_2}\delta q^{(3)}$$

Substituting these expressions into Equation 8.8.6 yields

$$\frac{A_2}{A_1 + A_2}\frac{s^{(1)}}{t^{(1)}}q^{(1)} - \frac{A_2}{A_1 + A_2}\frac{s^{(2)}}{t^{(2)}}q^{(2)} + \frac{s^{(3)}}{t^{(3)}}q^{(3)} = 0$$

Then, multiplying by $(A_1 + A_2)/2$ and rearranging terms leads to

$$\frac{1}{2A_1}\left(q^{(1)}\frac{s^{(1)}}{t^{(1)}} + q^{(3)}\frac{s^{(3)}}{t^{(3)}}\right) = \frac{1}{2A_2}\left(q^{(1)}\frac{s^{(2)}}{t^{(2)}} - q^{(3)}\frac{s^{(3)}}{t^{(3)}}\right)$$

According to Equation 8.5.3, this can be interpreted as

$$\left.\frac{d\theta_x}{dx}\right)_{\text{cell 1}} = \left.\frac{d\theta_x}{dx}\right)_{\text{cell 2}}$$

If the cross section has n cells, we therefore have

$$\left.\frac{d\theta}{dx}\right)_{\text{cell 1}} = \left.\frac{d\theta}{dx}\right)_{\text{cell 2}} = \cdots = \left.\frac{d\theta}{dx}\right)_{\text{cell } n}$$

8.9 Restraint Effects In Idealized Box Beams

Consider the box beam of uniform rectangular cross section shown in Figure 8.9.1.

As shown in section 8.7, the external complementary virtual work associated with these couples is

$$\delta W^*_{\text{ext}} = h\phi\delta P \qquad [8.9.1]$$

where ϕ is the warp angle. Since warping is to be restrained, $\delta W^*_{\text{ext}} = 0$.

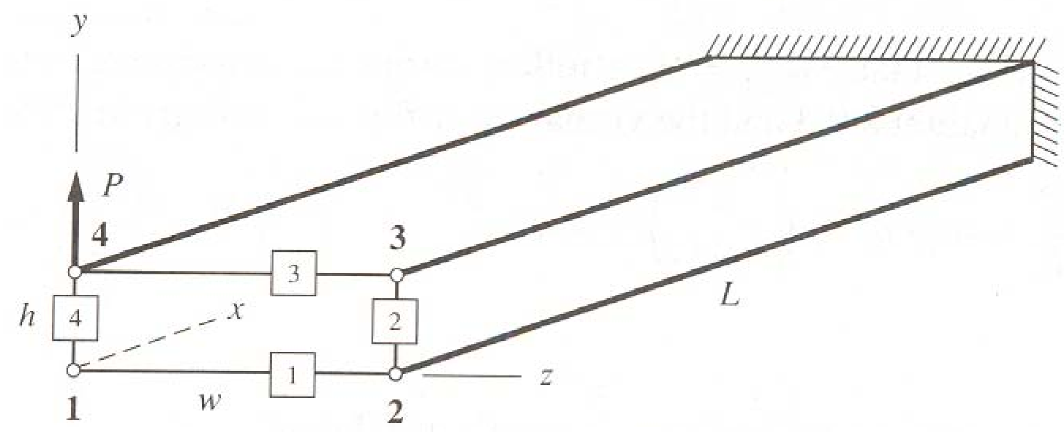


Figure 8.9.1 Idealized box beam of constant cross section. The webs have the same thickness t, and the cross-sectional area of each stringer is A.

8.9 Restraint Effects In Idealized Box Beams

The complementary internal virtual work is that of the panels alone and is given by (cf. Equation 8.2.6)

$$\delta W^*_{\text{int}} = \sum_{\text{panels}} \frac{A}{Gt} q \delta q = \frac{L}{Gt}\left[wq^{(1)}\delta q^{(1)} + hq^{(2)}\delta q^{(2)} + wq^{(3)}\delta q^{(3)} + hq^{(4)}\delta q^{(4)}\right] \qquad [8.9.2]$$

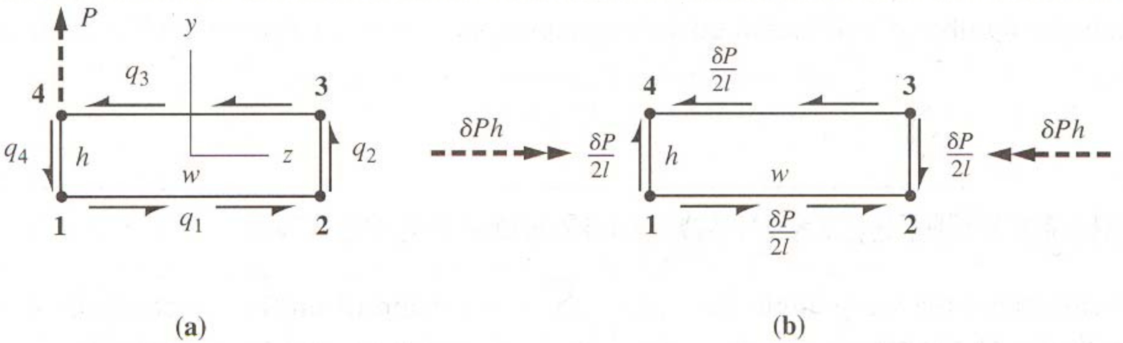


Figure 8.9.2 (a) Assumed directions for the shear flows due to P, with warping restrained. (b) Virtual shear flows shown in section 8.7 to accompany the virtual couples applied in the planes of the vertical webs.

we can impose the restriction that they be statically equivalent to the applied shear load P.

$$\sum F_y: \quad q^{(2)}h - q^{(4)}h = P$$

$$\sum F_z: \quad q^{(1)}w - q^{(3)}w = 0$$

$$\sum M_1: \quad q^{(2)}hw + q^{(3)}wh = 0$$

These imply that

$$q^{(2)} = -q^{(1)} \quad q^{(3)} = q^{(1)} \quad q^{(4)} = -q^{(1)} - \frac{P}{h} \qquad [8.9.3]$$

8.9 Restraint Effects In Idealized Box Beams

Substituting Equation 8.9.3 and the virtual shear flows into Equation 8.9.2, we therefore obtain

$$\frac{L}{Gt}\left[wq^{(1)}\frac{\delta P}{2L}+h\left(-q^{(1)}\right)\left(-\frac{\delta P}{2L}\right)+wq^{(1)}\frac{\delta P}{2L}+h\left(-q^{(1)}-\frac{P}{h}\right)\left(-\frac{\delta P}{2L}\right)\right]=0$$

which simplifies to

$$\frac{\delta P}{2Gt}\left[2\left(w+h\right)q^{(1)}+P\right]=0$$

Solving this for $q^{(1)}$ and substituting the result into Equation 8.9.3, we obtain all of the shear flows accompanying the warping restraint:

$$q^{(1)}=-\frac{P}{2(w+h)} \qquad q^{(2)}=\frac{P}{2(w+h)} \qquad q^{(3)}=-\frac{P}{2(w+h)} \qquad q^{(4)}=-\frac{h+2w}{2(w+h)h}P \qquad \textbf{[8.9.4]}$$

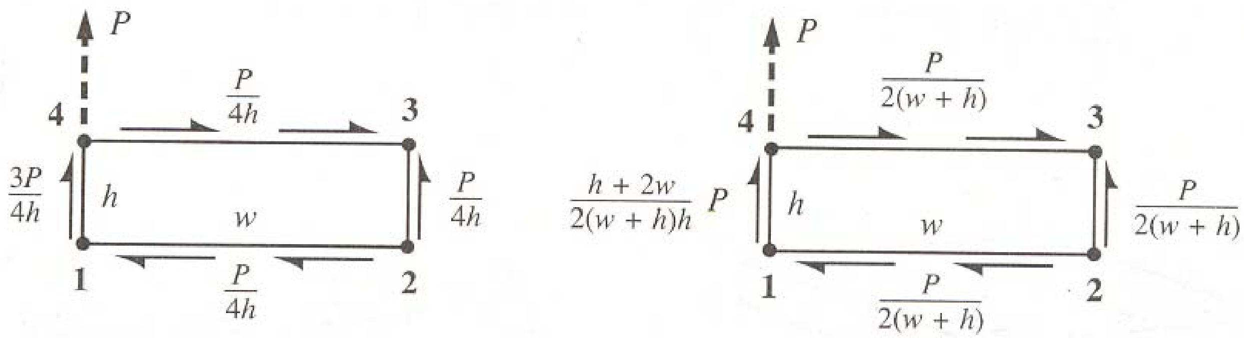


Figure 8.9.3 (a) Shear flows if the section of Figure 8.9.1 is free to warp. (b) Shear flows if warping is prohibited.

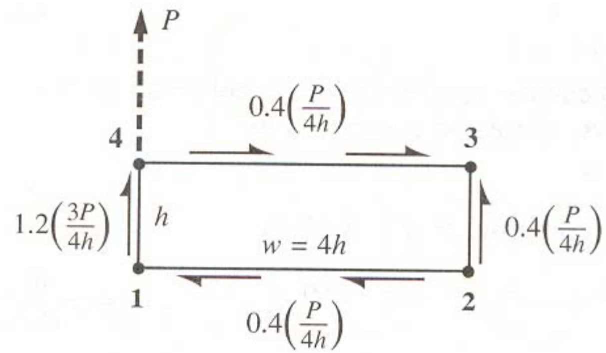


Figure 8.9.4 Shear flows in Figure 8.9.36 if $w = 4h$.

8.9 Restraint Effects In Idealized Box Beams

Example 8.9.1

If warping is restrained, calculate the shear flows in a beam with the cross section illustrated in Figure 8.9.5.

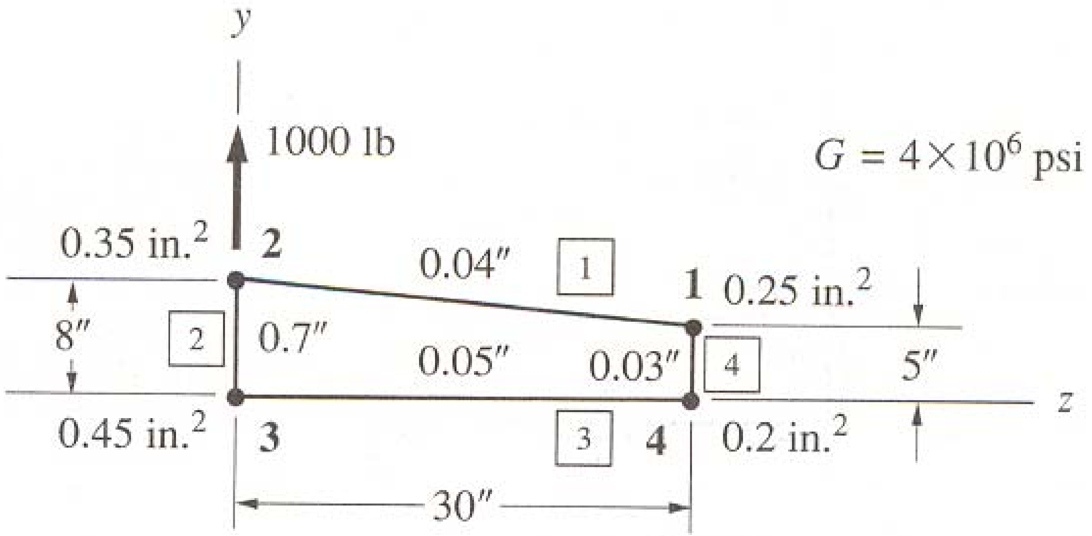


Figure 8.9.5 Load on a section of a box beam in which warping is restrained.

8.9 Restraint Effects In Idealized Box Beams

Example 8.9.1

Let us assume that the directions of the true shear flows are as sketched in Figure 8.9.7. The shear flows must be statically equivalent to the 1000 lb shear load directed upward through web 2-3. Therefore, the following three conditions apply:

$$\sum F_x = 0 \qquad \Rightarrow \; 30(q^{(1)} - q^{(3)}) = 0$$

$$\sum F_y = 1000 \qquad \Rightarrow \; (8-5)q^{(1)} - 8q^{(2)} + 5q^{(4)} = 1000$$

$$\sum M_{\text{flange 2}} = 0 \quad \Rightarrow \; (30q^{(3)}) \times 8 + (5q^{(4)}) \times 30 = 0$$

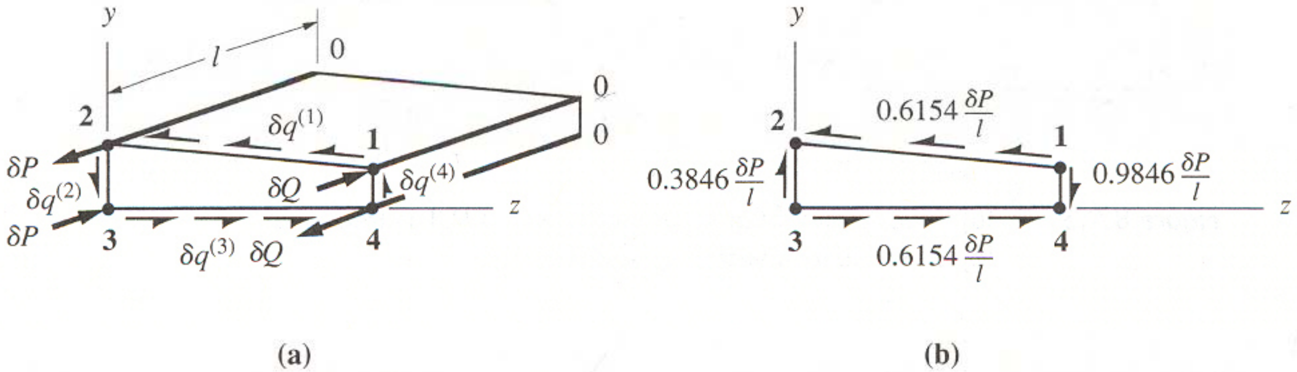


Figure 8.9.6 (a) Virtual couples applied to the vertical webs. (b) The corresponding virtual shear flows, calculated in Example 8.7.1.

Using these to express $q^{(2)}$, $q^{(3)}$, and $q^{(4)}$ in terms of $q^{(1)}$ we have

$$q^{(2)} = -\frac{5}{8}q^{(1)} - 125 \qquad q^{(3)} = q^{(1)} \qquad q^{(4)} = -\frac{8}{5}q^{(1)}$$

8.9 Restraint Effects In Idealized Box Beams

Example 8.9.1

The internal complementary virtual work is just that of the shear panels, which is

$$\delta W^*_{\text{int}} = \sum_{\text{panels}} \frac{A}{Gt} q\,\delta q = \frac{l}{G}\left(\frac{s^{(1)}}{t^{(1)}}q^{(1)}\delta q^{(1)} + \frac{s^{(2)}}{t^{(2)}}q^{(2)}\delta q^{(2)} + \frac{s^{(3)}}{t^{(3)}}q^{(3)}\delta q^{(3)} + \frac{s^{(4)}}{t^{(4)}}q^{(4)}\delta q^{(4)}\right)$$

Thus,

$$\delta W^*_{\text{int}} = \frac{l}{G}\left[\frac{\sqrt{30^2+3^2}}{0.04}q^{(1)}\left(0.6153\frac{\delta P}{l}\right) + \frac{8}{0.07}\left(-\frac{5}{8}q^{(1)}-125\right)\left(-0.3846\frac{\delta P}{l}\right)\right.$$
$$\left. + \frac{30}{0.05}q^{(1)}\left(0.6154\frac{\delta P}{l}\right) + \frac{5}{0.03}\left(-\frac{8}{5}q^{(1)}\right)\left(-0.9846\frac{\delta P}{l}\right)\right]$$

or $$\delta W^*_{\text{int}} = \frac{\delta P}{G}\left(1123q^{(1)} + 5494\right)$$

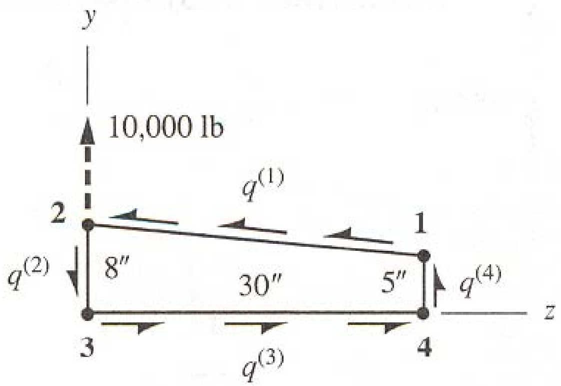


Figure 8.9.7 Assumed directions of the true shear flows.

8.9 Restraint Effects In Idealized Box Beams

Example 8.9.1

All of which are illustrated in Figure 8.9.8b, alongside the shear flows computed in Example 8.7.1, in which warping was unstrained.

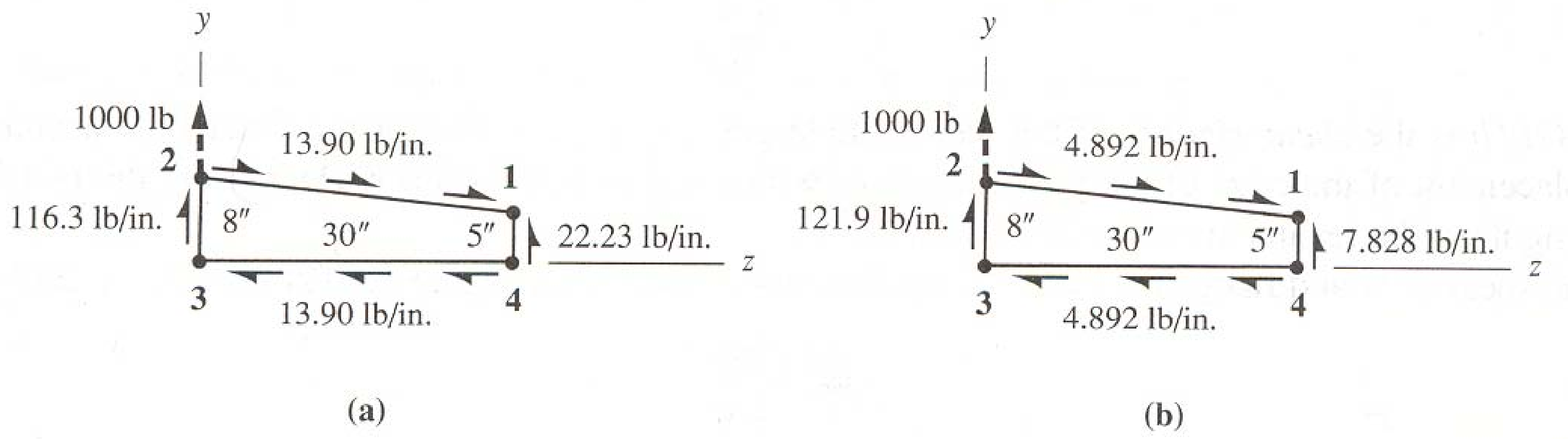


Figure 8.9.8 (a) Shear flows when warping is free to occur (cf. Example 8.7.1). (b) Shear flows when warping is prevented (the present case).

8.10 Shear Lag

Consider Figure 8.10.1, which shows a shear web bonded to two rigid walls and attached to a stiffener by means of which a point load P_0 is applied to the system.

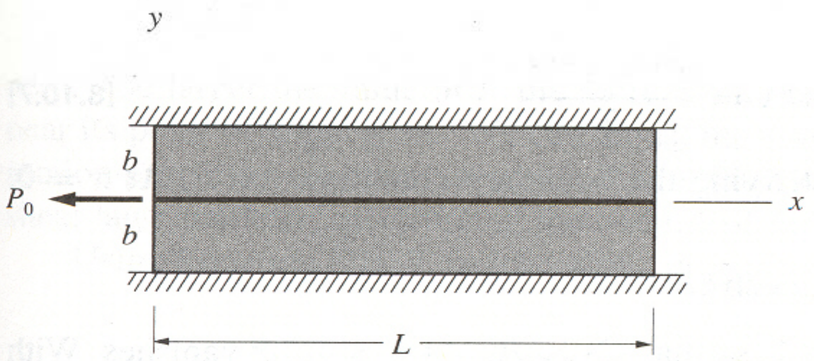


Figure 8.10.1 Load transfer to a shear web by means of a stiffener.

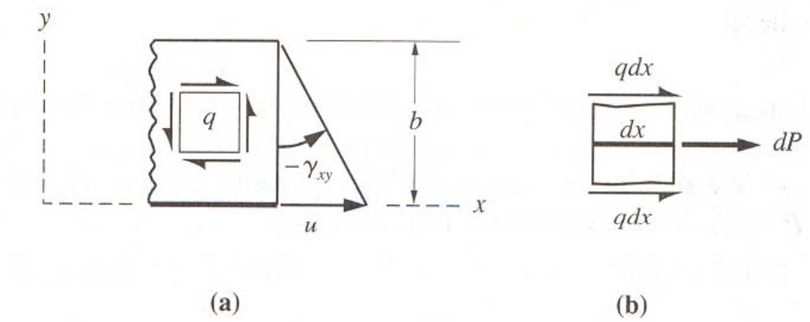


Figure 8.10.2 (a) Relationship between stiffener displacement and web shear in the upper panel. (b) Free-body diagram of a differential length of the stiffener.

the shear strain in the web is

$$\gamma_{xy} = -\frac{u}{b} \qquad [8.10.1]$$

The minus sign reflects the fact that the initial right angle between the vertical edge of the panel and the horizontal stiffener increases.

8.10 Shear Lag

The shear strain is related to the shear stress by Hook's l. $\tau_{xy} = G\gamma_{xy}$. **Since the shear flow equals the shear stress times the panel thickness t, we have**

$$q = -\frac{Gt}{b}u \qquad [8.10.2]$$

For the shear flow and flange load to be in equilibrium, we see from Figure 8.10.2b that $dP + 2q\,dx = 0$, or

$$q = -\frac{1}{2}\frac{dP}{dx} \qquad [8.10.3]$$

the normal strain (cf. Chapter 3) is

$$\frac{du}{dx} = \frac{P}{AE}$$

Differentiating Equation 8.10.2 with respect to x and substituting Equation 8.10.3 and 8.10.4 into the result yields a second-order differential equation involving just P,

$$\frac{1}{2}\frac{d^2P}{dx^2} = \frac{Gt}{b}\frac{P}{AE}$$

which we can write as

$$\frac{d^2P}{dx^2} - k^2P = 0 \qquad \left(k^2 = \frac{2Gt}{AEb}\right)$$

8.10 Shear Lag

The solution of the homogeneous differential equation, 8.10.5,

$$P = C_1 \sinh kx + C_2 \cosh kx$$

where

$$\sinh kx = \frac{e^{kx} - e^{-kx}}{2} \qquad \cosh kx = \frac{e^{kx} + e^{-kx}}{2}$$

We determine the integration constants C_1 and C_2 by satisfying the boundary conditions on P. At $x = 0$, $P = P_0$. From Equation 8.10.6, we have

$$P_0 = C_1 \sinh k(0) + C_2 \cosh k(0)$$

from Equation 8.10.7, $\sinh k\,(0) = 0$ and $\cosh k\,(0) = 1$, so that $C_2 = P_0$. At $x = L$, P vanishes.

$$0 = C_1 \sinh kL + P_0 \cosh kL$$

which means that $C_1 = -P_0 \left(\cosh kL / \sinh kL\right)$.

Therefore, the stiffener load P as a function of x is

$$P = -P_0 \frac{\cosh kL}{\sinh kL} \sinh kx + P_0 \cosh kx = P_0 \left(\frac{\sinh kL \cosh kx - \cosh kL \sinh kx}{\sinh kL} \right)$$

8.10 Shear Lag

Using the definitions of hyperbolic sine and cosine given in Equation 8.10.7,

$$\sinh(kL - kx) = \sinh kL \cosh kx - \cosh kL \sinh kx$$

Therefore, the expression for the stiffener force can be written more compactly as

$$P = P_0 \frac{\sinh k(L-x)}{\sinh kL} = P_0 \frac{\sinh kL(1 - x/L)}{\sinh kL}$$

Substituting this into Equation 8.10.3, we get the shear flow as a function of x, which is

$$q = \frac{P_0 k}{2} \frac{\cosh k(L-x)}{\sinh kL}$$

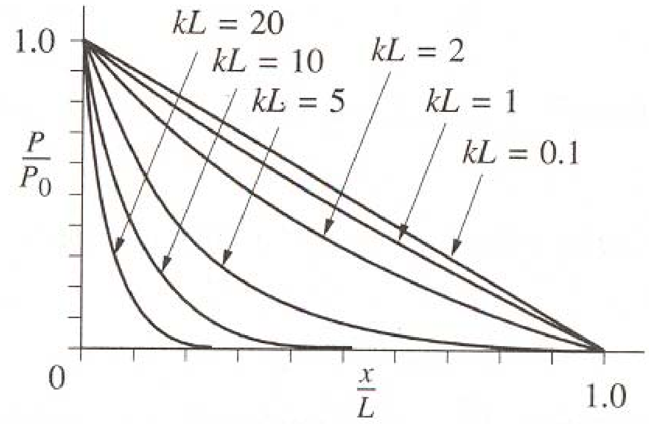


Figure 8.10.3 Stiffener load as a function of position for the stiffened web of Figure 8.10.1, according to Equation 8.10.8.

8.10 Shear Lag

Using Equation 8.10.7, we can write the expression for stiffener force,

$$P = P_0 \frac{e^{kL}e^{-kx} - e^{-kL}e^{kx}}{e^{kL} - e^{-kL}} = P_0 \frac{e^{-kx} - e^{-2kL}e^{kx}}{1 - e^{-2kL}}$$

From this, we see that if x remains finite while kL increases without bound, then, in the limit,

$$P = P_0 e^{-kx} \qquad [8.10.10]$$

It follows from Equation 8.10.3 that

$$q = \frac{P_0 k}{2} e^{-kx}$$

These expressions demonstrate the exponential nature of the decay of stiffener load and shear flow in the vicinity of the applied load.

8.10 Shear Lag

Example 8.10.1

Using the shear lag approach, find the formulas for the stiffener loads and panel shear flows in the plane, stiffened web structure shown in Figure 8.10.4. The top and bottom stiffeners have the same cross-sectional area, and all other properties are uniform throughout. Plot the results for the special case $A_1 = A_2 = 0.5$ in.^{2}, L = 40in., t = 0.1 in., b = 2 in., Q = 1000 lb, and G = 0.4E

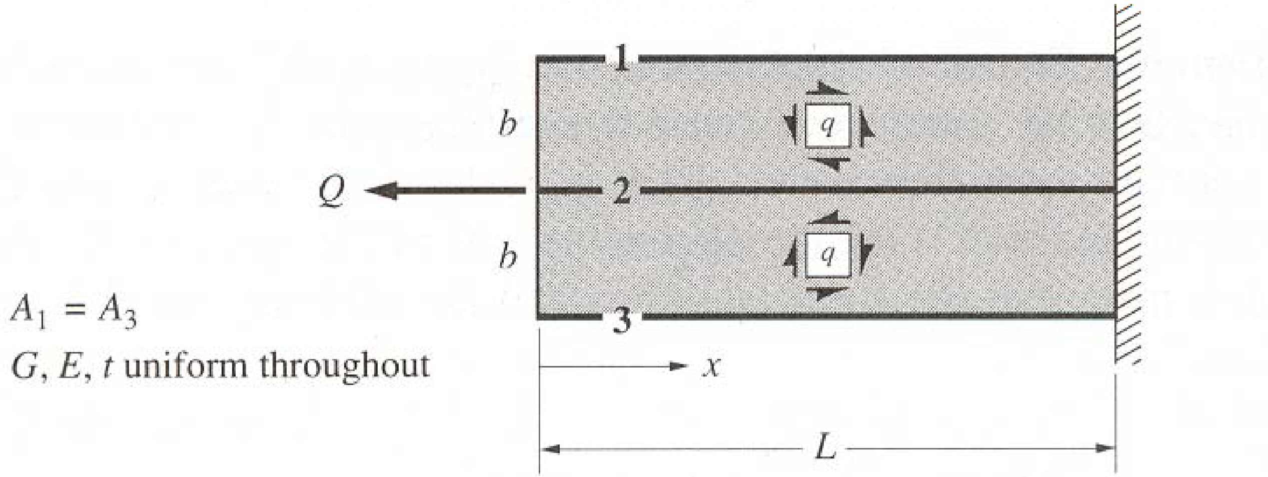


Figure 8.10.4 Stiffened panel with axial load applied to center stiffener.

8.10 Shear Lag

Example 8.10.1

From the free-body diagram in Figure 8.10.5a,

$$2P_1 + P_2 = Q \qquad [a]$$

We see that $q\,dx = dP_1$, or

$$q = \frac{dP_1}{dx} \qquad [b]$$

from Hooke's law,

$$q = \frac{Gt}{b}(u_1 - u_2) \qquad [c]$$

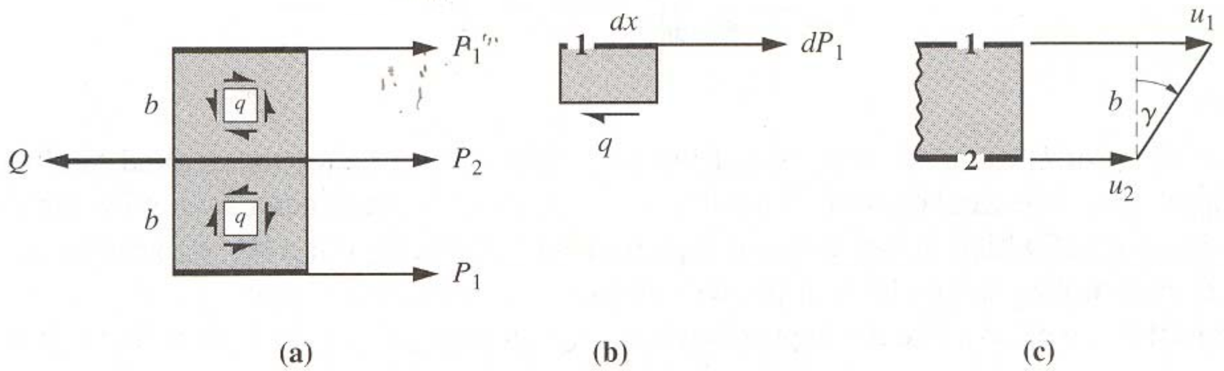


Figure 8.10.5 (a) Free-body diagram of a portion of the stiffened panels, revealing the flange loads. The upper and lower flange loads are equal by symmetry. (b) Free-body diagram of a differential length of the upper stiffener. (c) Relationship of the stiffener displacements to the web shear strain.

8.10 Shear Lag

Example 8.10.1

Differentiating both sides of Equation c with respect to x yields

$$\frac{dq}{dx} = \frac{Gt}{b}\left(\frac{du_1}{dx} - \frac{du_2}{dx}\right)$$

since the normal stress equals the axial load P divided by the cross-sectional area A,

$$\frac{dq}{dx} = \frac{Gt}{b}\left(\frac{P_1}{A_1 E} - \frac{P_2}{A_2 E}\right) = \frac{Gt}{Eb}\left(\frac{P_1}{A_1} - \frac{P_2}{A_2}\right)$$

Substituting Equations a and b into this expression,

$$\frac{d^2 P_1}{dx^2} = \frac{Gt}{Eb}\left[\frac{P_1}{A_1} - \frac{(Q-2P_1)}{A_2}\right] = \frac{Gt}{Eb}\left(\frac{2A_1 + A_2}{A_1 A_2}\right) P_1 - \frac{Gt}{Eb}\frac{Q}{A_2}$$

which can be written more compactly as

$$\frac{d^2 P_1}{dx^2} - k^2 P_1 = -\frac{Gt}{Eb}\frac{Q}{A_2} \qquad \text{where} \qquad k^2 = \frac{Gt}{Eb}\left(\frac{2A_1 + A_2}{A_1 A_2}\right)$$

8.10 Shear Lag

Example 8.10.1

The general solution of the second-order differential equation, Equation g, is

$$P_1 = \overbrace{C_1 \sinh kx + C_2 \cosh kx}^{\text{complementary solution}} + \overbrace{\frac{1}{k^2}\frac{Gt}{Eb}\frac{Q}{A_2}}^{\text{particular solution}}$$

$$P_1 = C_1 \sinh kx + C_2 \cosh kx + \frac{QA_1}{2A_1 + A_2} \qquad \text{[j]}$$

the shear flow is the derivative of P_1 with respect to x, Equation j implies that

$$q = kC_1 \cosh kx + kC_2 \sinh kx \qquad \text{[k]}$$

At $x = 0$, we know that $P_1 = 0$. Setting P_1 and x equal to zero in Equation j, we see that

$$C_2 = -\frac{QA_1}{2A_1 + A_2} \qquad \text{[l]}$$

At $x = L$, $u_1 = u_2 = 0$, so Equation c implies that $q = 0$ at $x = L$. Therefore, from Equations k and l,

$$C_1 = \frac{QA_1}{2A_1 + A_2}\frac{\sinh kL}{\cosh kL}$$

we get the expression for the flange load P_1,

$$P_1 = \frac{QA_1}{2A_1 + A_2}\frac{\sinh kL}{\cosh kL}\sinh kx - \frac{QA_1}{2A_1 + A_2}\cosh kx + \frac{QA_1}{2A_1 + A_2} \qquad \text{[n]}$$

$$= \frac{QA_1}{2A_1 + A_2}\frac{1}{\cosh kL}[(\sinh kL \sinh kx - \cosh kL \cosh kx) + \cosh kL]$$

8.10 Shear Lag

Example 8.10.1

Using the definition of the hyperbolic sine and cosine given in Equation 8.10.7,

$$\cosh kL \cosh kx - \sinh kL \sinh kx = \cosh k(L-x)$$

Therefore, Equation n can be written more compactly as

$$P_1 = \frac{QA_1}{2A_1 + A_2}\left[1 - \frac{\cosh k(L-x)}{\cosh kL}\right] \qquad \text{[o]}$$

Next, we obtain the formula for P_2 by substituting Equation o into Equation a, which leads to

$$P_2 = \frac{2QA_1}{2A_1 + A_2}\left[\frac{A_2}{2A_1} + \frac{\cosh k(L-x)}{\cosh kL}\right]$$

Finally, from Equation b, we obtain the shear flow in the webs of the structure, as follows:

$$q = \frac{QA_1 k}{2A_1 + A_2}\frac{\sinh k(L-x)}{\cosh kL}$$

the normal stress in each of the flanges would be $\sigma = Q/(2A_1 + A_2)$. Accordingly,

$$P_1 = \sigma A_1 = \frac{QA_1}{2A_1 + A_2} \qquad P_2 = \sigma A_2 = \frac{QA_2}{2A_1 + A_2} \qquad q = 0 \qquad \text{elementary solution}$$

8.10 Shear Lag

Example 8.10.1

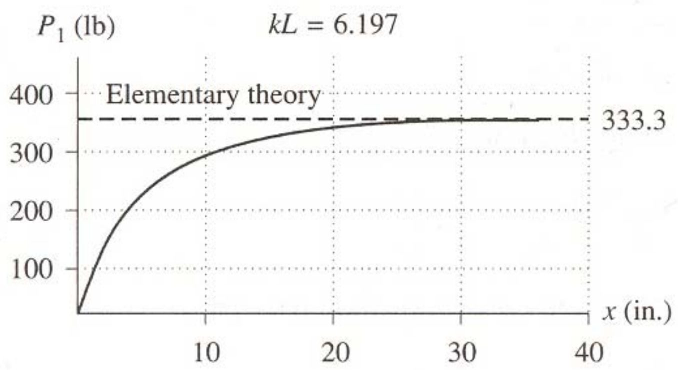


Figure 8.10.6 Axial load distribution in the upper and lower stiffeners of the structure in Figure 8.10.4, for the specified data.

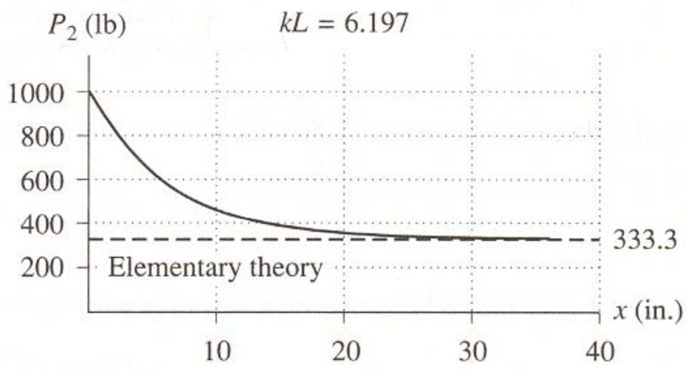


Figure 8.10.7 Axial load distribution in the center stiffener.

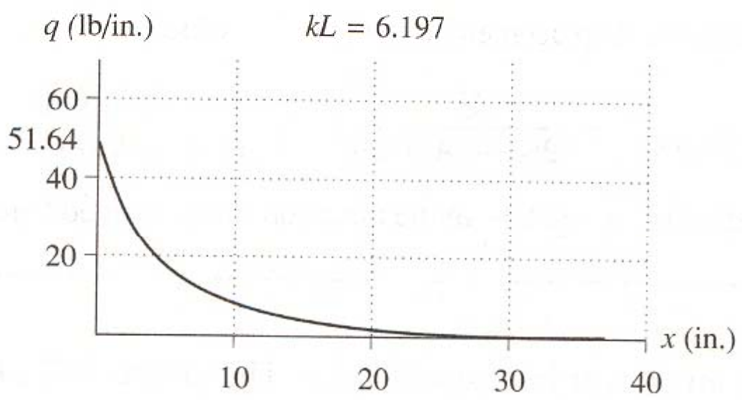


Figure 8.10.8 Shear flow in the panels of the structure in Figure 8.10.12.

8.10 Shear Lag

Example 8.10.2

Using the shear lag approach, obtain an expression for the displacement of the left end of the center stiffener of the previous example.

$$P_2 = \frac{2QA_1}{2A_1 + A_2}\left[\frac{A_2}{2A_1} + \frac{\cosh k(L-x)}{\cosh kL}\right]$$

According to Equation 8.10.4,

$$\frac{du_2}{dx} = \frac{P_2}{A_2 E}$$

Therefore,

$$\frac{du_2}{dx} = \frac{1}{A_2 E}\frac{2QA_1}{2A_1 + A_2}\left[\frac{A_2}{2A_1} + \frac{\cosh k(L-x)}{\cosh kL}\right]$$

Integrating this equation with respect to x,

$$u_2 = \frac{1}{A_2 E}\frac{2QA_1}{2A_1 + A_2}\left[\frac{A_2}{2A_1}x - \frac{1}{k}\frac{\sinh k(L-x)}{\cosh kL}\right] + C$$

8.10 Shear Lag

Example 8.10.2

The constant of integration, C, is found by applying the boundary condition $u_2(L) = 0$.
Setting $x = L$ and $u_2 = 0$ in Equation d, we see that

$$C = -\frac{QL}{(2A_1 + A_2)E}$$

we find that the displacement of any point on the center stiffener is

$$u_2 = -\frac{Q}{(2A_1 + A_2)\,E}\left[L - x + \frac{2A_1}{A_2}\frac{1}{k}\frac{\sinh k\,(L - x)}{\cosh kL}\right]$$

Evaluating this expression at $x = 0$ yields the displacement at the left end, which is

$$u_{2_{\text{left end}}} = -\frac{Q}{(2A_1 + A_2)\,E}\left[L + \frac{2A_1}{A_2}\frac{1}{k}\frac{\sinh kL}{\cosh kL}\right]$$

The minus sign means that the displacement is to the left, in the direction of the applied load Q.

8.10 Shear Lag

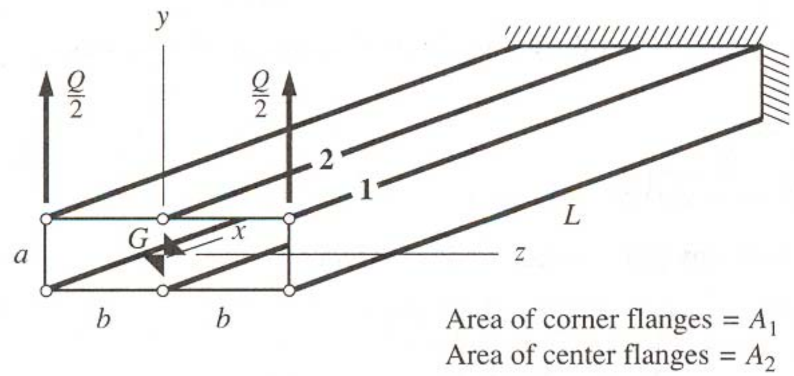


Figure 8.10.9 Idealized, cantilevered box beam of uniform, symmetrical cross section.

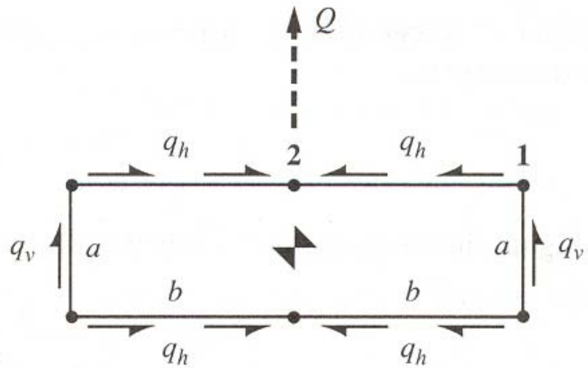


Figure 8.10.10 Shear flows in the cross section of the box beam.

Consider the idealized box beam shown in Figure 8.10.9.

To find the flange loads using beam theory, we use Equation 4.6.8, which in this case reduces to

$$\sigma_x = -\frac{M_z y}{I_z}$$

the area moment of inertia I_z is that of the six concentrated flange areas relative to G,

$$I_z = 2\left[2A_1\left(\frac{a}{2}\right)^2 + A_2\left(\frac{a}{2}\right)^2\right] = \frac{a^2}{2}(2A_1 + A_2)$$

8.10 Shear Lag

The bending moment as a function of spanwise coordinate x is $M_z = Qx$. Therefore, the axial stress in the top stringers, at $y = a/2$, is

$$\sigma_x = -\frac{(Qx)(a/2)}{(a^2/2)(2A_1 + A_2)} = -\frac{Q}{2A_1 + A_2}\frac{x}{a}$$

for stringers 1 and 2 on the top of the beam,

$$P_1 = -\frac{QA_1}{2A_1 + A_2}\frac{x}{a} \qquad P_2 = -\frac{QA_2}{2A_1 + A_2}\frac{x}{a} \qquad \text{[8.10.12]}$$

By symmetry, the shear flows in the vertical webs are both equal to q_v. From Figure 8.10.10 we see that

$$q_v = \frac{Q}{2a}$$

Using Equations 8.10.12 and 8.10.13,

$$q_h = \frac{Q}{2a} + \left(-\frac{QA_1}{2A_1 + A_2}\frac{1}{a}\right)$$

or $$q_h = \frac{Q}{2a}\frac{A_2}{2A_1 + A_2} \qquad \text{[8.10.14]}$$

8.10 Shear Lag

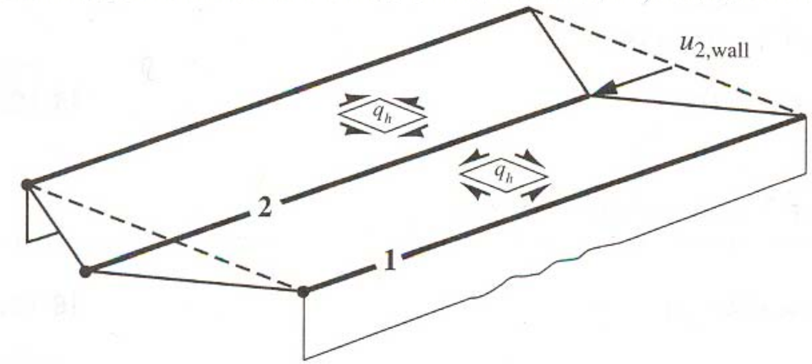


Figure 8.10.11 Free distortion of top panels due to a state of constant shear strain.

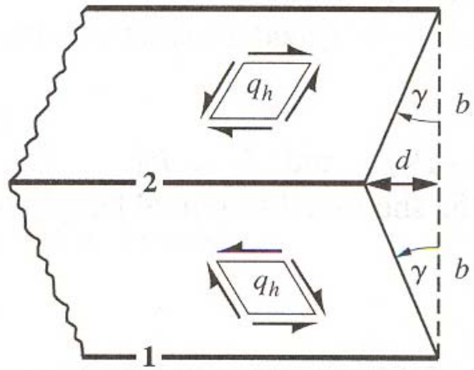


Figure 8.10.12 Relationship between the shear strain γ in the top panels and the displacement of the center stringer at the wall.

The term γ is related to the shear stress by Hooke's law, $\gamma = \dfrac{(q_h/t)}{G}$, which means that

$$d = \frac{q_h b}{Gt} \qquad [8.10.15]$$

8.10 Shear Lag

Consider the top panel of the box beam, as illustrated in Figure 8.10.13.

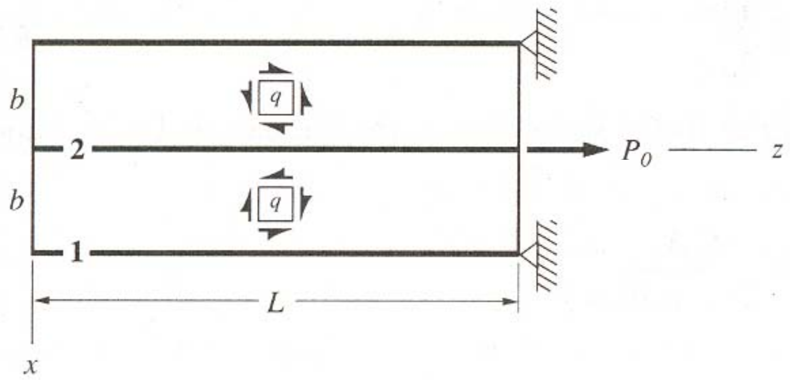


Figure 8.10.13 Top panel of the box beam in Figure 8.10.2.

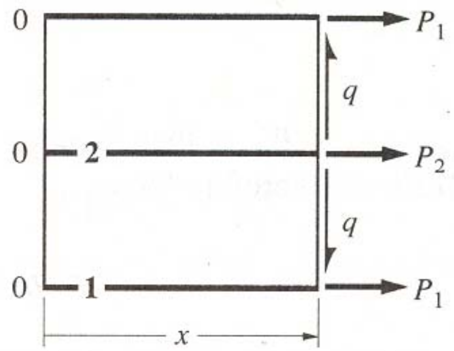


Figure 8.10.14 Free-body diagram of a portion of the top panel.

The relationship between the central stringer load P_1 and the corner stringer load P_2 is

$$2P_1 + P_2 = 0 \qquad \textbf{[8.10.16]}$$

At $x = L$, $P_1 = -P_0/2$ and $P_2 = P_0$.

Figure 8.10.15a shows a differential length of stringer 2, from which we infer

$$\frac{dP_2}{dx} = -2q \qquad \textbf{[8.10.17]}$$

8.10 Shear Lag

Part b of Figure 8.10.15 shows the axial displacements u_1 and u_2 of stringers 1 and 2 at any station x.

Assuming that $u_1 > u_2$, the shear strain is

$$\gamma_{xz} = \frac{(u_1 - u_2)}{b}$$

from Hooke's law,

$$q = \frac{Gt}{b}(u_1 - u_2) \qquad [8.10.18]$$

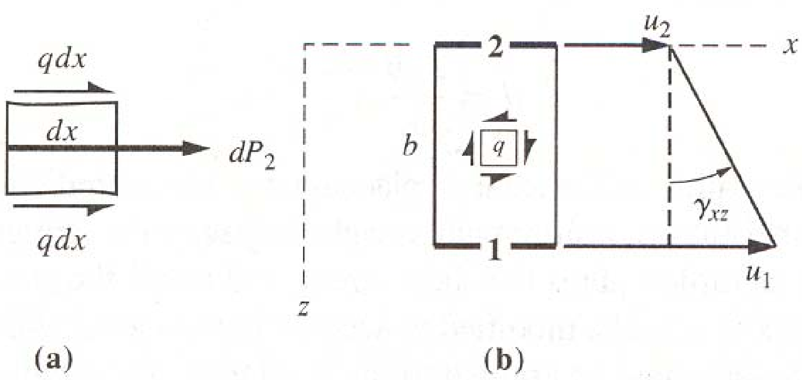


Figure 8.10.15 (a) Free-body diagram of a differential length of the center stringer. (b) Displacements of the center and corner stringers at station x.

8.10 Shear Lag

For stringers 1 and 2 Hooke's law also yields

$$\frac{du_1}{dx} = \frac{P_1}{A_1 E} \qquad \frac{du_2}{dx} = \frac{P_2}{A_2 E} \qquad [8.10.19]$$

Differentiating Equation 8.10.18 with respect to x and then substituting Equation 8.10.19,

$$\frac{dq}{dx} = \frac{Gt}{b}\left(\frac{P_1}{A_1 E} - \frac{P_2}{A_2 E}\right) \qquad [8.10.\ 20]$$

Substituting the shear flow $q = -\frac{1}{2} dP_2/dx$ from Equation 8.10.17 into the left side of this equation, and the stringer load $P_2 = -P_1/2$ from Equation 8.10.16 into the right side, we again get a second-order differential equation, in this case involving just P_2

$$\frac{d^2 P_2}{dx^2} - k^2 P_2 = 0 \qquad [8.10.\ 21a]$$

where

$$k^2 = \frac{Gt}{Eb}\left(\frac{1}{A_1} + \frac{2}{A_2}\right) \qquad [8.10.21b]$$

k is the shear lag parameter.

8.10 Shear Lag

The solution of this homogeneous differential equation is

$$P_2 = C_1 \sinh kx + C_2 \cosh kx \qquad \text{[8.10. 22]}$$

At $x = 0$, the unsupported end of the panel, $P_2 = 0$. From Equation 8.10.22,

$$0 = C_1 \sinh k(0) + C_2 \cosh k(0)$$

At $x = L$, P_2 equals the applied load P_0. Therefore, Equation 8.10.22 with $C_2 = 0$ yields

$$P_0 = C_1 \sinh kL$$

the stringer load P_2 as a function of x is

$$P_2 = P_0 \frac{\sinh kx}{\sinh kL} \qquad \text{[8.10. 23]}$$

With this we can obtain P_1 from Equation 8.10.16,

$$P_1 = -\frac{P_2}{2} = -\frac{P_0}{2} \frac{\sinh kx}{\sinh kL} \qquad \text{[8.10. 24]}$$

8.10 Shear Lag

the shear flow q from Equation 8.10.17,

$$q = -\frac{1}{2}\frac{dP_2}{dx} = -\frac{P_0 k}{2}\frac{\cosh kx}{\sinh kL} \qquad [8.10.\ 25]$$

where we used the fact that $d \sinh kx / dx = k \cosh kx$.

Evaluating Equation 8.10.18 at $x = L$,

$$q(L) = \frac{Gt}{b}\left[u_1(L) - u_2(L)\right]$$

with the aid of Equation 8.10.25,

$$-\frac{P_0 k}{2}\frac{\cosh kL}{\sinh kL} = \frac{Gt}{b}\left[0 - \frac{q_h b}{Gt}\right]$$

which yields the force P_0 required to keep the center stringer attached to the wall, as follows:

$$P_0 = \frac{2q_h}{k}\frac{\sinh kL}{\cosh kL} \qquad [8.10.\ 26]$$

8.10 Shear Lag

Substituting this back into the above expressions for P_2, P_1, and q, we get

$$P_2 = \frac{2q_h}{k}\frac{\sinh kx}{\cosh kL} \quad [8.10.\ 27]$$

$$P_1 = -\frac{q_h}{k}\frac{\sinh kx}{\cosh kL} \quad [8.10.\ 28]$$

$$q = -q_h\frac{\cosh kx}{\cosh kL} \quad [8.10.\ 29]$$

These loads must be superimposed on those obtained from elementary beam theory, Equation 8.10.12 and 8.10.14,

$$P_1 = \overbrace{\left[-\frac{QA_1}{2A_1 + A_2}\frac{x}{a}\right]}^{\text{pure bending}} + \overbrace{\left[-\left(\frac{Q}{2a}\frac{A_2}{2A_1 + A_2}\right)\left(\frac{1}{k}\right)\frac{\sinh kx}{\cosh kL}\right]}^{\text{shear lag}}$$

$$P_2 = \overbrace{\left[-\frac{QA_2}{2A_1 + A_2}\frac{x}{a}\right]}^{\text{pure bending}} + \overbrace{\left[2\left(\frac{Q}{2a}\frac{A_2}{2A_1 + A_2}\right)\left(\frac{1}{k}\right)\frac{\sinh kx}{\cosh kL}\right]}^{\text{shear lag}}$$

$$q = \overbrace{\left[\frac{Q}{2a}\frac{A_1}{2A_1 + A_2}\right]}^{\text{pure bending}} + \overbrace{\left[-\left(\frac{Q}{2a}\frac{A_2}{2A_1 + A_2}\right)\frac{\cosh kx}{\cosh kL}\right]}^{\text{shear lag}}$$

or

$$P_1 = -Q\frac{A_1}{2A_1 + A_2}\left(\frac{x}{a} + \frac{A_2}{A_1}\frac{1}{2ka}\frac{\sinh kx}{\cosh kL}\right) \quad \text{top corner stringers} \quad [8.10.30]$$

$$P_2 = -Q\frac{A_2}{2A_1 + A_2}\left(\frac{x}{a} - \frac{1}{ka}\frac{\sinh kx}{\cosh kL}\right) \quad \text{top center stringer} \quad [8.10.31]$$

$$q = \frac{Q}{2a}\frac{A_2}{2A_1 + A_2}\left(1 - \frac{\cosh kx}{\cosh kL}\right) \quad \text{horizontal webs} \quad [8.10.32]$$

8.10 Shear Lag

Example 8.10.3

Let the following numerical data apply to the box beam in Figure 8.10.9: $G = 0.4E$, $A_1 = A_2 = 1\text{in.}^2$, $t = 0.1\text{in.}$, $L = 40\text{in.}$, $a = 2\text{in.}$, $b = 4\text{in.}$, and $Q = 1000$ lb. Plot the flange loads and web shear flow versus span for the shear lag solution just obtained and compare them with elementary beam theory.

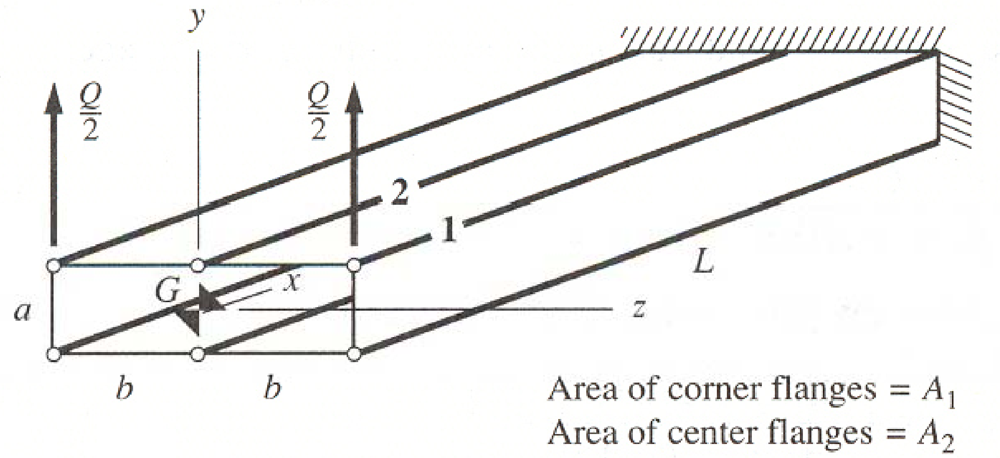


Figure 8.10.9 Idealized, cantilevered box beam of uniform, symmetrical cross section.

8.10 Shear Lag

Example 8.10.3

For the given data, the shear lag parameter, Equation 8.10.21b. is.

$$k^2 = \frac{Gt}{Eb}\left(\frac{1}{A_1} + \frac{2}{A_2}\right) = 0.4 \times 0.025\,(1+2) = 0.03$$

Substituting the numbers into Equation 8.10.30, 8.10.31, and 8.10.32,

$$P_1 = -166.7\,[x + 0.005657\sinh(0.1732x)]$$

$$P_2 = -166.7\,[x - 0.01131\sinh(0.1732x)]$$

$$q = 83.33\,[1 - 0.001960\cosh(0.1732x)]$$

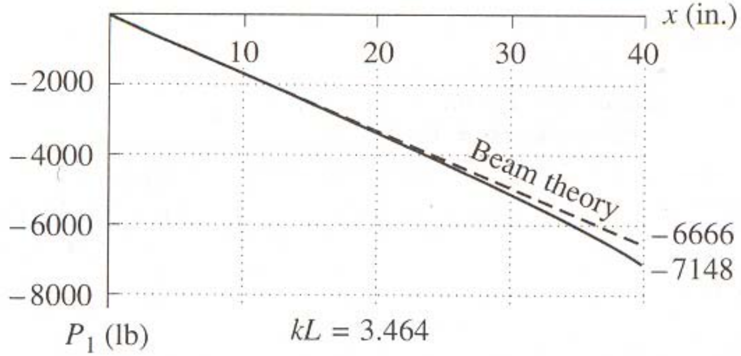


Figure 8.10.16 Corner stringer load distribution.

These are plotted in Figures 8.10.16, 17, and 18.

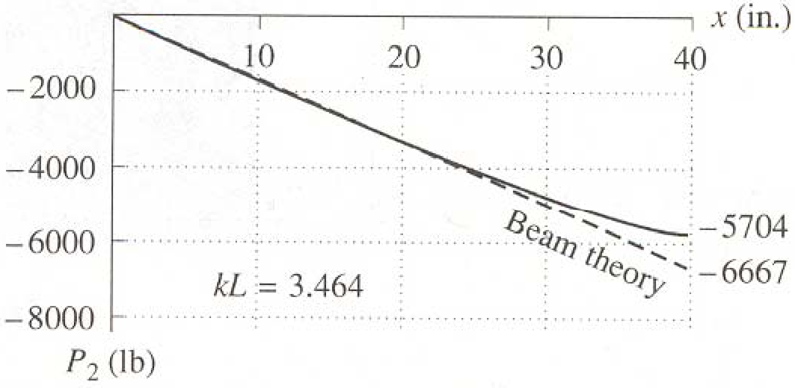


Figure 8.10.17 Center stringer load distribution.

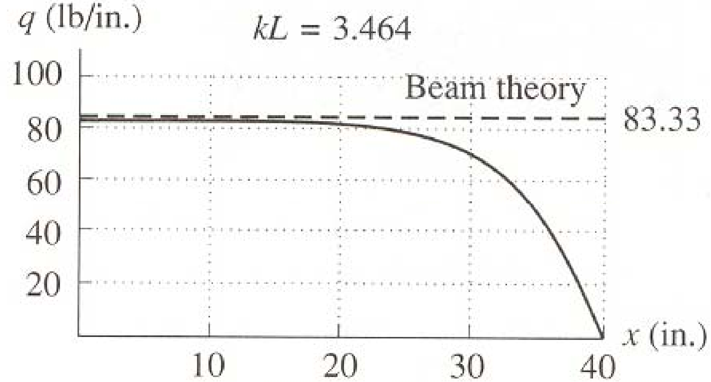


Figure 8.10.18 Shear flow in the horizontal webs.