Aircraft Structural Analysis

Chapter 12

Structural Stability

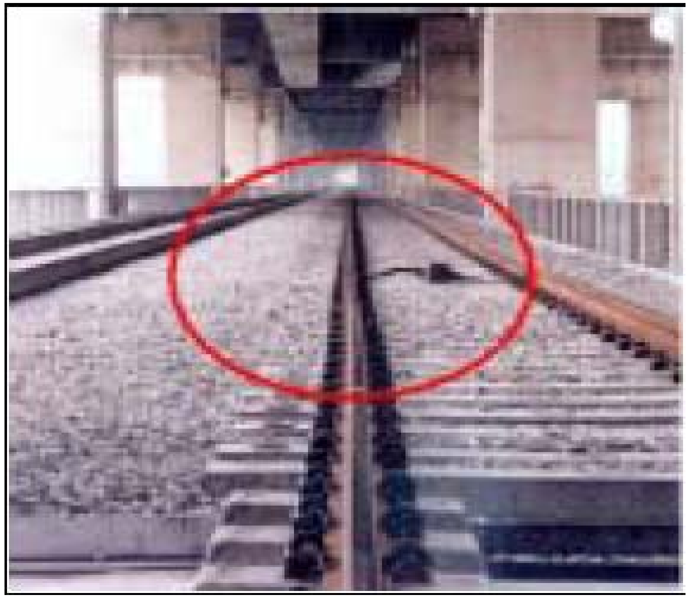
12.1 Introduction

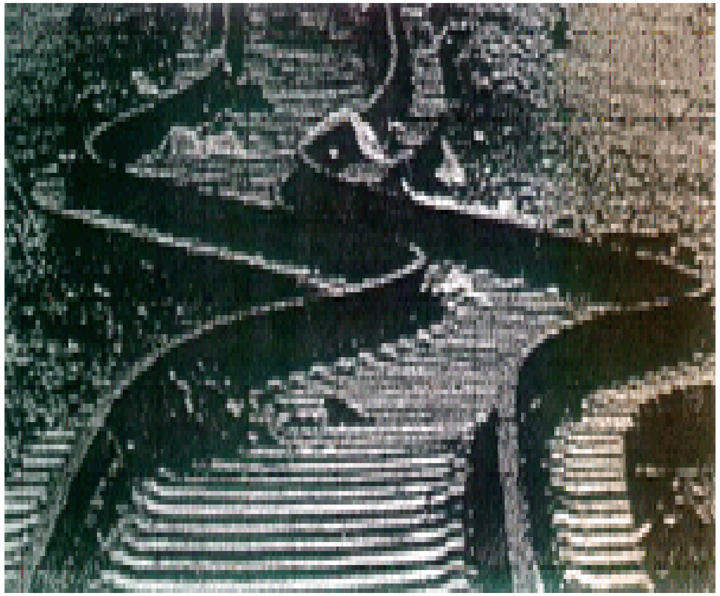
- Buckling is the finite bowing, warping, wrinkle, or twisting deformation that accompanies the development of excessive compressive stresses throughout a structure or some portion thereof. Primary buckling deformation extends over the major dimensions of a structure ; secondary buckling is confined to localized regions, such as the cross sections of individual members.

Buckling Behavior

Buckling of bridge structures
under earthquake

Buckling Behavior

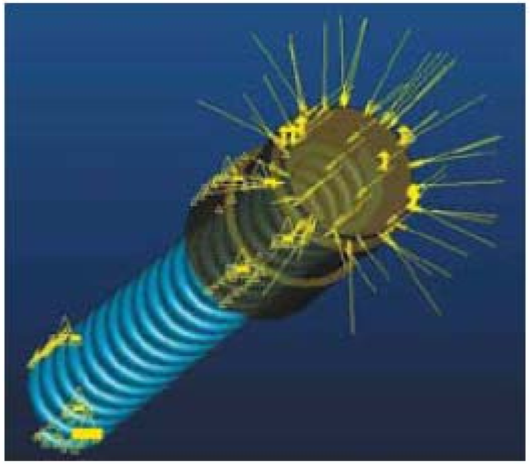




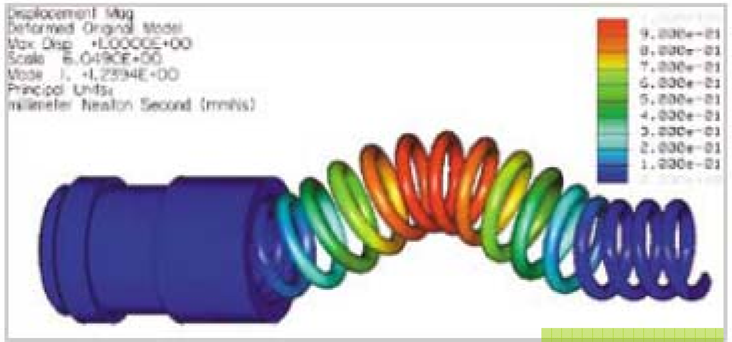
Buckling of long rail

Buckling Analysis

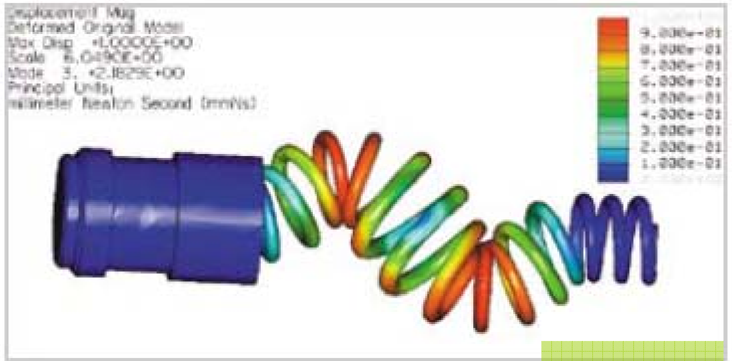
- Release valve



Modeling in Pro/Engineer



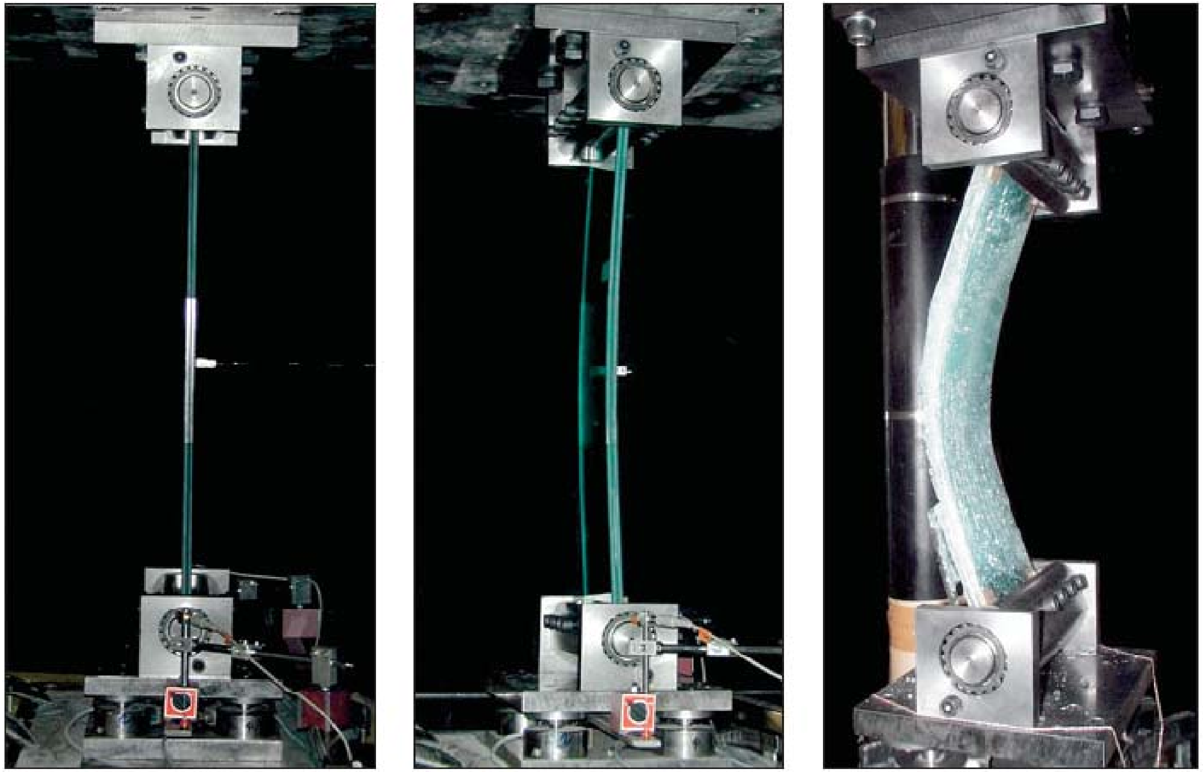
Mode 1



Mode 2

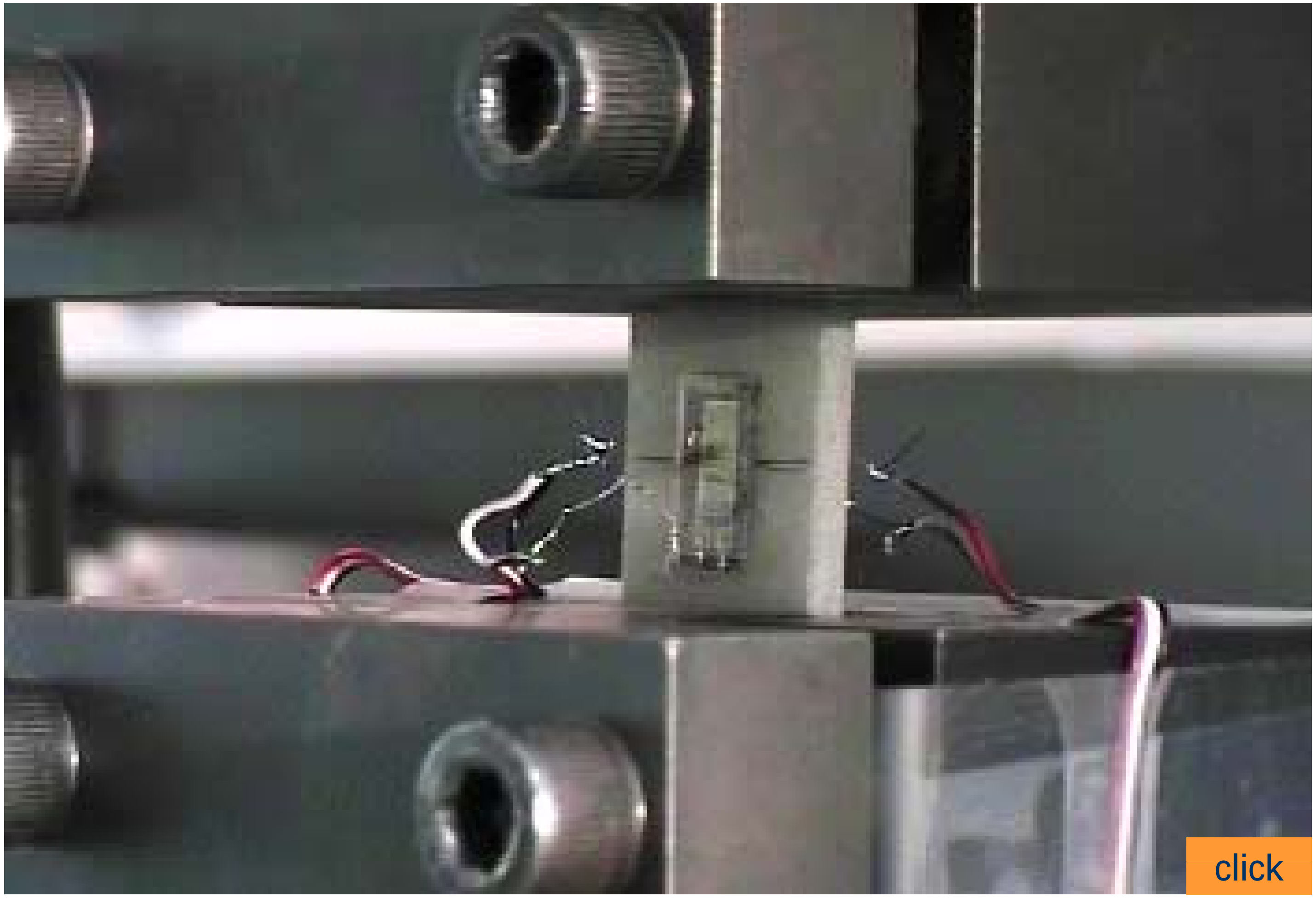
Buckling Test in Test-bed

- Glass structure



Buckling test

Buckling Test in Test-bed



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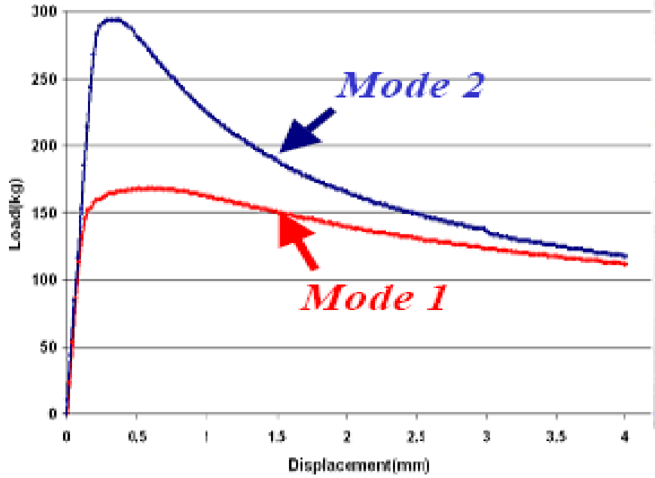
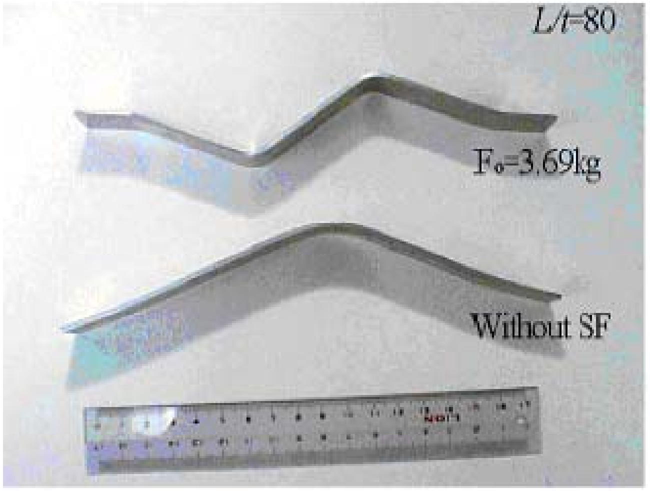
Buckling Test in Test-bed

- Al 6061 & TiNi

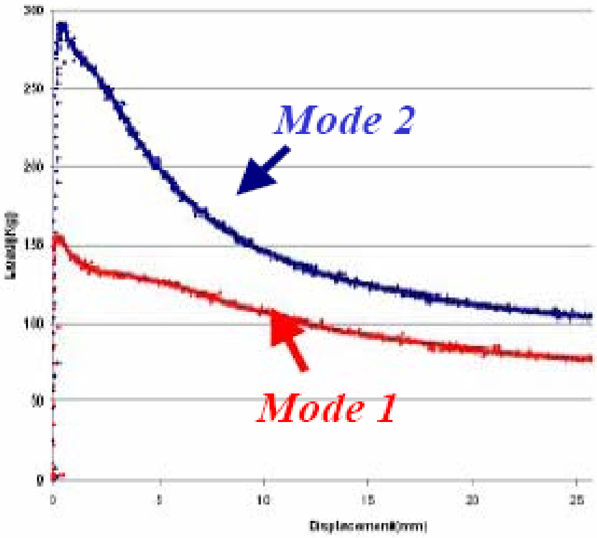


Buckling result for Al 6061

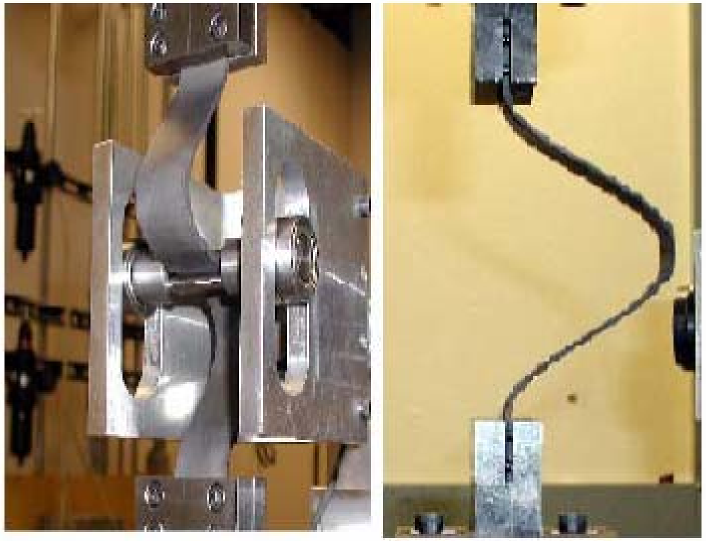


Buckling deformation for Al 6061

Buckling Test in Test-bed



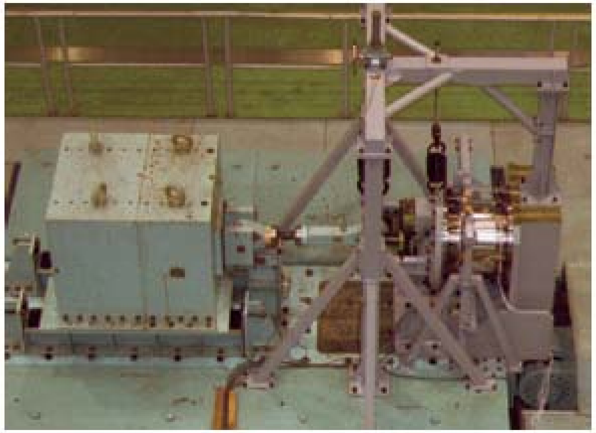
Buckling result
for TiNi



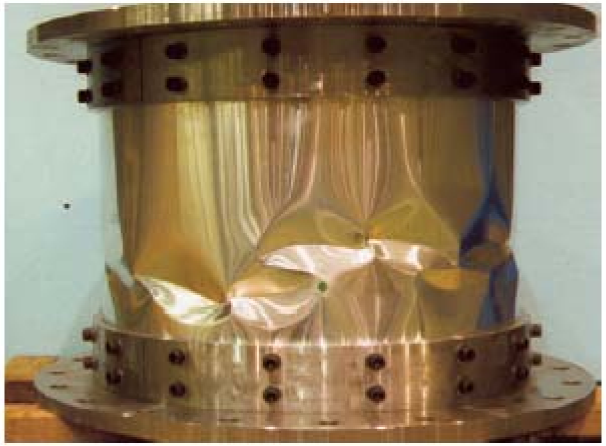
TiNi specimens after buckling test

Buckling Test in Test-bed

- Thin-walled cylindrical shell (1)



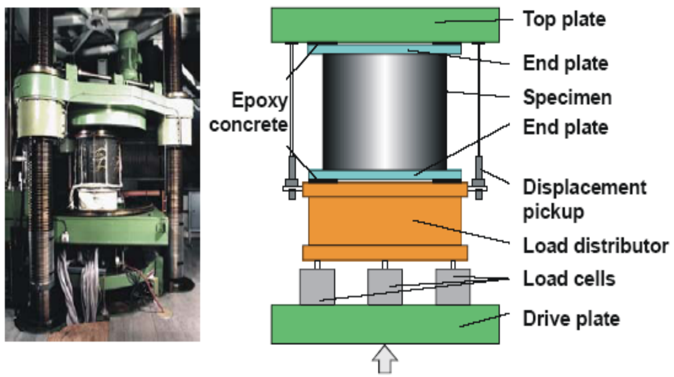
Test facility



Buckling deformation

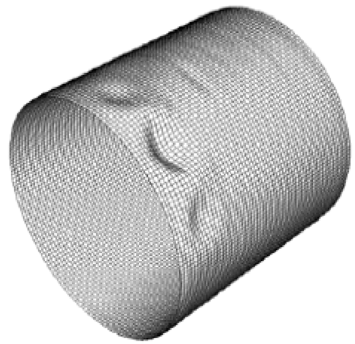
Buckling Test in Test-bed

- Thin-walled cylindrical shell (2)

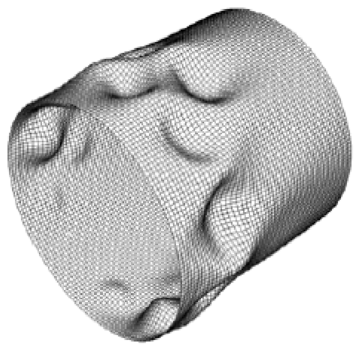


Test facility

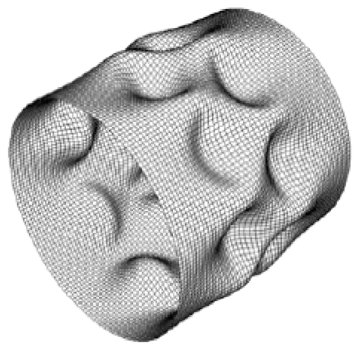
prebuckling deformation



initial buckling deformation



transient buckling deformation

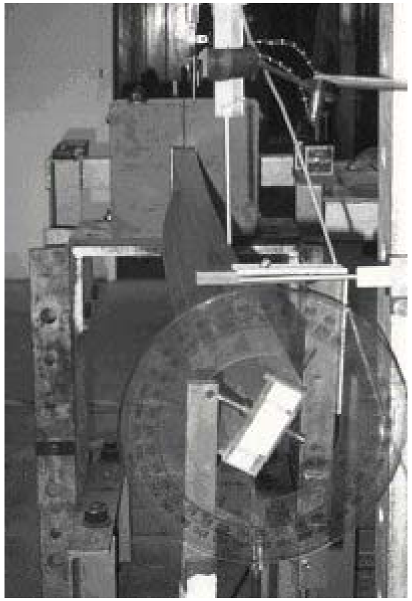


postbuckling deformation

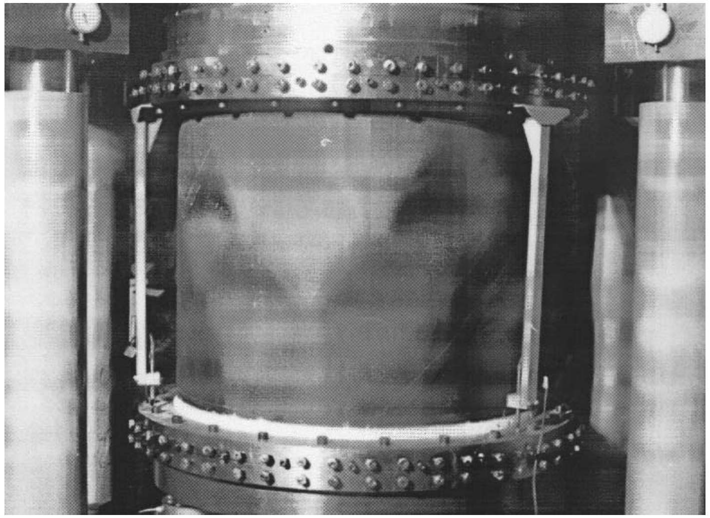
Buckling deformation by Numerical Analysis

Buckling Test in Test-bed

- Composite shell



GRP beam



Carbon fabric

12.2 Unstable Behavior

Consider the behavior of a simplified system that characterizes some of the primary stability phenomena. Figure 12.2.1 shows a perfectly straight, rigid rod that is vertical when the linearly elastic spring is unstretched. Neglecting gravity, write the equation of motion. The single coordinate required to specify the configuration of this system.

$$M_O = I_O \ddot{\theta}$$

where I_0 is the mass moment of inertial of the rod about O, and is the angular acceleration, measured positive clockwise, as in the net moment M_0 about O. Summing the moments of P and the spring force in Figure 12.2.1b, we obtain

$$P \times (L \sin\theta) - k(L \sin\theta) \times (L \cos\theta) = I_O \ddot{\theta}$$

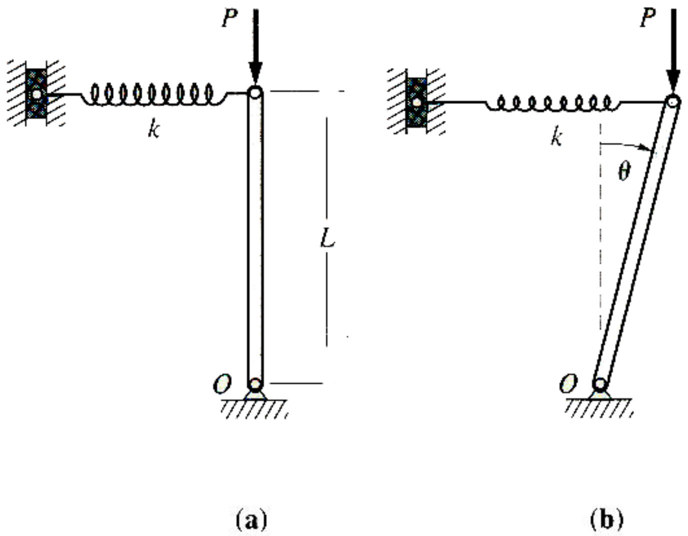


Figure 12.2.1 Rigid rod and linear spring assembly. (a) Before deformation. (b) After deformation

12.2 Unstable Behavior

$$P \times (L \sin\theta) - k(L \sin\theta) \times (L \cos\theta) = I_O \ddot{\theta}$$

$$\boxed{\ddot{\theta} + \frac{L}{I_O}(kL \cos\theta - P) \sin\theta = 0}$$

θ is small enough that the approximations $\sin\theta \cong \theta$ and $\cos\theta \cong 1$ are accurate

$$\ddot{\theta} + \frac{L}{I_O}(kL - P)\theta = 0$$

Let us also assume that at time $t = 0$, $\theta = \theta_0$, and $\dot{\theta} = 0$. That is, the rod is *perturbed* slightly from its vertical orientation by the amount of the small angular deflection θ_0. The form of the solution of Equation 12.2.3 depends on the sign of the coefficient of θ.

If $P < kL$, the coefficient of θ is positive and the solution is sinusoidal, as follows:

$$\theta = \theta_0 \cos \sqrt{\frac{L}{I_O}(kL - P)}t \qquad P < kL$$

The rod oscillates about the undeformed configuration, with the small amplitude of the disturbance θ_0. Eventually, ever-present friction, however small, will bring the rod to rest at equilibrium in the vertical, or nearly vertical, orientation. So, for $P < kL$, the system of Figure 12.2.1(a) is in *stable* equilibrium, much like a marble in the bottom of a teacup.

12.2 Unstable Behavior

If $P > kL$, the solution is expressed in terms of the hyperbolic cosine:

$$\theta = \theta_0 \cosh \sqrt{\frac{L}{I_O}(P - kL)}t \qquad P > kL$$

Here, the rod deviation from the vertical does not remain small, but increases exponentially in time, rapidly exceeding the small perturbation θ_0 and never returning to the vertical. Thus, if $P > kL$, the system is in *unstable* equilibrium, like a marble perched on a basketball.

Finally, if $P = kL$, we have

$$\theta = \theta_0 \qquad P = kL$$

In this case, the system simply remains at rest in its slightly perturbed configuration, neither oscillating nor diverging, like a marble on a flat table top if nudged to a neighboring location. This is called *neutral* equilibrium. For an elastic system, the load corresponding to neutral equilibrium is called the *critical* load P_{cr}.

If the load in Figure 12.2.1a exceeds P_{cr}, the system becomes unstable and seeks another equilibrium configuration, if it exists. Observe that setting $\ddot{\theta} = 0$ in Equation 12.2.1 yields

$$P = kL \cos \theta_{eq}$$

That is, the rod can be in equilibrium at the inclination $\theta > 0$ *if* the load is reduced to the value given in Equation 12.2.5. However, this "post-buckled" equilibrium state is unstable, since

$$\frac{dP}{d\theta_{eq}} = -kL \sin \theta_{eq} < 0$$

12.2 Unstable Behavior

A small increase $\Delta\theta_{eq}$ in the angular orientation is accompanied by a *decrease* in the load, which in turn causes an additional increment of θ_{eq}, and so on. When this situation occurs, we say the structure has *negative stiffness*. We can also check the stability of the post-buckled configuration (Figure 12.2.1b) by holding θ_{eq} fixed and applying a small perturbation, replacing θ_{eq} by $\theta_{eq} + \delta\theta$ in Equation 12.2.2,

$$\delta\ddot{\theta} + \frac{L}{I_O}\left[kL\cos\left(\theta_{eq} + \delta\theta\right) - P\right]\sin\left(\theta_{eq} + \delta\theta\right) = 0$$

Since $\delta\theta << 1$, a Taylor series expansion to the first order yields

$$\cos(\theta_{eq} + \delta\theta) = \cos\theta_{eq} - \sin\theta_{eq}\delta\theta$$

and

$$\sin(\theta_{eq} + \delta\theta) = \sin\theta_{eq} + \cos\theta_{eq}\delta\theta$$

$$\delta\ddot{\theta} - \frac{L}{I_O}\left[kL\sin^2\theta_{eq}\right]\delta\theta = 0$$

Since the coefficient of $\delta\theta$ is clearly negative, $\delta\theta$ increases exponentially with time, confirming that the post-buckled configuration is unstable.

12.2 Unstable Behavior

We can summarize our results on a load–deflection diagram, such as that shown in Figure 12.2.2. As the load P increases from zero, the rod remains vertical and in stable equilibrium until the critical load P_{cr} is reached. At that point, the load path *bifurcates*, or divides into two branches. On one, the rod remains vertical, in unstable equilibrium as the load increases. On the other, the post-buckling path, the rod leans away from vertical, in unstable equilibrium, with diminished load-carrying capacity.

The post-buckled configuration *can* be stable. If we replace the linear spring in Figure 12.2.1 by a torsional spring at pin O, we have the situation illustrated in Figure 12.2.3.

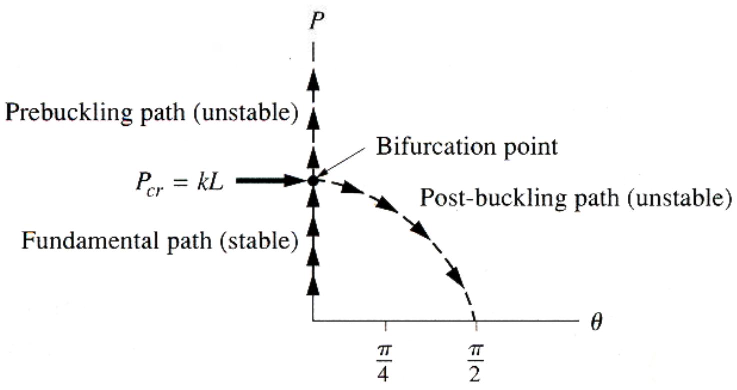


Figure 12.2.2 Load–deflection curve for the structure in Figure 12.2.1.

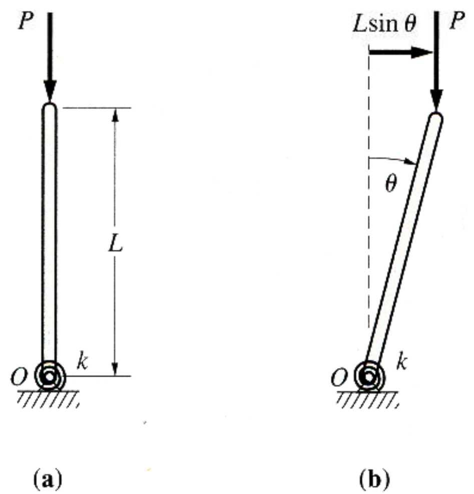


Figure 12.2.3 Rigid rod and torsional spring assembly. (a) Fundamental configuration. (b) Post-buckled configuration.

12.2 Unstable Behavior

To investigate the stability of the vertically oriented rod, we proceed as before. Summing the moments about O leads to the following differential equation for small θ

$$\ddot{\theta} + \frac{k - PL}{I_O}\theta = 0$$

It is clear that in this case,

$$P_{cr} = \frac{k}{L}$$

For equilibrium of the post-buckled configuration, $PL\sin\theta - k\theta = 0$, or

$$P = \frac{k}{L}\frac{\theta}{\sin\theta}$$

This load–deflection curve is plotted in Figure 12.2.4. Observe that in this case, $dP/d\theta$ is never negative, so the post-buckling path is stable. At the onset of buckling, the stiffness starts at zero and increases very slightly with increasing θ, until the rod approaches the horizontal orientation, and increases evermore rapidly thereafter. The small, positive, initial post-buckling stiffness means that for P even *slightly* greater than P_{cr}, a large angular deflection will occur after buckling and before the structure assumes its stable post-buckled orientation.

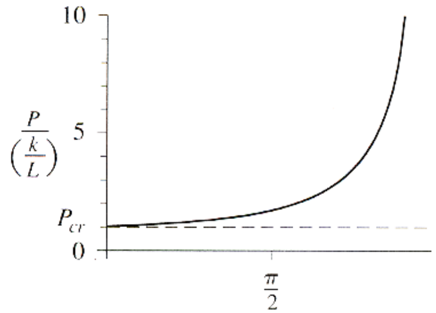


Figure 12.2.4 Load–deflection curve for the post-buckling path (Figure 12.2.3b).

12.2 Unstable Behavior

Another type of instability is illustrated by a shallow, symmetric truss, loaded in compression, as shown in Figure 12.2.5. During the application of load *P*, the point of application C moves through a displacement *v* from its initial vertical position y_0 to point C' with coordinate *y* ($y<y_0$). Let us use the principle of virtual work to find the relationship between the load P and displacement *v*.

"Shallow" implies that $y_0/a << 1$. The initial length L0 and final length L of the two elastic rods are

$$L_o = (a^2 + y_o^2)^{\frac{1}{2}} = a\left[1 + \left(\frac{y_o}{a}\right)^2\right]^{\frac{1}{2}} \cong a\left[1 + \frac{1}{2}\left(\frac{y_o}{a}\right)^2\right]$$

$$L = (a^2 + y^2)^{\frac{1}{2}} = a\left[1 + \left(\frac{y}{a}\right)^2\right]^{\frac{1}{2}} \cong a\left[1 + \frac{1}{2}\left(\frac{y}{a}\right)^2\right]$$

Where the higher order term neglected.

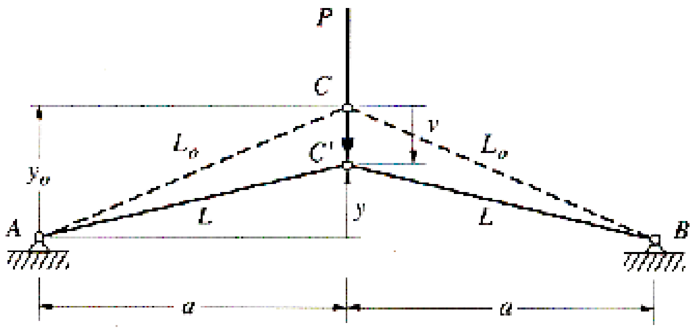


Figure 12.2.5 Shallow truss.

12.2 Unstable Behavior

The axial strain in each rod is therefore

$$\varepsilon = \frac{L - L_o}{L_o} = \frac{L}{L_o} - 1 = \frac{\left[1 + \frac{1}{2}\left(\frac{y}{a}\right)^2\right]}{\left[1 + \frac{1}{2}\left(\frac{y_o}{a}\right)^2\right]} - 1$$

Again, using the fact that $(y_o/a)^2 << 1$, we get

$$\varepsilon = \left[1 + \frac{1}{2}\left(\frac{y}{a}\right)^2\right]\left[1 - \frac{1}{2}\left(\frac{y_o}{a}\right)^2\right] - 1 = \frac{1}{2}\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y_o}{a}\right)^2\right]$$

From this expression for the true strain ε, we obtain the virtual strain $\delta\varepsilon$ by treating δ as the differential operator, so get

$$\delta\varepsilon = \frac{y}{a^2}\delta y$$

The internal virtual work for a linearly-elastic rod in uniaxial stress is

$$\delta W_{\text{int}} = \iiint_V \sigma_x \delta\varepsilon_x dV = \int_0^L (E\varepsilon_x)\delta\varepsilon_x A dx$$

Therefore, for this truss, in which the axial rigidity AE and the axial strain ε are constant, we have

$$\delta W_{\text{int}} = 2 \times AEL\varepsilon\delta\varepsilon$$

$$\delta W_{\text{int}} = 2AEa\left[1 + \frac{1}{2}\left(\frac{y}{a}\right)^2\right]\frac{1}{2}\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y_o}{a}\right)^2\right]\frac{y}{a^2}\delta y$$

12.2 Unstable Behavior

Neglecting the products of y_0/a and y/a that are higher than order 2 reduces this to

$$\delta W_{\text{int}} = \frac{AE}{a^3}\left(y^2 - y_o^2\right) y\,\delta y$$

which can be written in terms of the true and virtual displacements by observing that $y = y_0 - v$ (and therefore $\delta y = -\delta v$), so that

$$\delta W_{\text{int}} = \frac{AE}{a^3}\,(2y_o - v)\,(y_o - v)\,v\,\delta v$$

The external virtual work is simply

$$\delta W_{\text{ext}} = P\,\delta v$$

Setting δW_{ext} equal to δW_{int}, we find that

$$P = \frac{AE}{a^3}(2y_o - v)(y_o - v)v$$

which is the load-deflection relationship we seek. Since the load is a cubic function of the displacement v, the equation is nonlinear. If we nondimensionalize both sides of the equation, it can be written

$$\bar{P} = (2 - \bar{v})(1 - \bar{v})\bar{v}$$

where $\bar{P} = (P/AE)\,(a/y_o)^3$ and $\bar{v} = v/y_o$

12.2 Unstable Behavior

Observe that the structure is stable as the load is first applied; however, when the value of P reaches 0.385 at point A, the stiffness goes to zero and the unstable structure "snaps through," undergoing a five-fold increase in deflection, reaching point C on the second stable, positive-stiffness portion, which starts at B. Examples of elastic structures in which snap-through buckling can occur include slender shallow arches and thin-walled shallow domes, such as the bottom of an oilcan. Hence, the term "oilcanning" is commonly used to refer to the snap-through phenomenon.

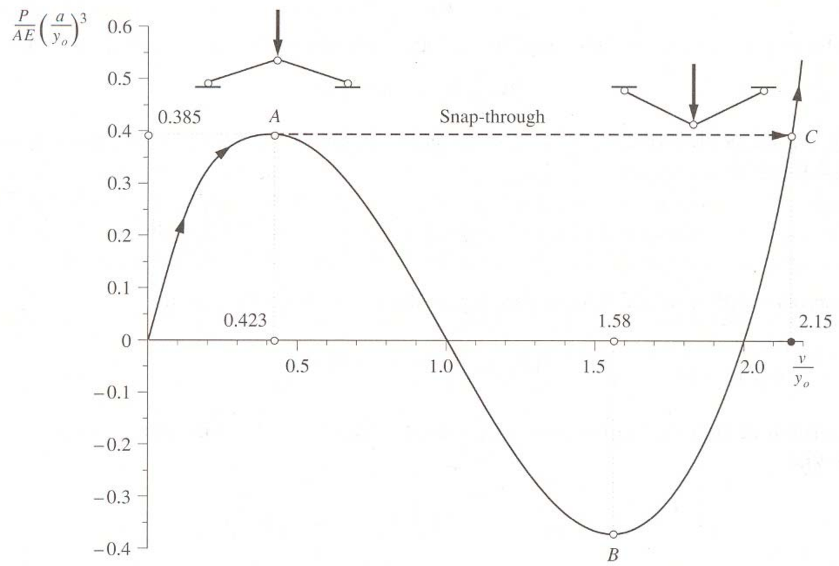


Figure 12.2.6 Load–deflection curve for the shallow truss in Figure 12.2.5. Inserts depict the configuration of the structure at *A* and *C*.

12.3 Beam Columns

A *column* is a straight bar subjected to compressive axial load. A beam column is a bar subjected to compressive axial load, as well as transverse load. Figure 12.3.1a shows a simply-supported, linearly-elastic, simple beam with a transverse load Q applied at its midspan and a compressive load P directed along its centroidal axis. The xy plane in which bending occurs is a plane of symmetry of the cross section. Let us calculate the maximum lateral deflection of the beam, which symmetry dictates is at point C, where the load Q acts.

Applying statics to the free body in Figure 12.3.1b and summing the moments around the neutral axis at the cut, we get

$$M_z + Pv + \frac{Q}{2}x = 0$$

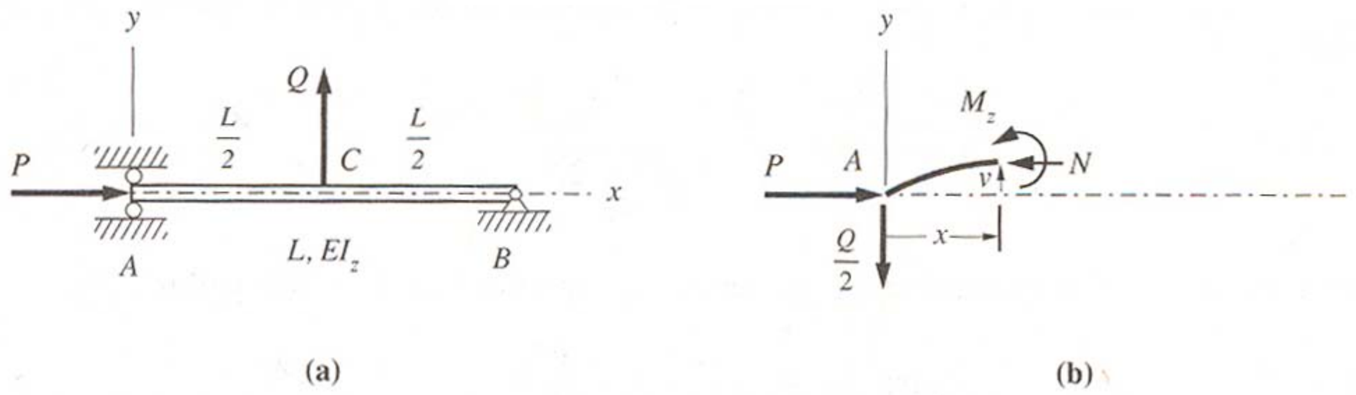


Figure 12.3.1 (a) Simply-supported beam column. (b) Free-body diagram of a portion of the deformed column.

12.3 Beam Columns

Maintaining the small strain/small curvature assumptions that we have used throughout our study of beams allows us to use Equation 10.3.19 to relate the bending moment M_z to the curvature d^2v/dx^2. The equilibrium equation can then be written

$$EI_z \frac{d^2 v}{dx^2} + Pv + \frac{Q}{2}x = 0 \qquad \frac{d^2 v}{dx^2} + \frac{P}{EI_z}v = -\frac{Q}{2EI_z}x$$

It is easy to verify that the general solution of this differential equation is

$$v = \overbrace{A \sin\left(\sqrt{\frac{P}{EI_z}}x\right) + B \cos\left(\sqrt{\frac{P}{EI_z}}x\right)}^{\text{complementary solution}} + \overbrace{\left(-\frac{Q}{2P}x\right)}^{\text{particular solution}}$$

To evaluate the constants of integration, A and B, we apply the boundary conditions. Requiring v=0 at x=0 implied B=0, so we are left with

$$v = A \sin\left(\sqrt{\frac{P}{EI_z}}x\right) - \frac{Q}{2P}x$$

Then, taking the first derivative yields the slope of the elastic curve, which is

$$\frac{dv}{dx} = A\sqrt{\frac{P}{EI_z}} \cos\left(\sqrt{\frac{P}{EI_z}}x\right) - \frac{Q}{2P}$$

By symmetry, the tangent to the elastic curve must be horizontal at the midspan. Setting dv/dx=0 at x=$L/2$ requires that

$$A = \frac{1}{\sqrt{\frac{P}{EI_z}} \cos\left(\sqrt{\frac{P}{EI_z}}\frac{L}{2}\right)} \frac{Q}{2P}$$

12.3 Beam Columns

Substituting this into Equation 12.3.3 and evaluating the resulting expression at *x=L/2* yields

$$q = \frac{\frac{1}{2P}\tan\left(\sqrt{\frac{P}{EI_z}}\frac{L}{2}\right) - \frac{1}{2P}\sqrt{\frac{P}{EI_z}}\frac{L}{2}}{\sqrt{\frac{P}{EI_z}}}Q$$

where q is the displacement at the point of application of the load Q. Simplifying this equation, we get

$$q = \frac{\tan\xi - \xi}{\xi^3}\frac{QL^3}{16EI_z}$$

Where ξ is the dimensionless quantity

$$\xi = \sqrt{\frac{P}{EI_z}}\frac{L}{2}$$

Solving Equation 12.3.4 for the load Q in terms of the displacement q, we get

$$Q = Kq$$

where K is the flexural stiffness coefficient and is given by

$$K = \frac{\xi^3}{\tan\xi - \xi}\frac{16EI_z}{L^3}$$

12.3 Beam Columns

The term K is a measure of the resistance of the beam column to lateral displacement. For a given column, K is a function of the compressive axial load P. Figure 12.3.2 is a plot of K versus ξ. Observe that the flexural stiffness decreases with increasing axial load, finally going to zero at $\xi=\pi/2$ (and becoming negative thereafter). A nonpositive flexural stiffness means that the beam is unstable: it cannot resist even the slightest tendency to nudge it away from its straight, equilibrium configuration. A lateral perturbation, no matter how small, will precipitate buckling, a large lateral deflection of the column.

The magnitude of the buckled deflection cannot be calculated using small displacement theory. Nevertheless, we can predict the onset of buckling by noting when the flexural stiffness vanishes. The axial load at which this occurs is the critical load P_{cr}. Figure 12.3.2 reveals that for a simply-supported elastic column.

$$P_{cr} = \frac{\pi^2 E I_z}{L^2}$$

This equation is known as the *Euler column formula* and P_{cr} is the Euler buckling load.

12.3 Beam Columns

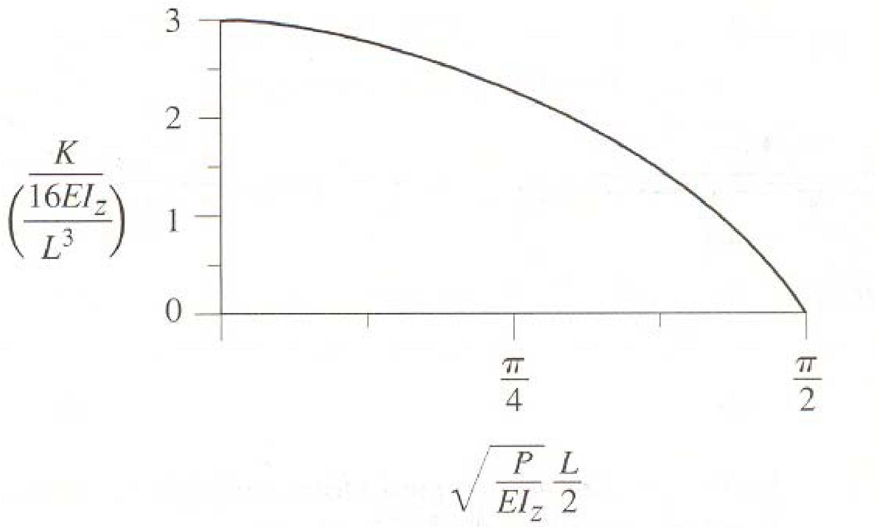


Figure 12.3.2 Bending stiffness of the simply-supported column as a function of compressive axial load.

To explore the notions of bending stiffness, elastic instability, and buckling load, we have chosen to apply the transverse load at the midspan of the beam column. However, it must be noted that the formula for the critical load of a simply-supported column is independent of the nature and location of the transverse disturbance

12.4 Slender Column Buckling

12.4.1 Pinned-Pinned Column

Figure 12.4.1a shows the column, with no transverse load, but with an axial load directed precisely along the centroidal axis, at its critical value. As before, we treat the column as we would a simple beam in bending, so that the plane in which bending occurs contains a symmetry axis of the cross section. Applying statics to the deformed free body in Figure 12.4.1b, we are led again to Equation 12.3.1 with Q=0, that is,

$$\frac{d^2v}{dx^2} + \frac{P_{cr}}{EI_z}v = 0$$

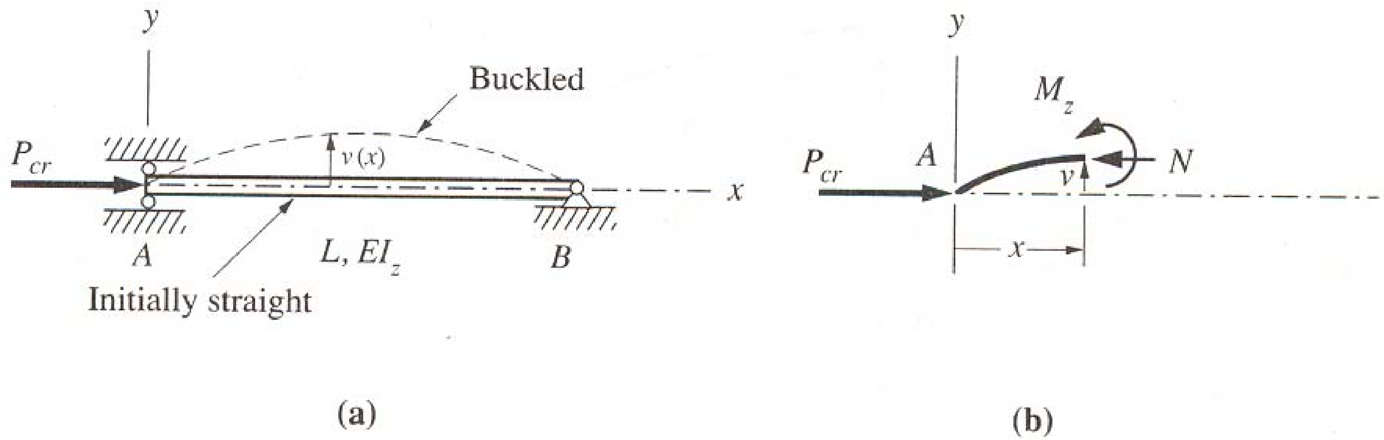


Figure 12.4.1 (a) Pinned-pinned column before and after buckling. (b) Free-body diagram of the elastically bent column.

12.4 Slender Column Buckling

It is customary to introduce the notation $$\lambda^2 = \frac{P_{cr}}{EI_z}$$

So that
$$\frac{d^2v}{dx^2} + \lambda^2 v = 0$$

The solution of this homogeneous differential equation is

$$v = A \sin \lambda x + B \cos \lambda x$$

To satisfy the boundary conditions, v must vanish at each end of the column, Setting v=0 at x=0 means that B=0, leaving us with $v = A \sin \lambda x$

Then, requiring v=0 at x=L implies that

$$A \sin \lambda L = 0$$

This equation can be satisfied by requiring A=0. However since we also have B=0, that would mean v=0 everywhere in the column. In other words, the column has not buckled. Since we are not interested in this "trivial solution", we instead require that

$$\sin \lambda L = 0$$

12.4 Slender Column Buckling

This equality holds if

$$\lambda = \frac{n\pi}{L}$$

where n is a positive integer (but not zero, since that would again lead to the trivial solution). Substituting this expression for λ into Equation 12.4.2, we deduce

$$P_{cr} = n^2 \frac{\pi^2 E I_z}{L^2} \qquad n = 1, 2, 3, \cdots$$

Apparently, there are a countably infinite number of buckling laods. For the *n*th such load, the deformed shape of the column, or *the mode shape* is given:

$$v_n = A_n \sin \lambda_n x = A_n \sin \frac{n\pi x}{L}$$

The first three of these sine-wave modes is shown in Figure 12.4.2. In reality, if buckling occurs, it happens at the lowest possible mode. Thus, for the pinned-pinned condition, the buckling load is the Euler load found in the previous section.

$$P_{cr} = \frac{\pi^2 E I_z}{L^2} \qquad \text{Pinned-pinned column}$$

12.4 Slender Column Buckling

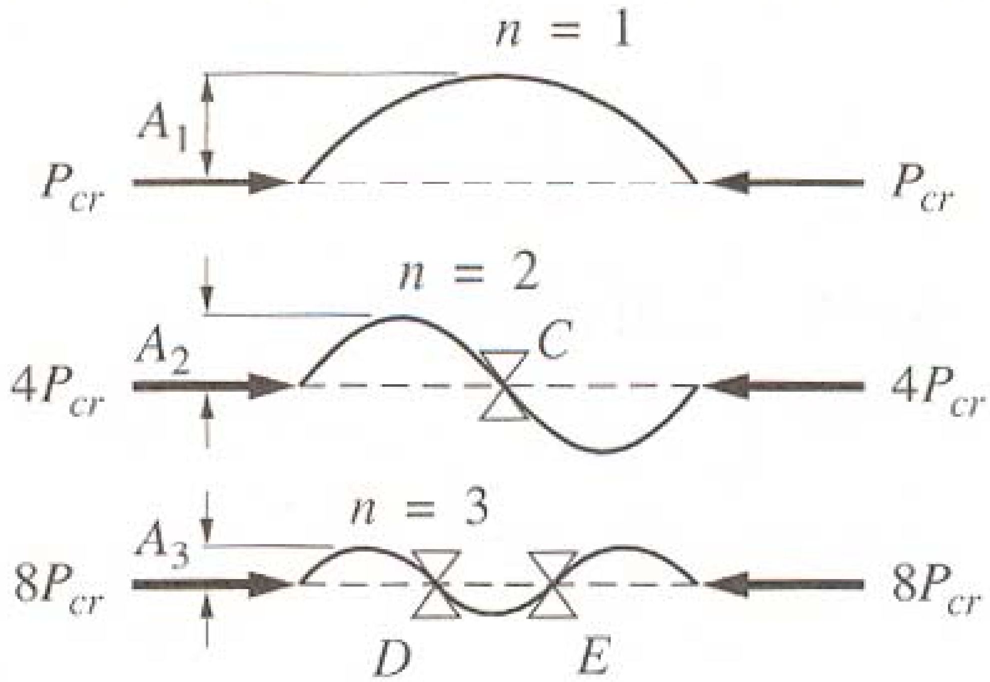


Figure 12.4.2 First three buckling modes of a pinned-pinned column.

12.4 Slender Column Buckling

12.4.2 Fixed-Fixed Column

The structure is clearly statically indeterminate. Using the free-body diagram of Figure 12.4.3b, we obtain the moment equilibrium equation,

$$M_z + P_{cr}v = M_A$$

Substituting the moment-curvature equation, $M_z = EI_z d^2v/dx^2$ and dividing by the flexural rigidity yields

$$\frac{d^2v}{dx^2} + \lambda^2 v = \frac{M_A}{EI_z}$$

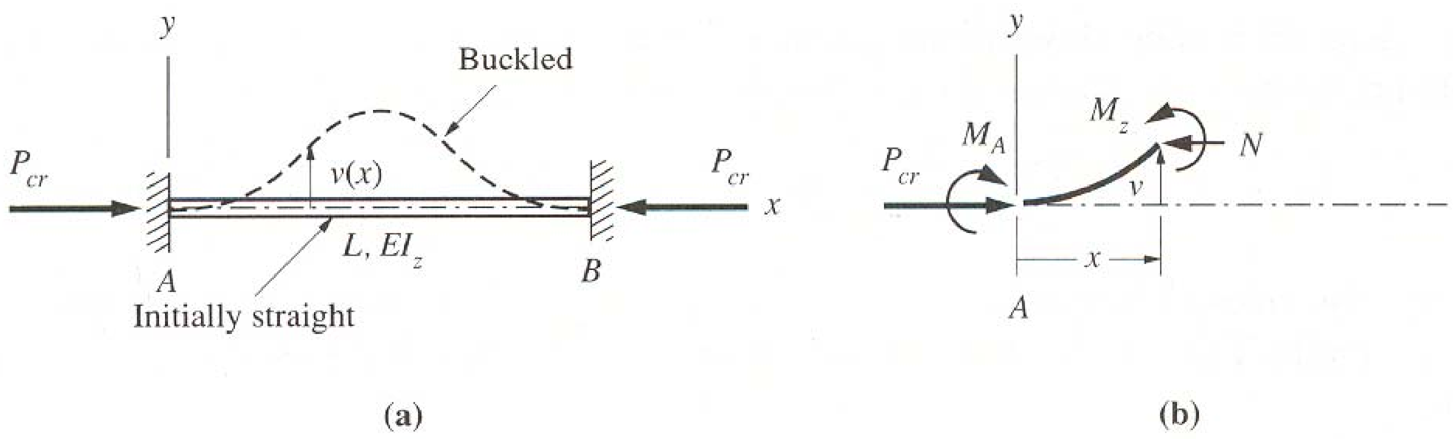


Figure 12.4.3 (a) Fixed-fixed column. (b) Free-body diagram of a portion of the deformed column.

12.4 Slender Column Buckling

There are four boundary conditions, two at each end of the column. At $x=0$, we have $v=0$ and $dv/dx=0$, which requires that $A = 0$ and $B = -M_A/\lambda^2 EI_z$, so we are left with

$$v = \frac{M_A}{\lambda^2 E I_z}(1 - \cos\lambda x) \qquad \frac{dv}{dx} = \frac{M_A}{\lambda E I_z}\sin\lambda x$$

For dv/dx to vanish at $x=L$, $\sin\lambda L=0$. This in turn implies that

$$\lambda L = n\pi, \quad n = 1, 2, 3, \cdots$$

Finally, $v=0$ at $x=L$ only if $\cos\lambda L=1$, and this is true if

$$\lambda L = n\pi, \quad n = 2, 4, 6, \cdots$$

Thus, we conclude that

$$\frac{P_{cr}}{EI_z} = \lambda^2 = \frac{n^2\pi^2}{L^2}, \quad n = 2, 4, 6, \cdots$$

12.4 Slender Column Buckling

Taking the lowest possible value of *n* yields the critical buckling load,

$$P_{cr} = 4\frac{\pi^2 E I_z}{L^2} \qquad \text{Fixed-fixed column}$$

The mode shape is

$$v = \frac{M_A}{P_{cr}}\left(1 - \cos\frac{2\pi x}{L}\right)$$

Since we have no means of computing a value for M_A, the amplitude is indeterminate.

Observe that restraining the rotations at the supports increases the buckling load by a factor of four over that of the simply-supported beam. Another way of looking at this is that the buckling load of a fixed-fixed column equals that of a pinned-pinned column half as long, that is,

$$P_{cr} = \frac{\pi^2 E I_z}{(L/2)^2}$$

We say that the *effective length* $\mathbf{L}_e$ of a fixed-fixed column is *L/2*.

In general, for a long column of actual length *L*, we can express the critical load as

$$P_{cr} = \frac{\pi^2 E I}{L_e^2} = c\frac{\pi^2 E I}{L^2} \qquad L_e = \frac{L}{\sqrt{c}}$$

The constant *c* is the *coefficient of constraint* or *end fixity factor*, which, as we have seen, depends on the manner in which the column is restrained at each end.

12.4 Slender Column Buckling

12.4.3 Pinned-Fixed Column

This type of column has a simple support at one end and is built in at the other. Figure 12.4.4b shows a free-body diagram of a deformed column portion lying between the simple support and station x. The diagram is identical to Figure 12.3.1 if we replace P with P_{cr} and $Q/2$ with $-Y_A$. Doing so in Equation 12.3.2 yields

$$v = \overbrace{A \sin \lambda x + B \cos \lambda x}^{\text{complementary solution}} + \overbrace{\frac{Y_A}{P_{cr}} x}^{\text{particular solution}}$$

The boundary condition at $x=0$ is $v=0$. Therefore, $B=0$ and

$$v = A \sin \lambda x + \frac{Y_A}{P_{cr}} x \qquad \frac{dv}{dx} = \lambda A \cos \lambda x + \frac{Y_A}{P_{cr}}$$

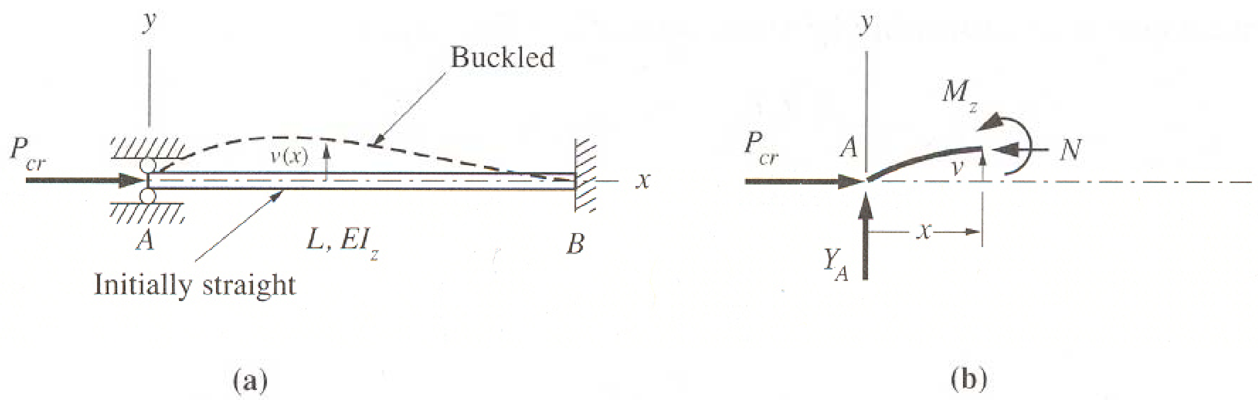


Figure 12.4.4 (a) Pinned-fixed column. (b) Free-body diagram of a portion of the deformed column, starting at the simple support.

12.4 Slender Column Buckling

Since $v = 0$ and $dv/dx = 0$ at $x = L$, we get

$$A \sin \lambda L = -\frac{Y_A}{P_{cr}} L \qquad \text{and} \qquad \lambda A \cos \lambda L = -\frac{Y_A}{P_{cr}}$$

Dividing the first of these two equations by the second yields

$$\frac{\sin \lambda L}{\lambda \cos \lambda L} = L$$

or

$$\tan \lambda L = \lambda L$$

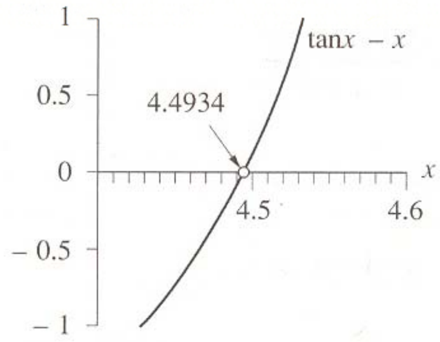


Figure 12.4.5 Graph of $\tan x - x$ versus x in the region of the first positive root, which is shown.

The smallest root of this transcendental equation can be found by using an iterative algorithm, such as the Newton-Raphson method, or by simply plotting the function $f(x) = \tan x - x$, as in Figure 12.4.5, which shows that

$$kL = 4.4934 = 1.4303\pi$$

Since $P_{cr} = EI_z\lambda^2$, the critical load for this set of constraints is

$$P_{cr} = 2.046\frac{\pi^2 EI_z}{L^2} = \frac{\pi^2 EI_z}{(0.6992L)^2} \qquad \text{Pinned-fixed column}$$

12.4 Slender Column Buckling

12.4.4 Free-Fixed Column

As indicated in Figure 12.4.6a, a *free-fixed* column has one end built in and the other end free of displacement contraints.

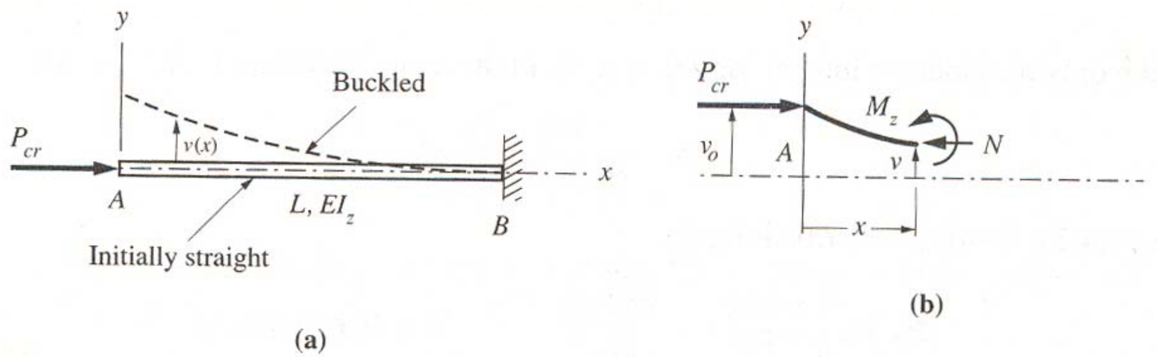


Figure 12.4.6 (a) Free-fixed column. (b) Free-body diagram of a portion of the deformed column, beginning with the free end.

Summing moments about the neutral axis at the right end of the free body in Figure 12.4.6b yields

$$M_z + P_{cr} v = P_{cr} v_o$$

12.4 Slender Column Buckling

$$\frac{d^2v}{dx^2} + \lambda^2 v = \lambda^2 v_o \qquad \lambda^2 = \frac{P_{cr}}{EI_z}$$

The solution to the differential equation is

$$v = \overbrace{A \sin \lambda x + B \cos \lambda x}^{\text{complementary solution}} + \overbrace{v_o}^{\text{particular solution}}$$

Since $v = v_o$ at $x = 0$, it follows that $B = 0$; therefore,

$$v = A \sin \lambda x + v_o$$

Accordingly,

$$\frac{dv}{dx} = \lambda A \cos \lambda x$$

Since the column is built in at $x = L$, we have $v = dv/dx = 0$ at that point, which means that

$$A \sin \lambda L + v_o = 0$$

and

$$\lambda A \cos \lambda L = 0$$

12.4 Slender Column Buckling

For a nontrivial solution, $\cos \lambda L = 0$, the roots of which are

$$\lambda L = \frac{n\pi}{2}, \quad n = 1, 3, 5, \cdots$$

As usual, we take only the smallest integer, or $\lambda L = \pi/2$. In this case,

$$A = -\frac{v_o}{\sin(\pi/2)} = -v_o$$

From Equation 12.4.19, we find the critical load,

$$P_{cr} = \frac{1}{4}\frac{\pi^2 EI_z}{L^2} = \frac{\pi^2 EI_z}{(2L)^2} \qquad \text{Free-fixed column}$$

With free-fixed constraints, a column buckles at *one-fourth* the critical load of a pinned-pinned column; therefore, $L_e = 2L$ and $c = 0.25$.

The shape of the fundamental mode is

$$v = v_o\left(1 - \sin\frac{\pi x}{2L}\right)$$

The amplitude v_o remains undetermined.

12.5 Column Imperfections and Load Misalignment

In the previous section, we assumed that the column was initially straight and that axial load was aligned perfectly with the centroidal axis. Let us investigate the effects of slight deviations from these conditions. We will first consider a column with initial imperfections.

Figure 12.5.1a shows a pinned-pinned column that is not straight, but is bent into an initial, unloaded shape. Unlike for the straight column, bending will occur immediately upon application of the axial load *P*, regardless of its magnitude, due to its offset from the slightly curved centerline of the bar. Figure 12.5.1b shows a free-body diagram of a portion of the column between the left end *A* and the cut at station *x*. Notice that the total deflection v_{tot} of the column at any point is the sum of its initial deviation v_0 from a straight line and the additional deflection *v* due to the applied load *P*.

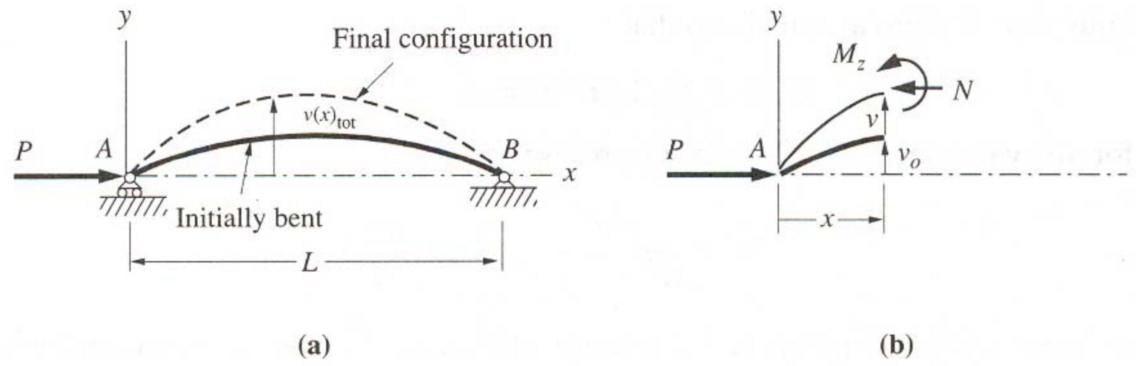


Figure 12.5.1 (a) Initially bent simply-supported column. (b) Free-body diagram of a portion of the column.

For moment equilibrium around the cut,

$$M_z + Pv_{\text{tot}} = 0 \qquad v_{\text{tot}} = v_o + v \qquad M_z + Pv = -Pv_o$$

12.5 Column Imperfections and Load Misalignment

Let us assume that the initial shape of the bar is that of a sine function with amplitude $a_0 L$, where a_0, the dimensionless imperfection amplitude, is a very small number:

$$v_o = a_o L \sin \frac{\pi x}{L}$$

Substituting this and the moment-curvature relation into the equilibrium equation yields

$$\frac{d^2 v}{dx^2} + \lambda^2 v = -\lambda^2 a_o L \sin \frac{\pi x}{L} \qquad \lambda^2 = \frac{P}{EI_z}$$

The general solution is

$$v = \overbrace{A \sin \lambda x + B \cos \lambda x}^{\text{complementary solution}} + \overbrace{\frac{\lambda^2 L^2}{\pi^2 - \lambda^2 L^2} a_o L \sin \frac{\pi x}{L}}^{\text{particular solution}}$$

At x=0, v must vanish, which requires that B=0. Therefore,

$$v = A \sin \lambda x + \frac{\lambda^2 L^2}{\pi^2 - \lambda^2 L^2} a_o L \sin \frac{\pi x}{L}$$

12.5 Column Imperfections and Load Misalignment

The deflection v must also be zero at $x = L$, so that

$$A \sin \lambda L = 0$$

This can be true for *any* value of λ if $A = 0$, in which case

$$v = \frac{\lambda^2 L^2}{\pi^2 - \lambda^2 L^2} a_o L \sin \frac{\pi x}{L}$$

Substituting this expression and Equation 12.5.2 into Equation 12.5.1 yields the equation for the total deflection of the column:

$$v_{\text{tot}} = \left(a_o L \sin \frac{\pi x}{L} \right) + \left(\frac{\lambda^2 L^2}{\pi^2 - \lambda^2 L^2} a_o L \sin \frac{\pi x}{L} \right) = \frac{\pi^2}{\pi^2 - \lambda^2 L^2} a_o L \sin \frac{\pi x}{L}$$

At $x = L/2$, the lateral deflection takes on its maximum value,

$$\delta = \frac{\pi^2}{\pi^2 - \lambda^2 L^2} a_o L$$

Noting Equation 12.5.4, and recalling the Euler column formula, we can write this as

$$\delta = \frac{\pi^2}{\pi^2 - \frac{PL^2}{EI_z}} a_o L = \frac{\pi^2 E I_z}{\pi^2 E I_z - P L^2} a_o L = \frac{P_{cr} L^2}{P_{cr} L^2 - P L^2} a_o L$$

12.5 Column Imperfections and Load Misalignment

Setting $a = \delta/L$, we therefore obtain

$$a = \frac{1}{1 - P/P_{cr}} a_o \qquad \text{or} \qquad \frac{P}{P_{cr}} = 1 - \frac{a_o}{a}$$

where a is the dimensionless midspan deflection.

The second of these two equations is plotted in Figure 12.5.2 for several values of a_o. Excessive lateral deflection of a slender column—$a \approx 0.1$ (ten percent of the column length)—occurs below the Euler buckling load, regardless of the size of the inevitable imperfection. However, the chart also shows that if a_o is sufficiently small, one can use the Euler buckling formula to predict the load capacity of a column, provided suitable safety factors are used (e.g., approximately 0.5 for steel).

Let us next consider a column that is geometrically perfect, but has an axial load applied off the centroidal axis, as illustrated in Figure 12.5.3a. The amount of the offset is called the *eccentricity*, e. To produce a symmetric deflection curve, thereby simplifying the analysis, the eccentricity must be the same at each end of the column. Using the free-body diagram in Figure 12.5.3b we obtain the following equilibrium equation:

$$M_z + P(e + v) = 0$$

Using the moment–curvature formula and Equation 12.5.4, we obtain

$$\frac{d^2 v}{dx^2} + \lambda^2 v = -\lambda^2 e$$

The general solution to this differential equation is

$$v = A \sin \lambda x + B \cos \lambda x - e$$

12.5 Column Imperfections and Load Misalignment

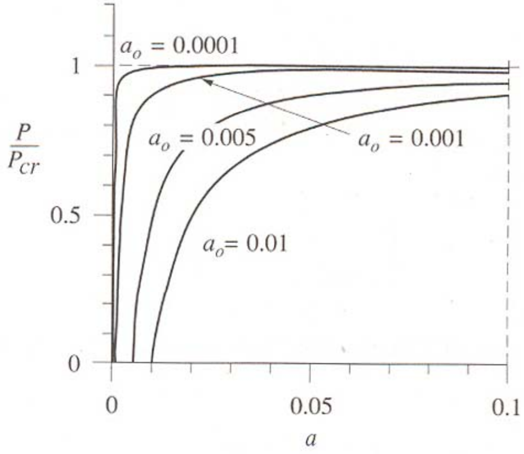


Figure 12.5.2 Axial load versus total (dimensionless) midspan deflection of a pinned-pinned column, for different amounts of initial imperfection.

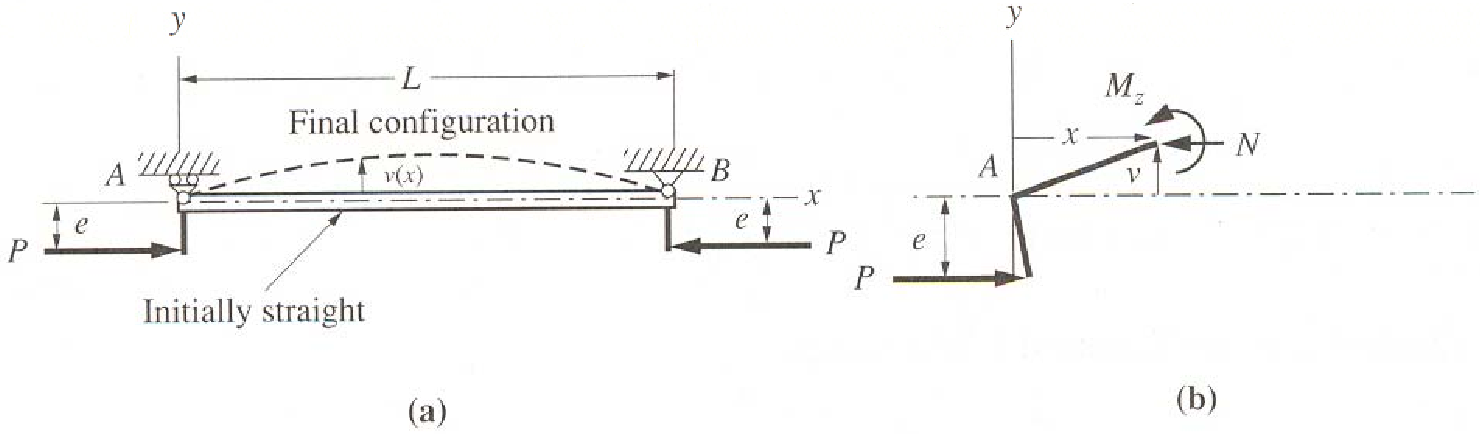


Figure 12.5.3 (a) Eccentrically loaded pinned-pinned column. (b) Free-body diagram of a portion of the column, beginning at the left support.

12.5 Column Imperfections and Load Misalignment

Setting the displacement equal to zero at $x=0$, we find that $B=e$. Therefore,

$$v = A \sin \lambda x + e(\cos \lambda x - 1)$$

The displacement must also vanish at the right support, $x=L$. Therefore,

$$A = \frac{1 - \cos \lambda L}{\sin \lambda L} e$$

Simplifying, we finally obtain

$$v = \frac{\sin \lambda (L - x) + \sin \lambda x - \sin \lambda L}{\sin \lambda L} e$$

Since the lateral deflection is symmetric about the midspan of the column [that is, $v(x)=v(L-x)$], the maximum value δ occurs at $x=L/2$ and is given by

$$\delta = \frac{2 \sin \frac{\lambda L}{2} - \sin \lambda L}{\sin \lambda L} e$$

12.5 Column Imperfections and Load Misalignment

Using the trigonometric identity

$$\sin \lambda L = 2 \sin \frac{\lambda L}{2} \cos \frac{\lambda L}{2}$$

we reduce this to the following expression

$$a = \left(\frac{1}{\cos(\lambda L/2)} - 1 \right) a_e$$

Where $a = \delta / L$ and $a_e = e/L$ are the dimensionless midspan deflection and eccentricity, respectively

$$\frac{\lambda L}{2} = \frac{L}{2} \sqrt{\frac{P}{E I_z}}$$

And the Euler column formula as

$$E I_z = \frac{L^2}{\pi^2} P_{cr}$$

12.5 Column Imperfections and Load Misalignment

we find that

$$\frac{\lambda L}{2} = \frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}$$

Substituting this expression into Equation 12.5.6, we get

$$\frac{a}{a_e} = \frac{1}{\cos\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}} - 1$$

Solving this for P/P_{cr} yields

$$\frac{P}{P_{cr}} = \left[\frac{2}{\pi}\cos^{-1}\left(\frac{a_e}{a}\right)\right]^2$$

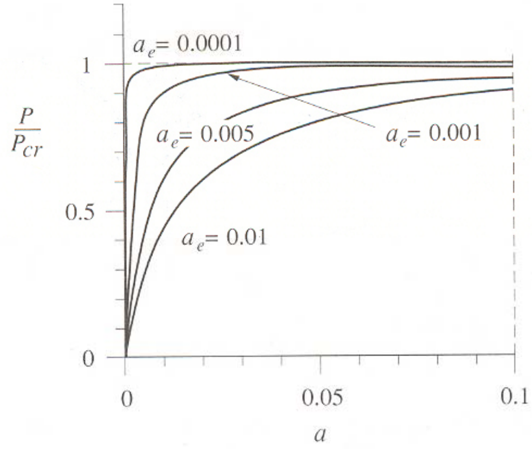


Figure 12.5.4 Load versus deflection for a pinned-pinned column with different eccentricities.

This load–deflection relation is plotted in Figure 12.5.4, which shows that if a column has even the smallest eccentricity, its load capacity is decreased. However, if the eccentricity is sufficiently small, the Euler buckling load can be used, given an appropriate safety factor, as previously discussed for a column with minor imperfections.

12.12 Static Wing Divergence

The lift production by a wing is accompanied by a twisting moment about the elastic axis of the wing box, causing an increase in the angle of attack, which in turn increases the lift and the twisting moment. At a speed called the divergence velocity, the effect becomes large enough to cause failure of the wing. Let us investigate this instability phenomena by considering the somewhat oversimplified case of a straight wing with no built-in twist and with a uniform lifting load distribution due to subsonic flow

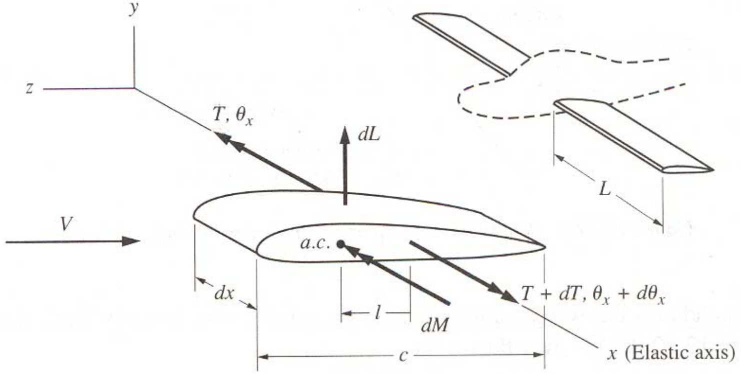


Figure 12.12.1 Differential wing section acted on by external aerodynamic loads (dL and dM) and the internal torsional stress resultants. (*a.c.* is the aerodynamic center.)

Figure 12.12.1 shows a free-body diagram of a spanwise differential section of a straight wing of constant chord c. The differential aerodynamic lift dL is shown acting at the aerodynamic center of the wing, which is a distance l in fromt of the elastic axis. The lift is found in terms of the dimensionless section lift coefficient c_l, the differential area cdx. And the dynamic pressure q, by the formula

$$dL = c_l q c dx$$

12.12 Static Wing Divergence

Lift is generally accompanied by a pitching moment, so the external differential moment *dM* is shown acting in the conventionally positive nose-up (clockwise) direction. If, as in the figure, *dM* is measured about the aerodynamic center, then in terms of the dimensionless moment coefficient $c_{m_{a.c.}}$, we have

$$dM = c_{m_{a.c.}} q c^2 dx$$

The torque *T* shown acting on the differential wing section arises, as we know, from the internal stresses required to equilibrate the externally-applied aerodynamic loads. For the free body in Figure 12.12.1 to be in equilibrium, the net moment about the elastic axis must vanish. Thus,

$$(T + dT) - T - l dL - dM = 0$$

Substituting Equations 12.12.1 and 2, cancelling terms, and dividing through by *dx*, we get

$$\frac{dT}{dx} - c_l q c l - c_{m_{a.c.}} q c^2 = 0$$

The lift coefficient depends on the wing's angle of attack *α*, and varies linearly from the zero lift angle to the onset of stall (cf. Figure 12.12.2). We may thus write

$$c_l = \frac{\partial c_l}{\partial \alpha}(\alpha - \theta_x)$$

where $\partial c_l / \partial \alpha$ is the slope of the lift curve, c_l versus α, and α is measured clockwise (here, in radians).

12.12 Static Wing Divergence

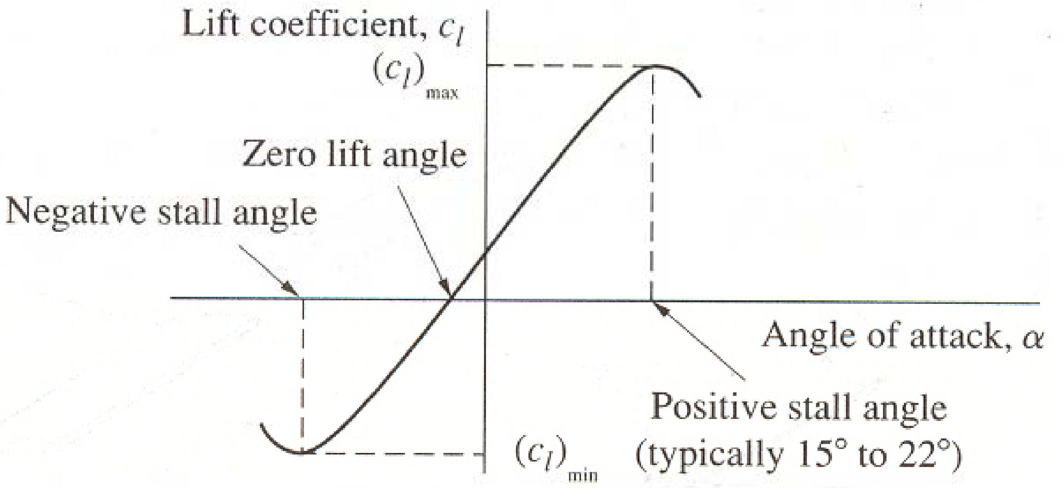


Figure 12.12.2 Variation of wing lift coefficient with angle of attack.

12.12 Static Wing Divergence

If the wing were rigid, then $\theta_x = 0$; that is, there would be no twisting of the structure associated with lift. For the flexible wing under consideration, however, we must subtract the aerodynamically-induced twist θ_x at the section from the angle of attack α which the wing would have if it were rigid. We subtract rather than add to remain consistent with our sign convention for positive twist, which, as indicated in Figure 12.12.1, is counterclockwise about the positive x axis, opposite to the direction of increasing α. By definition of the aerodynamic center, the moment coefficient $c_{m_{a.c.}}$ about that point does not depend on the angle of attack: $\partial c_{m_{a.c.}}/\partial\alpha = 0$. Substituting Equation 12.12.4 into Equation 12.12.3 yields

$$\frac{dT}{dx} + qcl\frac{\partial c_l}{\partial\alpha}\theta_x = qcl\frac{\partial c_l}{\partial\alpha}\alpha + c_{m_{a.c.}}qc^2$$

According to Equation 4.4.1 and 4.4.14, $T=GJd\theta_x/dx$. If GJ is constant, as in the present case, Equation 12.125 becomes a second-order ordinary differential equation in θ_x with constant coefficients:

$$\frac{d^2\theta_x}{dx^2} + \frac{qcl}{GJ}\frac{\partial c_l}{\partial\alpha}\theta_x = \frac{qcl}{GJ}\frac{\partial c_l}{\partial\alpha}\alpha + \frac{c_{m_{a.c.}}qc^2}{GJ}$$

12.12 Static Wing Divergence

Introducing the notation

$$\lambda^2 = \frac{qcl}{GJ}\frac{\partial c_l}{\partial \alpha}$$

and

$$\theta_a = \alpha + \frac{c_{m_{a.c.}} c}{l(\partial c_l/\partial \alpha)}$$

the differential equation can be written if the form

$$\frac{d^2\theta_x}{dx^2} + \lambda^2\theta_x = \lambda^2\theta_a$$

From earlier in this chapter, we recognize the solution of this equation to be

$$\theta_x = \overbrace{A\cos\lambda x + B\sin\lambda x}^{\text{complementary solution}} + \overbrace{\theta_a}^{\text{particular solution}}$$

12.12 Static Wing Divergence

One boundary condition on the twist angle is that it must vanish at the wing root, where $x=0$. For that to be true, we must have

$$A = -\theta_a$$

so that

$$\theta_x = B \sin \lambda x + \theta_a (1 - \cos \lambda x)$$

The derivative of this expression is

$$\frac{d\theta_x}{d\theta} = \lambda B \cos \lambda x + \theta_a \lambda \sin \lambda x$$

At the wing tip the torque T is zero. Since $d\theta_x/dx=T/GL$, it follows that $d\theta_x/dx=0$ at $x=L$. This, together with Equation 12.12.9, requires that

$$B = -\theta_a \frac{\sin \lambda L}{\cos \lambda L}$$

Substituting this into Equation 12.12.8, we find that

$$\theta_x = \theta_a \left(1 - \cos \lambda x - \frac{\sin \lambda L}{\cos \lambda L} \sin \lambda x\right)$$

12.12 Static Wing Divergence

Rearranging terms, substituting Equation 12.12.7, and noting the trigonometric identity

$$\cos\lambda L\cos\lambda x + \sin\lambda L\sin\lambda x = \cos\lambda\,(L-x)$$

we reduce this expression for the twist angle θ_x to

$$\theta_x = \left[\alpha + \frac{c_{m_{a.c.}}c}{l(\partial c_l/\partial\alpha)}\right]\left[1 - \frac{\cos\lambda\,(L-x)}{\cos\lambda L}\right]$$

From this, we can also obtain the torque versus span expression, which is

$$T = GJ\frac{d\theta_x}{dx} = GJ\left[\alpha + \frac{c_{m_{a.c.}}c}{l(\partial c_l/\partial\alpha)}\right]\frac{\lambda\sin\lambda\,(L-x)}{\cos\lambda L}$$

The maximum twist occurs at the wing tip, x=L, and the greatest torque is at the root, x=0:

$$\theta_x)_{\max} = \left[\alpha + \frac{c_{m_{a.c.}}c}{l(\partial c_l/\partial\alpha)}\right]\left(\frac{\cos\lambda L - 1}{\cos\lambda L}\right) \quad T)_{\max} = \left[\alpha + \frac{c_{m_{a.c.}}c}{l(\partial c_l/\partial\alpha)}\right]GJ\lambda\tan\lambda L$$

Observe that the deflection and torque approach infinity (and the wing fails) when $\cos\lambda L$ approaches zero ($\tan\lambda L$ approaches infinity), This occurs when

$$\lambda L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$$

Obviously, the smallest λ for which the twist and torque become unbounded is $\pi/2L$. At this critical value of λ, Equation 12.12.6 becomes

$$\lambda^2 = \frac{qcl}{GJ}\frac{\partial c_l}{\partial\alpha} = \frac{\pi^2}{4L^2}$$

12.12 Static Wing Divergence

Setting $q=\rho V^2/2$ and solving for V yields the divergence speed, which is

$$V_D = \sqrt{\frac{\pi^2}{(\partial c_l/\partial\alpha)}\frac{J}{clL^2}}\sqrt{\frac{G}{\rho}}$$

The divergence speed can be raised by increasing the wing torsional rigidity *GJ*, moving the elastic axis closer to the aerodynamic center (decreasing *l*), and flying at higher altitudes where the density of the atmosphere is low.

Let us next consider the swept wing shown in Figure 12.12.2. The sweep angle is *Ω*, and the wing in this case is swept back. At any section of the wing, the total angular deflection vector of the flexible wing is

$$\boldsymbol{\theta} = \theta_x \mathbf{i} + \theta_z \mathbf{k}$$

where θ_z is the bending rotation of the elastic axis at the section and θ_x is the twist about the elastic axis. Alternatively, θ may be resolved into components along the body axes $\bar{x}\bar{z}$ of the airplane and written as

$$\boldsymbol{\theta} = \bar{\theta}_x \bar{\mathbf{i}} + \bar{\theta}_z \bar{\mathbf{k}}$$

12.12 Static Wing Divergence

Thus,

$$\bar{\theta}_x \bar{\mathbf{i}} + \bar{\theta}_z \bar{\mathbf{k}} = \theta_x \mathbf{i} + \theta_z \mathbf{k}$$

Taking the dot product of both sides of this equation with the unit vector $\bar{\mathbf{i}}$ in the spanwise direction, we get

$$\bar{\theta}_x = \theta_x \left(\mathbf{i} \cdot \bar{\mathbf{i}}\right) + \theta_z \left(\mathbf{k} \cdot \bar{\mathbf{i}}\right) = \theta_x \cos \Omega + \theta_z \sin \Omega$$

The magnitude of $\bar{\theta}_x$ is the change $\Delta\bar{\alpha}$ in the angle of attack at the section, measured in the plane parallel to the free stream velocity V. thus,

$$\Delta\bar{\alpha} = -\bar{\theta}_x$$

The minus sign accounts for the different sign conventions for angle of attack (clockwise positive) and twist (counterclockwise positive). The change $\Delta\bar{\alpha}$ in angle of attack, as measured in the plane normal to the elastic axis, is

$$\Delta\alpha = \Delta\bar{\alpha} \cos \Omega$$

12.12 Static Wing Divergence

Substituting Equations 12.12.15 and 12.12.14 into Equation 12.12.16 yields

$$\Delta\alpha = \overbrace{(-\theta_x)}^{\Delta\alpha)_{\text{torsion}}} + \overbrace{(-\theta_z \tan\Omega)}^{\Delta\alpha)_{\text{bending}}}$$

Since θ_z is generally positive and $\tan\Omega > 0$ in Figure 12.12.3, we see that the wing deflection due to bending decreases the angle of attack and therefore acts to prevent static wing divergence. On the other hand, if the wing is swept forward, then $\tan\Omega < 0$, and flexure of the wing due to lift acts to promote divergence.

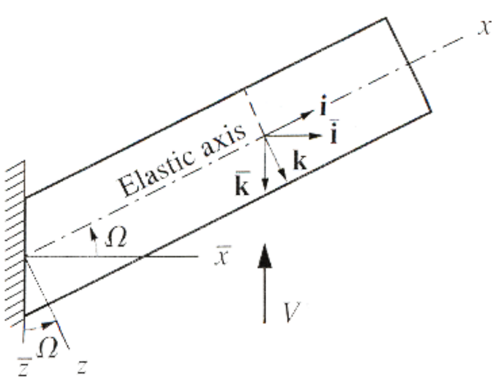


Figure 12.12.3 Plan view of a swept (back) wing.