Chapter 3 Review of 3 Dimensional Nonlinear Elasticity

Contents:

3.1 Definition of strain

Green strain, Almansi strain (Hamel strain)

1st Piolar-Kirchhoff strain, 2nd Piolar-Kirchhoff strain (PK-strains),

Cauchy stress..

We are going to propose means of expressing the deformation of a body.

Let us consider the motion of a body as shown in Figure 3.1.

Consider two points P and Q in the body before deformation.

When external forces are applied, the undeformed body will deform so that points P and Q move to points p and q in the deformed body,

respectively.

Figure 3.1 : The kinematics of a body motion.

The change in length of this line segment can serve as a <u>measure of the change of shape and size, i.e., deformation</u> of the body.

We define *<u>strain</u>* as a measure of the deformation of the body.

Let

$\vec{X} = X_k \vec{i}_k$: **Position vector of point P before deformation**

$\vec{x} = x_k \vec{i}_k$: **Position vector of point p after deformation**

$\vec{u} = u_k \vec{i}_k$: **Displacement vector of point P** **(3.1)**

$d\vec{X} = dX_k \vec{i}_k$: **Position vector of line segment PQ**

$d\vec{x} = dx_k \vec{i}_k$: **Position vector of line segment pq**

Clearly,

$$\vec{x} = \vec{X} + \vec{u} \tag{3.2}$$

In addition, $d\vec{x}$ **is expressed as**

$$d\vec{x} = \frac{\partial \vec{x}}{\partial X_k} dX_k = \frac{\partial x_i}{\partial X_k} dX_k \vec{i}_i \tag{3.3}$$

We introduce a second order quantity, $\vec{F}$ **called <u>deformation gradient tensor</u>**

such that

$$\vec{F} = \frac{\partial \vec{x}}{\partial X_k}\vec{i}_k = \frac{\partial x_i}{\partial X_k}\vec{i}_k\vec{i}_i \equiv (\nabla_{\vec{X}}\vec{x})^T \qquad \textbf{(3.4)}$$

where $\nabla_{\vec{X}}$ **is the gradient vector with respect to** X **coordinate system.**

Then we can rewrite Eq. (3.3) as

$$d\vec{x} = \vec{F} \bullet d\vec{X} \qquad \textbf{(3.5)}$$

Noting that $d\vec{x}$ **and** $d\vec{X}$ **are vectors, we easily recognize** $\vec{F}$ **is indeed a second order tensor and we name it the deformation gradient tensor.**

On the other hand, the $\vec{F}^T$ **is written as**

$$\vec{F}^T = \left(\frac{\partial \vec{x}}{\partial X_k}\vec{i}_k\right)^T = \vec{i}_k \frac{\partial \vec{x}}{\partial X_k} = \frac{\partial x_i}{\partial X_k}\vec{i}_k\vec{i}_i = \frac{\partial x_k}{\partial X_i}\vec{i}_i\vec{i}_k \tag{3.6}$$

Therefore, we can also rewrite Eq. (3.3) as

$$d\vec{x} = d\vec{X} \bullet F^T \tag{3.7}$$

Let us write lengths of line segments PQ **and** pq **as** dS **and** ds**, respectively.**

Using Eqs. (3.5) and (3.7), we can obtain the following expression.

$$(ds)^2 = d\vec{x} \bullet d\vec{x} = d\vec{X} \bullet \vec{F}^T \bullet \vec{F} \bullet d\vec{X} \tag{3.8}$$

Moreover, we can express $(dS)^2$ **such as**

$$(dS)^2 = d\vec{X} \bullet d\vec{X} = d\vec{X} \bullet \vec{\delta} \bullet d\vec{X} \tag{3.9}$$

where $$\vec{\delta} = \delta_{ij} \vec{i}_i \bullet \vec{i}_j \tag{3.10}$$

Then the difference between two scalar quantities in Eqs. (3.8) and (3.9) is written as

$$(ds)^2 - (dS)^2 = d\vec{X} \bullet (\vec{F}^T \bullet \vec{F} - \vec{\delta}) \bullet d\vec{X} \tag{3.11}$$

Note that this quantity can be used as a measure of the deformation of the body.

We introduce a second order quantity $\vec{E}$ **such as**

$$(ds)^2 - (dS)^2 = d\vec{X} \bullet 2\vec{E} \bullet d\vec{X} \tag{3.12}$$

where

$$\vec{E} = \frac{1}{2}(\vec{F}^T \bullet \vec{F} - \vec{\delta}) \tag{3.13}$$

Remembering $(ds)^2 - (dS)^2$ is a scalar and $d\vec{X}$ is a vector, we can <u>conclude</u> $\vec{E}$ is <u>a second order tensor</u> defined with respect to $\vec{X}$ coordinate system.

This is called the Green strain tensor.

From Eqs. (3.2) and (3.4), we can get

$$\vec{F} = [\nabla_{\vec{X}}(\vec{X} + \vec{u})]^T = \vec{\delta} + (\nabla_{\vec{X}}\vec{u})^T \tag{3.14}$$

Then the $\vec{F}^T$ is written as

$$\vec{F}^T = \vec{\delta} + (\nabla_{\vec{X}}\vec{u})$$

Therefore we are able to write Green strain tensor $\vec{E}$ as

$$\vec{E} = \frac{1}{2}(\vec{F}^T \bullet \vec{F} - \vec{\delta})$$

$$= \frac{1}{2}[(\nabla_{\vec{X}}\vec{u})^T + \nabla_{\vec{X}}\vec{u} + \nabla_{\vec{X}}\vec{u} \bullet (\nabla_{\vec{X}}\vec{u})^T)] \quad \textbf{(3.15)}$$

In order to obtain the expression for components of Green strain tensor $\vec{E}$, remember the expression for $\nabla_{\vec{X}}$.

Then we obtain

$$\nabla_{\vec{X}}\vec{u} = \frac{\partial}{\partial X_i}\vec{i}_i(u_j\vec{i}_j) = \frac{\partial u_j}{\partial X_i}\vec{i}_i\vec{i}_j$$

$$(\nabla_{\vec{X}}\vec{u})^T = \frac{\partial u_j}{\partial X_i}\vec{i}_j\vec{i}_i = \frac{\partial u_i}{\partial X_j}\vec{i}_i\vec{i}_j$$

Also,

$$\nabla_{\vec{X}}\vec{u}\bullet(\nabla_{\vec{X}}\vec{u})^T = \left(\frac{\partial u_k}{\partial X_i}\vec{i}_i\vec{i}_k\right)\bullet\left(\frac{\partial u_l}{\partial X_j}\vec{i}_l\vec{i}_j\right) = \frac{\partial u_k}{\partial X_i}\vec{i}_i\delta_{kl}\frac{\partial u_l}{\partial X_j}\vec{i}_j$$

$$= \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\vec{i}_i\vec{i}_j$$

Writing

$$\vec{E} = E_{ij}\vec{i}_i\vec{i}_j$$

We obtain the expression for E_{ij} **in index notation as**

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \sim \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \quad \textbf{(3.16)}$$

Note that Green strain tensor $\vec{E}$ **is symmetric, i.e.,** $E_{ij} = E_{ji}$ **..**

Without using index notation, components of $\vec{E}$ **are written as follows :**

$$E_{XX} = \frac{\partial u}{\partial X} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2 + \left(\frac{\partial w}{\partial X}\right)^2\right] = u,_X + \frac{1}{2}(u,_X^2 + v,_X^2 + w,_X^2)$$

$$E_{YY} = \frac{\partial v}{\partial Y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2 + \left(\frac{\partial w}{\partial Y}\right)^2\right] = v,_Y + \frac{1}{2}(u,_Y^2 + v,_Y^2 + w,_Y^2)$$

$$E_{ZZ} = \frac{\partial w}{\partial Z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Z}\right)^2 + \left(\frac{\partial v}{\partial Z}\right)^2 + \left(\frac{\partial w}{\partial Z}\right)^2\right] = w,_Z + \frac{1}{2}(u,_Z^2 + v,_Z^2 + w,_Z^2)$$

(3.17)

$$E_{XY} = \frac{1}{2}(\frac{\partial v}{\partial X} + \frac{\partial u}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\frac{\partial v}{\partial Y} + \frac{\partial w}{\partial X}\frac{\partial w}{\partial Y}] = \frac{1}{2}(v,_X + u,_Y + u,_X\, u,_Y + v,_X\, v,_Y + w,_X\, w,_Y)$$

$$E_{YZ} = \frac{1}{2}(\frac{\partial w}{\partial Y} + \frac{\partial v}{\partial Z} + \frac{\partial u}{\partial Y}\frac{\partial u}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y}\frac{\partial w}{\partial Z}] = \frac{1}{2}(w,_Y + v,_Z + u,_Y\, u,_Z + v,_Y\, v,_Z + w,_Y\, w,_Z)$$

$$E_{ZX} = \frac{1}{2}(\frac{\partial u}{\partial Z} + \frac{\partial w}{\partial X} + \frac{\partial u}{\partial Z}\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Z}\frac{\partial v}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial w}{\partial X}] = \frac{1}{2}(u,_Z + w,_X + u,_Z\, u,_X + v,_Z\, v,_X + w,_Z\, w,_X)$$

$$E_{YX} = E_{XY}, E_{YZ} = E_{ZY}, E_{XZ} = E_{ZX}$$

Note

(i) **Green strain tensor E is referred to the initial un-deformed geometry, and indicate what must occur during a given deformation.**

(ii) **We have no restriction on the strain-displacement relation in Eq. (3.16) or**

Eq. (3.17). This relation includes nonlinear terms in displacement components.

(iii) If we use Eqs. (3.4), (3.6), and (3.13), we can get another expression for Green strain tensor E. This expression is convenient for the physical interpretation of E.

$$\vec{E} = \frac{1}{2}[\vec{i}_i(\frac{\partial\vec{x}}{\partial X_i}\bullet\frac{\partial\vec{x}}{\partial X_j})\vec{i}_j - \vec{\delta}] = \frac{1}{2}[(\frac{\partial\vec{x}}{\partial X_i}\bullet\frac{\partial\vec{x}}{\partial X_j})\vec{i}_i\vec{i}_j - \vec{\delta}] \qquad \textbf{(3.18)}$$

Without using index notation, the components of E are also written as follows :

$$E_{XX} = \frac{1}{2}\left(\frac{\partial\vec{x}}{\partial X}\bullet\frac{\partial\vec{x}}{\partial X} - 1\right), E_{YY} = \frac{1}{2}\left(\frac{\partial\vec{x}}{\partial Y}\bullet\frac{\partial\vec{x}}{\partial Y} - 1\right), E_{ZZ} = \frac{1}{2}\left(\frac{\partial\vec{x}}{\partial Z}\bullet\frac{\partial\vec{x}}{\partial Z} - 1\right)$$

$$E_{XY} = \frac{1}{2}\frac{\partial\vec{x}}{\partial X}\bullet\frac{\partial\vec{x}}{\partial Y} = E_{YX}, E_{YZ} = \frac{1}{2}\frac{\partial\vec{x}}{\partial Y}\bullet\frac{\partial\vec{x}}{\partial Z} = E_{ZY}, E_{ZX} = \frac{1}{2}\frac{\partial\vec{x}}{\partial Z}\bullet\frac{\partial\vec{x}}{\partial X} = E_{XZ}$$

(3.19)

3.2 Physical Meaning of The Green Strain Terms

Consider a small rectangular parallelepiped at point P in a body, as in Fig. 2. If the body is <u>a rigid body</u>, <u>there is no translation and/or rotation in the body</u>. Therefore,

$$(ds)^2 - (dS)^2 = 0 .$$

This means that all strain components E_{ij} are zero in the rigid body.

Figure 3.2: The motion of a rectangular parallelepiped

Let us imagine next that this body has some deformation and let us focus on line elements PA, PB, and PC. After deformation, the body in general becomes non-rectangular and these line elements change to pa, pb, and pc, respectively,

Recalling Eq. (3.3).

$$d\vec{x} = \frac{\partial \vec{x}}{\partial X} dX + \frac{\partial \vec{x}}{\partial Y} dY + \frac{\partial \vec{x}}{\partial Z} dZ \qquad \textbf{(3.20)}$$

Note that $PA, PB,$ and PC are orthogonal to each other and vectors for line elements $pa, pb,$ and pc consist of $d\vec{x}$.

Then

$$p\vec{a} = \frac{\partial \vec{x}}{\partial X} dX$$
$$p\vec{b} = \frac{\partial \vec{x}}{\partial Y} dY$$
$$p\vec{c} = \frac{\partial \vec{x}}{\partial Z} dZ \qquad \textbf{(3.21)}$$

Now we consider changes in the line element lengths.

~ First look at the line element PA.

Define relative elongation **E_x as the ratio of the change in length of PA with**

respect to the original length.

That is

$$E_x = \frac{|pa| - |PA|}{|PA|} = \frac{|pa|}{|PA|} - 1$$

Then

$$|pa| = (1 + E_x)|PA|$$

From Eqs. (3.19) and (3.21), we know that

$$\left|pa\right|^2 = \frac{\partial \vec{x}}{\partial X} dX \bullet \frac{\partial \vec{x}}{\partial X} dX = (1+2E_{xx})dX^2$$

Similarly defining the relative elongations for PB and PC, we can obtain the following relations.

$$E_X = \sqrt{1+2E_{XX}} - 1$$
$$E_Y = \sqrt{1+2E_{YY}} - 1$$
$$E_Z = \sqrt{1+2E_{ZZ}} - 1 \qquad \textbf{(3.22)}$$

Therefore, E_{XX} **,** E_{YY} **, , and** E_{ZZ} **are related to the relative elongations**

E_{XX}, E_{YY}, and E_{ZZ}, **respectively and are called <u>extensional strains</u>.**

Next consider changes in the angles between adjacent line elements. The angle between PA **and** PB **is** 90^{o} **. For the sake of convenience, denote the angle between** pa **and** pb **as** $\frac{\pi}{2} - \phi_{XY}$. **Then,** ϕ_{XY} **is the angle change.**

The scalar product vectors $p\vec{a}$ **and** $p\vec{b}$ **with angle** ϕ_{XY} **is given as**

$$\frac{\partial \vec{x}}{\partial X} dX \bullet \frac{\partial \vec{x}}{\partial Y} dX = \left| \frac{\partial \vec{x}}{\partial X} dX \right| \left| \frac{\partial \vec{x}}{\partial Y} dX \right| \cos\left(\frac{\pi}{2} - \phi_{XY}\right)$$

Or

$$\cos\left(\frac{\pi}{2}-\phi_{XY}\right)=\sin\phi_{XY}=\frac{\dfrac{\partial\vec{x}}{\partial X}\bullet\dfrac{\partial\vec{x}}{\partial Y}}{\left|\dfrac{\partial\vec{x}}{\partial X}\right|\bullet\left|\dfrac{\partial\vec{x}}{\partial Y}\right|}$$

Noting that

$$\left|\frac{\partial\vec{x}}{\partial X}\right|=\sqrt{1+2E_{XX}}=1+E_X$$

$$\left|\frac{\partial\vec{x}}{\partial Y}\right|=\sqrt{1+2E_{YY}}=1+E_Y$$

Repeating the above procedures for changes in angles between pb **and** pc **and between** pc **and** pa**, we obtain similar expressions.**

Then using Eq. (3.19),

$$\sin\phi_{XY} = \frac{2E_{XY}}{(1+E_X)(1+E_Y)}$$

$$\sin\phi_{YZ} = \frac{2E_{YZ}}{(1+E_Y)(1+E_Z)}$$

$$\sin\phi_{ZX} = \frac{2E_{ZX}}{(1+E_Z)(1+E_X)} \qquad \textbf{(3.23)}$$

Thus the angular changes between adjacent line elements are related to the strain components $E_{XY}, E_{YZ},$, and E_{ZX} as well as to the elongation $E_X, E_Y,$ and E_Z. The twice strain components $E_{XY}, E_{YZ},$, and E_{ZX} are called <u>shear strains.</u>

2.3 Small Strain Assumption

In <u>many engineering problems</u> the strain components are small.

Then

$$E_X = \sqrt{1+2E_{XX}} - 1$$

$$\simeq 1 + \frac{1}{2}(2E_{XX}) - 1 = E_{XX}$$

Repeating for other elongations, we obtain

$$\begin{aligned} E_X &= E_{XX} \\ E_Y &= E_{YY} \\ E_Z &= E_{ZZ} \end{aligned} \qquad \textbf{(3.24)}$$

Therefore, the relative elongations are also small under small strain assumption.

For angle changes,

$$\sin\phi_{XY} = \frac{2E_{XY}}{(1+E_X)(1+E_Y)} \simeq 2E_{XY} \qquad \textbf{(3.25)}$$

Also we get

$$\begin{aligned} \sin\phi_{XY} &= 2E_{XY} \\ \sin\phi_{YZ} &= 2E_{YZ} \\ \sin\phi_{ZX} &= 2E_{ZX} \end{aligned} \qquad \textbf{(3.26)}$$

The shear strains are independent of the angle changes under the small strain assumption.

Note that under the small strain assumption, rotation can still be large.

2.4 Linear Strain Assumption

In addition to the small strain assumption, we add an assumption of small rotation of volume element. : The combination of these two assumptions is called linear strain assumption.

~ Under the linear strain assumption, we can neglect all the nonlinear terms in the strain-displacement relations Eq. (3.16) or Eq. (3.17).

In most cases of this course, we take the linear strain assumption.

In addition, we may use ε for strain tensor instead of E.

Moreover, in the case of this infinitesimal strains, the deformed state is very close to the undeformed state.

Therefore x is very close to X. Hereafter we will use x as the coordinate of the

undeformed body instead of X.

Then Eq. (3.16) becomes

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \sim \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \textbf{(3.27)}$$

In unabridged notation we have

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = u,_x$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = v,_y$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = w,_z \qquad \textbf{(3.28)}$$

$$\varepsilon_{xy} = \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) = \frac{1}{2}(v,_x + u,_y)$$

$$\varepsilon_{yz} = \frac{1}{2}(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}) = \frac{1}{2}(w,_y + v,_z)$$

$$\varepsilon_{zx} = \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) = \frac{1}{2}(u,_z + w,_x)$$

$$\varepsilon_{yx} = \varepsilon_{xy}, \varepsilon_{yz} = \varepsilon_{zy}, \varepsilon_{xz} = \varepsilon_{zx}$$

In engineering problems, we frequently use engineering shear strains such that

$$\begin{aligned} \gamma_{xy} &= 2\varepsilon_{xy} \\ \gamma_{yz} &= 2\varepsilon_{yz} \\ \gamma_{zx} &= 2\varepsilon_{zx} \end{aligned} \tag{3.29}$$

The justification for neglecting the nonlinear terms is given as follows :

Figure 3.3: Change in the segment dX

Consider the small line element vector dX i which changes to $\frac{\partial \vec{x}}{\partial X} dX$ where

$$\vec{x} = \vec{X} + \vec{u} = (X + u)\vec{i} + (Y + v)\vec{j} + (Z + w)\vec{k}$$

Let θ_1 **be the angle between** $\dfrac{\partial \vec{x}}{\partial X}$ **and** Y **axis.**

Then

$$\cos\theta_1 = \frac{\dfrac{\partial \vec{x}}{\partial X} \bullet \vec{j}}{\left|\dfrac{\partial \vec{x}}{\partial X}\right| \bullet |1|} = \frac{\left|\left(1 + \dfrac{\partial u}{\partial X}\right)\vec{i} + \dfrac{\partial v}{\partial X}\vec{j} + \dfrac{\partial w}{\partial X}\right| \bullet \vec{j}}{|1 + E_X|}$$

$$= \frac{\dfrac{\partial v}{\partial X}}{|1 + E_X|}$$

Similarly, for the angle, θ_2**, between** $\dfrac{\partial \vec{x}}{\partial X}$ **and Z axis,**

$$\cos\theta_2 = \frac{\dfrac{\partial \vec{x}}{\partial X}\bullet \vec{k}}{\left|\dfrac{\partial \vec{x}}{\partial X}\right|\bullet |1|} = \frac{\dfrac{\partial w}{\partial X}}{|1+E_X|}$$

For small strains, $E_X \ll 1.$ **For** $\dfrac{\partial \vec{x}}{\partial X}$ **close to X axis, i.e., small rotation of dX,**

$$\cos\theta_1, \cos\theta_2 \ << 1 \sim \frac{\partial v}{\partial X}, \frac{\partial w}{\partial X} << 1$$

In addition, the deformed coordinate x **is close to** X **.**

Similarly, if we consider the small rotation of the line element

dy **and** dz, $\frac{\partial u}{\partial Y}, \frac{\partial w}{\partial Y} << 1$ **and** $\frac{\partial u}{\partial Z}, \frac{\partial w}{\partial Z} << 1$.

Therefore, all nonlinear terms can be neglected since

$$\frac{\partial v}{\partial X} >> \left(\frac{\partial v}{\partial X} \right)^2$$

Etc.

2.5 Strain Transformation Law

We will try to find the relationship between two strain components expressed with respect to two different coordinate systems, $\vec{x}$**, and** $\tilde{x}$**.**

The position vector of a point P **can be written as**

Figure 3.4 : Change in segment PQ

The position vector of a point P **can be written as**

$$\vec{x} = \tilde{x} + a$$

where a is the vector between origins of two coordinate systems. Or using unabridged notation,

$$\vec{x} = x\vec{i} + y\vec{j} + z\vec{k} = \tilde{x}\tilde{\vec{i}} + \tilde{y}\tilde{\vec{j}} + \tilde{z}\tilde{\vec{k}} + \vec{a}$$

Taking a dot product with $\vec{i}$ **,**

$$x = \vec{x} \bullet \vec{i} = (x\vec{i} + y\vec{j} + z\vec{k}) \bullet \vec{i} = \tilde{x}\vec{\tilde{i}} \bullet \vec{i} + \tilde{y}\vec{\tilde{j}} \bullet \vec{i} + \tilde{z}\vec{\tilde{k}} \bullet \vec{i} + \vec{a} \bullet \vec{i}$$

$$= \tilde{x}(\vec{\tilde{i}} \bullet \vec{i}) + \tilde{y}(\vec{\tilde{j}} \bullet \vec{i}) + \tilde{z}(\vec{\tilde{k}} \bullet \vec{i}) + \vec{a} \bullet \vec{i}$$

$$= \tilde{x}c_{\tilde{x}x} + \tilde{y}c_{\tilde{y}x} + \tilde{z}c_{\tilde{z}x} + \vec{a} \bullet \vec{i}$$

where $c_{\tilde{x}x}, c_{\tilde{y}x}$ **and** $c_{\tilde{z}x}$ **are direction consines.**

From the above equation, we can obtain

$$c_{\tilde{x}x} = \frac{\partial x}{\partial \tilde{x}}, c_{\tilde{y}x} = \frac{\partial x}{\partial \tilde{y}}, c_{\tilde{z}x} = \frac{\partial x}{\partial \tilde{z}}$$

Taking a dot product with $\hat{j}$ or $\hat{k}$, we get similar expressions for other direction cosines.

Using index notation,

$$\frac{\partial x_j}{\partial \tilde{x}_i} = c_{ij} \tag{3.30}$$

Consider the quantity $ds^2 - dS^2$ defined in Eq. (3.12).

Using index notation (instead of E and X, we use $\vec{\varepsilon}$ and $\vec{X}$ hereafter,),

$$ds^2 - dS^2 = 2\varepsilon_{ij} dx_i dx_j \tag{3.31}$$

Or in matrix form,

$$ds^2 - dS^2 = 2[dx, dy, dz]\begin{bmatrix} \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{xz} \\ \varepsilon_{xy}, \\ \varepsilon_{xz} .. \end{bmatrix}\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \tag{3.32}$$

The quantity $ds^2 - dS^2$ is a scalar and thus invariant under coordinate transformation.

In $\tilde{X}$ coordinate system,

$$ds^2 - dS^2 = d\tilde{x} \bullet \tilde{\varepsilon} \bullet d\tilde{x}$$

$$= 2[d\tilde{x}, d\tilde{y}, d\tilde{z}] \begin{bmatrix} \tilde{\varepsilon}_{xx}, \tilde{\varepsilon}_{xy}, \tilde{\varepsilon}_{xz} \\ \tilde{\varepsilon}_{xy}, \\ \tilde{\varepsilon}_{xz} .. \end{bmatrix} \begin{bmatrix} d\tilde{x} \\ d\tilde{y} \\ d\tilde{z} \end{bmatrix} \qquad \textbf{(3.33)}$$

According to the chain rule of differentiation

$$dx = \frac{\partial x}{\partial \tilde{x}} d\tilde{x} + \frac{\partial x}{\partial \tilde{y}} d\tilde{y} + \frac{\partial x}{\partial \tilde{z}} d\tilde{z} = c_{\tilde{x}x} d\tilde{x} + c_{\tilde{y}x} d\tilde{y} + c_{\tilde{z}x} d\tilde{z}$$

$$dy = ...$$

$$dz = ... \qquad \textbf{(3.34)}$$

In matrix form

$$\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = [?] \begin{Bmatrix} d\tilde{x} \\ d\tilde{y} \\ d\tilde{z} \end{Bmatrix} = T \begin{Bmatrix} d\tilde{x} \\ d\tilde{y} \\ d\tilde{z} \end{Bmatrix} \qquad \textbf{(3.35)}$$

where

$$T = [?] \qquad \textbf{(3.36)}$$

In addition

$$[dx, dy, dz] = \begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix}^T = [d\tilde{x}, d\tilde{y}, d\tilde{z}]T^T \qquad \textbf{(3.37)}$$

After substituting Eqs. (3.35) and (3.37) into Eq. (3.32), setting it equal to Eq. (3.33), we get

$$\begin{bmatrix} \tilde{\varepsilon}_{xx}, \tilde{\varepsilon}_{xy}, \tilde{\varepsilon}_{xz} \\ \tilde{\varepsilon}_{xy}, \\ \tilde{\varepsilon}_{xz}.. \end{bmatrix} = T^T \begin{bmatrix} \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{xz} \\ \varepsilon_{xy}, \\ \varepsilon_{xz}.. \end{bmatrix} T \qquad \textbf{(3.38)}$$

In index notation, we can express as

$$\tilde{\varepsilon}_{ij} = c_{ik} c_{jl} \varepsilon_{ij} \tag{3.39}$$

Expanding Eq. (3.38),

$$\tilde{\gamma} = T_{\varepsilon} \gamma \tag{3.40}$$

where $\tilde{\gamma}$ is the engineering strain vector such that

$$\gamma = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} = 2\varepsilon_{xy} \\ \gamma_{yz} = 2\varepsilon_{yz} \\ \gamma_{zx} = 2\varepsilon_{zx} \end{Bmatrix} \quad \textbf{(3.41)}$$

and $\tilde{\gamma}$ is the engineering strain vector defined with respect to $\tilde{x}$ coordinate system.

In addition, the 6 X 6 transformation matrix $\underline{T_{\varepsilon}}$ **is given as**

$$\underline{T_{\varepsilon} =}$$

$$\begin{bmatrix} c^{2}{}_{\tilde{x}x}, \ldots\ldots\ldots \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

2,6 Compatibility Equations

Let us consider the strain-displacement relations Eq. (3.27)

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \sim \frac{1}{2}(u_{i,j} + u_{j,i})$$

(i) If displacements u_i (i = 1..3) are given, we can readily determine all strain components by substituting u_i into the above equation.

(ii) Inversely, when strains are given, we should determine three displacement components by integration of six differential equations given by the above expression. Then we cannot expect single-valued strains. Furthermore,

displacements of interest to us will be continuous. The resulting equations are called the compatibility equations.

Differentiating Eq. (3.27) twice and rearranging free indices, we can have

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} = \frac{1}{2}\left(\frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l} + \frac{\partial^3 u_j}{\partial x_i \partial x_k \partial x_l} \right)$$

$$\frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} = \frac{1}{2}\left(\frac{\partial^3 u_k}{\partial x_l \partial x_i \partial x_j} + \frac{\partial^3 u_l}{\partial x_k \partial x_i \partial x_j} \right)$$

$$\frac{\partial^2 \varepsilon_{lj}}{\partial x_k \partial x_i} = \frac{1}{2}\left(\frac{\partial^3 u_j}{\partial x_l \partial x_i \partial x_k} + \frac{\partial^3 u_l}{\partial x_j \partial x_i \partial x_k} \right)$$

$$\frac{\partial^2 \varepsilon_{ki}}{\partial x_l \partial x_j} = \frac{1}{2}\left(\frac{\partial^3 u_k}{\partial x_i \partial x_j \partial x_l} + \frac{\partial^3 u_i}{\partial x_k \partial x_j \partial x_l} \right)$$

By adding the first two equations and then subtracting the last two equations, we eliminate u_i components and thus obtain a set of relations involving only strains.

That is

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{lj}}{\partial x_k \partial x_i} - \frac{\partial^2 \varepsilon_{ki}}{\partial x_l \partial x_j} = 0 \qquad \textbf{(3.43)}$$

Actually, only 6 of these 81 equations of compatibility are independent, These are given as follows in unabridged notation :

(3.44)

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} = \gamma_{xy,xy}$$

$$\varepsilon_{yy,zz} + \varepsilon_{zz,yy} = \gamma_{yz,yz}$$

$$\varepsilon_{zz,xx} + \varepsilon_{xx,zz} = \gamma x_{zx,zx}$$

$$2\varepsilon_{xx,yz} = (-\gamma_{yz,x} + \gamma_{zx,y} + \gamma_{xy,z})$$

$$2\varepsilon_{yy,zx} = (-\gamma_{zx,y} + \gamma_{xy,z} + \gamma_{yz,x})$$

$$2\varepsilon_{zz,xy} = (-\gamma_{xy,z} + \gamma_{yz,x} + \gamma_{zx,y})$$

2.7 Principal strains and Principle Directions

As the coordinate system changes, the values of strains change according to the strain transformation law. Now we like to find those directions for which the relative elongations or extensional strains attain extrema (i.e., maxima or minima).

Those directions are called principal directions and the corresponding strains are called principal strains.

Suppose $\varepsilon_{ij}\,(i, j = 1...3)$ **are given at a material point of a body in** xyz **-coordinate system. We like to seek new coordinates** $\tilde{x}, \tilde{y}$**, and** $\tilde{z}$ **, in which** $\tilde{\varepsilon}_{xx}$ **is the principal strain. From the strain transformation law Eq. (3.40),**

$$\tilde{\varepsilon}_{xx} = c_{\tilde{x}x}^2 \varepsilon_{xx} + c_{\tilde{x}y}^2 \varepsilon_{yy} + c_{\tilde{x}z}^2 \varepsilon_{zz} + 2c_{\tilde{x}x} c_{\tilde{x}y} \varepsilon_{xy} + 2c_{\tilde{x}y} c_{\tilde{x}z} \varepsilon_{yx} + 2c_{\tilde{x}z} c_{\tilde{x}x} \varepsilon_{zx} \quad \textbf{(3.45)}$$

For simplicity, we introduce new notations $\nu_x, \nu_y,$ and ν_z such as

$$\nu_x = c_{\tilde{x}x}, \nu_y = c_{\tilde{x}y}, \nu_z = c_{\tilde{x}z} \quad \textbf{(3.46)}$$

Then Eq. (3.45) can be written as

$$\tilde{\varepsilon}_{xx} = f(\nu_x, \nu_y, \nu_z) = \varepsilon_{xx} \nu_{xx}^2 + \ldots ? \quad \textbf{(3.47)}$$

Now , we have the following relation or constraint

$$g(\nu_x, \nu_y, \nu_z) = 1 - (\nu_x^2 + \nu_y^2 + \nu_2^2) = 0 \quad \textbf{(3.48)}$$

Now we will find the extremum of $\tilde{\varepsilon}_{xx}$ by constructing a function such that

$$F(\nu_x, \nu_y, \nu_z, \lambda) \tilde{\varepsilon}_{xx} = f(\nu_x, \nu_y, \nu_z) + \lambda g(\nu_x, \nu_y, \nu_z) \quad \textbf{(3.49)}$$

According to the Lagrangian multiplier method, the values of ν^* for the extremum of F are obtained from

$$\frac{\partial F}{\partial \nu_x} = 2(\varepsilon_{xx} - \lambda)\nu_x + 2\varepsilon_{xy}\nu_y + 2\varepsilon_{xz}\nu_z = 0$$

$$\frac{\partial F}{\partial \nu_y} = ?$$

$$\frac{\partial F}{\partial \nu_z} = ?$$

$$\frac{\partial F}{\partial \lambda} = ?$$

In matrix form

$$\begin{bmatrix} \varepsilon_{xx} - \lambda, \varepsilon_{xy}, \varepsilon_{xz} \\ \varepsilon_{xy}, ?, ? \\ \varepsilon_{xz}, ?, ? \end{bmatrix} \begin{Bmatrix} \nu_x \\ \nu_y \\ \nu_z \end{Bmatrix} = 0 \tag{3.50}$$

If we introduce the following shorthand notation

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{xz} \\ \varepsilon_{xy}, ?, ? \\ \varepsilon_{xz}, ?, ? \end{bmatrix} = 0 \tag{3.51}$$

$$\nu = \begin{Bmatrix} \nu_x \\ \nu_y \\ \nu_z \end{Bmatrix} \tag{3.52}$$

$$I = \begin{bmatrix} 1,0,0 \\ 0,1,0 \\ 0,0,1 \end{bmatrix} = Unity.Matrix$$

(3.53)

$$(\varepsilon - \lambda I)v = 0$$

(3.54)

In order to have nontrivial solutions for v **, the determinant** $\left| \ (\varepsilon - \lambda I) \ \right| = 0$

Or ? (3.55)

This equation holds for special values of λ**. These special values are called eigenvalues.**

Expanding Eq. (3.55), we will have a cubic equation for λ **;**

$$-\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 = 0 \qquad \textbf{(3.56)}$$

where

$$J_1 \;=\; ? \qquad \textbf{(3.57)}$$

$$J_2 \;=\; ? \qquad \textbf{(3.58)}$$

$$J_3 \;=\; ? \qquad \textbf{(3.59)}$$

First strain invariant J_1, second strain invariant J_2 and third strain invariant J_3 do not change under coordinate transformation.

Since ε is symmetric, these exist three real eigenvalues which are obtained by solving the cubic equation (3.56). These eigenvalues will be denoted as

λ_I, λ_{II}, and λ_{III}. Now suppose that we know λ_I and let ν^I be ν associated with λ_I.

Then we have from Eq. (3.50),

$$(\varepsilon - \lambda I)\nu^I = 0 \qquad (3.60)$$

or in matrix form

? (3.61)

Only two of the above equation are independent and the third equation is provided by the constraint equation

$$(\nu_x^I)^2 + (\nu_y^I)^2 + (\nu_z^I)^2 = 1 \qquad (3.62)$$

From the three equations, we calculate ν_x^I, ν_y^I, and ν_z^I.

In a similar manner, we can calculate ν^{II} and ν^{III}.

Now we will show that λ_I is actually a principal strain. Pre-multiplying Eq. (3.60) with $(\nu^I)^T$,

$$(\vec{\nu}^I)^T\ (\vec{\varepsilon} - \lambda_I \vec{I})\vec{\nu}^I = 0$$

Or

$$(\vec{\nu}^I)^T\ \vec{\varepsilon}\vec{\nu}^I - \lambda_I (\vec{\nu}^I)^T \vec{\nu}^I = 0 \qquad \textbf{(3.63)}$$

From the orthonormality condition Eq. (3.62),

$$(\vec{\nu}^I)^T \vec{\nu}^I = 1$$

On the other hand, by noticing Eqs. (3.42) and (3.46), we can show

$$\tilde{\varepsilon}^I_{xx} = (\vec{\nu}^I)^T \vec{\varepsilon}\vec{\nu}^I$$

Therefore we conclude by introducing these two equations into Eq. (3.63) that

$$\lambda_I = \tilde{\varepsilon}^I_{xx} \tag{3.64}$$

In order to indicate λ_I, λ_{II}, **and** λ_{III} **are actually principal strains, we use** $\varepsilon_I, \varepsilon_{II}$, **and** ε_{III} **such that**

$$\begin{aligned} \varepsilon_I &= \lambda_I \\ \varepsilon_{II} &= \lambda_{II} \\ \varepsilon_{III} &= \lambda_{III} \end{aligned} \tag{3.65}$$

With $\varepsilon_I, \varepsilon_{II}$, **and** ε_{III} **, the cubic equation (3.56) can be expressed as**

$$-\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 = -(\lambda - \varepsilon_I)(\lambda - \varepsilon_{II})(\lambda - \varepsilon_{III}) = 0$$

Expanding, we have

$$J_1 = ?$$
$$J_2 = ?$$
$$J_3 = ? \qquad \textbf{(3.66)}$$

Therefore, since the principal strains for the given state of strain are unique, $J_1, J_2,$ and J_3 are invariant.

Orthogonality of Principal Directions

The principal directions are orthogonal with each other. For example, if $\lambda_I \neq \lambda_{II}$, then eigenvectors $\vec{v}^I$ and $\vec{v}^{II}$ are orthogonal.

That is

$$(\vec{v}^I)^T \vec{v}^{II} = 0 \qquad \textbf{(3.67)}$$

In unabridged form,

$$\lambda_x^I \lambda_x^{II} + \lambda_y^I \lambda_y^{II} + \lambda_z^I \lambda_z^{II} = 0 \qquad \textbf{(2.68)}$$

(Proof !)

2.8 Volume Change

We will determine the volume change of the small parallelepiped as shown in Fig. 2. The initial volume before deformation is $dV_0 = dxdydz$. The volume after deformation is

$$dV = \frac{\partial \vec{x}}{\partial X} dX \bullet \left(\frac{\partial \vec{x}}{\partial Y} dY \times \frac{\partial \vec{x}}{\partial Z} dZ \right) \qquad \textbf{(3.69)}$$

Using the formula for the product of two triple scalar product such that

$$\left[\vec{u}\bullet(\vec{v}\times\vec{w})\right]\left[\vec{a}\bullet(\vec{b}\times\vec{c})\right]=\begin{bmatrix}\vec{u}\bullet\vec{a},\vec{u}\bullet\vec{b},\vec{u}\bullet\vec{c}\\ \vec{v}\bullet\vec{a},\vec{v}\bullet\vec{b},\vec{v}\bullet\vec{c}\\ \vec{w}\bullet\vec{a},\vec{w}\bullet\vec{b},\vec{w}\bullet\vec{c}\end{bmatrix} \qquad \textbf{(3.70)}$$

Then

$$(dV)^2=[\frac{\partial\vec{x}}{\partial X}dX\bullet(\frac{\partial\vec{x}}{\partial Y}dY\times\frac{\partial\vec{x}}{\partial Z}dZ)][\frac{\partial\vec{x}}{\partial X}dX\bullet(\frac{\partial\vec{x}}{\partial Y}dY\times\frac{\partial\vec{x}}{\partial Z}dZ)]$$

$$=[\frac{\partial\vec{x}}{\partial X}\bullet(\frac{\partial\vec{x}}{\partial Y}\times\frac{\partial\vec{x}}{\partial Z})][\frac{\partial\vec{x}}{\partial X}\bullet(\frac{\partial\vec{x}}{\partial Y}\times\frac{\partial\vec{x}}{\partial Z})](dXdYdZ)^2$$

$$=\begin{vmatrix}\ldots\\ ..?..\\ \ldots\end{vmatrix}(dV_o)^2$$

$$= \begin{vmatrix} 1+2E_{XX}, E_{XY}, E_{XZ} \\ E_{XY}, 1+2E_{YY}, E_{YZ} \\ E_{XZ}, E_{YZ}, 1+2E_{ZZ} \end{vmatrix} (dV_o)^2 = G(dV_o)^2$$

Therefore

$$dV = \sqrt{G}\, dV_o \qquad \textbf{(3.71)}$$

Expanding,

$$G = 1 + 2J_1 + 4J_2 + 8 \qquad \textbf{(3.72)}$$

The relative volume change of the element is defined as

$$\frac{dV - dV_0}{dV_0} = \frac{dV}{dV_0} - 1 = \sqrt{G} - 1 = \sqrt{1 + 2J_1 + 4J_2 + 8J_3} - 1 \qquad \textbf{(3.73)}$$

For small strains, $1 \gg J_1 \gg J_2 \gg J_3$.

Then the relative change becomes

$$\frac{dV - dV_0}{dV_0} \simeq \sqrt{1+2J_1} - 1 \simeq 1 + 2J_1 - 1 = J_1$$

Therefore, for small strains,

$$\frac{dV - dV_0}{dV_0} \simeq J_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon \qquad \textbf{(3.74)}$$

On the other hand, we can express the volume change in terms of the determinant of Jacobian matrix J **.**

Rewriting Eq. (3.69),

$$dV = \frac{\partial \vec{x}}{\partial X} \bullet \left(\frac{\partial \vec{x}}{\partial Y} \times \frac{\partial \vec{x}}{\partial Z} \right) dXdYdZ = |J| dV_o \qquad \textbf{(3.75)}$$

where J **is Jacobian matrix between** X **and** x **coordinate systems and is**

equivalent to F **if** F **is written as a 3 x 3 matrix such that**

$$J = F = \begin{bmatrix} \frac{\partial x}{\partial X}, \frac{\partial x}{\partial Y}, \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X}, \frac{\partial y}{\partial Y}, \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X}, \frac{\partial z}{\partial Y}, \frac{\partial z}{\partial Z} \end{bmatrix} \tag{3.76}$$

Comparing Eqs. (3.71) and (3.75), we know that

$$J = |\mathbf{J}| = \sqrt{G} = \det(\mathbf{F}) \tag{3.77}$$

2,9 Change of Area

Let us consider a small triangle PAB in the undeformed body with two side vectors $dX^{(1)}$ and $dX^{(2)}$. This triangle becomes the triangle *pab* with two side vectors $dX^{(1)}$ and $dX^{(2)}$ after deformation. dS° and dS are areas of these triangles. In addition, let N and n be the unit normal vectors of them. We will determine the area change in these triangles. Then

Figure 2.6: Change of a triangle element

$$NdS^{\circ} = \frac{1}{2} dX^{(1)} \times dX^{(2)} = \frac{1}{2} P_{rst} dX_s^{(1)} dX_t^{(2)} i_r$$

$$ndS = \frac{1}{2} dX^{(1)} \times dX^{(2)} = \frac{1}{2} P_{ijk} dx_j^{(1)} dx_k^{(2)} i_i$$

From Eq. (2.3),

$$dx_j{}^{(1)} = \frac{\partial x_j}{\partial X_s} dX_s{}^{(1)}$$

$$dx_k{}^{(2)} = \frac{\partial x_j}{\partial X_t} dX_t{}^{(2)}$$

Then

$$ndS \cdot F = P_{ijk} \frac{\partial x_j}{\partial X_s} \frac{\partial x_k}{\partial X_t} dX_s{}^{(1)} dX_t{}^{(2)} i_i \frac{\partial x_l}{\partial X_r} i_l i_r$$

$$= P_{ijk} \frac{\partial x_i}{\partial X_r} \frac{\partial x_k}{\partial X_s} \frac{\partial x_k}{\partial X_t} dX_s{}^{(1)} dX_t{}^{(2)} i_r$$

$$= P_{rst} (\det F) dX_s^{(1)} dX_t^{(2)} i_r$$

$$= (\det F) NdS^{\circ}$$

Therefore, using Eq. (2.76)

$$NdS^{\circ} = \frac{1}{J} n \cdot F dS \qquad \textbf{(2.78)}$$

This relationship is called Nanson's formula.