Chapter 7
Energy Storage Elements

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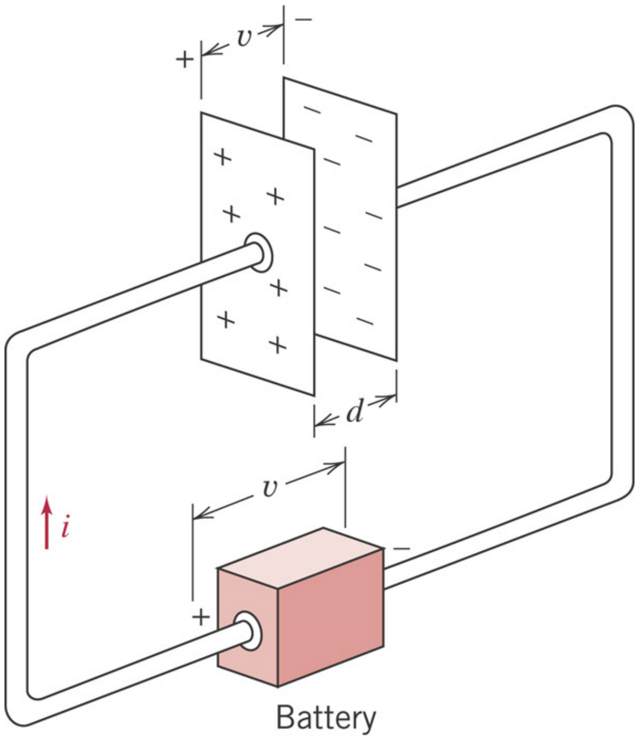
Capacitors

- A capacitor is a two terminal elements that is a model of a device consisting of two conducting plates separated by a nonconducting material.



$$C = \varepsilon A/d, \quad q = CV$$

where $\begin{cases} \varepsilon : \text{the dielectric constant} \\ A : \text{the area of the plates} \\ d : \text{the space between plates} \end{cases}$

Table 7.2-1 Relative Dielectric Constant	
Material	$\varepsilon_r = \varepsilon/\varepsilon_o$
Glass	7
Nylon	2
Bakelite	5

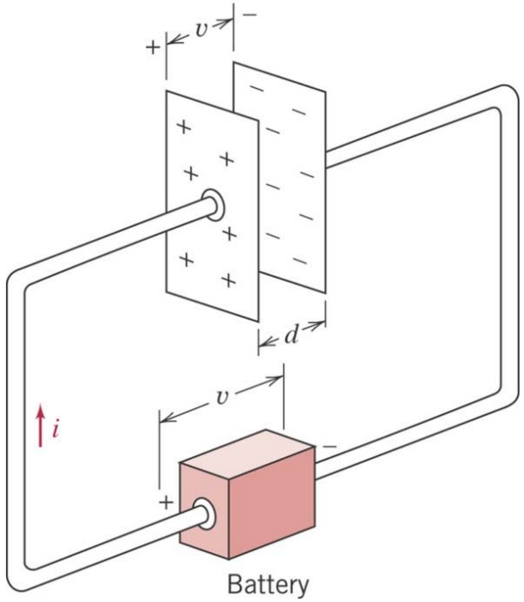
Capacitors (cont'd)

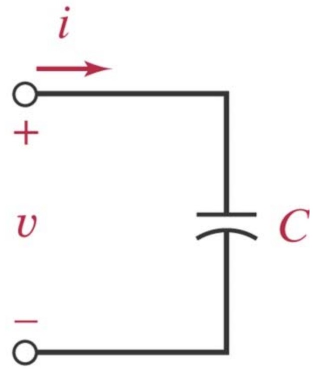
- A capacitor is a measure of the ability of a device to store energy in the form of separated charge of an electric field.

- C (capacitance)
  - The constant of proportionality
  - The unit is coulomb per volt and is called farad (F) in honor of Faraday.

- The current i is

$$i = \frac{dq}{dt} = C\frac{dv}{dt}$$





Displacement Current

James Clerk Maxwell noticed a logical inconsistency when applying Ampère's law to a charging or discharging capacitor. If surface *S* passes between the plates of the capacitor, and not through any wires, then even though there is no conduction current, magnetic field is induced. He concluded that Ampere's law law had to be incomplete. To resolve the problem, he came up with the concept of displacement current and made a generalized version of Ampère's law which was incorporated into Maxwell's equations.

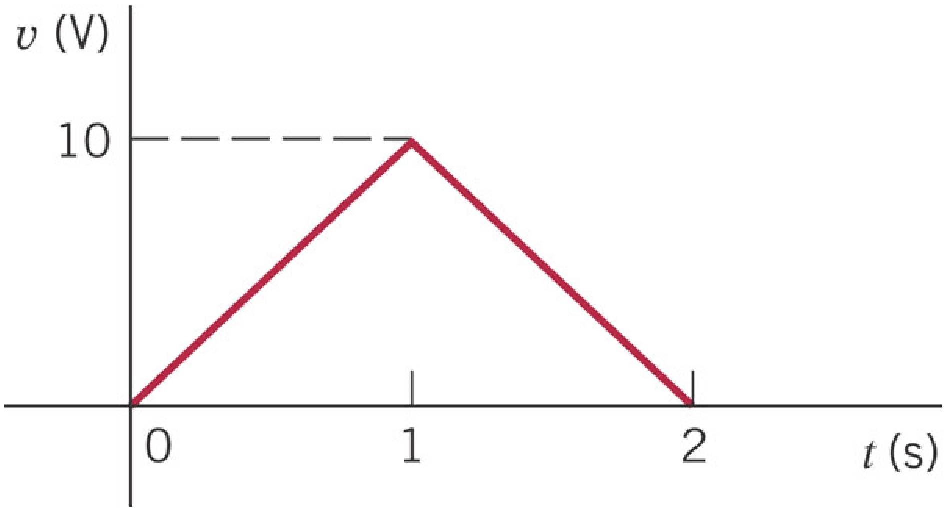
$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \oint_C \vec{H} \cdot \mathrm{d}\vec{l} = \iint_S \vec{J} \cdot \mathrm{d}\vec{A} + \frac{\mathrm{d}}{\mathrm{d}t} \iint_S \vec{D} \cdot \mathrm{d}\vec{A}$$

$$\vec{D} = \varepsilon \vec{E}$$

In linear media, this is the displacement flux density (in coulombs per square meter).

Example 7.2-1 **Capacitor current and voltage**

- Find the current for a capacitor C=1mF when the voltage across the capacitor is represented by the signal shown in Figure 7.2-6



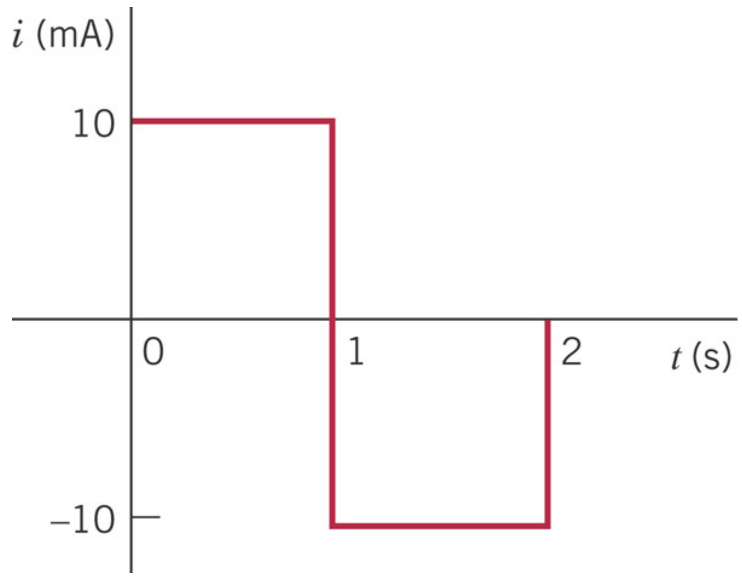
Solution

- The voltage is given by

$$
\begin{aligned}
v &= 0 & t &\le 0 \\
&= 10t & 0 \le t &\le 1 \\
&= 20 - 10t & 1 \le t &\le 2 \\
&= 0 & t &\ge 2
\end{aligned}
$$

Then, since i=Cdv/dt, where C=10^{-3} F, we obtain

$$
\begin{aligned}
i &= 0 & t &< 0 \\
&= 10^{-2} & 0 < t &< 1 \\
&= -10^{-2} & 1 < t &< 2 \\
&= 0 & t &> 2
\end{aligned}
$$



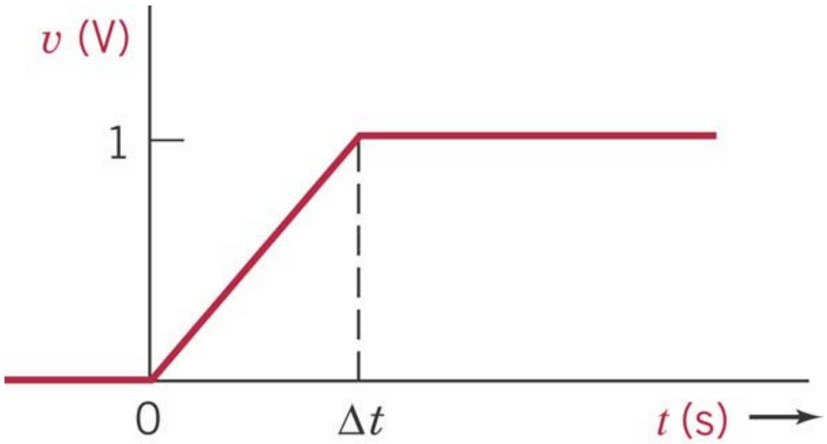
Therefore, the resulting current is a series of two pulses of magnitudes 10^{-2}A and 10^{-2}A, respectively, as shown in Figure 7.2-6

Capacitors (cont'd)

- Now consider the waveform shown in Figure 7.2-3. Since i=Cdv/dt, we obtain

$$
\begin{aligned}
i &= 0 && t < 0 \\
&= C(1/\Delta t) && 0 < t < \Delta t \\
&= 0 && t > \Delta t
\end{aligned}
$$

Thus, we obtain a pulse of height equal to C/Δt.
Clearly, Δt cannot decline to zero or we experience an infinite current.
An infinite current is an impossibility.



$\Delta t \neq 0$

- Therefore, the voltage across a capacitor cannot change instantaneously.

Capacitors (cont'd)

- The capacitor voltage can be found by integrating the capacitor current from time -∞ until time t.

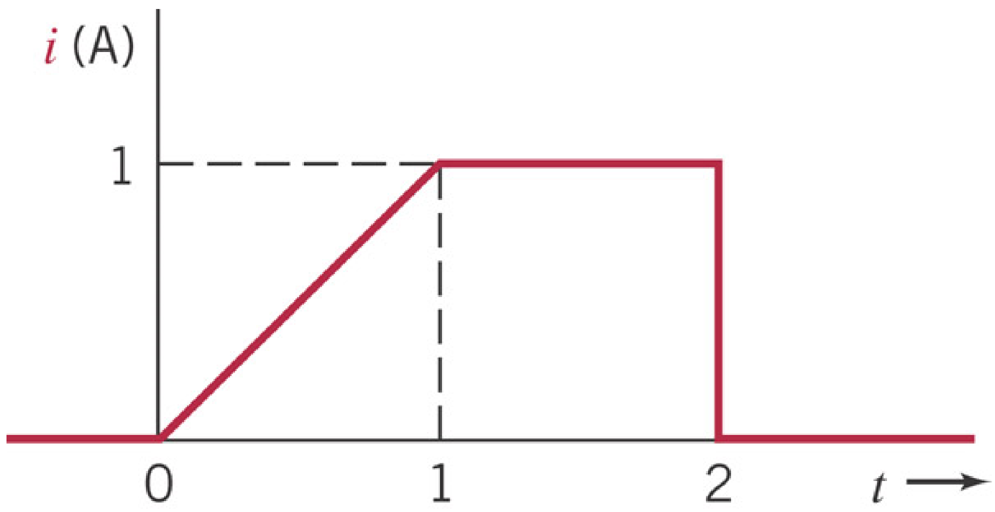
$$v = \frac{1}{C}\int_{-\infty}^{t} i\,d\tau$$

$$= \frac{1}{C}\int_{t_0}^{t} i\,d\tau + \frac{1}{C}\int_{-\infty}^{t_0} i\,d\tau = \frac{1}{C}\int_{t_0}^{t} i\,d\tau + v(t_0)$$

$$\text{where} \begin{cases} t_0 : & \text{initial time} \\ v(t_0): & \text{initial condition} \end{cases}$$

Example 7.2-2 **Capacitor Current and Voltage**

- Find the voltage v for a capacitor C=1/2 F when the current is as shown in Figure 7.2-8.

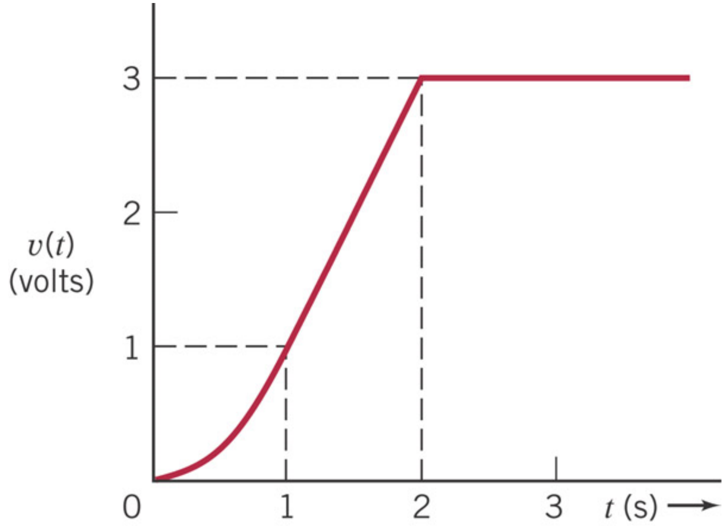


Solution

- The current is given by

$$
\begin{aligned}
i &= 0 & t < 0 \\
&= t & 0 < t < 1 \\
&= 1 & 1 < t < 2 \\
&= 0 & t > 2
\end{aligned}
$$

Then since $v = \frac{1}{C}\int_0^t i\, d\tau$

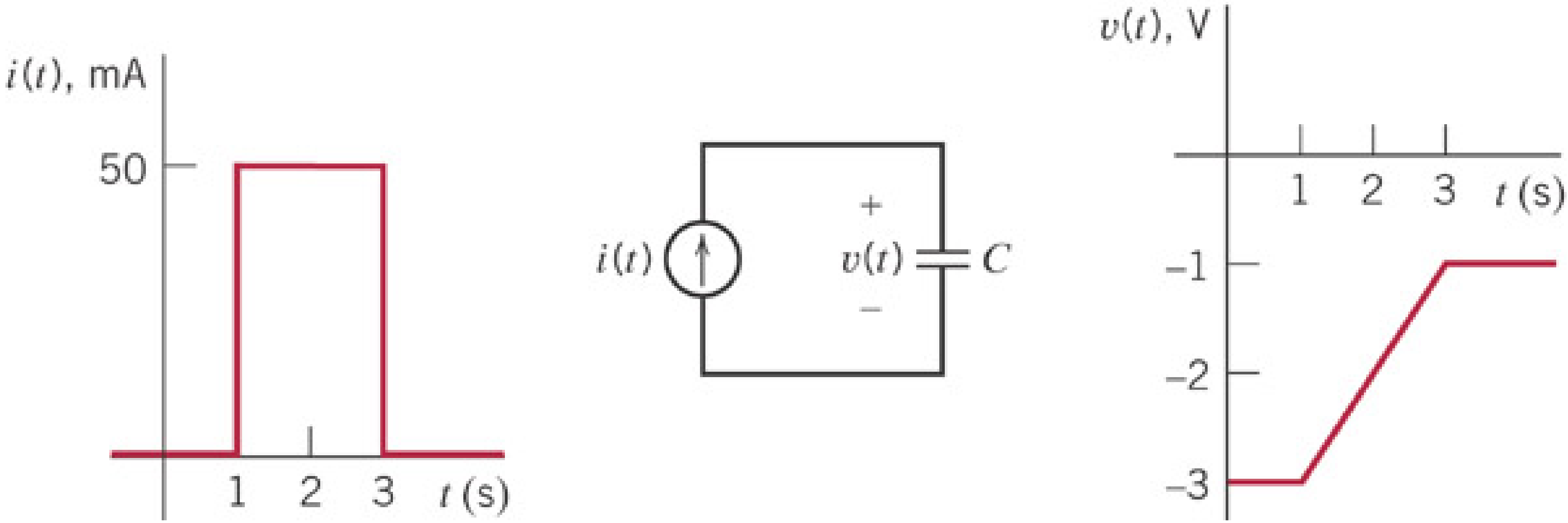


And C=1/2, we have

$$
\begin{aligned}
v &= 0 & t \le 0 \\
&= 2\int_0^t \tau d\tau = t^2 & 0 \le t \le 1 \\
&= 2\int_1^t (1) d\tau + v(1) = 2(t-1) + 1 = 2t - 1 & 1 \le t \le 2 \\
&= v(2) = 3 & t \ge 2
\end{aligned}
$$

Example 7.2-3 Capacitor current and voltage

- Figure 7.2-11 shows a circuit together with two plots. The plots represent the current and voltage of the capacitor in the circuit. Determine the values the constants, a and b, used to label the plot of the capacitor current.



Solution

- The current and voltage of the capacitor are related by

$$v(t) = \frac{1}{C}\int_{t0}^{t} i(\tau)d\tau + v(t_0)$$

$$v(t) - v(t_0) = \frac{1}{C}\int_{t0}^{t} i(\tau)d\tau \qquad (7.2\text{–}6)$$

$v(t) - v(t_0) =$ The difference between the values of voltage at times t and t_0

$\int_{t0}^{t} i(\tau)d\tau =$ The area under the plot of i(t) versus t, for times between t and t_0

Pick convenient values t and t0, for example, t_0=1s and t=3s. Then

$$v(t) - v(t_0) = -1 - (-3) = 2\text{V}$$

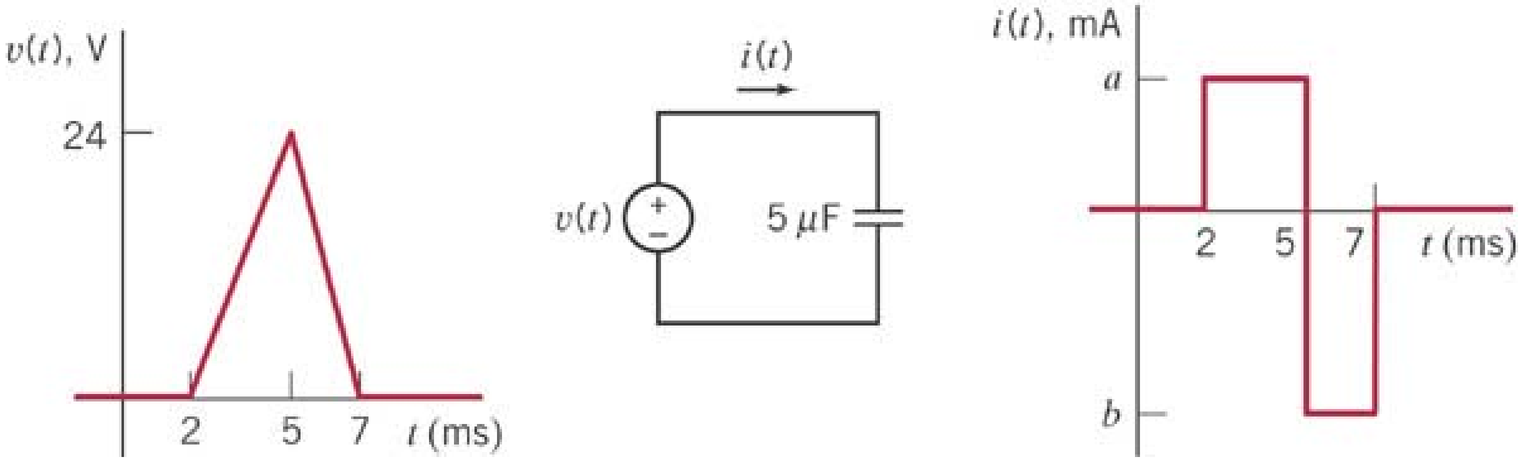
$$\int_{t0}^{t} i(\tau)d\tau = \int_{1}^{3} 0.05 d\tau = (0.05)(3-1) = 0.1\text{A}\cdot\text{s}$$

Using Eq. 7.2-6 gives

$$2 = \frac{1}{C}(0.1) \quad \Rightarrow \quad C = 0.05\frac{\text{A}\cdot\text{s}}{\text{V}} = 0.05\text{F} = 50\text{mF}$$

Example 7.2-4 **Capacitor Current and Voltage**

- Figure 7.2-11 shows a circuit together with two plots. The plots represent the current and voltage of the capacitor in the circuit. Determine the values the constants, a and b, used to label the plot of the capacitor current.



Solution

- The current and voltage of the capacitor are related by

$$i(t) = C\frac{d}{dt}v(t) \qquad (7.2\text{-}7)$$

To determine the value of a pick t=3ms

$$\frac{d}{dt}v(0.003) = \frac{0-24}{0.002-0.005} = 8000\frac{\text{V}}{\text{s}}$$

Using Eq. 7.2-7 gives

$$a = (5\times10^{-6})(8000) = 40\text{mA}$$

To determine the value of b, pick t=6ms;

$$\frac{d}{dt}v(0.006) = \frac{24-0}{0.005-0.007} = 12\times10^{3}\frac{\text{V}}{\text{s}}$$

Using Eq. 7.2-7 gives

$$b = (5\times10^{-6})(12\times10^{3}) = 60\text{mA}$$

Example 7.2-5 **Capacitor Current and Voltage**

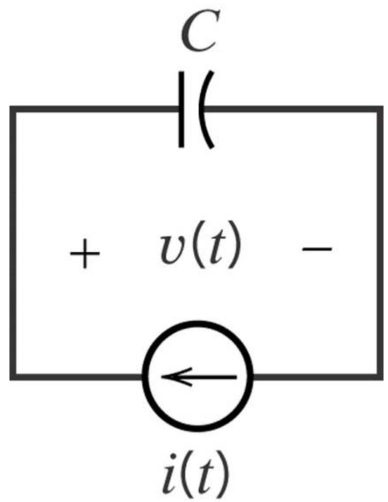
- The input to the circuit shown in Figure 7.2-12 is the current

$$i(t) = 3.75e^{-1.2t}\,\text{A} \quad \text{for } t > 0$$

The output is the capacitor voltage

$$v(t) = 4 - 1.25e^{-1.2t}\,\text{V} \quad \text{for } t > 0$$

Find the value of the capacitance, C



Solution

- The capacitor voltage is related to the capacitor current by

$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v(0)$$

That is,

$$4 - 1.25e^{-1.2t} = \frac{1}{C}\int_0^t 3.75e^{-1.2\tau}d\tau + v(0) = \frac{3.75}{C(-1.2)}e^{-1.2\tau}\Big|_0^t + v(0) = \frac{-3.125}{C}(e^{-1.2t} - 1) + v(0)$$

Equating the coefficients of $e^{-1.2t}$ gives

$$1.25 = \frac{3.125}{C} \quad \Rightarrow \quad C = \frac{3.125}{1.25} = 2.5\text{F}$$

Energy Storage in a Capacitor

- The energy stored in a capacitor is

$$w_c(t) = \int_{-\infty}^{t} vi\,d\tau$$

Since $$i = C\frac{dv}{dt}$$

we have $$w_c(t) = \int_{-\infty}^{t} vC\frac{dv}{dt}\,d\tau = C\int_{v(-\infty)}^{v(t)} v\,dv$$

$$= \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)}$$

Since the capacitor was uncharged at t=−∞, set v(−∞)=0
Therefore,

$$w_c(t) = \frac{1}{2}Cv^2(t)\ \text{(J)}$$

Energy Storage in a Capacitor (cont'd)

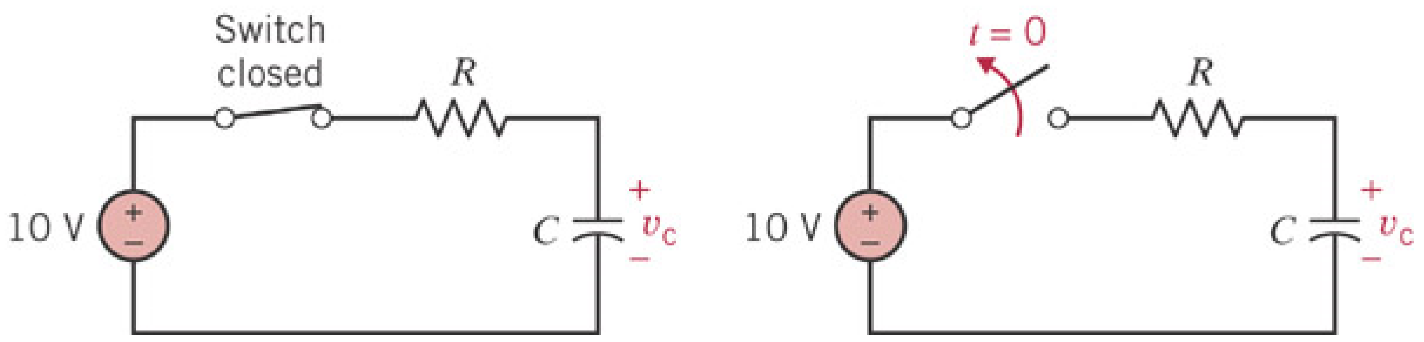
- Since q=Cv,

  we have $w_c(t) = \frac{1}{2C} q^2(t) \text{ (J)}$

- The voltage and charge on a capacitor cannot change instantaneously.

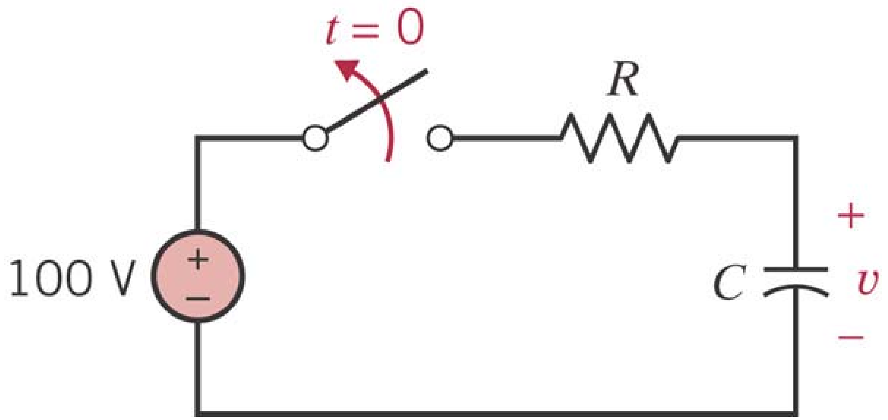
- In Figure 7.3-1

$$v_c(0^+) = v_c(0^-) = 10 \text{ V}$$



Example 7.3-1 **Energy stored by a capacitor**

- A 10-mF capacitor is charged to 100V, as shown in the circuit of Figure 7.3-2. Find the energy stored by the capacitor and the voltage of the capacitor at $t=0^+$ after the switch is opened



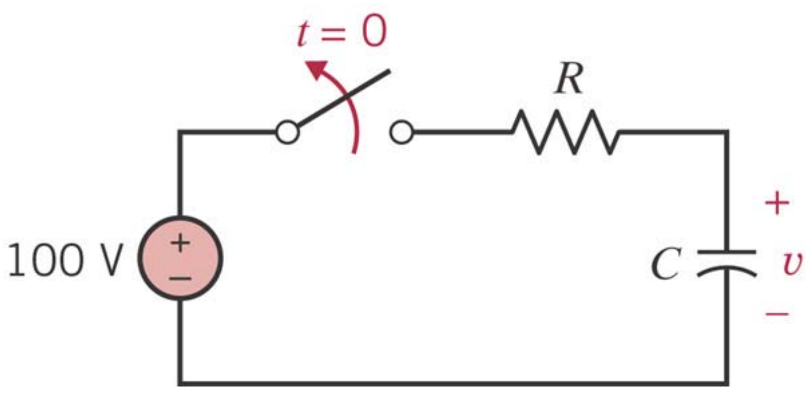
Solution

- The voltage of the capacitor is v=100V at t=0^-. Since the voltage at t=0^+ cannot change from the voltage at t=0^-, we have

$$v(0^+) = v(0^-) = 100\text{V}$$

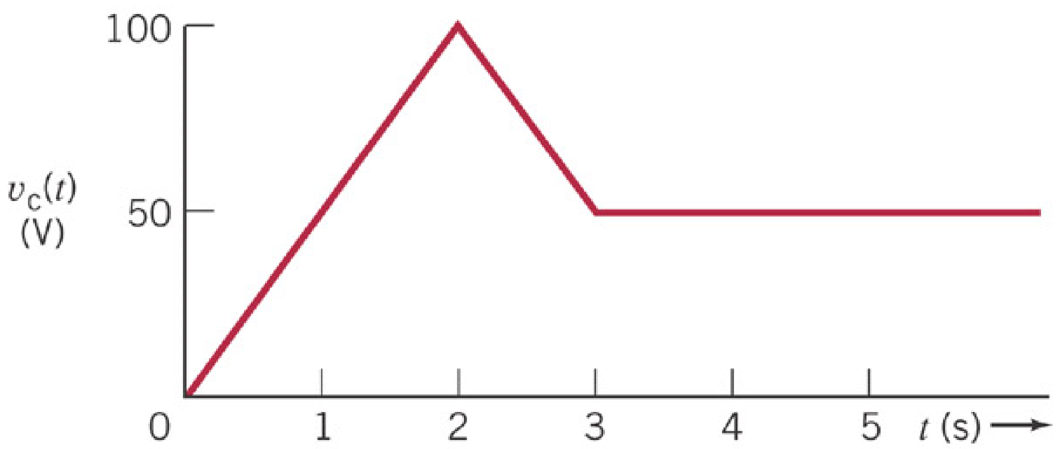
- The energy stored by the capacitor at t=0^+ is

$$w_c = \frac{1}{2}Cv^2 = \frac{1}{2}(10^{-2})(100^2) = 50\text{J}$$



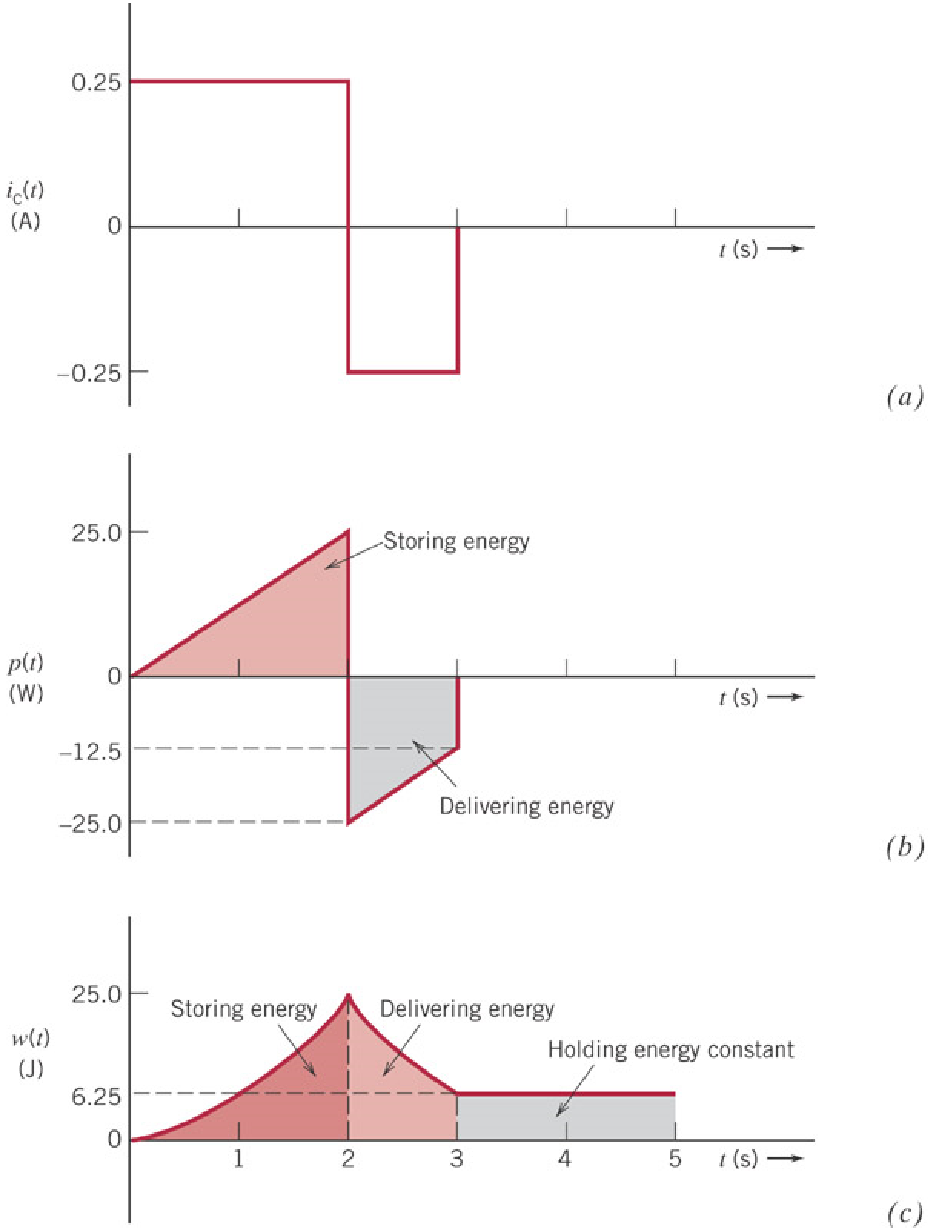
Example 7.3-2 **Power and Energy for a Capacitor**

- The voltage across a 5-mF capacitor varies as shown in Figure 7.3-3. Determine and plot the capacitor current, power and energy



Solution

- Figure 7.3-4a
  - Current
  - $i_c = C dv/dt$
- Figure 7.3-4b
  - Power
  - $P(t)=v(t)i(t)$
- Figure 7.3-4c
  - Energy
  - $\omega = \int p\, dt$



Series and Parallel Capacitors

- Consider Parallel connection of N capacitors as shown in Figure 7.4-1. Using KCL, we have

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

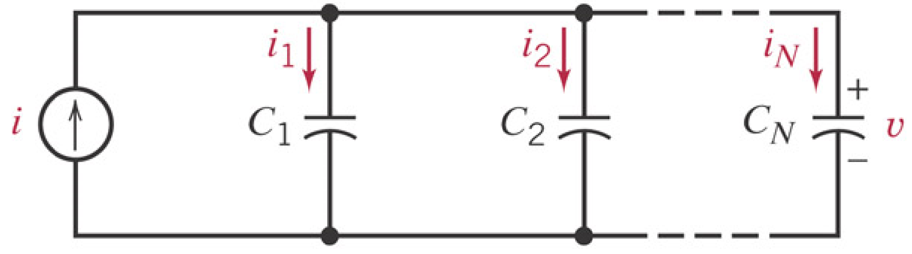


Fig 7.4-1

Since $$i_n = C_n \frac{dv}{dt}$$

$$i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \cdots + C_N \frac{dv}{dt}$$

$$= \left(C_1 + C_2 + C_3 + \cdots + C_N\right) \frac{dv}{dt}$$

$$= \left(\sum_{n=1}^{N} C_n\right) \frac{dv}{dt}$$

Series and Parallel Capacitors

- For the equivalent circuit shown in Figure 7.4-2

$$i = C_p \frac{dv}{dt}$$

It is clear that

$$C_p = C_1 + C_2 + C_3 + \cdots + C_N = \sum_{n=1}^{N} C_n$$

- The equivalent capacitance of a set of N parallel capacitors is simply the sum of the individual capacitances. It must be noted that all the parallel capacitors will have the same initial condition, v(0)

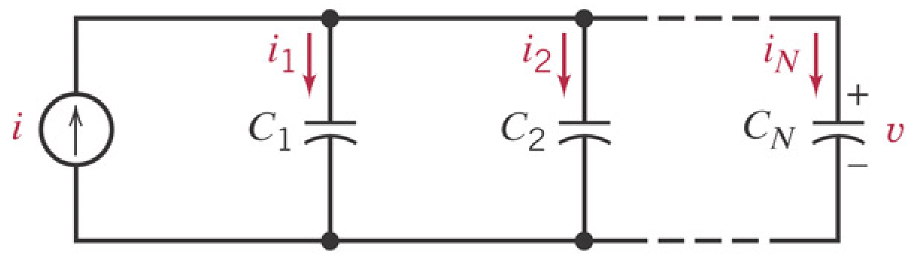


Fig 7.4-1

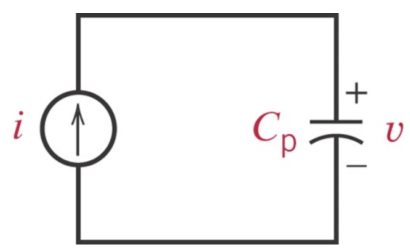


Fig 7.4-2

Series and Parallel Capacitors

- Consider series connection of N capacitors as shown in Figure 7.4-3. Using KVL, we have

$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

Since $$v_n(t) = \frac{1}{C_n}\int_{t_0}^{t} i\,d\tau + v_n(t_0)$$

$$v = \frac{1}{C_1}\int_{t_0}^{t} i\,d\tau + v_1(t_0) + \cdots + \frac{1}{C_2}\int_{t_0}^{t_0} i\,d\tau + v_N(t_0)$$

$$= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i\,d\tau + \sum_{n=1}^{N} v_n(t_0)$$

$$= \sum_{n=1}^{N}\frac{1}{C_n}\int_{t_0}^{t} i\,d\tau + \sum_{n=1}^{N} v_n(t_0)$$

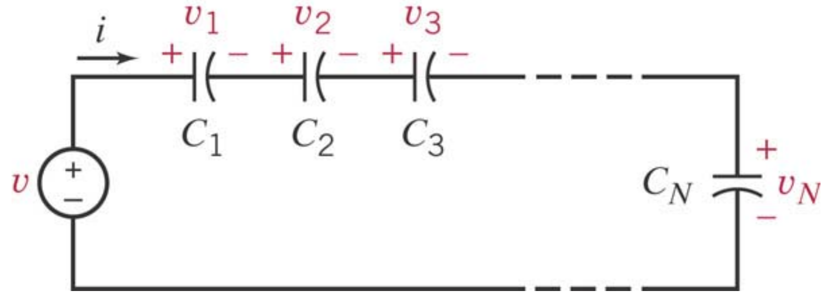


Fig 7.4-3

Series and Parallel Capacitors

- Since

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0) = \sum_{n=1}^{N} v_n(t_0)$$

we obtain

$$v = \sum_{n=1}^{N} \frac{1}{C_n} \int_{t_0}^{t} i\, d\tau + v(t_0)$$

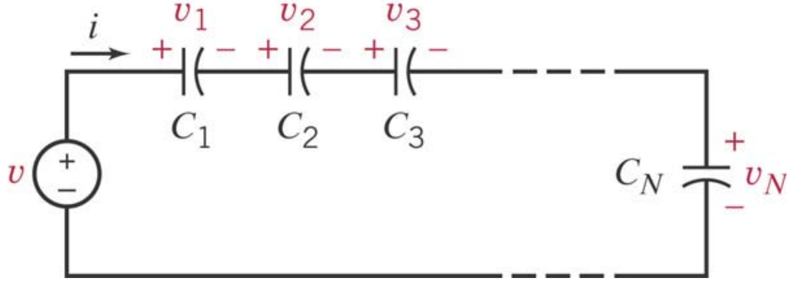


Fig 7.4-3

For the equivalent circuit shown in Figure 7.4-4

$$v(t) = \frac{1}{C_S} \int_{t_0}^{t} i\, d\tau + v(t_0)$$

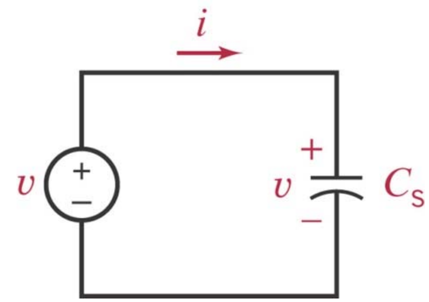


Fig 7.4-4

we find that

$$\frac{1}{C_s} = \sum_{n=1}^{N} \frac{1}{C_n}$$

Example 7.4-1 **Parallel and Series Capacitors**

- Find the equivalent capacitance for the circuit of Figure 7.4-5 when $C_1=C_2=C_3=2mF$, $v_1(0)=10V$, and $v_2(0)=v_3(0)=20V$

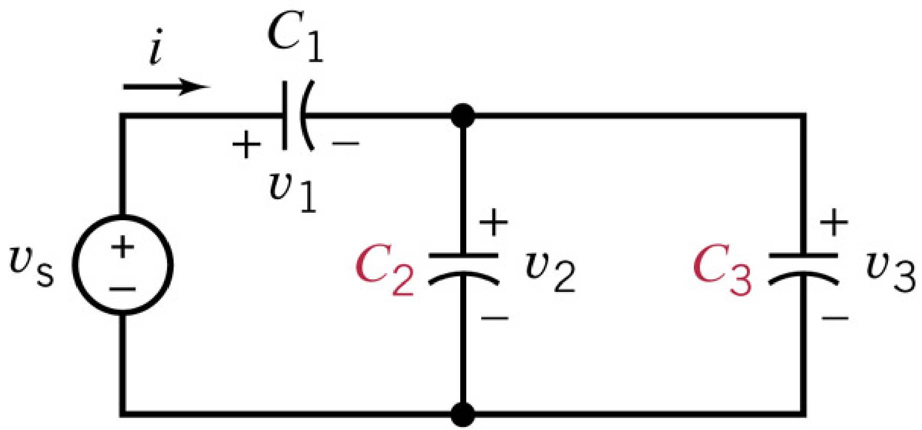


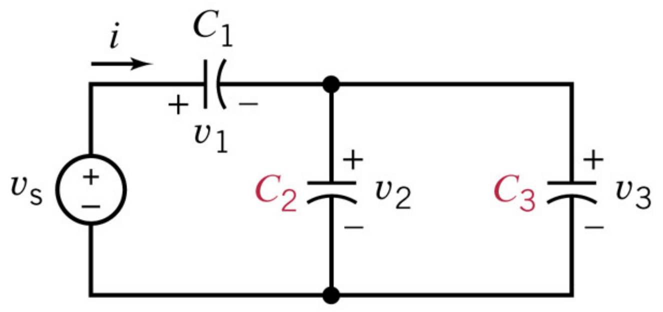
Figure 7.4-5

Solution

- Cp and Cs

$$C_P = C_2 + C_3 = 4 \text{ mF}$$

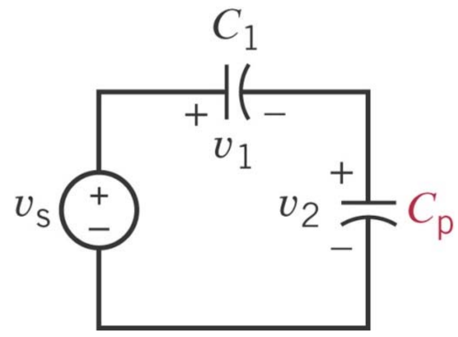
$$C_S = \frac{C_1 C_P}{C_1 + C_P} = \frac{(2\times10^{-3})(4\times10^{-3})}{(2\times10^{-3})+(4\times10^{-3})} = \frac{8}{6} \text{ mF}$$



7.4−5

The voltage at t=0 across Cs is

$$v(0)=v_1(0)+v_p(0)$$

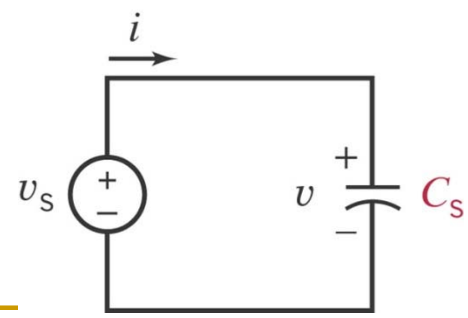


7.4−6

Where vp(0)=20V, the voltage across the capacitance Cp at t=0.

$$v(0)=10+20=30V$$

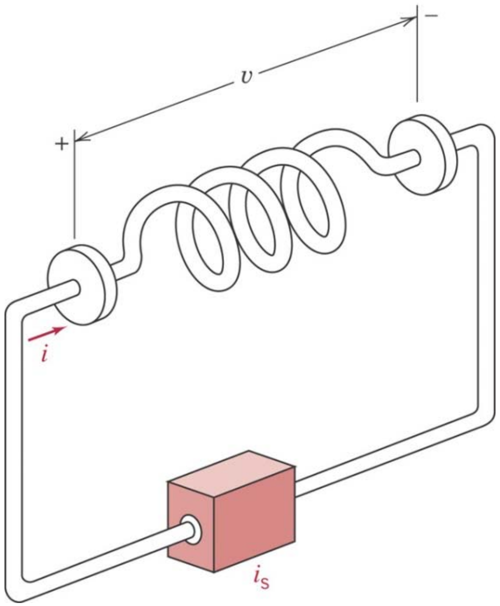
Thus, we obtain the equivalent circuit shown in Figure 7.4-7



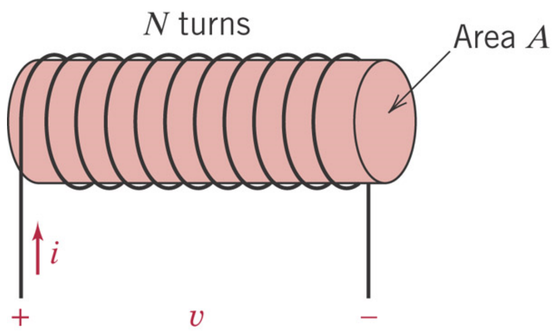
7.4−7

Inductors

- A inductor is a two terminal element consisting of a winding of N turns for introducing inductance into an electric circuit.
- Inductance is the property of an electric device by which a time-varying current through the device produces a voltage across it.



Coil of wire connected to a current source

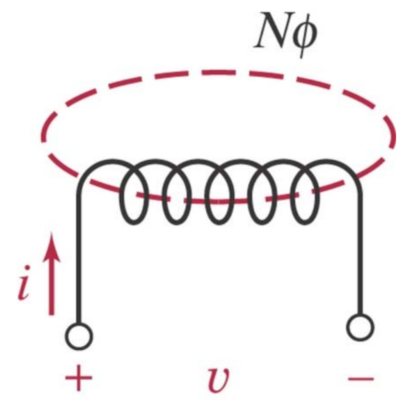


Coil wound as a tight helix on a core of area A

Inductors

- The force experienced by two neighboring current-carrying wires can be described in terms of the existence of a magnetic field, which can be described in terms of magnetic flux the forms a loop around the coil, as shown in Figure 7.5-3. A magnetic flux Φ(t) is associated with a current i in a coil. In this case we have an N-turn coil, and each flux line passes through all turns. Then the total flux is said to be NΦ.
- According to Faraday, the changing flux creates an induced voltage in each turn equal to the derivative of the flux Φ, so the voltage v across N turn is

$$v = N\frac{d\phi}{dt}$$



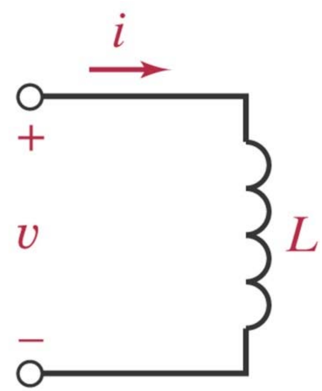
Inductors

- Since the total flux NΦ si proportional to the current i in a coil, we have

$$N\phi = Li$$

where L, inductance, is the constant of proportionality. Then the voltage v is

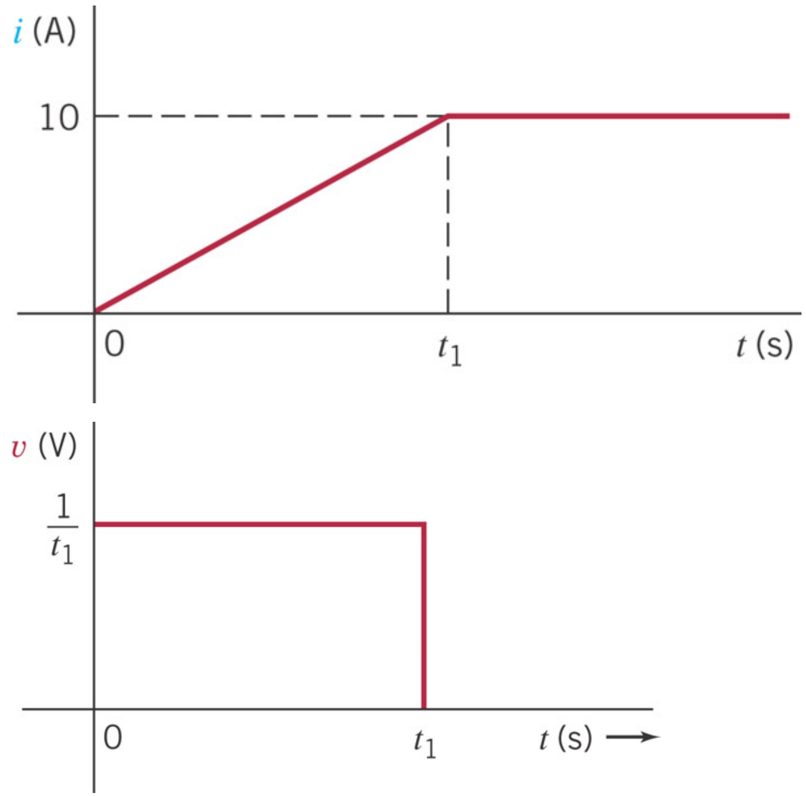
$$v = L\frac{di}{dt}$$



Inductors

- Now consider the current waveform shown in Figure 7.5-7
  Since v=Ldi/dt, we obtain

$$
\begin{aligned}
v &= 0 & t &< 0 \\
&= 1/t_1 & 0 < t &< t_1 \\
&= 0 & t &> t_1
\end{aligned}
$$



Clearly, t1 cannot decline to zero or we experience an infinite voltage and we would require infinite power at the terminals of the inductor.

- **The current in an inductor cannot change instantaneously.**

Inductors

- The inductor current can be found by integrating the inductor voltage from time -∞ until time t.

$$i = \frac{1}{L}\int_{-\infty}^{t} v\,d\tau$$
$$= \frac{1}{L}\int_{t_0}^{t} v\,d\tau + \frac{1}{L}\int_{-\infty}^{t_0} v\,d\tau = \frac{1}{L}\int_{t_0}^{t} v\,d\tau + i(t_0)$$

$$\text{where} \begin{cases} t_0 : & \text{initial time} \\ i(t_0) : & \text{initial condition} \end{cases}$$

Example 7.5-1 **Inductor Current and Voltage**

- Find the voltage across an inductor, L=0.1H, when the current in the inductor is

$$i = 20te^{-2t} \text{ A}$$

for t>0 and i(t)=0

Solution

- The voltage for t>0 is

$$v = L\frac{di}{dt} = (0.1)\frac{d}{dt}(20te^{-2t})$$
$$= 2(-2te^{-2t} + e^{-2t}) = 2e^{-2t}(1-2t)\ \text{V}$$

The voltage is equal to 2V when i=0, as shown in Figure 7.5-11b. The current waveform is shown in Figure 7.5-11a. Note that the current reaches a maximum value and the voltage is zero at t=0.5s.

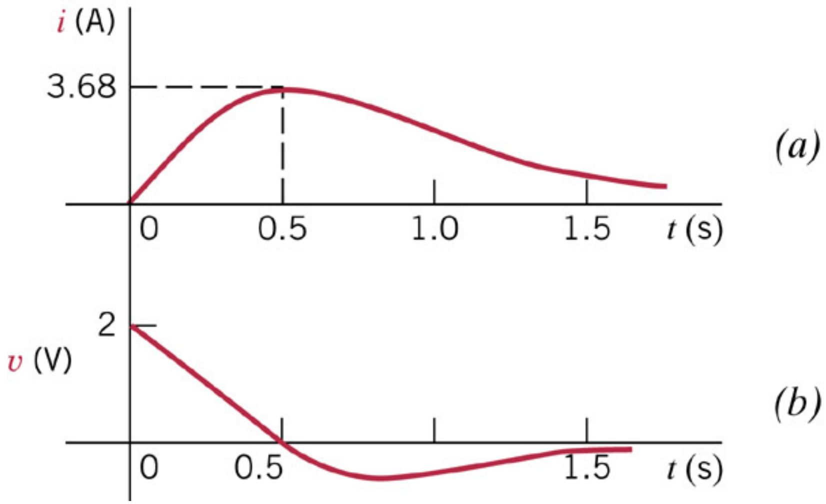
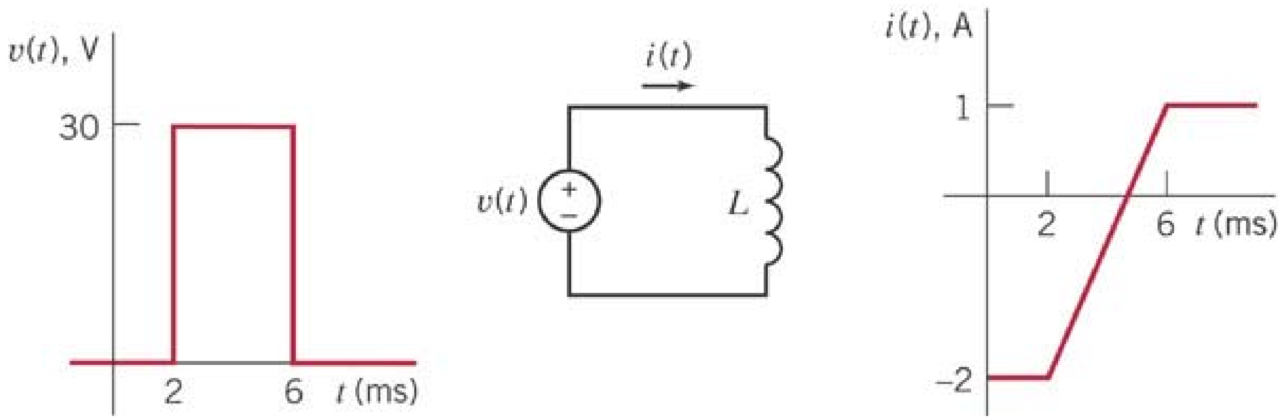


Fig. 7.5-11

Example 7.5-2 **Inductor Current and Voltage**

- Figure 7.5-12 shows a circuit together with two plots. The plots represent the current and voltage of the inductor in the circuit. Determine the value of the inductance of the inductor.



Solution

- The current and voltage of the inductor are related by

$$i(t) = \frac{1}{L}\int_{t0}^{t} v(\tau)d\tau + i(t_0)$$

$$i(t) - i(t_0) = \frac{1}{L}\int_{t0}^{t} v(\tau)d\tau \qquad (7.5\text{-}8)$$

$i(t) - i(t_0) =$ The difference between the values of current at times t and t_0

$\int_{t0}^{t} v(\tau)d\tau =$ The area under the plot of v(t) versus t, for times between t and t_0

Pick convenient values t and t0, for example, t_0=2ms and t=6ms. Then

$$i(t) - i(t_0) = 1 - (-2) = 3\text{A}$$

$$\int_{t0}^{t} v(\tau)d\tau = \int_{0.002}^{0.006} 30 d\tau = (30)(0.006 - 0.002) = 0.12\text{V}\cdot\text{s}$$

Using Eq. 7.5-8 gives

$$3 = \frac{1}{L}(0.12) \quad \Rightarrow \quad L = 0.040\frac{\text{V}\cdot\text{s}}{\text{A}} = 0.040\text{F} = 40\text{mH}$$

Example 7.5-3 **Inductor Current and Voltage**

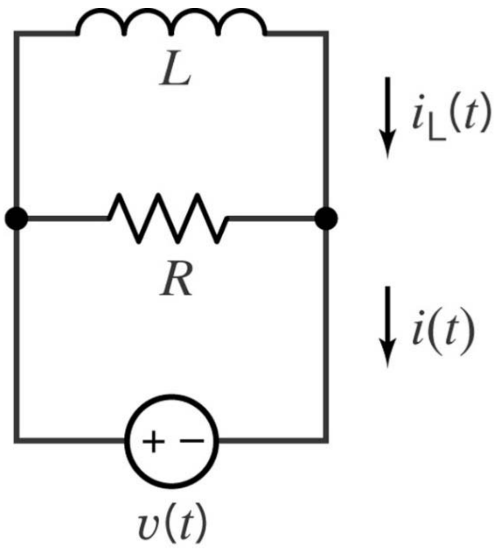
- The input to the circuit shown in Figure 7.5-13 is the voltage

$$v(t) = 4e^{-20t}\,\mathrm{V} \quad \text{for} \quad t > 0$$

The output is the current

$$i(t) = -1.2e^{-20t} - 1.5\,\mathrm{A} \quad \text{for} \quad t > 0$$

The initial inductor current is $i_L(0)$=-3.5A. Determine the values of the inductance, L and resistance, R.



Solution

- Apply KCL at either node to get

$$i(t) = \frac{v(t)}{R} + i_L(t) = \frac{v(t)}{R} + \left[\frac{1}{L} \int_0^t v(\tau) d\tau + i(0) \right]$$

That is

$$-1.2e^{-20t} - 1.5 = \frac{4e^{-20t}}{R} + \frac{1}{L} \int_0^t 4e^{-20t} d\tau - 3.5 = \frac{4e^{-20t}}{R} + \frac{1}{L(-20)} (e^{-20t} - 1) - 3.5$$

$$= \left(\frac{4}{R} - \frac{1}{5L} \right) e^{-20t} + \frac{1}{5L} - 3.5$$

Equating coefficients gives

$$-1.5 = \frac{1}{5L} - 3.5 \quad \Rightarrow \quad L = 0.1\text{H}$$

and $$-1.2 = \frac{4}{R} - \frac{1}{5L} = \frac{4}{R} - \frac{1}{5(0.1)} = \frac{4}{R} - 2 \quad \Rightarrow \quad R = 5\Omega$$

Energy storage in an inductor

- The energy stored in an inductor is

$$w = \frac{1}{2} L i^2$$

  - The energy stored in an inductor during the interval t_0 to t is given by

$$w(t) = \int_{t_0}^{t} p\,d\tau = L\int_{i(t_0)}^{i(t)} i\,di = \frac{L}{2}\left[i^2(t)\right]_{i(t_0)}^{i(t)} = \frac{L}{2} i^2(t) - \frac{L}{2} i^2(t_0)$$

  - Usually we select $t_0 = -\infty$ for the inductor and then the current $i(-\infty)=0$.
  - Also we used expression for power

$$p = vi = \left(L\frac{di}{dt}\right) i$$

Example 7.6-1 **Inductor Voltage and Current**

- Find the current in an inductor, L=0.1H, when the voltage across the inductor is

$$v = 10te^{-5t}\,\mathrm{V}$$

Assume that the current is zero for $t \le 0$.

Solution

- The voltage as a function of time is shown in figure 7.6-1a. Note that the voltage reaches a maximum at t=0.2s
  The current is

$$i = \frac{1}{L}\int_0^t v\,d\tau + i(t_0)$$

  Since the voltage is zero for t<0, the current in the inductor at t=0 is i(t)=0,
  Then we have

$$i = 10\int_0^t 10\tau e^{-5\tau} d\tau = 100\left[\frac{-e^{-5\tau}}{25}(1+5\tau)\right]_0^t = 4(1-e^{-5\tau}(1+5t))\text{A}$$

  The current as a function of time is shown in Figure 7.6-1b.

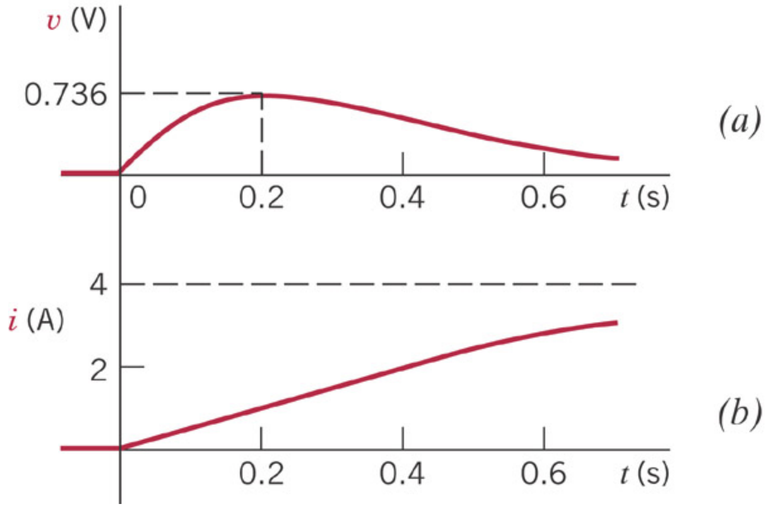
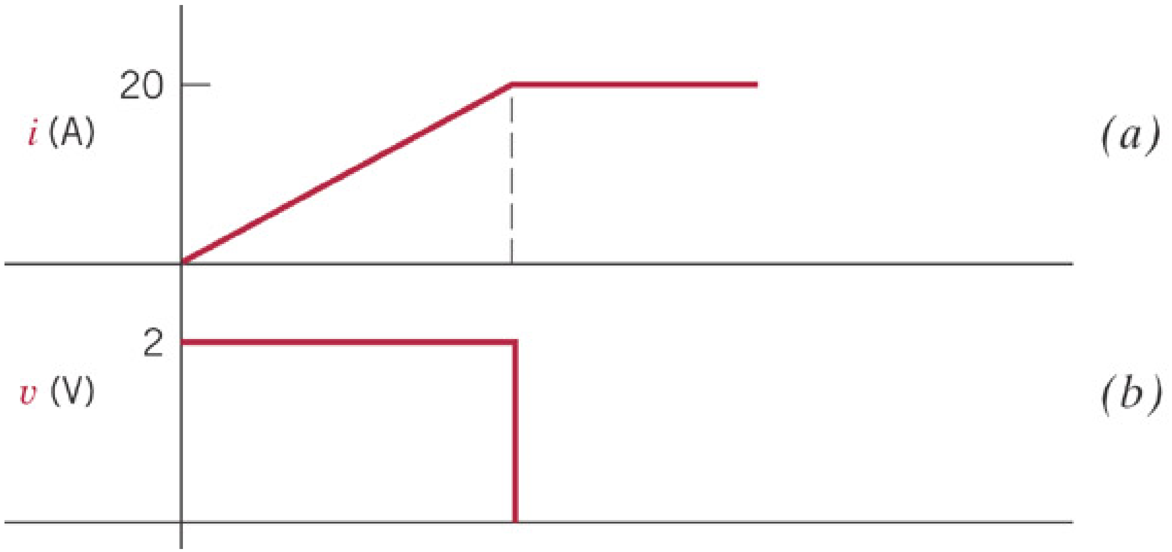


Fig 7.6-1

Example 7.6-2 **Power and Energy for an Inductor**

- Find the power and energy for an inductor of 0.1H when the current and voltage are as shown in Figures 7.6-2a,b



Solution

- Current

$$i = 0 \qquad t \le 0$$
$$= 20t \qquad 0 \le t \le 1$$
$$= 20 \qquad 1 \le t$$

- Voltage

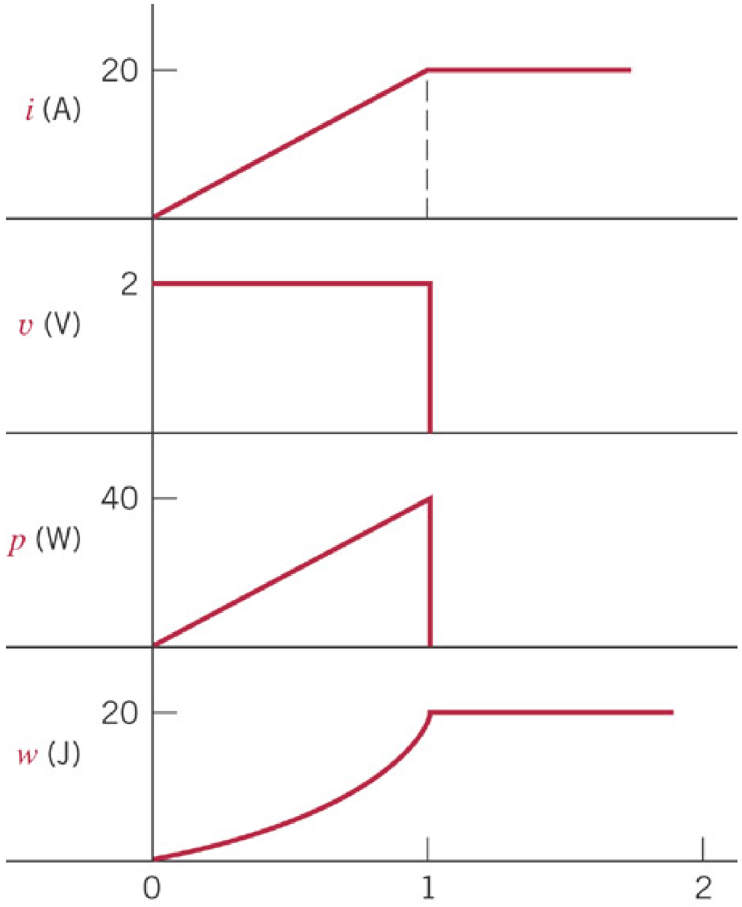
$$v = 0 \qquad t < 0$$
$$= 2 \qquad 0 < t < 1$$
$$= 0 \qquad 1 < t$$

- Power

$$p = vi = 0 \qquad t \le 0$$
$$= 40t \qquad 0 \le t < 1$$
$$= 0 \qquad 1 < t$$

- Energy

$$w = \frac{1}{2}Li^2 = 0 \qquad t \le 0$$
$$= 0.05(20t)^2 \qquad 0 \le t \le 1$$
$$= 0.05(20)^2 \qquad 1 \le t$$



Example 7.6-3 **Power and Energy for an Inductor**

- Find the power and energy stored in a 0.1-H inductor when $i=20te^{-2t}$ A and $v=2e^{-2t}(1-2t)$ V for $t \geq 0$ and $i=0$ for $t<0$.

Solution

- The power is

$$p = iv = (20te^{-2t})\left[2e^{-2t}(1-2t)\right] = 40te^{-4t}(1-2t)\text{W} \quad \text{t} > 0$$

The energy is then

$$w = \frac{1}{2}Li^2 = 0.05(20te^{-2t})^2 = 20t^2e^{-4t}\text{J} \quad \text{t} > 0$$

w is positive for all values of t>0. The energy stored in the inductor is shown in Figure 7.6-3

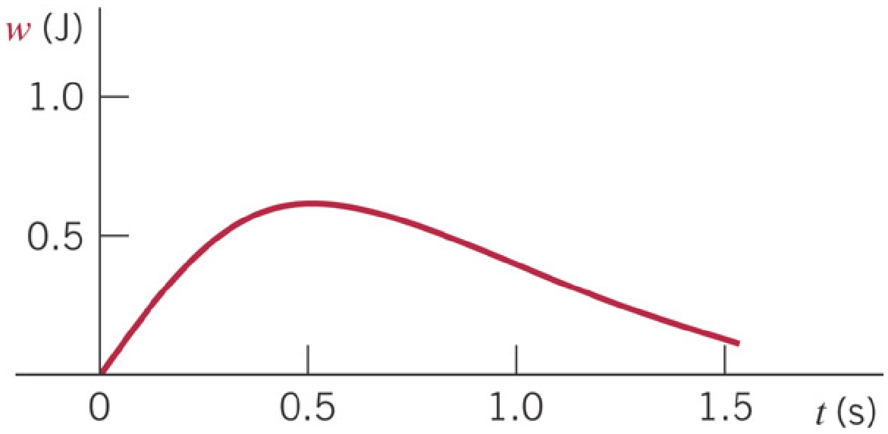


Fig 7.6-3

Series and Parallel Inductors

- Consider a series connection of N inductors as shown in Figure 7.7-1. The voltage across the series connection

$$v = v_1 + v_2 + \cdots + v_N$$

$$v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \cdots + C_N \frac{di}{dt}$$

$$= (L_1 + L_2 + \cdots + L_N) \frac{di}{dt}$$

$$= \left(\sum_{n=1}^{N} L_n \right) \frac{di}{dt}$$

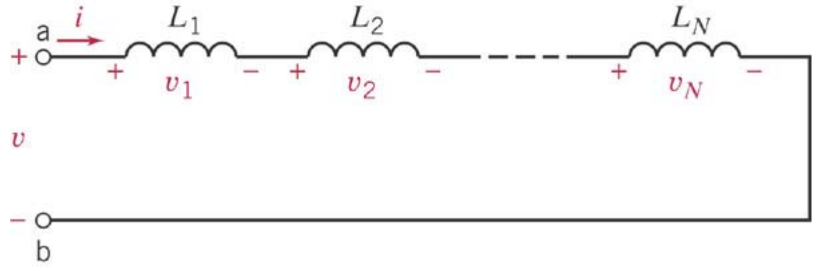


Figure 7.7-1

Series and Parallel Inductors

- Since the equivalent series inductor L_s, as shown in Figure 7.7-2

$$v = L_s \frac{di}{dt}$$

we require that

$$L_s = \sum_{n=1}^{N} L_n$$

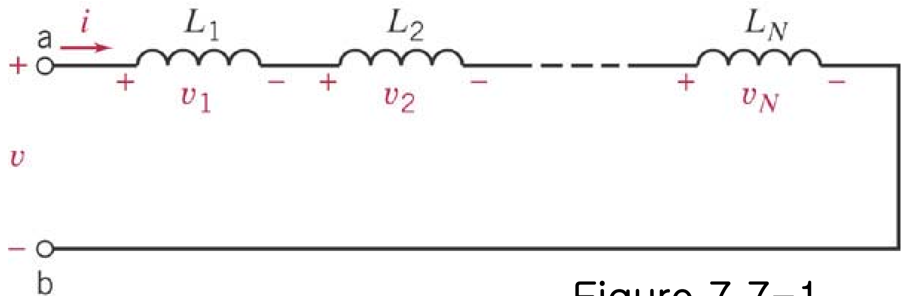


Figure 7.7−1

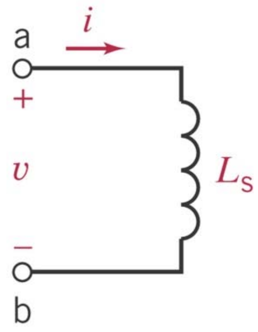


Figure 7.7−2

- Thus, an equivalent inductor for a series of inductors is the sum of the N inductors.

Series and Parallel Inductors

- Consider parallel connection of N capacitors as shown in Figure 7.7-3. Using KCL, we have

$$i = \sum_{n=1}^{N} i_n$$

Since

$$i_n(t) = \frac{1}{L_n}\int_{t_0}^{t} v\,d\tau + i_n(t_0)$$

we may obtain the expression

$$i = \sum_{n=1}^{N}\frac{1}{L_n}\int_{t_0}^{t} v\,d\tau + \sum_{n=1}^{N} i_n(t_0)$$

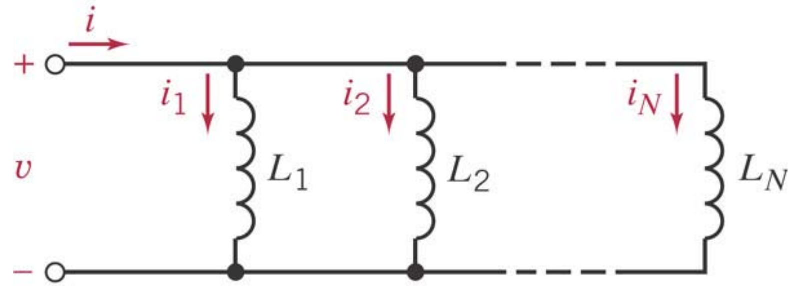


Figure 7.7−3

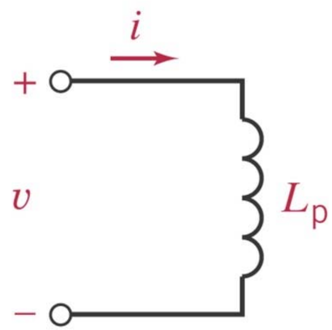


Figure 7.7−4

Series and Parallel Inductors

- For the equivalent circuit shown in Figure 7.7-4

$$i(t) = \frac{1}{L_P}\int_{t_0}^{t} v\,d\tau + i(t_0)$$

we have

$$\frac{1}{L_p} = \sum_{n=1}^{N} \frac{1}{L_n}$$

and

$$i(t_0) = \sum_{n=1}^{N} i_n(t_0)$$

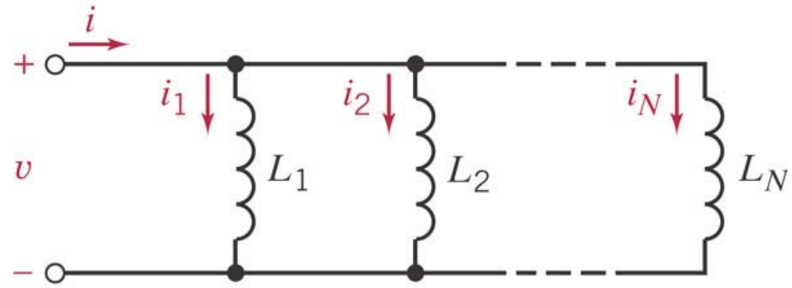


Figure 7.7−3

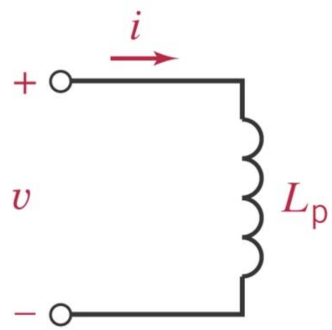
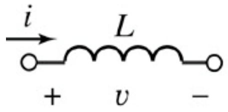
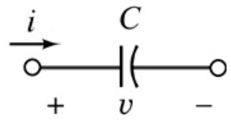


Figure 7.7−4

Characteristics of Energy Storage Elements

VARIABLE	INDUCTORS	CAPACITORS
Passive sign convention		
Voltage	$v = L\dfrac{di}{dt}$	$v = \dfrac{1}{C}\int i d\tau + v(0)$
Current	$i = \dfrac{1}{L}\int v d\tau + i(0)$	$i = C\dfrac{dv}{dt}$
Power	$p = Li\dfrac{di}{dt}$	$p = Cv\dfrac{dv}{dt}$
Energy	$w = \dfrac{1}{2}Li^2$	$w = \dfrac{1}{2}Cv^2$
An instantaneous change is not permitted for the element's:	current	Voltage
Will permit an instantaneous change in the element's:	voltage	Current
This element acts as a:	Short circuit to a constant current into its terminals	Open circuit to a constant voltage across its terminals

Initial conditions of switched circuits

- An inductor behaves as a short circuit to a dc current.

$$\frac{di}{dt} = 0 \quad \Rightarrow \quad v = L\frac{di}{dt} = 0$$

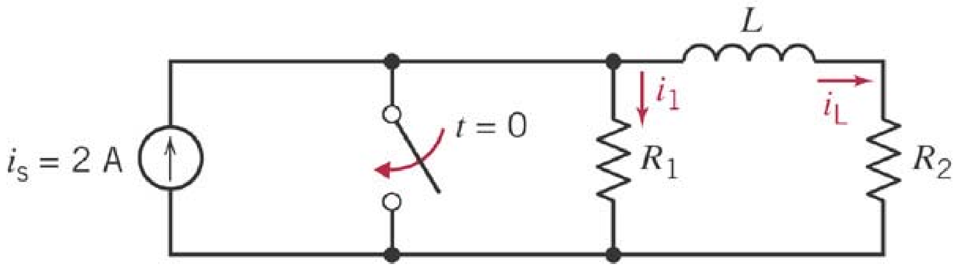
- An capacitor behaves as a open circuit to a dc voltage.

$$\frac{dv}{dt} = 0 \quad \Rightarrow \quad i = C\frac{dv}{dt}$$

Initial conditions of switched circuits

- Consider a circuit with an inductor as shown in Figure 7.8-1.
  At t=0-, i_L is constant current,
  so the inductor voltage is zero.

$$i_L(0^-) = \frac{R_1}{R_1 + R_2} i_S = \left(\frac{1}{2}\right) 2 = 1 \text{ A}$$



Since the current cannot change instantaneously for the inductor,

$$i_L(0^+) = i_L(0^-) = 1 \text{ A}$$

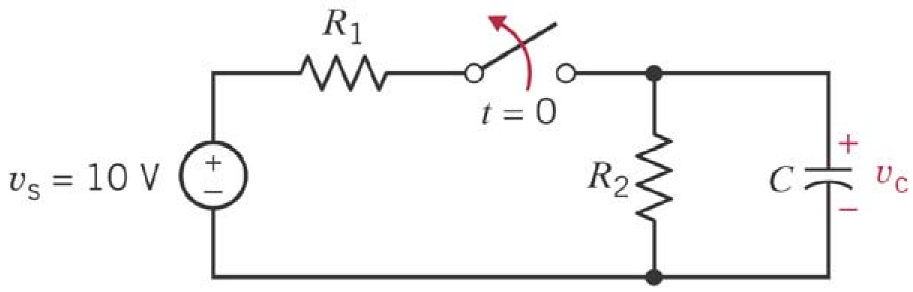
However, the current in the resistor can change instantaneously. After the switch is thrown, we require that the voltage across R_1 be equal to zero, and therefore

$$i_1(0^-) = 1 \text{ A} \quad \text{and} \quad i_1(0^+) = 0 \text{ A}$$

Initial conditions of switched circuits

- Consider a circuit with a capacitor as shown in Figure 7.8-2
  At t=0-, the capacitor appears as
  an open circuit.

$$v_C(0^-) = \frac{R_2}{R_1+R_2} v_S = \left(\frac{1}{2}\right) 10 = 5 \text{ V}$$



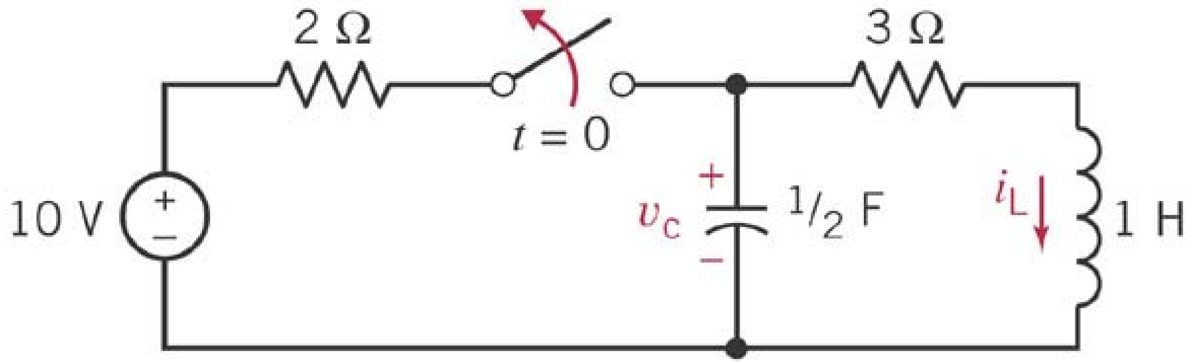
Since the voltage across a capacitor cannot change instantaneously,

$$v_C(0^+) = v_C(0^-) = 5 \text{ V}$$

When the switch is opened, the source is removed from the circuit but the voltage across the capacitor remains equal to 5 V.

Example 7.8-1 **Initial Conditions in a Switched Circuit**

- Consider the circuit Figure 7.8-1. Prior to t=0, the switch has been closed for a long time. Determine the values of the capacitor voltage and inductor current immediately after the switch opens at time t = 0.

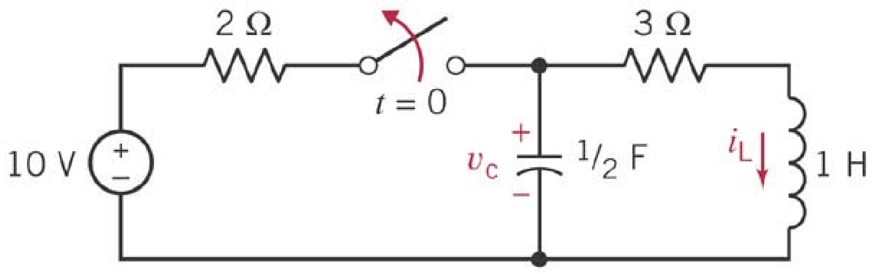


Solution

- At t=0-, we may replace the capacitor by an open circuit and the inductor by a short circuit, as shown in Figure 7.8-4.

$$i_L(0^-) = \frac{10}{5} = 2\ \text{A}$$

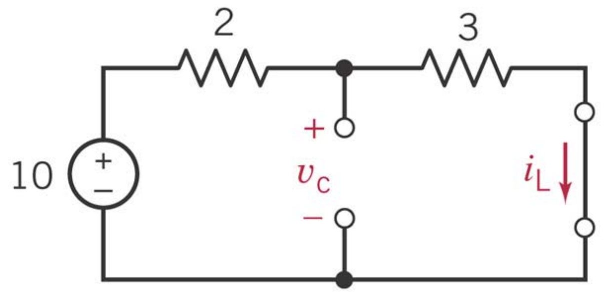
$$v_C(0^-) = \left(\frac{3}{5}\right)10 = 6\ \text{V}$$



Then we note that

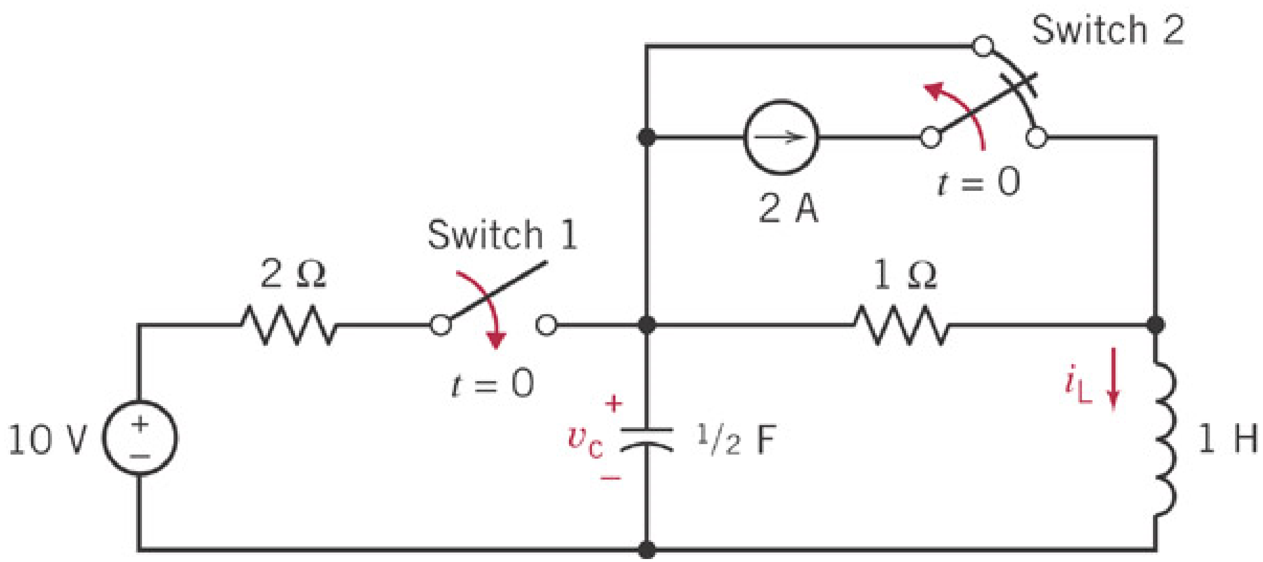
$$v_C(0^+) = v_C(0^-) = 6\ \text{V}$$

$$i_L(0^+) = i_L(0^-) = 2\ \text{A}$$



Example 7.8-2 **Initial Conditions in a Switched Circuit**

- Find $i_L(0+)$, $v_c(0+)$, $dv_c(0+)/dt$ and $di_L(0+)/dt$ for the circuit of Figure 7.8-5. Assume that switch 1 has been open and switch 2 has been closed for a long time and steady-state conditions prevail at t=0-.



Solution

- We redraw the circuit for t=0- by replacing the inductor with a short circuit and the capacitor with an open circuit, as shown in Figure 7.8-6

$$i_L(0^+) = i_L(0^-) = 0$$
$$v_c(0^+) = v_c(0^-) = -2\text{V}$$

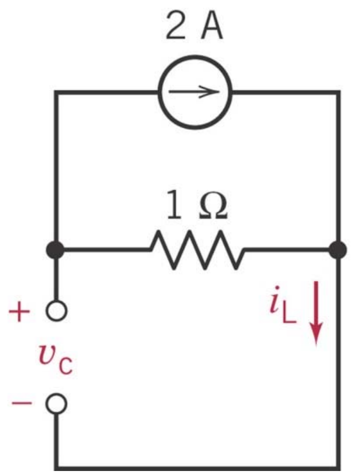


Fig 7.8-6

In order to find $dv_c(0+)/dt$ and $di_L(0+)/dt$, we throw the switch at t=0 and redraw the circuit of Figure 7.8-5, as shown in Figure 7.8-7
obtain $dv_c(0+)/dt$, $di_L(0+)/dt$

$$i_c = C\frac{dv_c}{dt}$$
$$\frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C}$$
$$v_L = L\frac{di_L}{dt}$$
$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

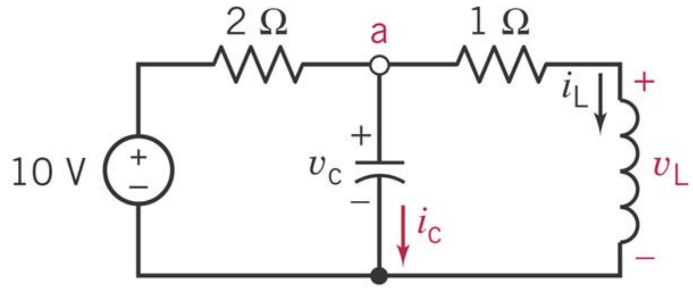


Fig 7.8-7

Solution

- Using KVL for the right-hand mesh of Figure 7.8-7,

$$v_L - v_c + 1i_L = 0$$

$$v_L(0^+) = v_c(0^+) - i_L(0^+) = -2 - 0 = -2\text{V}$$

$$\frac{di_L(0^+)}{dt} = -2\text{A/s}$$

to find i_c we write KCL at node a to obtain

$$i_c + i_L + \frac{v_c - 10}{2} = 0$$

$$i_L(0^+) = \frac{10 - v_c(0^+)}{2} - i_L(0^+) = 6 - 0 = 6\text{A}$$

$$\frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = 12\text{V/s}$$

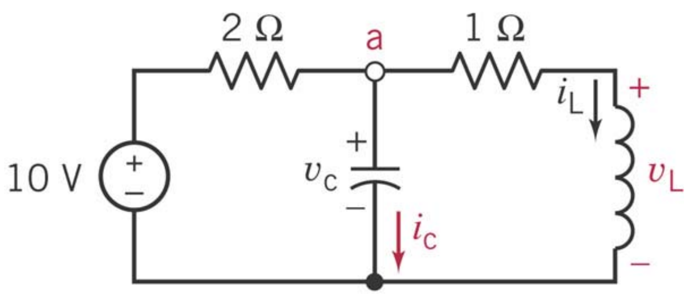


Fig 7.8−7

Operational amplifier circuits and linear differential equations

- This section describes a procedure for designing op amp circuits that implement linear differential equations such as

$$2\frac{d^3}{dt^3}y(t)+5\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)+3y(t)=6x(t) \qquad (7.9\text{−}1)$$

The solution of this equation is a function, y(t), that depends both on the function x(t) and on a set of initial conditions. It is convenient to use the initial conditions:

$$\frac{d^2}{dt^2}y(t)=0\ , \quad \frac{d}{dt}y(t)=0 \quad \text{and} \quad y(t)=0$$

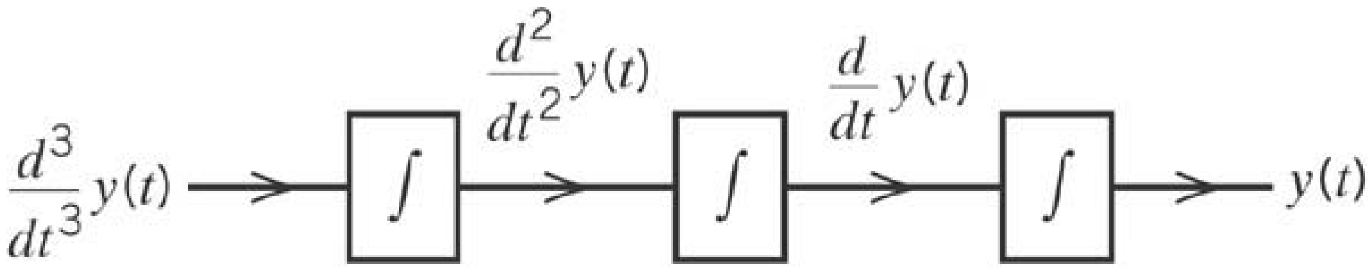
We will treat x(t) as the input to the differential equation and y(t) as the output.

Operational amplifier circuits and linear differential equations

- Suppose that we were some how to obtain $\frac{d^3}{dt^3}y(t)$

  We could then integrate three times to obtain $\frac{d^2}{dt^2}y(t)$ $\frac{d}{dt}y(t)$ $y(t)$

  as illustrated in Figure 7.9-2



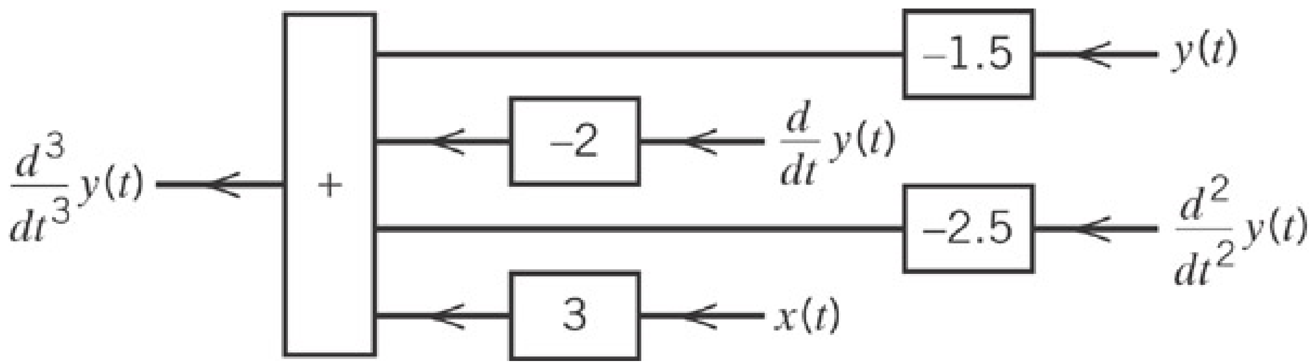
Operational amplifier circuits and linear differential equations

- Now we must obtain $\frac{d^3}{dt^3}y(t)$

  To do so, solve Eq. 7.9-1 for $\frac{d^3}{dt^3}y(t)$ to get

$$\frac{d^3}{dt^3}y(t) = 3x(t) - \left[2.5\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 1.5y(t)\right] \qquad (7.9\text{-}3)$$

  Next, represent this equation by a block diagram as shown in figure 7.9-3



Operational amplifier circuits and linear differential equations

- Finally the block diagrams can be combined as shown in Figure 7.9-4

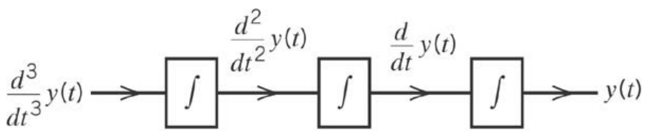


Fig. 7.9-2

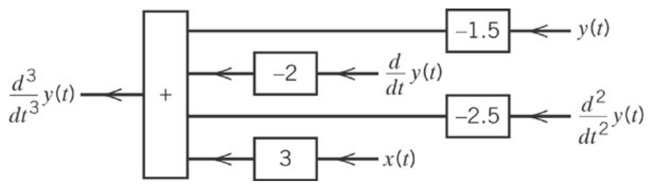


Fig. 7.9-3

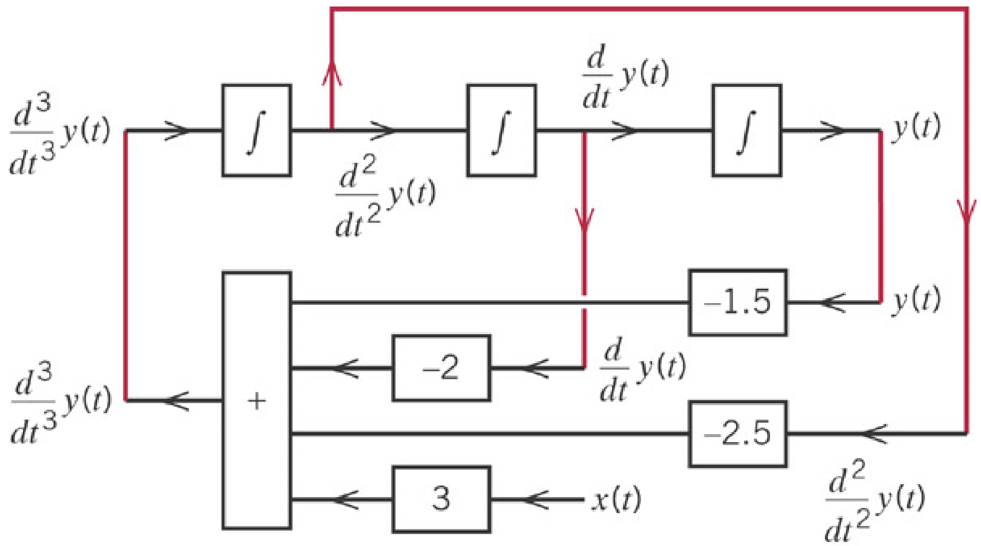
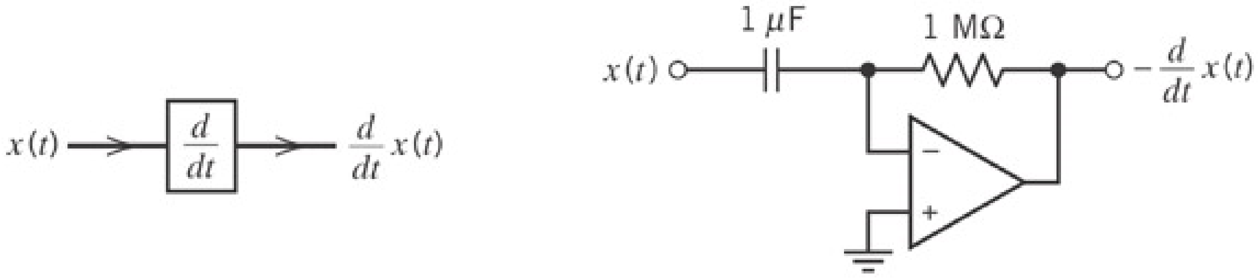


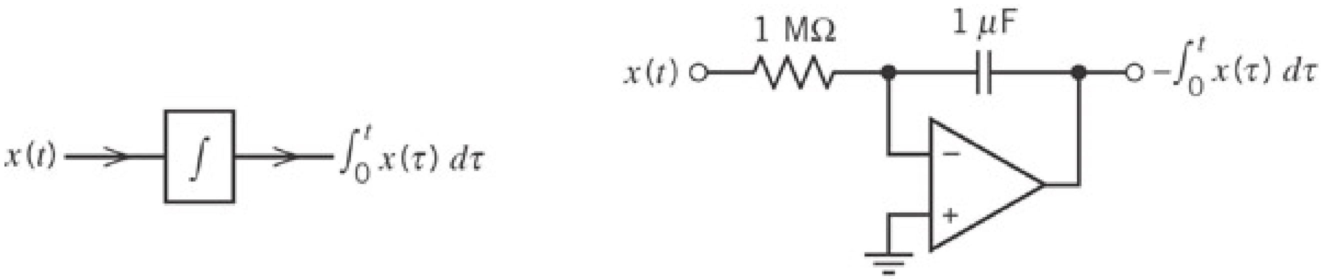
Fig. 7.9-4

Operational amplifier circuits and linear differential equations

- Op amp circuits to implement differentiation and integration.



(a) Differentiation (a)(b) (b)



(c) Integration (c)(d) (d)

Department of Electrical and Computer Engineering, SNU
Prof. SungJune Kim

Operational amplifier circuits and linear differential equations

- To see how the integrator works, consider Figure 7.9-6.
  The voltage across the resistor is

  $$v_R(t) = v_1(t) - v_2(t) = x(t) - 0 = x(t)$$

  Use Ohm's law to get

  $$i_R(t) = \frac{v_R(t)}{R} = \frac{x(t)}{R}$$

  Applying KCL to node 2 gives

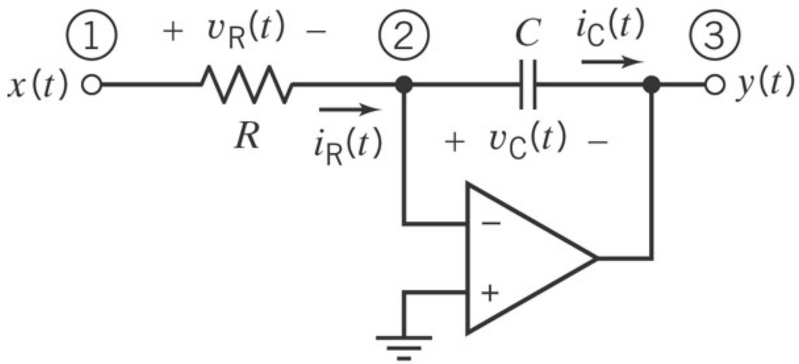
  $$i_C(t) = i_R(t) = \frac{x(t)}{R}$$

  The voltage across the capacitor is

  $$v_C(t) = v_2(t) - v_3(t) = 0 - y(t) = -y(t)$$

  The capacitor voltage is related to the capacitor current by

  $$v_c(t) = \frac{1}{C}\int_0^t i_C(\tau)d\tau + v_C(0)$$

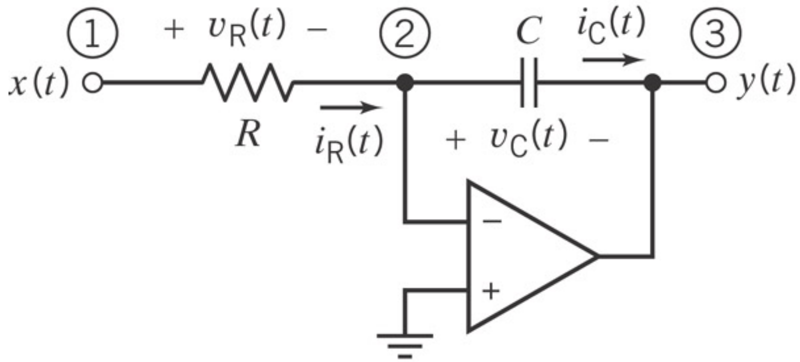


Operational amplifier circuits and linear differential equations

- Recall that y(0)=0. Thus $v_c(0)=0$ and

$$v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)d\tau = \frac{1}{C}\int_0^t \frac{x(\tau)}{R}d\tau = \frac{1}{RC}\int_0^t x(\tau)d\tau$$

Since $y(t) = -v_C(t)$



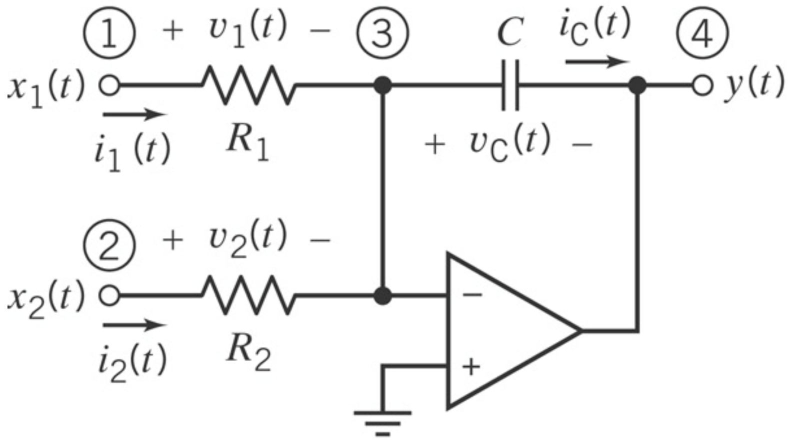
we obtain

$$y(t) = -\frac{1}{RC}\int_0^t x(\tau)d\tau = -k\int_0^t x(\tau)d\tau$$

where $k = \frac{1}{RC}$

Operational amplifier circuits and linear differential equations

- Figure 7.9-7 illustrates a summing integrator. Applying KCL at node 3 gives



$$i_C(t) = i_1(t) + i_2(t) = \frac{x_1(t)}{R_1} + \frac{x_2(t)}{R_2}$$

The voltage across the capacitor is

$$v_C(t) = v_3(t) - v_4(t) = 0 - y(t) = -y(t)$$

The capacitor voltage is related to the capacitor current by

$$v_c(t) = \frac{1}{C}\int_0^t i_C(\tau)d\tau + v_C(0)$$

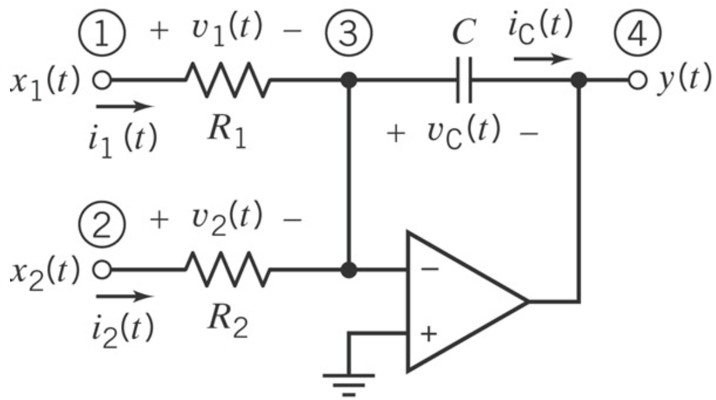
Operational amplifier circuits and linear differential equations

- Recall that y(0)=0. Thus $v_c(0)$=0 and

$$v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)d\tau = \frac{1}{C}\int_0^t \left(\frac{x_1(t)}{R_1} + \frac{x_2(t)}{R_2}\right)d\tau = \int_0^t \left(\frac{x_1(t)}{R_1 C} + \frac{x_2(t)}{R_2 C}\right)d\tau$$

Since $y(t) = -v_C(t)$

we obtain



$$y(t) = -\int_0^t \left(\frac{x_1(t)}{R_1 C} + \frac{x_2(t)}{R_2 C}\right)d\tau = -\int_0^t \left(k_1 x_1(t) + k_2 x_2(t)\right)d\tau$$

where

$$k_1 = \frac{1}{R_1 C} \quad \text{and} \quad k_2 = \frac{1}{R_2 C}$$

Operational amplifier circuits and linear differential equations

- To accommodate inverting integrators, it is necessary to modify the block diagram.

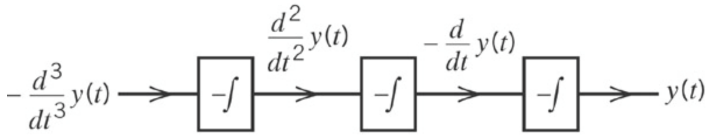


Fig. 7.9-8

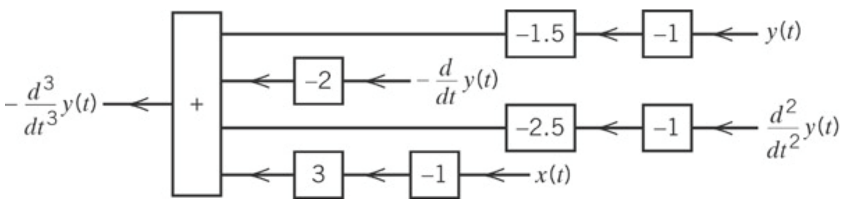


Fig. 7.9-9

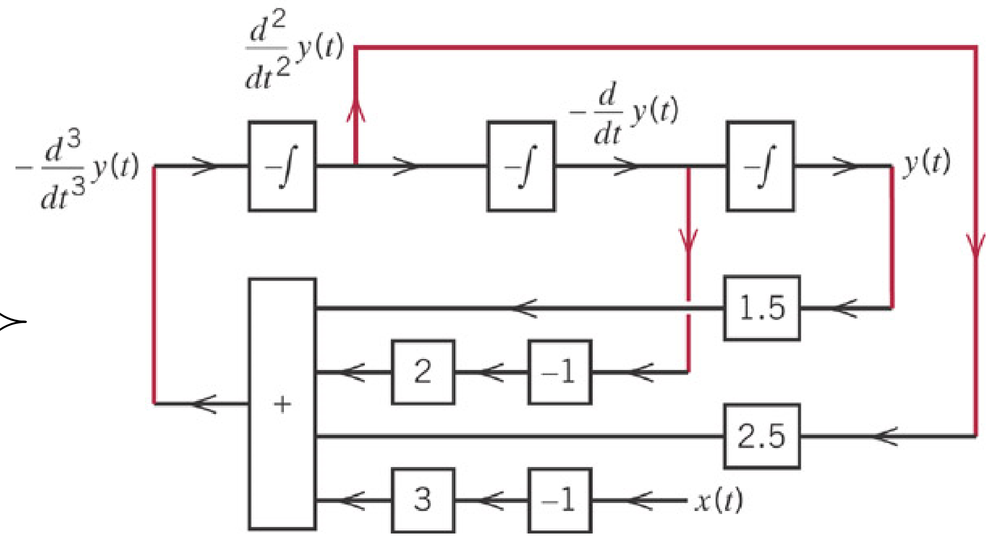


Fig. 7.9−10

Operational amplifier circuits and linear differential equations

- Figure 7.9-11 emphasize the blocks the can be implemented by a single four input summing integrator.
- Figure 7.9-12 shows the four-input summing integrator.
  The signal $d^2y(t)/dt^2$ is the output and is also one of the inputs.

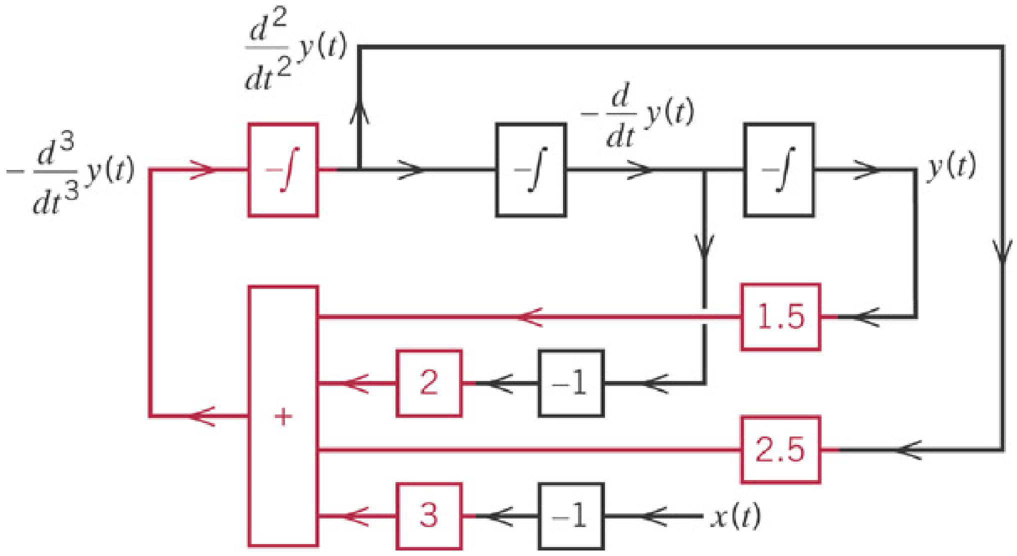


Fig. 7.9-11

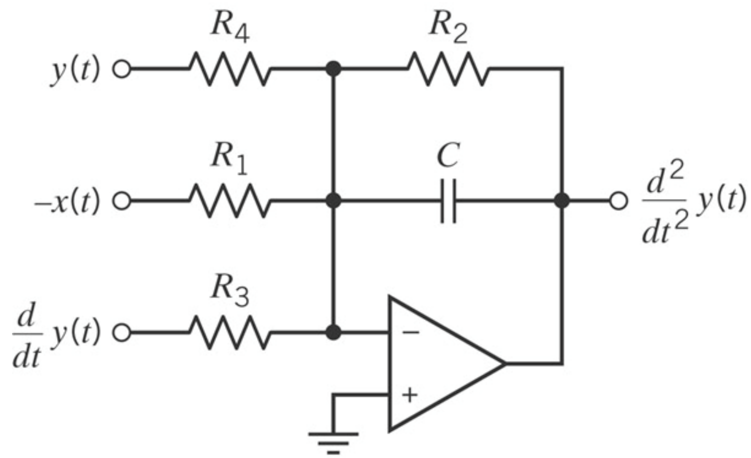


Fig. 7.9-12

Operational amplifier circuits and linear differential equations

- The summing integrator is represented by the equation

$$\frac{d^2}{dt^2}y(t) = -\int_0^t\left(\frac{1}{R_1C}[x(t)] + \frac{1}{R_2C}\frac{d^2}{dt^2}y(t) + \frac{1}{R_3C}\left[\frac{d}{dt}y(t)\right] + \frac{1}{R_4C}y(t)\right)d\tau$$

Integrating both sides of Eq. 7.9-3 gives

$$\frac{d^2}{dt^2}y(t) = -\int_0^t\left(3[x(t)] - 2.5\frac{d^2}{dt^2}y(t) + 2\left[\frac{d}{dt}y(t)\right] + 1.5y(t)\right)d\tau$$

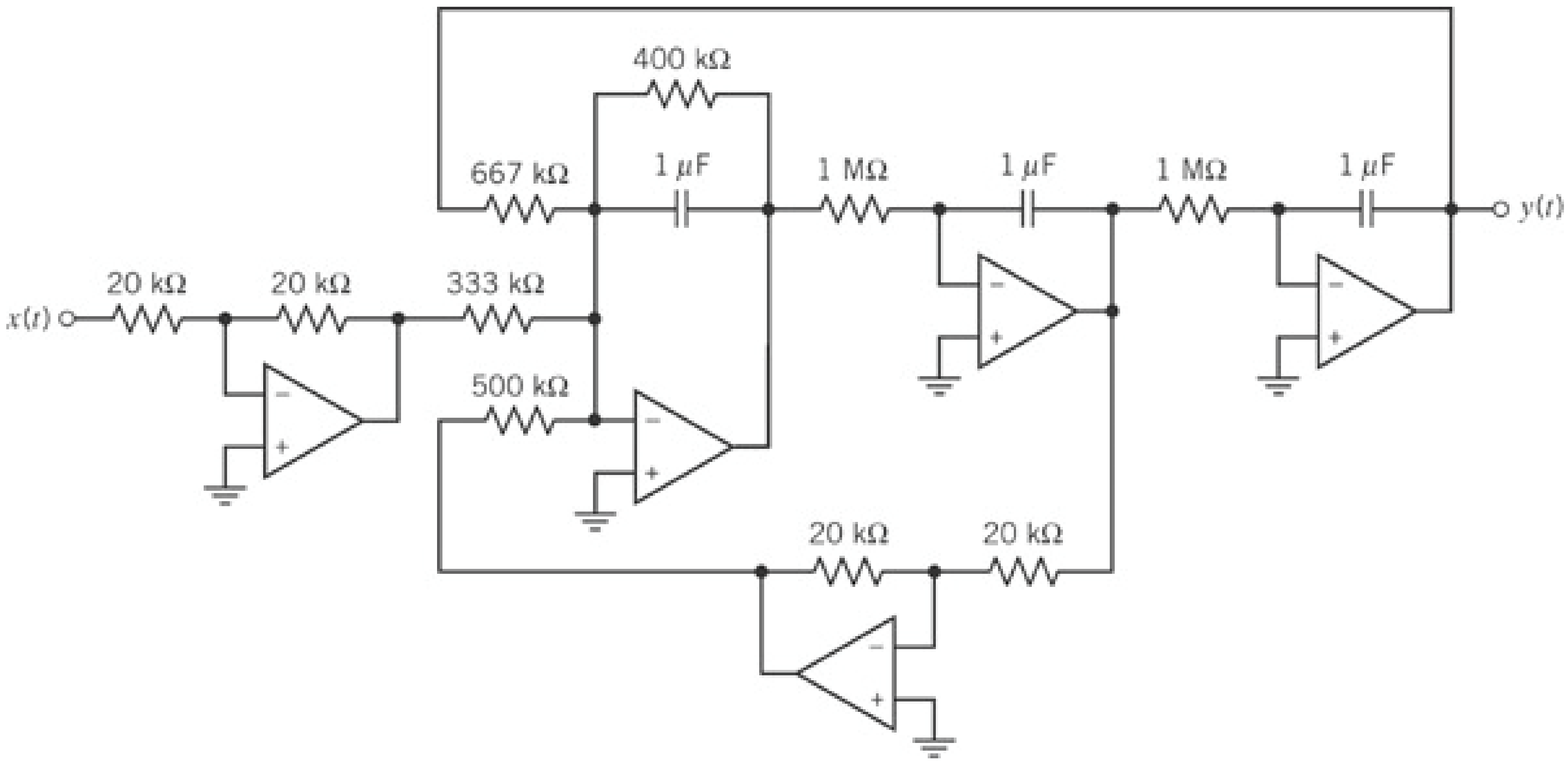
For convenience, pick $C = 1\ \mu\text{F}$

Then,

$$R_1 = 333\ \text{k}\Omega,\quad R_2 = 400\ \text{k}\Omega,\quad R_3 = 500\ \text{k}\Omega,\quad \text{and}\quad R_4 = 667\ \text{k}\Omega.$$

Operational amplifier circuits and linear differential equations

- The finished circuit is shown in figure 7.9-13



Using MATLAB to plot capacitor or inductor voltage and current

- Suppose that the current in a 2-F capacitor is

$$i(t) = \begin{cases} 4 & t \le 2 \\ t+2 & 2 \le t \le 6 \\ 20-2t & 6 \le t \le 14 \\ -8 & t \ge 14 \end{cases}$$

Where the units of current are A and the units of time are s.
When the initial capacitor voltage is -5V, the capacitor voltage can be calculated using

$$v(t) = \frac{1}{2}\int_0^t i(\tau)\,d\tau - 5$$