Iso-Parametric Formulation

Outlines

- It makes formulations for computer program simple

- It allows to create elements with a shape of a straight line or a curved surface.
  Make it possible to choose a variety of factors.

- We will derive the stiffness matrix of simple beam elements and rectangular elements using an iso-parametric formulation.

- Numerical integration: We will calculate the stiffness matrix of rectangular elements that is made using an iso-parametric formulation.

- Finally, we will consider several higher-order elements and shape functions.

1 Iso_parametric formulation: Stiffness matrix of a beam element

The term of iso-parametric formulation comes from the usage of shape functions [N] which is used to determine an element shape for approximation of deformation.

- If a deformation function is $u = a_1 + a_2 s$, use a node $x = a_1 + a_2 s$ on a beam element.
- It is formulated using the natural (or intrinsic) coordinate system, s, defined by geometry of elements. A transformation mapping is used for the element formulation between natural coordinate system, s, and global coordinate system, x.

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Step 1: Determination of element type

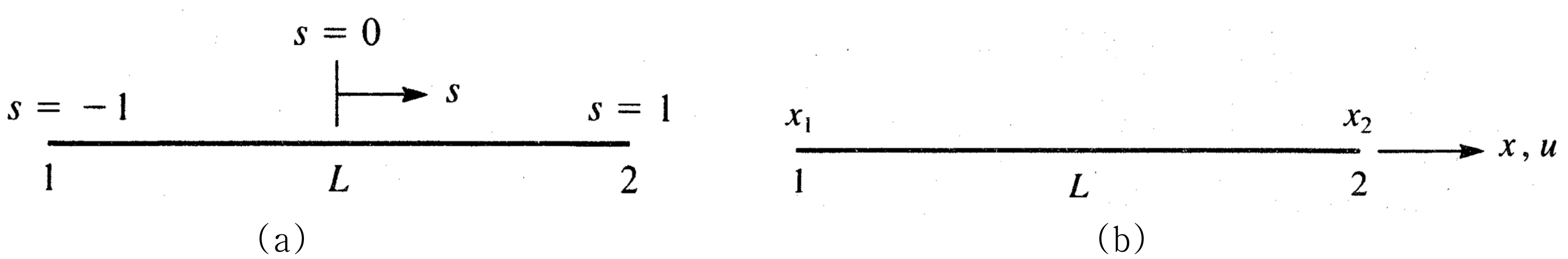


Fig. 1: Linear beam element at node x in (a) natural coordinate system, s, (b) global coordinate system, x.

Relation between s and x coordinate systems: (when s and x coordinate systems are parallel)

$$x = x_c + \frac{L}{2}s$$ x_c indicates center of element

x can be expressed as a function of x_1 and x_2

$$x = \frac{1}{2}[(1-s)x_1 + (1+s)x_2] = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Then shape functions are

$$N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$

Note : $N_1 + N_2 = 1$

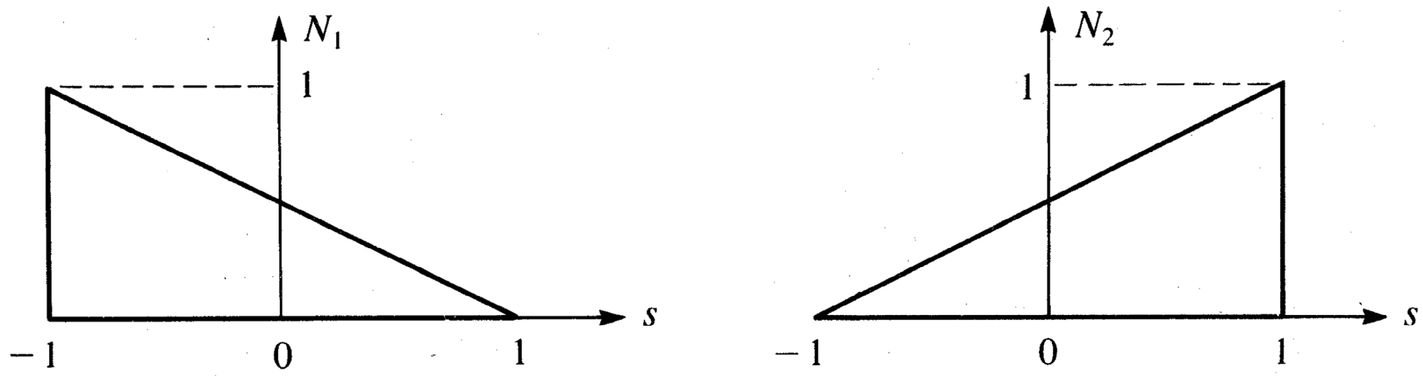


Fig. 2: Shape functions in natural coordinate system

Step 2: Determination of deformation function $\{u\} = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

u and x are called iso-parameter because they are defined by the same shape function at the same node.

Step 3: Definition of strain-displacement and stress-strain relations

Calculation of element matrix $[B]$:

- By chain rule $\dfrac{du}{ds} = \dfrac{du}{dx}\dfrac{dx}{ds} \Rightarrow \dfrac{du}{dx} = \dfrac{\left(\dfrac{du}{ds}\right)}{\left(\dfrac{dx}{ds}\right)} = \dfrac{= [-\frac{1}{2}, \ \frac{1}{2}]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}{\left(\dfrac{L}{2}\right)}$

$$\therefore \quad \{\varepsilon_x\} = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- Therefore, $\{\varepsilon\} = [B]\{d\}$ $\qquad [B] = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix}$

Step 4: Calculation of element stiffness matrix

Eelement stiffness matrix: $[k] = \int_0^L [B]^T [D][B] A\, dx$

- In general, matrix $[B]$ is a function of s: $\int_0^L f(x)\, dx = \int_{-1}^1 f(s) |\underline{J}|\, ds$

  where $\underline{J}$ is Jacobian.

  In case of 1-D, $|\underline{J}| = \underline{J}$. In case of simple beam element : $|\underline{J}| = \dfrac{dx}{ds} = \dfrac{L}{2}$

  Ratio of element's length between global and natural coordinate systems

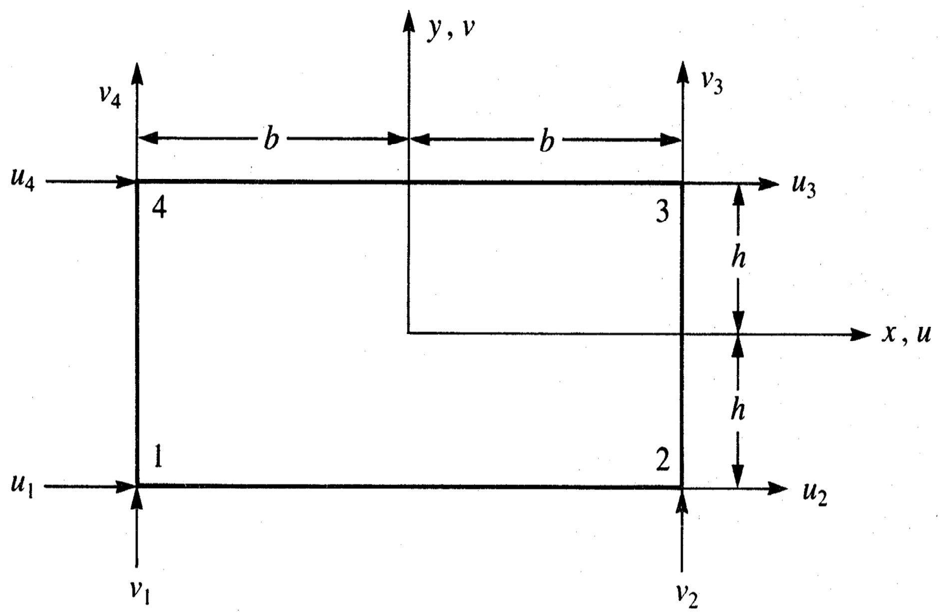
- Stiffness matrix in a natural coordinate system:

$$[k] = \frac{L}{2} \int_{-1}^1 [B]^T E [B] A\, ds = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2 Rectangular plane stress element

Characteristics of rectangular element:
- It is easy to input data, and it is simple to calculate stress.
- Physical boundary conditions are not well approximated at the edge of rectangle.

Step 1: Determination of element type – using natural coordinate (x, y)



$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad (11.2.1)$$

Four node rectangular element and nodal displacement

Step 2: Determination of deformation function – element deformation functions, u and v, are linear along the rectangular corner

$$u(x,y)=a_1+a_2x+a_3y+a_4xy$$
$$v(x,y)=a_5+a_6x+a_7y+a_8xy$$

$$\Rightarrow$$

$$u(x,y)=\frac{1}{4bh}[(b-x)(h-y)u_1+(b+x)(h-y)u_2+(b+x)(h+y)u_3+(b-x)(h+y)u_4]$$
$$v(x,y)=\frac{1}{4bh}[(b-x)(h-y)v_1+(b+x)(h-y)v_2+(b+x)(h+y)v_3+(b-x)(h+y)v_4]$$

$$\therefore\ \{\psi\}=\begin{Bmatrix}u\\v\end{Bmatrix}=[N]\{d\}=\begin{bmatrix}N_1&0&N_2&0&N_3&0&N_4&0\\0&N_1&0&N_2&0&N_3&0&N_4\end{bmatrix}\{d\}$$

where shape functions are

$$N_1=\frac{(b-x)(h-y)}{4bh}\qquad N_2=\frac{(b+x)(h-y)}{4bh}$$
$$N_3=\frac{(b+x)(h+y)}{4bh}\qquad N_4=\frac{(b-x)(h+y)}{4bh}$$

Step 3: Definition of strain-displacement and stress-strain relationships

Element strain in a 2-D stress state:

$$\{\varepsilon\} \equiv \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = [B]\{d\}$$

where

$$[B] = \frac{1}{4bh}\begin{bmatrix} -(h-y) & 0 & (h-y) & 0 & (h+y) & 0 & -(h+y) & 0 \\ 0 & -(b-x) & 0 & -(b+x) & 0 & (b+x) & 0 & (b-x) \\ -(b-x) & -(h-y) & -(b+x) & (h-y) & (b+x) & (h+y) & (b-x) & -(h+y) \end{bmatrix}$$

Step 4: Calculation of element stiffness matrix and element equation

Element stiffness matrix: $[k] = \int_{-h}^{h}\int_{-b}^{b}[B]^T[D][B]t\,dxdy$

Element force matrix: $\{f\} = \iiint_V [N]^T\{X\}\,dV + \{P\} + \iint_s [N]^T\{T\}dS$

Element equation: $\{f\} = [k]\{d\}$

Step 5,6, and 7

Step 5, 6, and 7 are constitution of global stiffness matrix, determinant of unknown deformation, calculation of stress. However, stress in each element varies in all directions of x and y.

3 Iso-parametric formulation: stiffness matrix of a plane element

A process of iso-parametric formulation is same in all elements

Step 1: Determination of element type

It is possible to numerically integrate the rectangular element defined in natural coordinate system $s-t$.

Transformation equation: $x = x_c + bs \qquad y = y_c + ht$

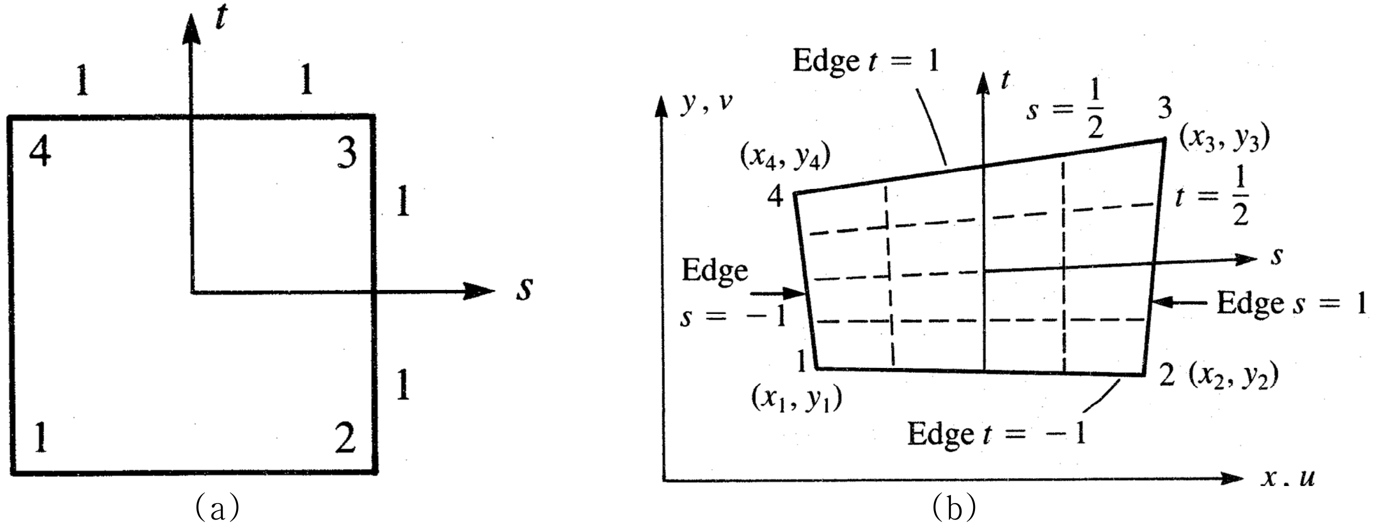


Fig. 4: (a) A linear rectangular element in a coordinate system, $s-t$, (b) A rectangular element in a coordinate system, $x-y$, The size and shape of the rectangular element are defined by coordinates of four nodes.

Transformation equation between a local coordinate system, $s-t$, and a global coordinate system, $x-y$:

$$\begin{aligned} x &= a_1 + a_2 s + a_3 t + a_4 st \\ y &= a_5 + a_6 s + a_7 t + a_8 st \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= \frac{1}{4}[(1-s)(1-t)x_1 + (1+s)(1-t)x_2 \\ &\quad + (1+s)(1+t)x_3 + (1-s)(1+t)x_4] \\ y &= \frac{1}{4}[(1-s)(1-t)y_1 + (1+s)(1-t)y_2 \\ &\quad + (1+s)(1+t)y_3 + (1-s)(1+t)y_4] \end{aligned}$$

In a matrix form:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

$$N_1 = \frac{(1-s)(1-t)}{4}$$

$$N_2 = \frac{(1+s)(1-t)}{4}$$

$$N_3 = \frac{(1+s)(1+t)}{4}$$

$$N_4 = \frac{(1-s)(1+t)}{4}$$

1. Shape function is linear.
2. Any point in rectangular element $(s,\ t)$ can be mapped to the quadrilateral element point $(x,\ y)$ in Fig. 4(b).
3. Note that for all values of s and t, $N_1 + N_2 + N_3 + N_4 = 1$.
4. N_i (i = 1, 2, 3, 4) is 1 for node i, and 0 for the other nodes.

Two general conditions of shape functions:

1. $\sum_{i=1}^{n} N_i = 1 \qquad (i = 1,2,\ldots,n)$
2. $N_i = 1$ for node i, $N_i = 0$ for the other nodes.

Additional conditions:

3. Continuity of deformation --- Lagrangian Interpolation
4. Continuity of slope --- Hermitian Interpolation

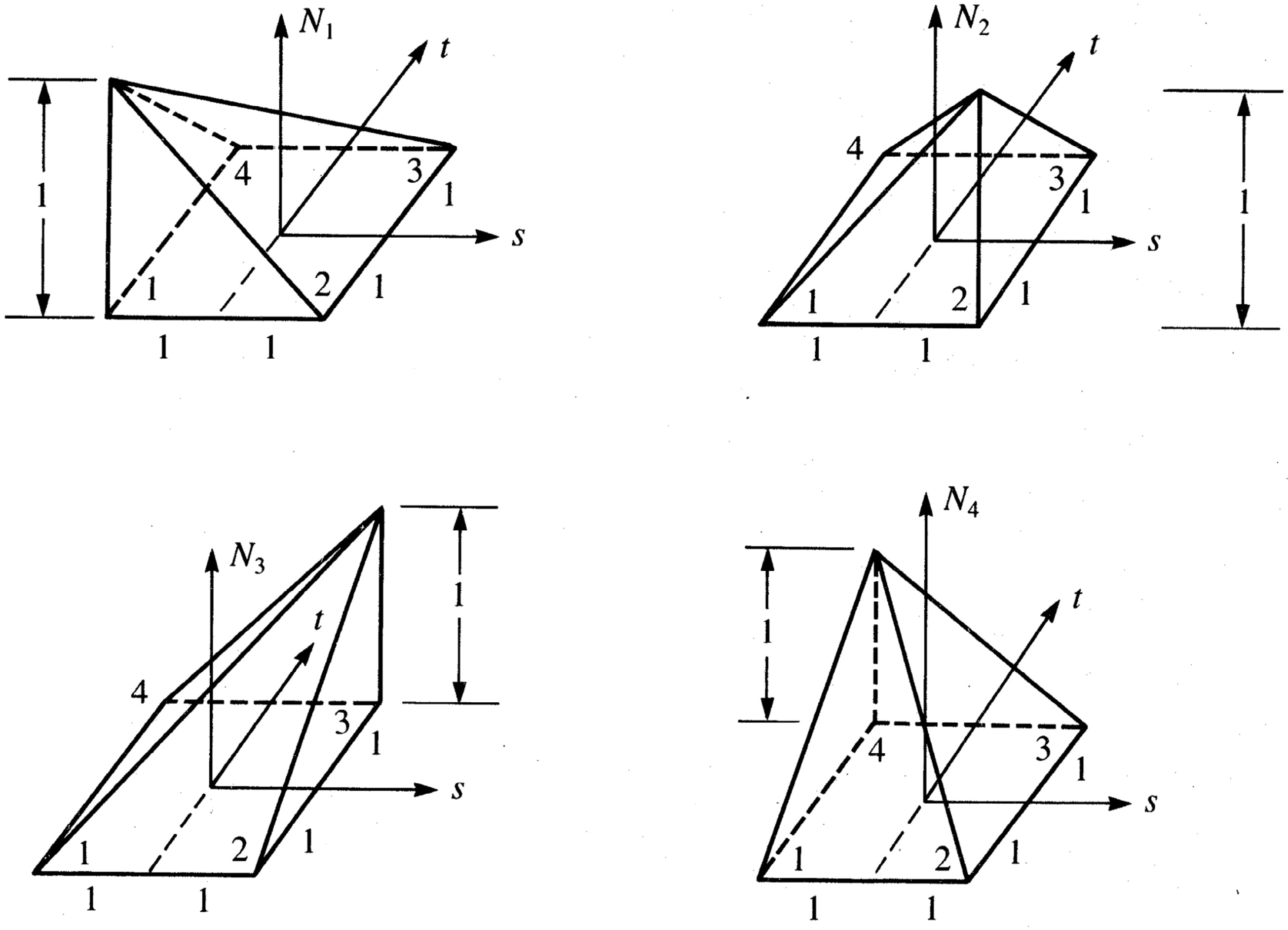


Fig. 5: Change of shape functions in a linear rectangular element

Step 2: Determination of deformation

Deformation functions in the element are defined by shape functions that are used to define element shape.

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Step 3: Strain-displacement and stress-strain relationships

The derivative of deformation u and v about x and y should be executed using a chain rule of derivation because the deformation function is expressed with s and t.

Reference: chain rule of f

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Calculating $(\partial f / \partial x)$ and $(\partial f / \partial y)$ using Cramer's lure (Appendix. B).

$$\frac{\partial f}{\partial x} = \frac{1}{|\underline{J}|}\begin{vmatrix} \frac{\partial f}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}, \quad \frac{\partial f}{\partial y} = \frac{1}{|\underline{J}|}\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial f}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t} \end{vmatrix} \quad \text{where} \quad |\underline{J}| = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} \qquad (*)$$

Element strain:

$$\underline{\varepsilon} \equiv \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial()}{\partial x} & 0 \\ 0 & \frac{\partial()}{\partial y} \\ \frac{\partial()}{\partial y} & \frac{\partial()}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \underline{Bd}$$

A formulation to obtain $\underline{B}$ is required.

Using the equation (*) in previous page (use u or v instead of f):

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial ()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial ()}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial ()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial ()}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial ()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial ()}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial ()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial ()}{\partial t} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

or $\underline{\varepsilon} = \underline{D'}\,\underline{Nd}$ where

$$\underline{D'} = \frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial ()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial ()}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial ()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial ()}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial ()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial ()}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial ()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial ()}{\partial t} \end{bmatrix}$$

Thus,

$$\underset{(3\times 8)}{\underline{B}} = \underset{(3\times 2)}{\underline{D'}}\ \underset{(2\times 8)}{\underline{N}}$$

Step 4: Derivation of element stiffness matrix and equation

Stiffness matrix in a coordinate system, $s-t$:

$$[k]=\iint_A [B]^T[D][B]tdxdy$$

Converge the integral region from $x-y$ to $s-t$:

$$[k]=\int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]t|J|dsdt$$

Determinent $|\underline{J}|$ is

$$|\underline{J}|=\frac{1}{8}\{X_c\}^T\begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix}\{Y_c\}$$

where $\{X_c\}^T=[x_1\ x_2\ x_3\ x_4]$, $\{Y_c\}=\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$

$|\underline{J}|$ **is a function of** s **,** t **in natural coordinate system, and** $x_1, x_2, \ldots, y_4$ **in the known global coordinate system**.

Calculation of $\underline{B}$: $$\underline{B}(s,t) = \frac{1}{|\underline{J}|}[\underline{B}_1 \ \ \underline{B}_2 \ \ \underline{B}_3 \ \ \underline{B}_4]$$

where

$$\underline{B}_i = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix} \qquad i = 1,\ 2,\ 3,\ 4$$

and

$$a = \frac{1}{4}[y_1(s-1) + y_2(-1-s) + y_3(1+s) + y_4(1-s)]$$

$$b = \frac{1}{4}[y_1(t-1) + y_2(1-t) + y_3(1+t) + y_4(-1-t)]$$

$$c = \frac{1}{4}[x_1(t-1) + x_2(1-t) + x_3(1+t) + x_4(-1-t)]$$

$$d = \frac{1}{4}[x_1(s-1) + x_2(-1-s) + x_3(1+s) + x_4(1-s)]$$

For example, $N_{1,s} = \frac{1}{4}(t-1) \qquad N_{1,t} = \frac{1}{4}(s-1) \qquad (etc.)$

Element body force matrix:

$$\underset{(8\times 1)}{\{f_b\}} = \int_{-1}^{1}\int_{-1}^{1} \underset{(8\times 2)}{[N]^T} \ \underset{(2\times 1)}{\{X\}} \ t|\underline{J}| \, ds\, dt$$

Element surface force matrix: Length is L , an edge $t=1$ (See. Fig. 4(b))

$$\underset{(4\times 1)}{\{f_s\}} = \int_{-1}^{1} \underset{(4\times 2)}{[N]^T} \ \underset{(2\times 1)}{\{T\}} \ t\frac{L}{2} ds \quad , \quad \begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4r} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{bmatrix}^T \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} t\frac{L}{2} ds$$

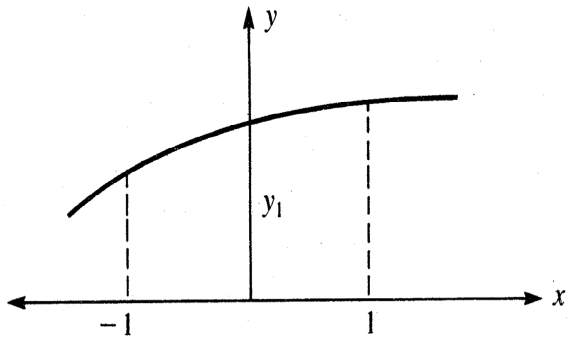
For $N_1 = 0$ and $N_2 = 0$ along the edge $t = 1$, the nodal force is zero at nodes 1 and 2.

4 Gaussian Quadrature (Numerical integration)

One node Gaussian quadrature

$$I = \int_{-1}^{1} y dx \approx y_1 * \{(1)-(-1)\}$$
$$= 2y_1$$

If function y is straight line,
it has exact solution.



General equation: $$I = \int_{-1}^{1} y\,dx = \sum_{i=1}^{n} W_i y_i$$

- Gaussian quadrature using n nodes(Gaussian point) can exactly calculate polynomial equation which has integral term under $2n-1$ order.

When function $f(x)$ is not a polynomial, Gaussian quadrature is inaccurate. However, the more Gaussian points are used, the more accurate solution is. In general, the ratio of two polynomials is not a polynomial.

- Table 1 Gaussian points for integration from -1 to +1

Number of Points	*Locations*, x_i	*Associated Weights*, W_i
1	$x_1 = 0.000...$	2.000
2	$x_1, x_2 = \pm 0.57735026918962$	1.000
3	$x_1, x_3 = \pm 0.77459666924148$ $x_2 = 0.000...$	$5/9 = 0.555...$ $8/9 = 0.888...$

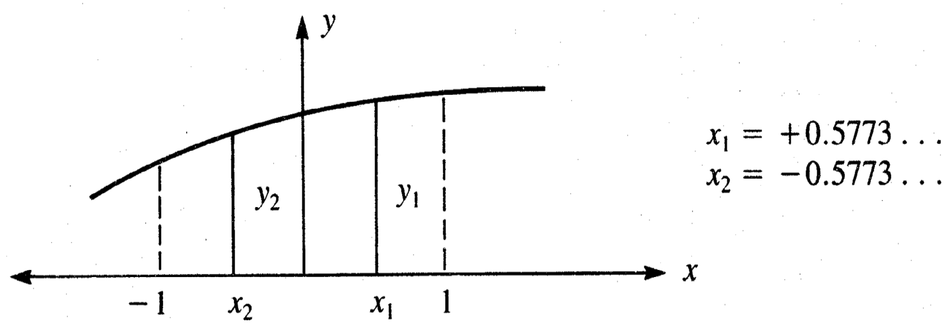


Fig. 7: Gaussian quadrature with two extraction points

2-D problem: Integrate about second coordinate after integrate about first coordinate.

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt = \int_{-1}^{1}\left[\sum_i W_i f(s_i,t)\right]$$

$$= \sum_j W_j\left[\sum_i W_i f(s_i,t_j)\right] = \sum_i\sum_j W_i W_j f(s_i,t_j)$$

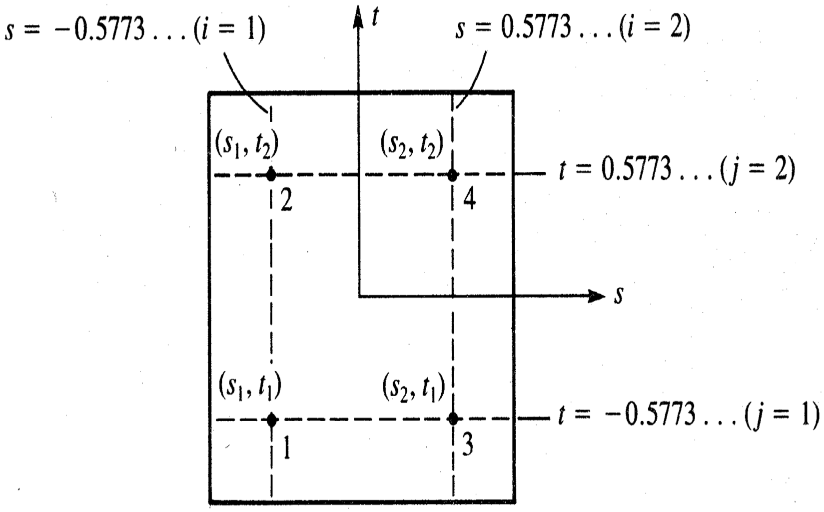
For 2×2: $I = W_1W_1 f(s_1,t_1) + W_1W_2 f(s_1,t_2) + W_2W_1 f(s_2,t_1) + W_2W_2 f(s_2,t_2)$

where the sample four points are located at

$s_i,\ t_i = \pm 0.5773....$

$= \pm 1/\sqrt{3}$

And the all weight factors are 1.000 . Thus, the two summation marks can be interpreted as one summation mark for four points of the rectangle.



3-D problem: $$I=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(s,t,z)\,ds\,dt\,dz=\sum_i\sum_j\sum_k W_iW_jW_k f(s_i,t_j,z_k)$$

NOTE: If the integration limit is $\int_0^1 f(x)dx=\sum_{i=1}^n W_i f(x_i)$, the weight factor W_i and the location x_i are different from that of the integration limit which is between -1 and 1 (See table 2).

Table 2. Gaussian points of the four node gaussian integration (integration from 0 to 1)

Locations, x_i	*Associated Weights*, W_i
0.0693185	0.1739274
0.3300095	0.3260725
0.6699905	0.3260725
0.9305682	0.1739274

Example 1: Calculate the integration of $\sin \pi x$ using numerical integration.

$$I = \int_0^1 \sin \pi x \, dx$$

Using table 2, the following can be obtained.

$$\begin{aligned} I &= \sum_{i=1}^{4} W_i \sin \pi x_i \\ &= W_1 \sin \pi x_1 + W_2 \sin \pi x_2 + W_3 \sin \pi x_3 + W_4 \sin \pi x_4 \\ &= 0.1739 \sin \pi(0.0694) + 0.3261 \sin \pi(0.3300) \\ &\quad 0.3261 \sin \pi(0.6700) + 0.1739 \sin \pi(0.9306) \\ &= 0.6366 \end{aligned}$$

Use four decimal places. The exact value of direct integration is 0.6366. Note that location x_i and weight factor W_i are different from that in table 2 if we use the 3-points Gaussian integration.

5 Calculation of stiffness matrix by Gaussian integration

Element stiffness matrix in 2-D:

$$\underline{k} = \iint_A \underline{B}^T(x,y)\,\underline{D}\,\underline{B}(x,y)\,t\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} \underline{B}^T(s,t)\,\underline{D}\,\underline{B}(s,t)\,|\underline{J}|\,t\,ds\,dt$$

The integral term $\underline{B}^T\underline{D}\underline{B}|\underline{J}|$, which is a function of (s,t) , is calculated by the numerical integration.

Using four-points Gaussian integration,

$$\begin{aligned}\underline{k} = {} & \underline{B}^T(s_1,t_1)\underline{D}\underline{B}(s_1,t_1)|\underline{J}(s_1,t_1)|tW_1W_1 \\ & + \underline{B}^T(s_2,t_2)\underline{D}\underline{B}(s_2,t_2)|\underline{J}(s_2,t_2)|tW_2W_2 \\ & + \underline{B}^T(s_3,t_3)\underline{D}\underline{B}(s_3,t_3)|\underline{J}(s_3,t_3)|tW_3W_3 \\ & + \underline{B}^T(s_4,t_4)\underline{D}\underline{B}(s_4,t_4)|\underline{J}(s_4,t_4)|tW_4W_4\end{aligned}$$

where $s_1 = t_1 = -0.5773, s_2 = -0.5773, t_2 = 0.5773, s_3 = 0.5773, t_3 = -0.5773,$
$s_4 = t_4 = 0.5773$, $W_1 = W_2 = W_3 = W4 = 1.000$.

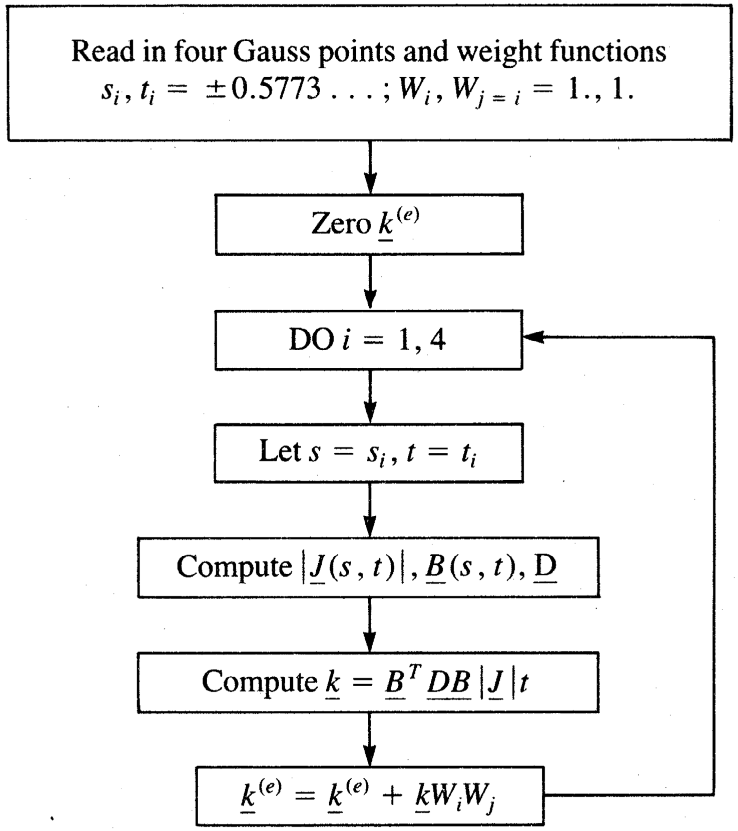
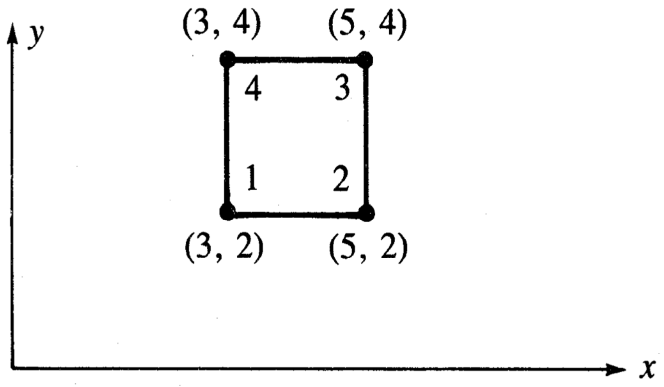


Fig. 9: Flow chart for obtaining $\underline{k}^{(e)}$ using Gaussian integration

Example 2



Calculate the stiffness matrix of rectangular element using four-point Gaussian integration.

$E = 30 \times 106\, psi$, $\nu = 0.25$.

The unit of length in global coordinate system is inch, and $t = 1\ in$.

Fig. 10: Quadrilateral elements for calculation of stiffness

Using 4-points rule:

$$
\begin{aligned}
(s_1, t_1) &= (-0.5733,\ -0.5773) \qquad & W_1 &= 1.0 \\
(s_2, t_2) &= (-0.5733,\ 0.5773) & W_2 &= 1.0 \\
(s_3, t_3) &= (0.5733,\ -0.5773) \quad , & W_3 &= 1.0 \quad . \\
(s_4, t_4) &= (0.5733,\ 0.5773) & W_4 &= 1.0
\end{aligned}
$$

Calculation of stiffness matrix:

$$\begin{aligned}
\underline{k} = \underline{B}^T&(-0.5773,-0.5733)\underline{D}\underline{B}(-0.5773,-0.5773)\\
&\times\left|\underline{J}(-0.5773,-0.5773)\right|(1)(1.000)(1.000)\\
&+\underline{B}^T(-0.5773,0.5773)\underline{D}\underline{B}(-0.573,0.5773)\\
&\times\left|\underline{J}(-0.5773,0.5773)\right|(1)(1.000)(1.000)\\
&+\underline{B}^T(0.5773,-0.5773)\underline{D}\underline{B}(0.573,-0.5773)\\
&\times\left|\underline{J}(0.5773,-0.5773)\right|(1)(1.000)(1.000)\\
&+\underline{B}^T(0.5773,0.5773)\underline{D}\underline{B}(0.573,0.5773)\\
&\times\left|\underline{J}(0.5773,0.5773)\right|(1)(1.000)(1.000)
\end{aligned}$$

We need to calculate $|\underline{J}|$ and $\underline{B}$ at Gaussian points
$(s_1,t_1)=(-0.5733,\ -0.5773)$, $(s_2,t_2)=(-0.5733,\ 0.5773)$,
$(s_3,t_3)=(0.5733,\ -0.5773)$, $(s_4,t_4)=(0.5733,\ 0.5773)$.

Calculation of $|\underline{J}|$:

$$|\underline{J}(-0.5773,-0.5773)| = \frac{1}{8}[3\ \ 5\ \ 5\ \ 3]$$

$$\times\begin{bmatrix} 0 & 1-(-0.5773) & -0.5773-(-0.5773) & -0.5773-1 \\ -0.5773-1 & 0 & -0.5773+1 & -0.5773-(-0.5773) \\ -0.5773-(-0.5773) & -0.5773-1 & 0 & -0.5773+1 \\ 1-(-0.5773) & -0.5773+(-0.5773) & -0.5773-1 & 0 \end{bmatrix}$$

$$\times\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix} = 1.000$$

Similarly,

$$|\underline{J}(-0.5733,-0.5733)| = 1.000$$
$$|\underline{J}(0.5733,-0.5733)| = 1.000$$
$$|\underline{J}(0.5733,0.5733)| = 1.000$$

Generally $|\underline{J}| \neq 1$, and it changes within the element.

Calculation of $\underline{B}$:

$$\underline{B}(-0.5733,-0.5733)=\frac{1}{\left|\underline{J}(-0.5733,-0.5733)\right|}\begin{bmatrix}\underline{B}_1 & \underline{B}_2 & \underline{B}_3 & \underline{B}_4\end{bmatrix}$$

Calculation of $\underline{B}_1$:

$$\underline{B}_1=\begin{bmatrix} aN_{1,s}-bN_{1,t} & 0 \\ 0 & cN_{1,t}-dN_{1,s} \\ cN_{1,t}-dN_{1,s} & aN_{1,s}-bN_{1,t}\end{bmatrix}$$

where

$$\begin{aligned} a &= \frac{1}{4}\left[y_1(s-1)+y_2(-1-s)+y_3(1+s)+y_4(1-s)\right] \\ &= \frac{1}{4}\big[2(-0.5773-1)+2(-1-0.5773)) \\ &\quad +4(1+(-0.5773))+4(1-(--0.5773))\big] \\ &= 1.00 \end{aligned}$$

The same calculation can be used to obtain b,c,d .

Also,

$$N_{1,s} = \frac{1}{4}(t-1) = \frac{1}{4}(-0.5773-1) = -0.3943$$

$$N_{1,t} = \frac{1}{4}(s-1) = \frac{1}{4}(-0.5773-1) = -0.3943$$

Similarly, $\underline{B}_2$, $\underline{B}_3$, $\underline{B}_4$ can be calculated at $(-0.5773,-0.5773)$. And calculate $\underline{B}$ repeatedly at other Gaussian points.

Generally a computer program is used to calculate $\underline{B}$ and $\underline{k}$.
Final form of $\underline{B}$ is.

$$\underline{B} = \begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0 & -0.1057 & 0 & -0.3943 \\ -0.1057 & -0.1057 & -0.3743 & 0.1057 & 0.3943 & 0 & -0.3943 & 0 \\ 0 & 0.3943 & 0 & 0.1057 & 0.3943 & 0.3943 & 0.1057 & -0.3943 \end{bmatrix}$$

Matrix $\underline{D}$:

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12 \end{bmatrix} \times 10^6\, psi$$

Finally, the stiffness matrix $\underline{k}$:

$$\underline{k} = 10^4 \begin{bmatrix} 1466 & 500 & -866 & -99 & -733 & -500 & 133 & 99 \\ 500 & 1466 & 99 & 133 & -500 & -733 & -99 & -866 \\ -866 & 99 & 1466 & -500 & 133 & -99 & -733 & 500 \\ -99 & 133 & -500 & 1466 & 99 & -866 & 500 & -733 \\ -733 & -500 & 133 & 99 & 1466 & 500 & -866 & -99 \\ -500 & -733 & -99 & -866 & 500 & 1466 & 99 & 133 \\ 133 & -99 & -733 & 500 & -866 & 99 & 1466 & -500 \\ 99 & -866 & 500 & -733 & -99 & 133 & -500 & 1466 \end{bmatrix}$$

6 Higher order shape function

- Higher order shape function can be obtained by adding additional nodes to the each side of the linear element.
- It has higher order strain distribution in element, and it converges to the exact solution rapidly with few elements.
- It can more accurately approximate the irregular boundary shape.

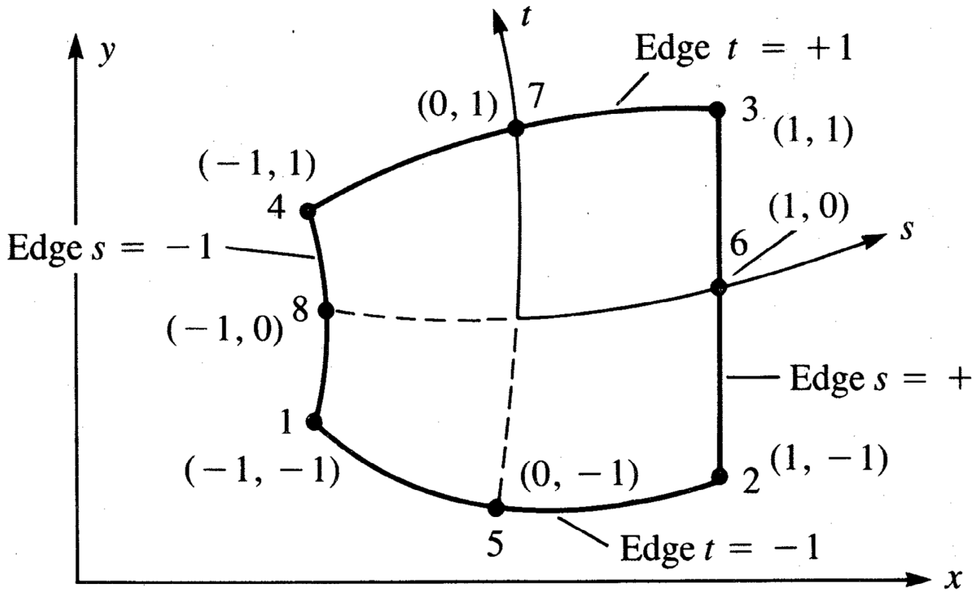


Fig. 11: 2nd order iso-parametric element

Second order iso-parametric element:

$$x = a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2 + a_6 t^2 + a_7 s^2 t + a_8 st^2$$
$$y = a_9 + a_{10} s + a_{11} t + a_{12} st + a_{13} s^2 + a_{14} t^2 + a_{15} s^2 t + a_{16} st^2$$

For the corner node (i =1, 2, 3, 4)

$$N_1 = \frac{1}{4}(1-s)(1-t)(-s-t-1)$$
$$N_2 = \frac{1}{4}(1+s)(1-t)(s-t-1)$$
$$N_3 = \frac{1}{4}(1+s)(1+t)(s+t-1)$$
$$N_4 = \frac{1}{4}(1-s)(1+t)(-s+t-1)$$

or

$$N_i = \frac{1}{4}(1+ss_i)(1+tt_i)(ss_i + tt_i - 1)$$
$$s_i = -1,\ 1,\ 1,\ -1 \qquad for\ i = 1,\ 2,\ 3,\ 4$$
$$t_i = -1, -1,\ 1,\ 1 \qquad for\ i = 1,\ 2,\ 3,\ 4$$

For the middle node (i =5, 6, 7, 8),

$$N_5 = \frac{1}{2}(1-t)(1+s)(1-s)$$

$$N_6 = \frac{1}{2}(1+s)(1+t)(1-t)$$

$$N_7 = \frac{1}{2}(1+t)(1+s)(1-s)$$

$$N_8 = \frac{1}{2}(1-s)(1+t)(1-t)$$

or

$$N_i = \frac{1}{2}(1-s^2)(1+tt_i) \qquad t_i = -1,1 \quad for\ i = 5,\ 7$$

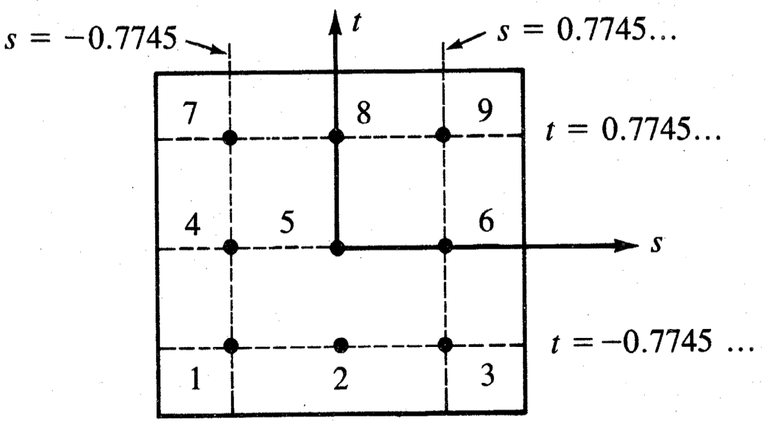
$$N_i = \frac{1}{2}(1-ss_i)(1-t^2) \qquad s_i = -1,1 \quad for\ i = 5,\ 7$$

When edge shape and displacement are function of s^2 (if t is constant) or t^2 (if s is constant), it satisfies the general shape function conditions.

Deformation function:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_8 \end{Bmatrix}$$

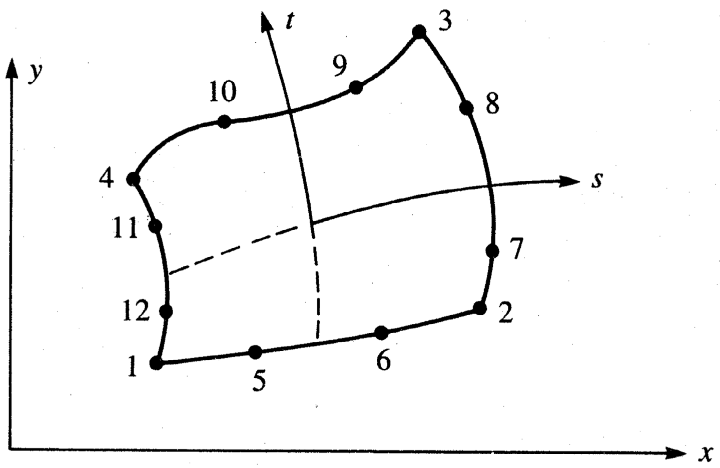
Strain matrix: $\varepsilon = \underline{B}d = \underline{D}'\underline{N}\underline{d}$



2^{nd} order iso-parameter with 8 nodes
For the calculation of $\underline{B}$ and $\underline{k}$,
9-points Gaussian rule is used
(3×3 rule). There is large difference
between 2×2 and 3×3 rule, and 3×3
rule is generally recommended.
(Bathe and Wilson[7])

]

3rd order iso-parametric element:



Shape function of a 3rd order element is based on incomplete 4th order polynomial (see reference [3]).

$$x = a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2 + a_6 t^2 + a_7 s^2 t + a_8 st^2 + a_9 s^3 + a_{10} t^3 + a_{11} s^3 t + a_{12} st^3$$

y also has same polynomial equation.

For the corner nodes ($i = 1, 2, 3, 4$): $$N_i = \frac{1}{32}(1+ss_i)(1+tt_i)[9(s^2+t^2)-10]$$

where $$s_i = -1, 1, 1, -1 \quad \textit{for } i = 1, 2, 3, 4$$ $$t_i = -1, -1, 1, 1 \quad \textit{for } i = 1, 2, 3, 4$$

For the nodes ($i = 7, 8, 11, 12$) when $s = \pm 1$: $$N_i = \frac{9}{32}(1+ss_i)(1+9tt_i)(1-t^2)$$

where $s_i = \pm 1$, $t_i = \pm\frac{1}{3}$

For the nodes ($i = 5, 6, 9, 10$) when $t = \pm 1$: $$N_i = \frac{9}{32}(1+tt_i)(1+9ss_i)(1-s^2)$$

where $t_i = \pm 1$, $s_i = \pm\frac{1}{3}$

When the shape function of coordinates has lower order than that of deformation, it is called Subparametric formulation (For example, x is linear, u is 2nd order function). The opposite way is called Superparametric formulation.