항공기 구조 역학

CHAPTER 4.
Engineering structural analysis

지도교수 : 신상준

4.3 Hyperstatic systems

❖ "Statically determinate" or "Isostatic"

Total No. of unknown internal forces **=** No. of equilibrium equations

- Reaction forces
- Forces acting in the members

➢ Unknown forces can be determined from the equations of equilibrium alone, without using the strain-displacement relation, constitutive laws

➢ Example 4.1

❖ "Statically indeterminate" or "hyperstatic" systems

Total No. of unknown forces **>** No. of equilibrium equations

❖ "degree of redundancy" N_R

N_R = Total No. of unknown internal forces - No. of equilibrium equations

➢ Example 4.2

➢ Simultaneous solution of the 3 fundamental groups of equations

4.3 Hyperstatic systems

❖ Difference between "isostatic" and "hyperstatic" systems

➢ Solution procedure

- isostatic
  - equations of equilibrium are only needed
- hyperstatic
  - Equilibrium equations cannot be solved independently of the other 2 sets of equations of elasticity
- 2 main approaches
  - the force method
  - the displacement method

➢ Nature of the solution for the unknown internal forces

- isostatic
  - internal forces can be expressed in terms of the externally applied forces.

    →internal force distribution is independent of the stiffness characteristics of the structure
- hyperstatic
  - internal forces depend on the applied loads, but also on the stiffness of the structure

    →internal force distribution depends on the stiffness characteristics of the structure

4.3 Hyperstatic systems

- Hyperstatic : “dual load paths”
  - Equilibrium equations are not sufficient to determine how much of the load will be carried by load path 1, 2, …
  - According to their relative stiffness, the stiffer load path will carry more load than the more compliant one
  - More damage tolerant
- Isostatic : “single load path”

4.3 Hyperstatic systems

4.3.2 The displacement or stiffness method

❖ Expressing the governing in terms of displacement

① Equilibrium equations of the system : free body diagrams

② Constitutive laws : express internal forces in terms of member deformations or strains

③ Strain-displacement equations : express system deformation in terms of displacements

④ Introduce ③→②: find the internal forces in terms of displacements

⑤ Introduce ④→①: yield the equations of equilibrium in terms of displacements

⑥ Solve ⑤: find the displacement of the system

⑦ Find system deformations : back-substitute the displacements into ③

⑧ Find system internal forces : back-substitute the deformations into ②

4.3 Hyperstatic systems

4.3.3 The force or flexibility method

❖ **Focuses on the solution of the system internal forces, strains and displacements are then recovered**

① Equilibrium equations of the system

② Determine N_R

③ Cut the system and N_R locations and define a single relative displacement for each of the cuts

→ originally hyperstatic system is transformed into an isostatic system

④ Apply N_R redundant forces, each along the relative displacements. Express all internal forces in terms of the applied loads and N_R redundant forces

⑤ Constitutive laws : express system deformations in terms of N_R redundant forces

⑥ Strain-displacement equations : express the relative displacements at N_R cuts in terms of N_R redundant forces

⑦ Impose vanishing of the relative displacements at N_R cuts

⑧ Recover system deformations and system displacements

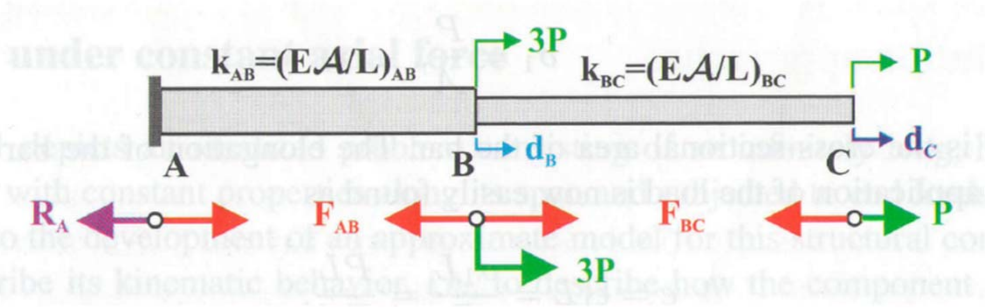
4.3 Hyperstatic systems

- Force method → a linear set of equations of size N_R can be applied effectively, using good engineering judgments
- Displacement method → a linear set of equations of size N_D(unknown displacements) more amenable to automated solution processes

Q & A

Example 4.1

❖ **Series connection of axially loaded bars**



- Axial force equilibrium condition for each joint - points B and C

$$F_{AB} = 4P \qquad F_{BC} = P \tag{4.4}$$

- Constitutive laws for each bar

$$e_{AB} = \frac{4P}{K_{AB}} \qquad e_{BC} = \frac{P}{K_{BC}}$$

- Overall extension of the bar, displacement of point C

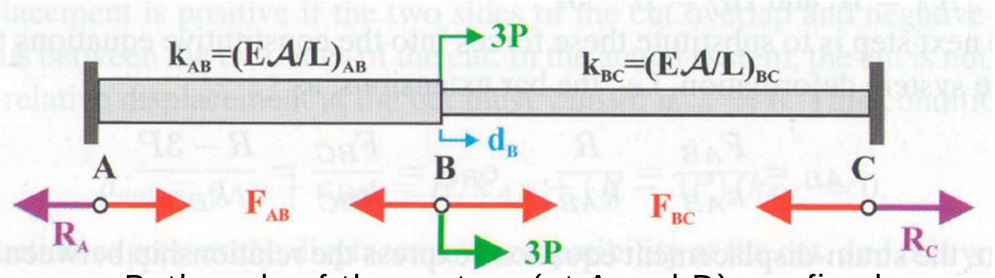
→compatibility condition, $d_c = e_{AB} + e_{BC}$

$$d_C = e_{AB} + e_{BC} = \left(\frac{4}{K_{AB}} + \frac{1}{K_{BC}}\right)P$$

→ internal forces in the bar and the reaction force at point A can be found from equilibrium conditions alone ("isostatic")

Example 4.2

❖ **Series connection of axially loaded bars (displacement approach)**



Both ends of the system (at A and D) are fixed

- Unknowns : 4 $\begin{cases} \text{2 reaction forces } R_A, R_C \\ \text{2 bar forces } F_{AB}, F_{BC} \end{cases}$
- Equilibrium eqns. : 3, one at each of the three joints

$$R_A = F_{AB}$$

$$F_{BC} - F_{AB} + 3P = 0$$

$$R_C = F_{BC}$$

✓ "hyperstatic", "statically indeterminate", "statically redundant"

- Constitutive laws

$$e_{AB} = \frac{F_{AB}}{K_{AB}} \qquad e_{BC} = \frac{F_{BC}}{K_{BC}}$$

→ equilibrium eqn. for point B

$$K_{AB}e_{AB} - K_{BC}e_{BC} = 3P$$

Example 4.2

- Kinematics of the system - bar extension in terms of the displacements

$$d_B = e_{AB} \qquad d_C = e_{AB} + e_{BC} \tag{4.5}$$

- Displacement at C = 0 → $d_c = 0$

$$e_{AB} = -e_{BC} \qquad d_B = e_{AB} = -e_{BC}$$

→ Eq.(4.5) - single eqn. of the unknown displacement at B

$$(K_{AB} + K_{AB})d_B = 3P$$

$$d_B = e_{AB} = -e_{BC} = \frac{3P}{K_{AB} + K_{BC}}$$

- Back substitution

$$F_{AB} = K_{AB} e_{AB} = \frac{3K_{AB}P}{K_{AB} + K_{BC}} \tag{4.6}$$

$$F_{BC} = -K_{BC} e_{BC} = -\frac{3K_{BC}P}{K_{AB} + K_{BC}}$$

✓ "isostatic" problem - internal forces only depend on the externally applied loads.

✓ "hyperstatic" problem - internal forces depend on the applied loads, but also on the stiffness of the structure

Example 4.3

- **Series connection of axially loaded bars (force approach)**
  - ✓ Example 4.2 - 4 unknowns, 3 equilibrium eqns.

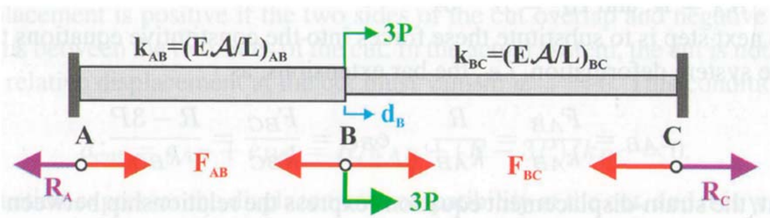
➢ If any one of the 4 internal forces is known, the 3 others can be directly determined from equilibrium eqns.

F_{AB} → denoted by R

Then, $F_{BC} = R - 3P$, $R_A = R$, $R_C = R - 3P$

➢ Substitute these forces into the constitutive eqns. to determine the system deformation (bar extensions)

$$e_{AB} = \frac{F_{AB}}{K_{AB}} = \frac{R}{K_{AB}} \qquad e_{BC} = \frac{F_{BC}}{K_{BC}} = \frac{R-3P}{K_{BC}}$$



➢ Strain-displacement equation - Fig. 4.3

$$d_A = d_C = 0 \qquad d_B = e_{AB}$$

compatibility of deformation between A and C → $e_{AB} + e_{BC} = 0$

→ necessary eqn. to solve for R

$$e_{AB} + e_{BC} = \frac{R}{K_{AB}} + \frac{R-3P}{K_{BC}} = 0 \qquad R = \frac{3K_{AB}}{K_{AB}+K_{BC}}P$$

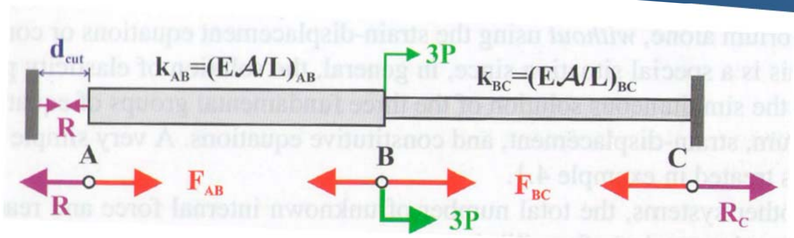
Example 4.3

- Equilibrium eqns.

$$F_{AB} = R = \frac{3K_{AB}}{K_{AB} + K_{BC}} P \qquad F_{BC} = R - 3P = -\frac{3K_{AB}}{K_{AB} + K_{BC}} P$$

→ Identical to the previous example (displacement approach)

Example 4.3



➢ "Force" method - determination of the unknown force, R is based on the enforcement of compatibility conditions for system deformations

✓ Step 1. - the system is assumed to be "cut" at a location of R

✓ Step 2. - think of force R as an externally applied load

$$e_{AB} = \frac{F_{AB}}{K_{AB}} = \frac{R}{K_{AB}} \qquad e_{BC} = \frac{F_{BC}}{K_{BC}} = \frac{R-3P}{K_{BC}}$$

✓ Step 3. - compatibility condition, $d_{cut} = e_{AB} + e_{BC}$

in the actual system the cut is not present, $d_{cut} = 0$

$$d_{cut} = e_{AB} + e_{BC} = \frac{R}{K_{AB}} + \frac{R-3P}{K_{BC}} = 0$$

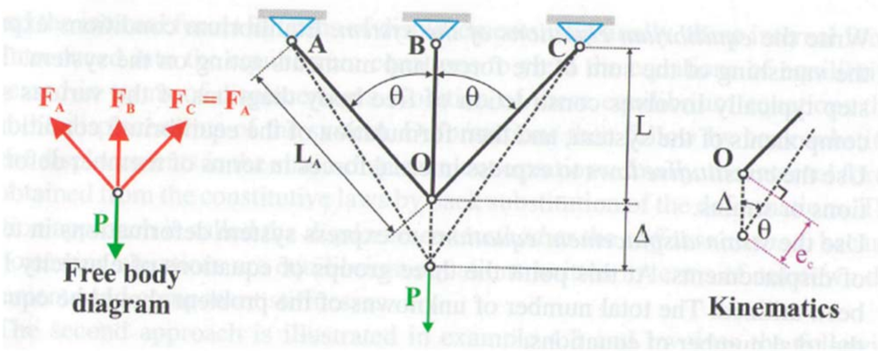
→ expresses the displacement compatibility at the cut, in terms of forces and flexibilities (inverse of stiffness)

✓ can be solved for the unknown force, R

$$R = \frac{3/K_{BC}}{1/K_{AB} + 1/K_{BC}} P = \frac{3K_{AB}}{K_{AB} + K_{BC}} P$$

Example 4.4

❖ **Hyperstatic 3-bar truss (displacement method)**



➢ 3 bars pinned together, vertical load P applied at 0

➢ Assumption : geometric and material symmetry about vertical axis OB

$$A_A = A_C, \quad E_A = E_C \quad \rightarrow \quad F_A = F_C$$

horizontal displacement component

✓ Step 1. - equilibrium eqn.

$$F_B + 2F_A \cos\theta = P \qquad (4.8)$$

{ 2 unknowns : F_A, F_B
  1 equilibrium eqn. → hyperstatic system of order 1

✓ Step 2. - constitutive laws

$$e_A = e_C = \frac{F_A L_A}{(EA)_A} = \frac{F_A L}{(EA)_A \cos\theta} \qquad e_B = \frac{F_B L}{(EA)_B} \qquad (4.9)$$

Example 4.4

- ✓ Step 3. - strain-displacement eqn.

  If small displacement, i.e., $\Delta \ll L$

  $\rightarrow$ θ changes little during deformation

  $\rightarrow$ $e_C \approx A\cos\theta$ (4.10)

  $\rightarrow$ $e_A = e_C = A\cos\theta, \qquad e_B = \Delta$

- ✓ Step 4. - express the internal forces in terms of displacements

$$\frac{F_A}{(EA)_B} = \frac{F_C}{(EA)_B} = \frac{\Delta}{L}\bar{k}_A\cos^2\theta \qquad \frac{F_B}{(EA)_B} = \frac{\Delta}{L} \tag{4.11}$$

where $\bar{k}_A = \dfrac{(EA)_A}{(EA)_B}$: non-dimensioned stiffness of bar A

- ✓ Step 5. - express the single equilibrium condition in terms of the single displacement Δ (substitute Eq. (4.11) into (4.8)

$$\frac{\Delta}{L} + 2\frac{\Delta}{L}\bar{k}_A\cos^3\theta = \frac{P}{(EA)_B} \tag{4.12}$$

- ✓ Step 6. - solves for Δ

$$\frac{\Delta}{L} = \frac{1}{1+2\bar{k}_A\cos^3\theta}\frac{P}{(EA)_B} \quad \text{or} \quad \Delta = \frac{P}{k} \quad \text{where } k = \frac{(EA)_B + 2(EA)_A\cos^3\theta}{L}$$

: equivalent vertical stiffness of the 3-bar truss

Example 4.4

- ✓ Step 7. - deformation recovery, Eq. (4.12) → (4.10)

$$\frac{e_A}{L}=\frac{e_C}{L}=\frac{\cos\theta}{1+2\bar{k}_A\cos^3\theta}\frac{P}{(EA)_B}\qquad \frac{e_B}{L}=\frac{1}{1+2\bar{k}_A\cos^3\theta}\frac{P}{(EA)_B}\tag{4.13}$$

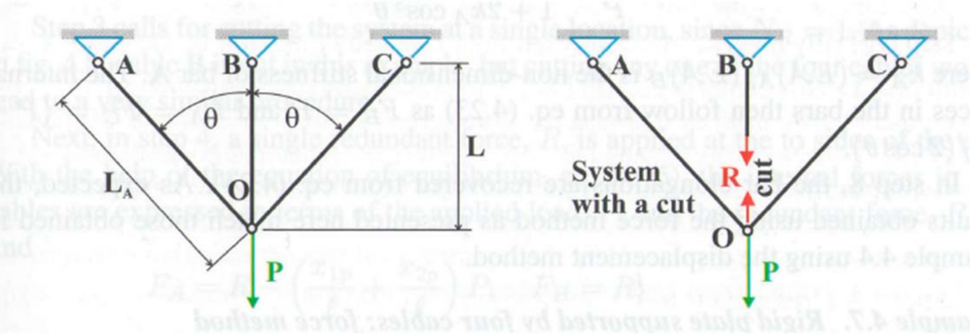
- ✓ Step 8. - internal force, Eq.(4.13) → (4.9)

$$\frac{F_A}{P}=\frac{F_C}{P}=\frac{\bar{k}_A\cos^2\theta}{1+2\bar{k}_A\cos^3\theta}\qquad \frac{F_B}{P}=\frac{1}{1+2\bar{k}_A\cos^3\theta}\tag{4.14}$$

$\frac{F_A}{F_B}=\bar{k}_A\cos^2\theta$: ratio of the internal forces is in proportion to the ratio of their stiffness

Example 4.6

❖ **Hyperstatic 3-bar truss (force method)**



✓ Step 1. - single equilibrium eqn.

$$F_B + 2F_A \cos\theta = P \qquad (4.22)$$

✓ Step 2. - system degree of redundancy $N_R = 2 - 1 = 1$

✓ Step 3. - cut the system at 1 location since $N_R = 1$

→ bar B is cut

✓ Step 4. - single redundant force R is applied at the sides of the cut with R treated as a known load, (4.23) →

$$F_A = F_C = \frac{P - R}{2\cos\theta} \qquad F_B = R \qquad (4.23)$$

Example 4.6

- ✓ Step 5. - bar extensions, expressed in terms of R,

  Constitutive laws, Eq.(4.9) →

$$\frac{e_C}{L} = \frac{e_A}{L} = \frac{F_A}{(EA)_A \cos\theta} = \frac{P-R}{2(EA)_A \cos^2\theta}$$

$$\frac{e_B}{L} = \frac{F_B}{(EA)_B} = \frac{R}{(EA)_B} \qquad (4.24)$$

- ✓ Step 6. - determination of the relative displacement at the cut

$$d_{cut} = \frac{e_A}{\cos\theta} - e_B = \frac{(P-R)L}{2(EA)_A \cos^3\theta} - \frac{RL}{(EA)_B}$$

- ✓ Step 7. - $d_{cut} = 0$, to find the redundant force R

$$\frac{R}{P} = \frac{1}{1 + 2\bar{k}_A \cos^3\theta}$$

- ✓ Step 8. - recovery of the bar elongation