458.308 Process Control & Design

Lecture 4b: Models for Control -- Complex Dynamics

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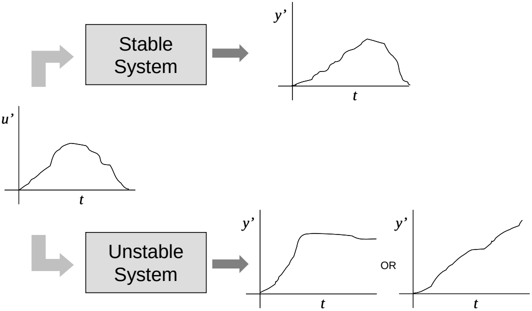
General Form of Transfer Function

$$\frac{Y(s)}{U(s)} = G(s) = \frac{N(s)}{D(s)} e^{-\theta s} = \gamma \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)} e^{-\theta s}$$

- Poles: Roots of the denominator polynomial $D(s)$.
- Zeros: Roots of the numerator polynomial $N(s)$.

$$G(s) = \frac{2s+1}{s^2+4s+3} \quad p_1 = -3,\ p_2 = -1, z_1 = -1/2$$
$$G(s) = \frac{5}{s^2-s+1} \quad p_1,\ p_2 = \frac{1 \pm \sqrt{3}j}{2}$$

Definition of Stability



- For linear systems, same as BIBO (Bounded-Input/Bounded-Output) stability.
- BIBO stability: All output variables are bounded when all input variables are bounded.

Stability of Linear (or Linearized) Systems

- If all poles have negative real part, the dynamics is stable.
- If any of the poles have positive or zero real part, the dynamics is unstable.

Transfer Function	Stability	Impulse Response
$\frac{1}{(s-1)(s+5)}$	Unstable	$Ae^{t} + Be^{-5t}$
$\frac{1}{s(s+5)}$	Unstable	$A + Be^{-5t}$
$\frac{1}{(s+2)(s+5)}$	Stable	$Ae^{-2t} + Be^{-5t}$

System Gain

$$\text{Gain} = \frac{\text{Output Change}}{\text{Input Change}} = \frac{y'(\infty)}{u'(\infty)}$$

- Step Change in the input of size $M \rightarrow$ Ultimate response in y?

$$y'(\infty) = \lim_{s \to 0} s \left(G(s) \frac{M}{s} \right) = \lim_{s \to 0} G(s) M$$

Hence, we get

$$\text{Gain} = \frac{y'(\infty)}{u'(\infty)} = \frac{\lim_{s \to 0} G(s) M}{M} = \lim_{s \to 0} G(s)$$

Upshot & Warning

- $G(0)$ is the gain!
- This works only when the dynamics is stable. For unstable dynamics, gain is ∞.

Examples

$$G(s) = \frac{1}{(s+2)(s+5)} \quad \text{Gain} = G(0) = \frac{1}{10}$$

$$G(s) = \frac{5s+2}{(6s+7)(7s^2+2s+5)} \quad \text{Gain} = G(0) = \frac{2}{35}$$

$$G(s) = \frac{1}{(s-2)(s+5)} \quad G(0) = -\frac{1}{10} \text{ but Gain}=\infty$$

Damping

- Underdamped dynamics:
  - Nonoscillatory input $\rightarrow$ Oscillatory response
- If the poles are complex numbers (w/ nonzero imaginary parts), the dynamics is underdamped.
- The imaginary part of the pole is the frequency of oscillation (rad/time).

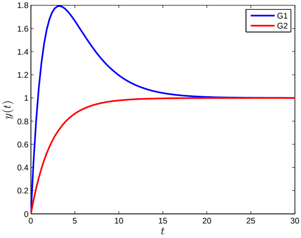
$G(s) = \frac{1}{s^2-2s+5}$	$p_1, p_2 = 1 \pm 2j$	Underdamped, unstable
$G(s) = \frac{1}{s^2+4s+3}$	$p_1 = -3,\ p_2 = -1$	Overdamped, stable
$G(s) = \frac{1}{s^2+2s+5}$	$p_1, p_2 = -1 \pm 2j$	Underdamped, stable

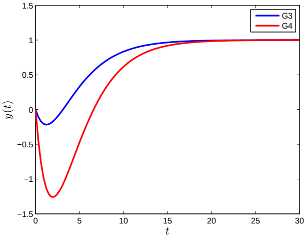
Overshoot and Inverse Response

- Existence of overshoot or inverse response can be determined from zeros of the transfer function.
  - **Overshoot**: a Left-Half-Plane (negative) zero closer to the origin than the dominant pole (the pole that's closest to the origin)
  - **Inverse response**: a Right-Half-Plane (positive) zero
  - The closer the RHP zero to the origin, the more pronounced the inverse response.

Examples

$G_1(s) = \frac{(10s+1)}{(3s+1)(2s+1)}$	$p_1 = -\frac{1}{3},\ p_2 = -\frac{1}{2},\ z_1 = -\frac{1}{10}$	LHP zero, Overshoot
$G_2(s) = \frac{(2.5s+1)}{(3s+1)(2s+1)}$	$p_1 = -\frac{1}{3},\ p_2 = -\frac{1}{2},\ z_1 = -\frac{1}{2.5}$	LHP zero, No Overshoot
$G_3(s) = \frac{(-2.5s+1)}{(3s+1)(2s+1)}$	$p_1 = -\frac{1}{3},\ p_2 = -\frac{1}{2},\ z_1 = \frac{1}{2.5}$	RHP zero, Inverse Response
$G_4(s) = \frac{(-10s+1)}{(3s+1)(2s+1)}$	$p_1 = -\frac{1}{3},\ p_2 = -\frac{1}{2},\ z_1 = \frac{1}{10}$	RHP zero, Bigger Inverse Response



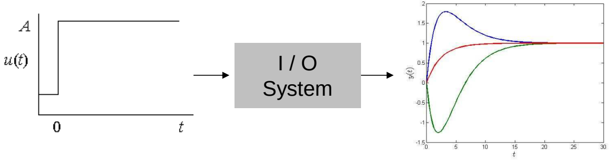


Speed of Response

- Speed of response is determined roughly by the dominant pole (the pole that's close to the origin), which corresponds to the slowest time constant.

$$\text{Settling time} \approx 3 \sim 5 \ \times \frac{1}{\text{dominant pole}}$$

2^{nd} Order System Plus a Zero



$$Y(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} U(s)$$

$$\leftrightarrow \tau_1 \tau_2 \frac{d^2 y}{dt^2} + 2(\tau_1 + \tau_2)\frac{dy}{dt} + y = K\left(\tau_a \frac{du}{dt} + u\right)$$

- Possible responses
  - Monotonic response (like the overdamped 2^{nd} order system)
  - Overshoot
  - Inverse response

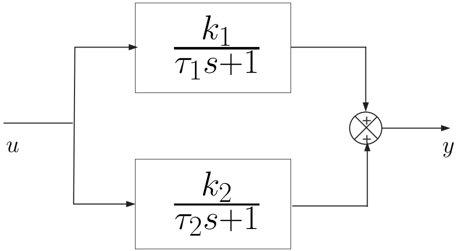
Effect of τ_a

- $\tau_a > \tau_1$: Overshoot
- $\tau_a \leq \tau_1$: Overdamped response with no overshoot
- $\tau_a < 0$: Inverse response (the initial response is the opposite direction to the final response).

Note: $\tau_1 > \tau_2$

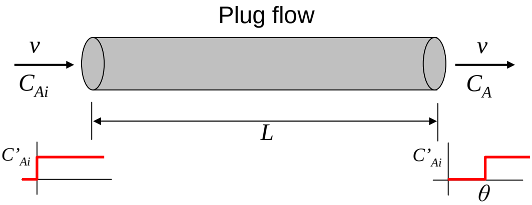
Exemplary Scenario?

Two first-order effects in parallel:



$$\frac{Y(s)}{U(s)} = \frac{k_1}{\tau_1 s + 1} + \frac{k_2}{\tau_2 s + 1} = \frac{(k_1 + k_2)\left(\frac{\tau_2 k_1 + \tau_1 k_2}{k_1 + k_2} s + 1\right)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

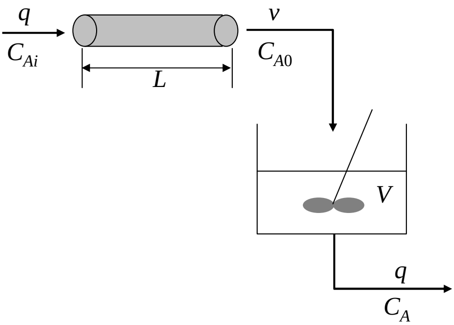
Transport Delays



$$C'_A(t) = C'_{Ai}(t-\theta) \xrightarrow{\mathcal{L}} C'_A(s) = e^{-\theta s} C'_{Ai}(s)$$

$\theta = \frac{L}{v}$: dead time or transport delay

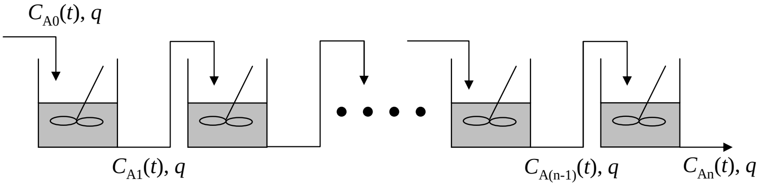
First-Order-Plus-Delay System



$$C'_A(s) = \frac{K}{\tau s + 1} C'_{A0}(s), \quad C'_{A0}(s) = e^{-\theta s} C'_{Ai}(s)$$

$$C'_A(s) = \frac{K}{\tau s + 1} e^{-\theta s} C'_{Ai}(s), \quad K = 1,\ \tau = \frac{V}{q},\ \theta = \frac{A \cdot L}{q}$$

Approximating a High Order System with a Delay



$$C'_{A(i+1)}(s) = \frac{1}{\tau s+1} C'_{Ai}(s) \quad \Rightarrow \quad C'_{An}(s) = \frac{1}{(\tau s+1)^n} C'_{A0}(s)$$

$$\approx \quad e^{-(n\tau)s} C'_{A0}(s) \qquad \text{(for large } n\text{)}$$

Note: $e^x \approx 1 + x$, $\tau = V/q$

Development of Empirical Models from Process Data

- In some situations, it is not feasible to develop a theoretical (physically-based) model due to:
  - Lack of information
  - Model complexity
  - Engineering effort required
- An attractive alternative: Develop an empirical dynamic model from input-output data
  - Advantage: less effort is required
  - Disadvantage: the model is only valid (at best) for the range of data used in its development
    ``Empirical models usually don't extrapolate very well."

Fitting First-Order / Second-Order Model Using Step Tests

- Simple TF models can be obtained graphically from step response data.
- Process reaction curve: a plot of the output response of a process to a step change input
- If the process of interest can be approximated by a first- or second-order linear model, the model parameters can be obtained by inspection of the process reaction curve.

First-Order Model

$$
\begin{aligned}
Y(s) &= \frac{K}{\tau s + 1} U(s) \\
y(t) &= KM(1 - e^{-t/\tau})
\end{aligned}
$$

1. Gain K: $\frac{\Delta y}{M}$ at steady state
2. Time constant τ:
   - $\frac{d}{dt}\left(\frac{y}{KM}\right)_{t=0} = \frac{1}{\tau}$
   - or $\tau = t\big|_{(y=0.632\times y_{ss})}$

First-Order Plus Time Delay Model

$$G(s) = \frac{Ke^{-\theta}s}{\tau s + 1}$$

For this FOPTD model, we note the following characteristics of its step response:

1. The response attains 63.2% of its final response at time, $t = \tau + \theta$
2. The line drawn tangent to the response at maximum slope ($t = \theta$) intersects the $y/KM = 1$ line at $t = \tau + \theta$.
3. The step response is essentially complete at $t = 5\tau$. In other words, the settling time is $t_s = 5\tau$.