Chapter 9
The Complete Response of Circuits with Two Energy Storage Elements

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Second-Order Circuit

- A second-order circuit is a circuit that is represented by a second-order differential equation.

$$\frac{d^2}{dt^2}x(t) + 2\alpha \frac{d}{dt}x(t) + \omega_0^2 x(t) = f(t)$$

x(t): output of the circuit (=response of the circuit)
f(t) : input to the circuit
α : damping coefficient
ω_0 : resonant frequency

Second-Order Circuit

- To find the response of the second-order circuit,
  - Represent the circuit by a second-order differential equation.
  - Find the general solution of the homogeneous differential equation. This solution is the natural response, $x_n(t)$. The natural response will contain two unknown constants that will be evaluated later.
  - Find a particular solution of the differential equation. This solution is the forced response, $x_f(t)$.
  - Represent the response of the second-order circuit as $x(t)=x_n(t) + x_f(t)$.
  - Use the initial conditions, for example, the initial values of the currents in inductors and the voltage across capacitors, to evaluate the unknown constants.

Direct Method

- Let us consider the circuit shown in Figure 9.2-1.
  Writing the nodal equation at the top node, we have

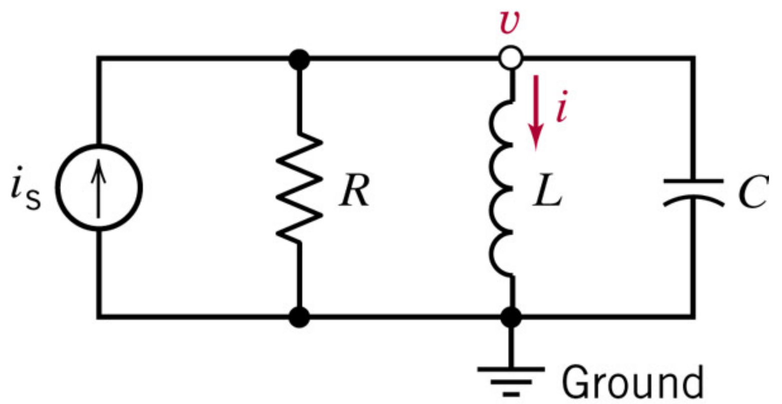
$$\frac{v}{R} + i + C\frac{dv}{dt} = i_S$$

We write the equation for the inductor as

$$v = L\frac{di}{dt}$$

Then

$$\frac{L}{R}\frac{di}{dt} + i + CL\frac{d^2 i}{dt^2} = i_S$$



This method of obtaining the second-order differential equation may be called the direct method and is summarized in Table 9.2-1.

Direct Method

- **Table 9.2-1**

The Direct Method for Obtaining the Second-Order Differential Equation of a Circuit	
Step1	Identify the first and second variables, x_1 and x_2. These variables are capacitor voltages and/or inductor currents.
Step2	Write one first-order differential equation, obtaining $\frac{dx_1}{dt} = f(x_1, x_2)$
Step3	Obtain an additional first-order differential equation in terms of the second variable so that $\frac{dx_2}{dt} = Kx_1$ or $x_1 = \frac{1}{K}\frac{dx_2}{dt}$
Step4	Substitute the equation of step3 into the equation of step2, thus obtaining a second-order differential equation in terms of x_2

Direct Method-example

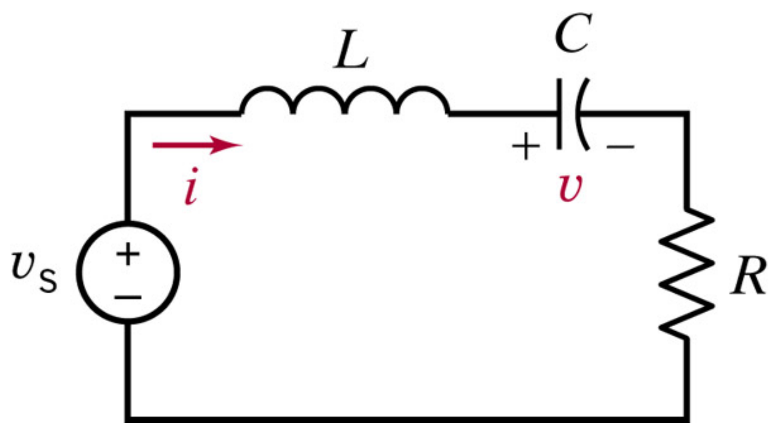
- Let us consider the circuit shown in Figure 9.2-2.
  Writing KVL around the loop, we have

$$L\frac{di}{dt}+v+Ri=v_S \qquad \text{or} \qquad \frac{di}{dt}+\frac{v}{L}+\frac{R}{L}i=\frac{v_S}{L}$$

We write the equation for the capacitor as

$$i=C\frac{dv}{dt}$$

Then $$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{v_S}{LC}$$



This method is the direct method.

Operator Method: using differential operator s

- Another method of obtaining the second-order equation describing a circuit is called the operator method.
- Consider the circuit shown in Figure 9.2-3

The mesh equations are

$$L_1 \frac{di_1}{dt} + R(i_1 - i_2) = v_S$$

and

$$R(i_2 - i_1) + L_2 \frac{di_2}{dt} = 0$$

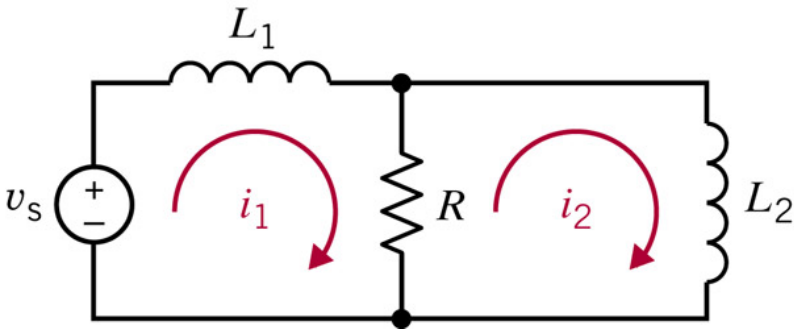
Now let us use $R = 1\ \Omega, \quad L_1 = 1\ \text{H}, \quad \text{and} \quad L_2 = 2\ \text{H}$

Then we have

$$\frac{di_1}{dt} + i_1 - i_2 = v_S$$

and

$$i_2 - i_1 + 2\frac{di_2}{dt} = 0$$



Operator Method: Form 2nd order algebraic equation using differential operator s, then back to the differential eq.

We may rearrange these equations as

$$\frac{di_1}{dt} + i_1 - i_2 = v_S \qquad \text{and} \qquad -i_1 + i_2 + 2\frac{di_2}{dt} = 0$$

The differential operator *s*, where *s=d/dt*, is used to transform differential equations into algebraic equations.

$$(s+1)i_1 - i_2 = v_S \qquad \text{and} \qquad -i_1 + (2s+1)i_2 = 0$$

We may use **Cramer's rule** to solve for i_2, obtaining

$$i_2 = \frac{1v_S}{(s+1)(2s+1)-1}$$

Therefore

$$(2s^2 + 3s)i_2 = v_S$$

and the differential equation is

$$2\frac{d^2 i_2}{dt^2} + 3\frac{di_2}{dt} = v_S$$

see Appendix A.4 "Cramer's rule" (page 830)

- **Cramer's rule**
  - A set of simultaneous equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\;\; \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

can be written in matrix form as $\mathbf{Ax} = \mathbf{b}$

  - Cramer's rule states that the solution for the unknown, x_k, is

$$x_k = \frac{\Delta_k}{\Delta}$$

where Δ is the determinant of $\mathbf{A}$ and Δ_k is Δ with the kth column replaced by the column vector $\mathbf{b}$.

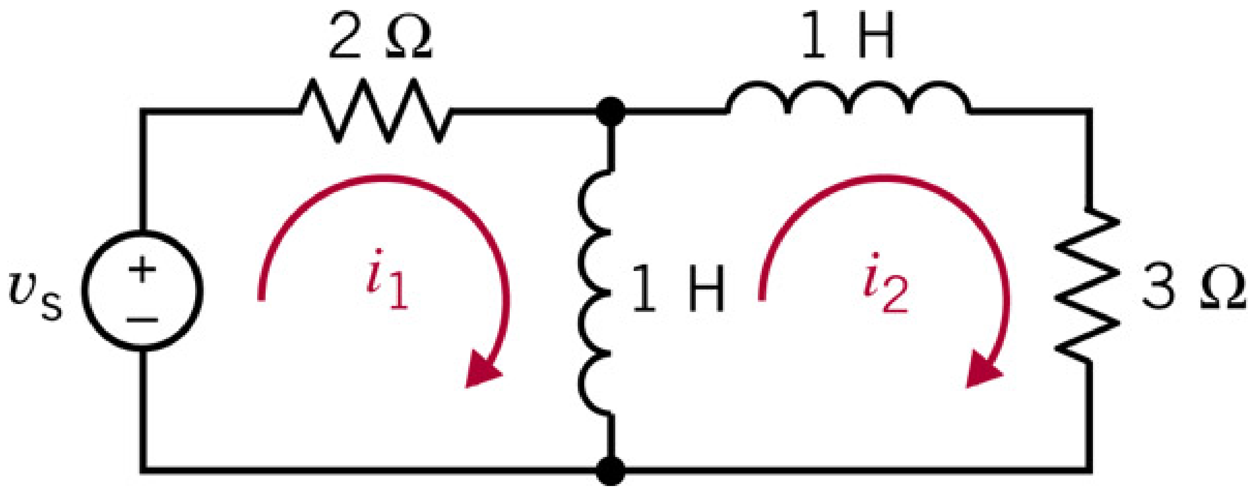
Operator Method

- Table 9.2-2

Operator Method for obtaining the second-order differential equation of a circuit	
Step1	Identify the variable x_1 for which the solution is desired
Step2	Write one differential equation in terms of the desired variable x_1 and a second variable x_2
Step3	Obtain an additional equation in terms of the second variable and the first variable
Step4	Use the operator $s=d/dt$ and $1/s=\int dt$ to obtain two algebraic equations in terms of s and the two variables x_1 and x_2.
Step5	Using Cramer's rule, solve for the desired variable so that $x_1=f(s,\text{source})=P(s)/Q(s)$, where P(s) and Q(s) are polynomials in s
Step6	Rearrange the equation of step 5 so that $Q(s)x_1=P(s)$
Step7	Convert the operators back to derivatives for the equation of step 6 to obtain the second-order differential equation

Operator Method: Example 9.2-1 **Representing a Circuit by a Differential Equation**

- Find the differential equation for the current i_2 for the circuit of Figure 9.2-4

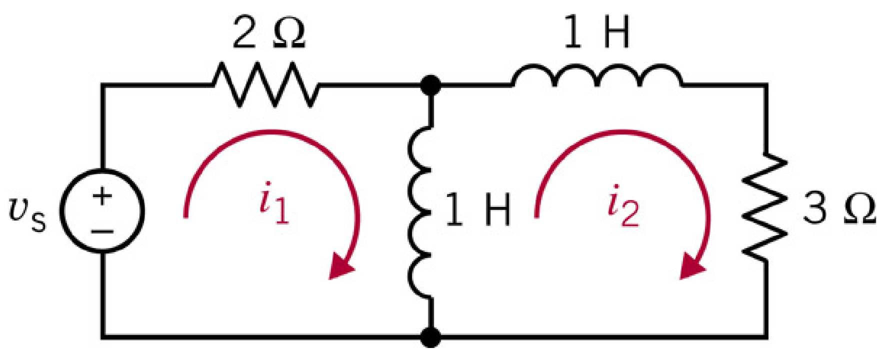


Solution

- Write the two mesh equations using KVL to obtain

$$2i_1 + \frac{di_1}{dt} - \frac{di_2}{dt} = v_s$$

$$-\frac{di_1}{dt} + 3i_2 + 2\frac{di_2}{dt} = 0$$



Using the operator s=d/dt, we have

$$(2+s)i_1 - si_2 = v_s$$

and

$$-si_1 + (3+2s)i_2 = 0$$

Using Cramer's rule to solve for i_2, we obtain

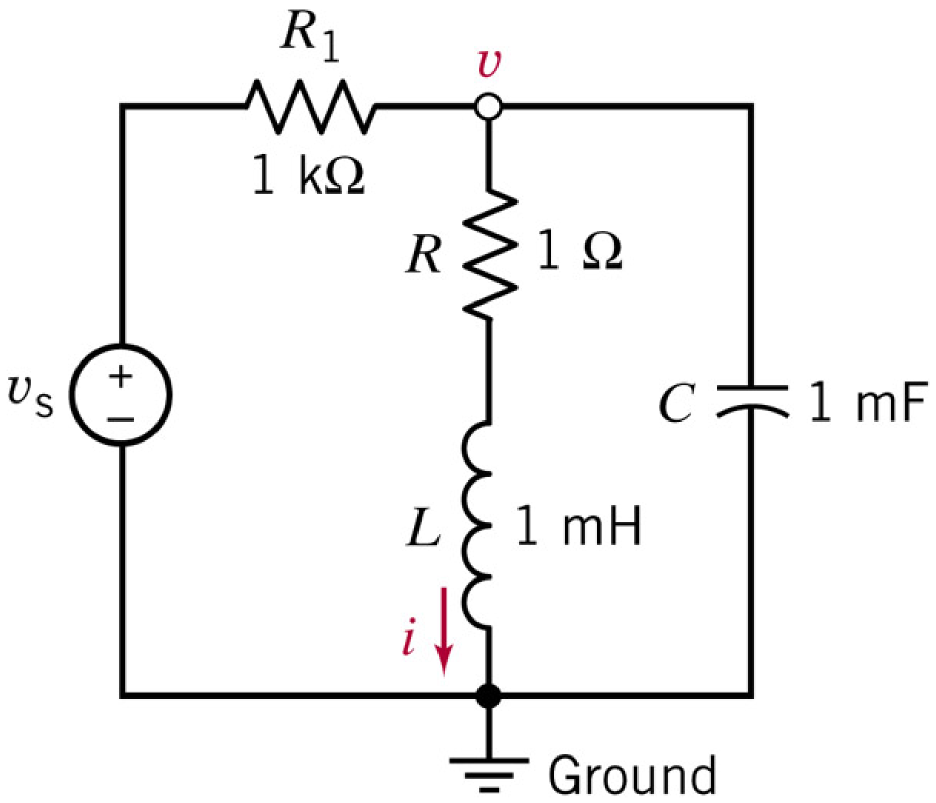
$$i_2 = \frac{sv_s}{(2+s)(3+2s)-s^2} = \frac{sv_s}{s^2+7s+6} \qquad (9.2\text{–}16)$$

Rearranging Eq. 9.2-16, we obtain $(s^2+7s+6)i_2 = sv_s$

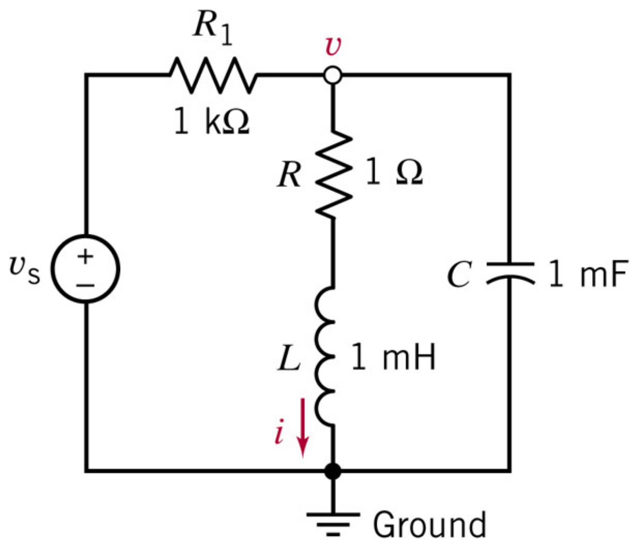
Therefore, the differential equation for i_2 is $\frac{d^2i_2}{dt^2} + 7\frac{di_2}{dt} + 6i_2 = \frac{dv_s}{dt}$

Operator Method Example 9.2-2 **Representing a Circuit by a Differential Equation**

- Find the differential equation for the voltage v for the circuit of Figure 9.2-5



Solution



- The KCL node equation at the upper node

$$\frac{v-v_s}{R_1}+i+C\frac{dv}{dt}=0$$

Write the equation for the current through the branch containing the inductor as $Ri+L\dfrac{di}{dt}=v$

Using the operator s=d/dt, we have the two equations

$$\frac{v}{R_1}+Csv+i=\frac{v_s}{R_1}$$

$$-v+Ri+Lsi=0$$

Substituting the parameter values and rearranging, we have

$$(10^{-3}+10^{-3}s)v+i=10^{-3}v_s$$

$$-v+(10^{-3}s+1)i=0$$

Using Cramer's rule, solve for v to obtain $v=\dfrac{(s+1000)v_s}{(s+1)(s+1000)+10^6}=\dfrac{(s+1000)v_s}{s^2+1001s+1001\times10^3}$

Therefore, $(s^2+1001s+1001\times10^3)v=(s+1000)v_s$

or the differential equation we seek is $\dfrac{d^2v}{dt^2}+1001\dfrac{dv}{dt}+1001\times10^3v=\dfrac{dv_s}{dt}+1000v_s$

Solution of the Second-Order Differential Equation – The Natural Response

- A circuit with two irreducible energy elements can be represented by a second-order differential equation of the form

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

where the constants a_2, a_1, a_0 are known and the forcing function $f(t)$ is specified.

- The complete response is given by

$$x = x_{\mathrm{n}} + x_{\mathrm{f}}$$

where x_n is natural response and x_f is forced response. The natural response satisfies the unforced differential equation when $f(t)=0$. The forced response x_f satisfies the differential equation with the forcing function present.

The Natural Response from Characteristic Equation

- The natural response of a circuit, x_n, will satisfy the equation

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \qquad (9.3\text{-}2)$$

- Since x_n and its derivatives must satisfy the equation, we postulate the exponential solution

$$x = Ae^{st} \qquad (9.3\text{-}3)$$

where A and s are to be determined.

- Substituting Eq. 9.3-3 in Eq. 9.3-2, we have

$$a_2 As^2 e^{st} + a_1 Ase^{st} + a_0 Ae^{st} = 0 \qquad (9.3\text{-}4)$$

Since $x = Ae^{st}$, we may rewrite Eq. 9.3-4 as

$$\left(a_2 s^2 + a_1 s + a_0\right) x = 0$$

- Since we do not accept the trivial solution, it is required that

$$a_2 s^2 + a_1 s + a_0 = 0 \qquad (9.3\text{-}5)$$

This equation is called a ***characteristic equation***.

Characteristic Equation

The **characteristic equation** is derived from the governing differential equation for a circuit by setting all independent source to zero value and assuming an exponential solution.

- The solution of Eq. 9.3-5 has two roots, s_1 and s_2, where

$$s_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \quad \text{and} \quad s_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

- When there are two distinct roots, there are two solutions such that

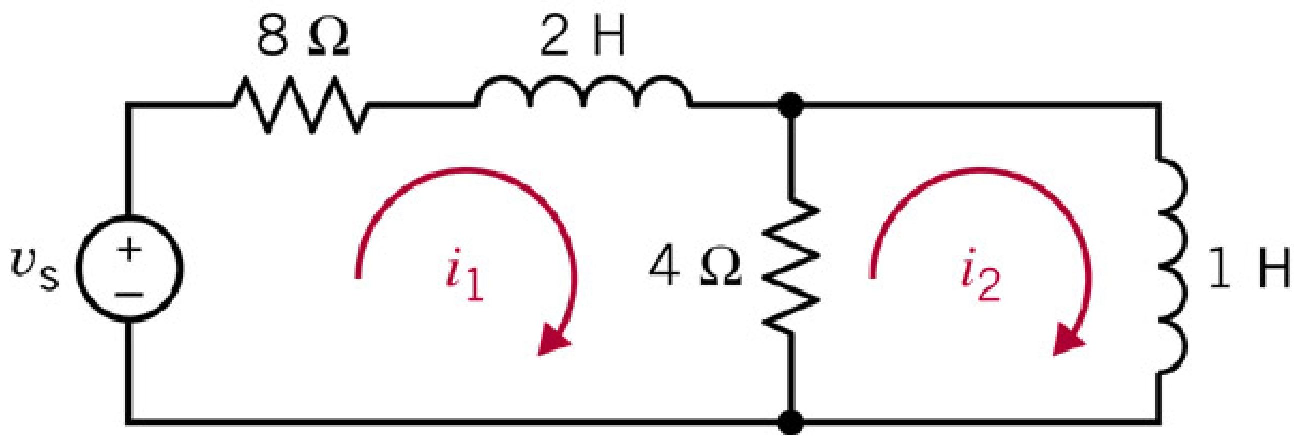
$$x_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where A1 and A2 are unknown constants that will be evaluated later.

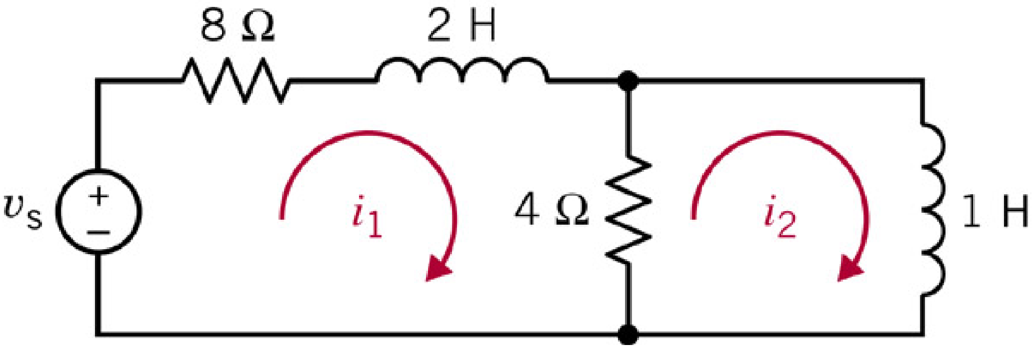
The **roots** of the characteristic equation contain all the information necessary for determining the character of the natural response.

Example 9.3-1 **Natural Response of a Second-Order Circuit**

- Find the natural response of the circuit current i_2 shown in Figure 9.3-2. Use operators to formulate the differential equation and obtain the response in terms of two arbitrary constants.



Solution



- Writing the two mesh equations

$$12i_1 + 2\frac{di_1}{dt} - 4i_2 = v_s$$

$$-4i_1 + 4i_2 + 1\frac{di_2}{dt} = 0$$

Using the operator s=d/dt, we obtain

$$(12+2s)i_1 - 4i_2 = v_s$$

$$-4i_1 + (4+s)i_2 = 0$$

Using Cramer's rule, solve for i_2

$$i_2 = \frac{4v_s}{(12+2s)(4+s)-16} = \frac{4v_s}{2s^2+20s+32} = \frac{2v_s}{s^2+10s+16}$$

Therefore, $$(s^2+10s+16)i_2 = 2v_s$$

Solution

- Note that $(s^2 + 10s + 16) = 0$ is the characteristic equation. Thus, the roots of the characteristic equation are s_1=-2 and s_2=-8. Therefore, the natural response is

$$x_n = A_1 e^{-2t} + A_2 e^{-8t}$$

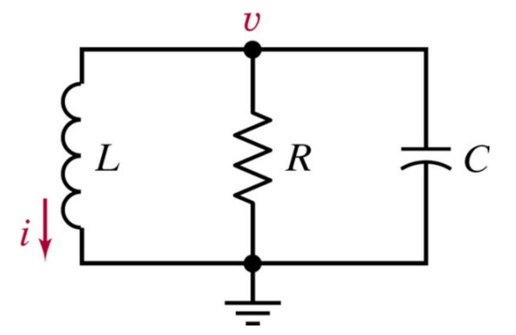
where $x=i_2$. The roots s_1 and s_2 are the characteristic roots and are often called the natural frequencies. The reciprocals of the magnitude of the real characteristic roots are the time constants. The time constants of this circuit are 1/2s and 1/8s.

Natural Response of the Unforced Parallel RLC Circuit

- Consider the circuit shown in Figure 9.4-1

  Write the KCL at the node to obtain

$$\frac{v}{R}+\frac{1}{L}\int_0^t v\,d\tau + i(0) + C\frac{dv}{dt} = 0 \qquad (9.4\text{-}1)$$



Taking the derivative, we have

$$C\frac{d^2v}{dt^2}+\frac{1}{R}\frac{dv}{dt}+\frac{1}{L}v=0 \qquad (9.4\text{-}2)$$

Using the parameter *s*, we obtain the characteristic equation

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

The two roots of the characteristic equation are

$$s_1=-\frac{1}{2RC}+\left[\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}\right]^{1/2} \quad \text{and} \quad s_2=-\frac{1}{2RC}-\left[\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}\right]^{1/2}$$

Natural Response of the Unforced Parallel RLC Circuit

- When s_1 is not equal to s_2, the solution to the second-order differential Eq. 9.4-2 for t>0 is

$$\boxed{v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \qquad (9.4\text{-}6)$$

- The roots of the characteristic equation may be rewritten as

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Where $\alpha = 1/(2RC)$ and $\omega_0^2 = 1/(LC)$

- The roots of the characteristic equation assume three possible conditions:
  1. Two real and distinct roots when $\alpha^2 > \omega_0^2$ → *overdamped*
  2. Two real equal roots when $\alpha^2 = \omega_0^2$ → *critically damped*
  3. Two complex roots when $\alpha^2 < \omega_0^2$ → *underdamped*

Natural Response of the Unforced Parallel RLC Circuit

- Let us determine the natural response for the overdamped RLC circuit of Figure 9.4-1 when the initial conditions are $v(0)$ and $i(0)$ for the capacitor and the inductor, respectively.

  At t=0 for Eq. 9.4-6, we have

$$v_n(0) = A_1 + A_2 \qquad (9.4\text{-}9)$$

  Since A_1 and A_2 are both unknown, we need one more equation at t=0. Rewriting Eq. 9.4-1 at t=0, we have

$$\frac{v(0)}{R} + i(0) + C\frac{dv(0)}{dt} = 0$$

  Since $i(0)$ and $v(0)$ are known, we have

$$\frac{dv(0)}{dt} = -\frac{v(0)}{RC} - \frac{i(0)}{C} \qquad (9.4\text{-}10)$$

Natural Response of the Unforced Parallel RLC Circuit

- Thus, we now know the initial value of the derivative of *v* in terms of the initial conditions.

Taking the derivative of Eq. 9.4-6 and setting t=0, we obtain

$$\frac{dv_{\mathrm{n}}(0)}{dt} = s_1 A_1 + s_2 A_2 \qquad (9.4\text{-}11)$$

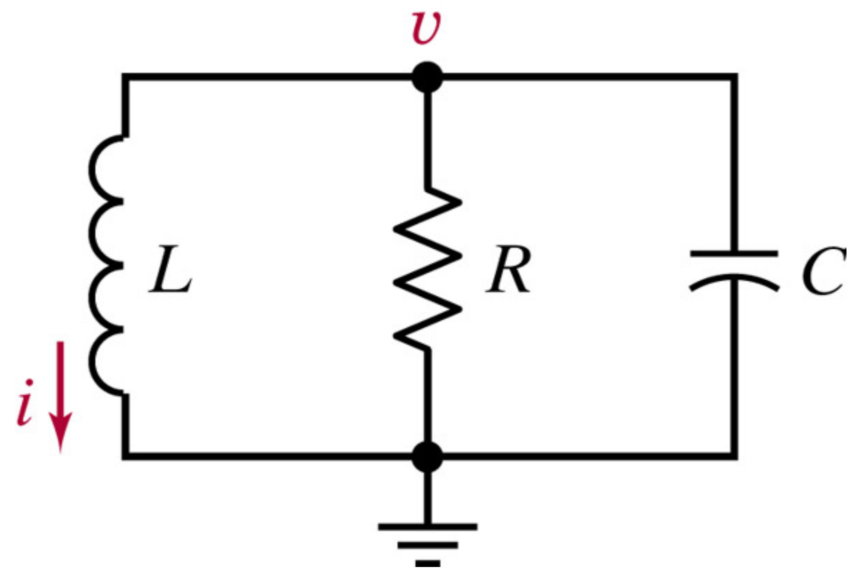
Using Eqs. 9.4-10 and 9.4-11, we obtain a second equation in terms of the two constants as

$$s_1 A_1 + s_2 A_2 = -\frac{v(0)}{RC} - \frac{i(0)}{C} \qquad (9.4\text{-}12)$$

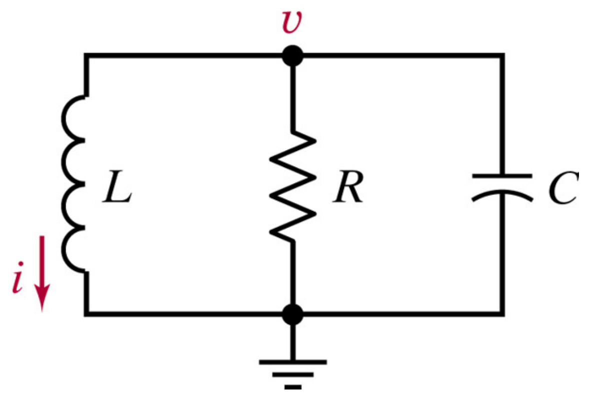
Using Eqs. 9.4-9 and 9.4-12, we may obtain A_1 and A_2.

Example 9.4-1 Natural Response of an Overdamped Second-Order Circuit

- Find the natural response of v(t) for t>0 for the parallel RLC circuit shown in Figure 9.4-1 when R=2/3Ω, L=1H, C=1/2F, v(0)=10V, and i(0)=2A.



Solution



- Using Eq. 9.4-3, the characteristic equation is

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

or

$$s^2 + 3s + 2 = 0$$

Therefore, the roots of the characteristic equation are

$$s_1 = -1 \quad \text{and} \quad s_2 = -2$$

Then the natural response is $v_n = A_1 e^{-t} + A_2 e^{-2t}$ (9.4−13)

The initial capacitor voltage is v(0)=10, so we have

$$v_n(0) = A_1 + A_2$$

or

$$10 = A_1 + A_2 \qquad (9.4\text{−}14)$$

We use Eq. 9.4-12 to obtain the second equation for the unknown constants.

$$s_1 A_1 + s_2 A_2 = -\frac{v(0)}{RC} - \frac{i(0)}{C}$$

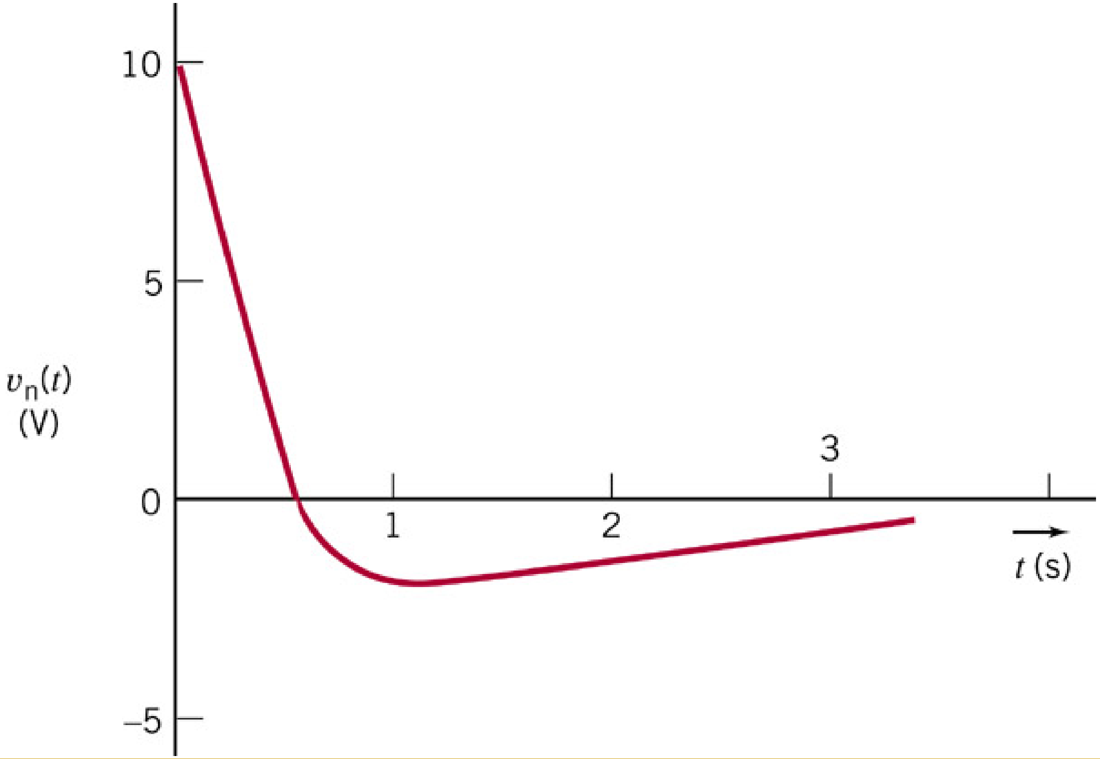
Therefore, $-A_1 - 2A_2 = -\frac{10}{1/3} - \frac{2}{1/2} = -34$ (9.4−15)

Solution

- Solving Eqs. 9.4-14 and 9.4-15 simultaneously, we obtain A2=24 and A1=-14. Therefore, the natural response is

$$v_n = (-14e^{-t} + 24e^{-2t})\text{V}$$

The natural response of the circuit is shown in Figure 9.4-2



Natural Response of the Critically Damped Unforced Parallel RLC Circuit

- We consider the parallel RLC circuit, and we will determine the special case when the characteristic equation has two equal real roots.
- Let us assume that $s_1 = s_2$ and proceed to find $v_n(t)$

$$v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_3 e^{s_1 t} \qquad (9.5\text{-}1)$$

Since the two roots are equal, we have only one undetermined constant, but we still have two initial conditions to satisfy. Clearly, Eq. 9.5-1 is not the total solution.

recap:Natural Response of the Unforced Parallel RLC Circuit

- When s_1 is not equal to s_2, the solution to the second-order differential Eq. 9.4-2 for t>0 is

$$\boxed{v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \qquad (9.4\text{-}6)$$

- The roots of the characteristic equation may be rewritten as

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Where $\alpha = 1/(2RC)$ and $\omega_0^2 = 1/(LC)$

- The roots of the characteristic equation assume three possible conditions:
  1. Two real and distinct roots when $\alpha^2 > \omega_0^2$ → *overdamped*
  2. Two real equal roots when $\alpha^2 = \omega_0^2$ → *critically damped*
  3. Two complex roots when $\alpha^2 < \omega_0^2$ → *underdamped*

Natural Response of the Critically Damped Unforced Parallel RLC Circuit

- We try the solution

$$\boxed{v_\mathrm{n} = e^{s_1 t}(A_1 t + A_2)} \qquad (9.5\text{-}2)$$

- Let us consider a parallel RLC circuit where

$$L = 1 \text{ H}, \quad R = 1\ \Omega, \quad C = 1/4 \text{ F}, \quad v(0) = 5 \text{ V}, \quad \text{and} \quad i(0) = -6 \text{ A}$$

The characteristic equation for the circuit is

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

or

$$s^2 + 4s + 4 = 0$$

The two roots are then $s_1 = s_2 = -2$

Using Eq. 9.5-2 for the natural response, we have

$$v_\mathrm{n} = e^{-2t}(A_1 t + A_2) \qquad (9.5\text{-}3)$$

Natural Response of the Critically Damped Unforced Parallel RLC Circuit

- Since $v_n(0)=5$, we have at $t=0$

$$A_2 = 5$$

Differentiate Eq. 9.5-3 to obtain

$$\frac{dv}{dt} = -2A_1te^{-2t} + A_1e^{-2t} - 2A_2e^{-2t} \qquad (9.5\text{-}4)$$

Evaluating Eq. 9.5-4 at t=0, we have

$$\frac{dv(0)}{dt} = A_1 - 2A_2$$

Again, we may use Eq. 9.4-10 so that

$$\frac{dv(0)}{dt} = -\frac{v(0)}{RC} - \frac{i(0)}{C}$$

or

$$A_1 - 2A_2 = \frac{-5}{1/4} - \frac{-6}{1/4} = 4$$

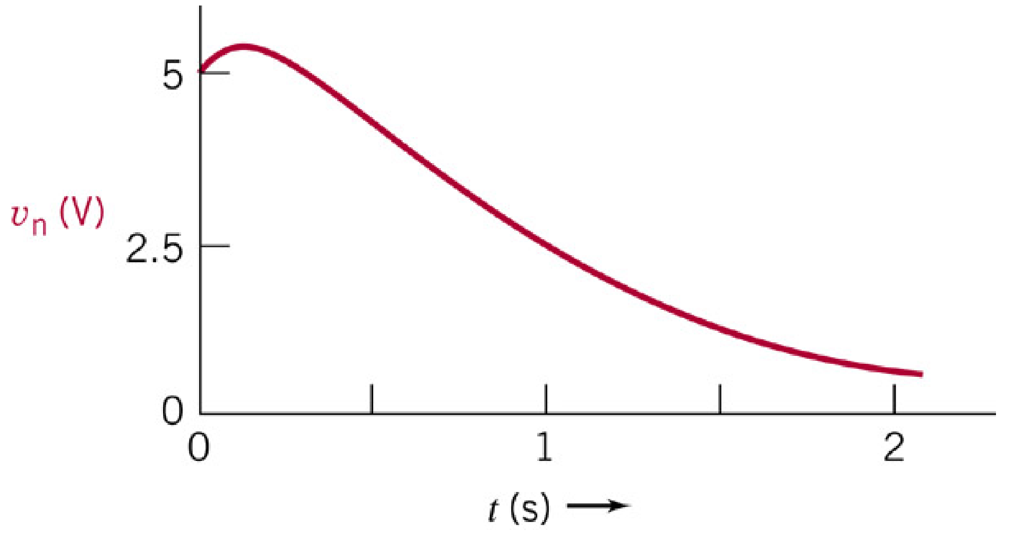
Therefore, $A_1 = 14$

Natural Response of the Critically Damped Unforced Parallel RLC Circuit

- The natural response is

$$v_n = e^{-2t}(14t+5)\ \text{V}$$

- The critically damped natural response of this RLC circuit is shown in Figure 9.5-1



Natural Response of an Underdamped Unforced Parallel RLC Circuit

- The characteristic equation of the parallel RLC circuit will have two complex conjugate roots when $\alpha^2 < \omega_0^2$. This condition is met when

$$LC < (2RC)^2$$

or when

$$L < 4R^2C$$

Recall that

$$v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

When

$$\omega_0^2 > \alpha^2$$

we have

$$s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

where

$$j = \sqrt{-1}$$

recap:Natural Response of the Unforced Parallel RLC Circuit

- When s_1 is not equal to s_2, the solution to the second-order differential Eq. 9.4-2 for t>0 is

$$v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad (9.4\text{-}6)$$

- The roots of the characteristic equation may be rewritten as

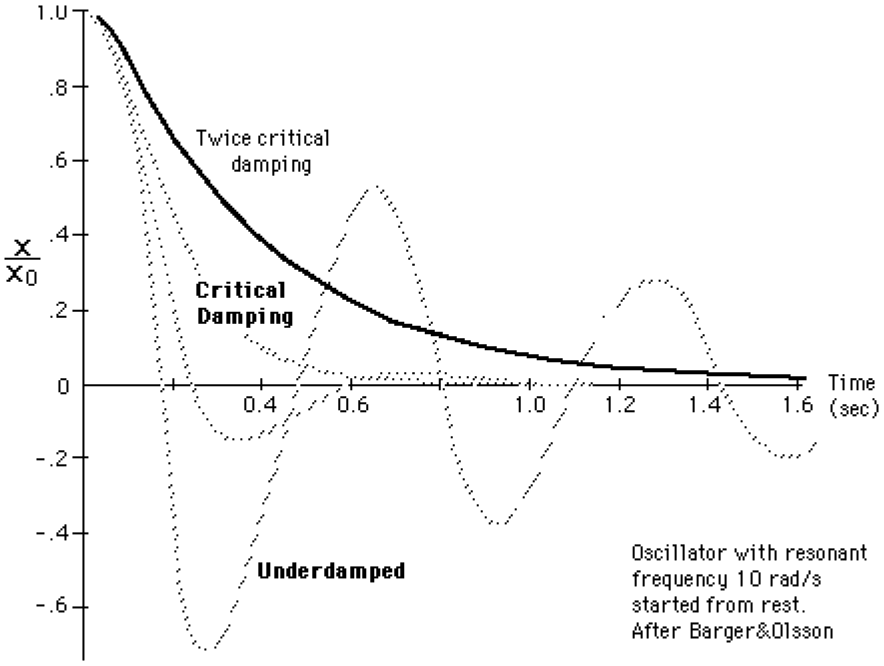
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Where $\alpha = 1/(2RC)$ and $\omega_0^2 = 1/(LC)$

- The roots of the characteristic equation assume three possible conditions:
  1. Two real and distinct roots when $\alpha^2 > \omega_0^2$ → *overdamped*
  2. Two real equal roots when $\alpha^2 = \omega_0^2$ → *critically damped*
  3. Two complex roots when $\alpha^2 < \omega_0^2$ → *underdamped*

Natural Response of the Unforced Parallel RLC Circuit

Critical damping provides the quickest approach to zero amplitude for a damped oscillator. With less damping (underdamping) it reaches the zero position more quickly, but oscillates around it. With more damping (overdamping), the approach to zero is slower. Critical damping occurs when the damping coefficient is equal to the undamped resonant frequency of the oscillator.



Natural Response of an Underdamped Unforced Parallel RLC Circuit

- The complex roots lead to an oscillatory-type response. We define the square root $\sqrt{\omega_0^2 - \alpha^2}$ as ω_d, which we will call the damped resonant frequency. The factor α, called the damping coefficient, determines how quickly the oscillator subside. Then the roots are

$$s_{1,2} = -\alpha \pm j\omega_\mathrm{d}$$

Therefore, the natural response is

$$v_\mathrm{n} = A_1 e^{-\alpha t} e^{j\omega_\mathrm{d} t} + A_2 e^{-\alpha t} e^{-j\omega_\mathrm{d} t}$$

or

$$v_\mathrm{n} = e^{-\alpha t}\left(A_1 e^{j\omega_\mathrm{d} t} + A_2 e^{-j\omega_\mathrm{d} t}\right)$$

using Euler identity

- Let us use the **Euler identity**

$$e^{\pm j\omega t} = \cos \omega t \pm j \sin \omega t$$

The natural response can be rewritten as

$$v_{\mathrm{n}} = e^{-\alpha t}\left(A_1 \cos \omega_d t + jA_1 \sin \omega_d t + A_2 \cos \omega_d t - jA_2 \sin \omega_d t\right)$$
$$= e^{-\alpha t}\left[\left(A_1 + A_2\right)\cos \omega_d t + j\left(A_1 - A_2\right)\sin \omega_d t\right]$$

- We replace (A_1+A_2) and $j(A_1–A_2)$ with new constants B_1 and B_2. A_1 and A_2 must be complex conjugates so that B_1 and B_2 are real numbers.

$$\boxed{v_{\mathrm{n}} = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right)} \qquad (9.6\text{-}5)$$

- The natural underdamped response is oscillatory with a decaying magnitude. The rapidity of decay depends on α , and the frequency of oscillation depends on ω_{d} .

Back to basics: in Appendix D **"Euler's Formula"**

- **Euler's formula**
  - Euler's formula is

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad \text{(D-1)}$$

An alternative form of Euler's formula is

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

To derive Euler's formula, let $f = \cos\theta + j\sin\theta$

Differentiating, we obtain

$$\frac{df}{d\theta} = -\sin\theta + j\cos\theta$$
$$= j(\cos\theta + j\sin\theta)$$
$$= if$$

When $f = e^{j\theta}$, we have $\frac{df}{d\theta} = if$

as required. Thus we obtain the result, Eq. D-1

Finding coefficients using initial conditions

- Let us find the general form of the solution for B_1 and B_2 in terms of the initial conditions when the circuit is unforced.

  Then at t=0 we have $v_n(0) = B_1$

  The derivative of v_n is

  $$\frac{dv_n}{dt} = e^{-\alpha t}\left[\left(\omega_d B_2 - \alpha B_1\right)\cos\omega_d t - \left(\omega_d B_1 + \alpha B_2\right)\sin\omega_d t\right]$$

  at t=0 we obtain $\frac{dv_n(0)}{dt} = \omega_d B_2 - \alpha B_1$

  Recall that we found earlier that Eq. 9.4-10 provides dv(0)/dt for the parallel RLC circuit as

  $$\frac{dv(0)}{dt} = -\frac{v(0)}{RC} - \frac{i(0)}{C} \qquad (9.6\text{-}7)$$

  Therefore, we use Eqs. 9.6-6 and 9.6-7

  $$\omega_d B_2 = \alpha B_1 - \frac{v(0)}{RC} - \frac{i(0)}{C} \qquad (9.6\text{-}8)$$

Example 9.6-1 Natural Response of an Underdamped Second-Order Circuit

- Consider the parallel RLC circuit when $R=25/3\Omega$, $L=0.1H$, $C=1mF$, $v(0)=10V$, and $i(0)=-0.6A$. Find the natural response $v_n(t)$ for $t>0$

Solution

- First, we determine α^2 and ω_0^2 to determine the form of the response. Consequently we obtain

$$\alpha = 1/(2RC) = 60 \quad \text{and} \quad \omega_0^2 = 1/(LC) = 10^4$$

Therefore, $\omega_0^2 > \alpha^2$ and the natural response is underdamped. The damped resonant frequency is

$$\omega_\mathrm{d} = \left(\omega_0^2 - \alpha^2\right)^{1/2} = \left(10^4 - 3.6\times 10^3\right)^{1/2} = 80 \text{ rad/s}$$

Hence, the characteristic roots are

$$s_1 = -\alpha + j\omega_d = -60 + j80$$

and

$$s_2 = -\alpha - j\omega_d$$

Consequently, the natural response is obtained as

$$v_\mathrm{n}(t) = B_1 e^{-60t}\cos 80t + B_2 e^{-60t}\sin 80t$$

Solution

- Since v(0)=10, we have

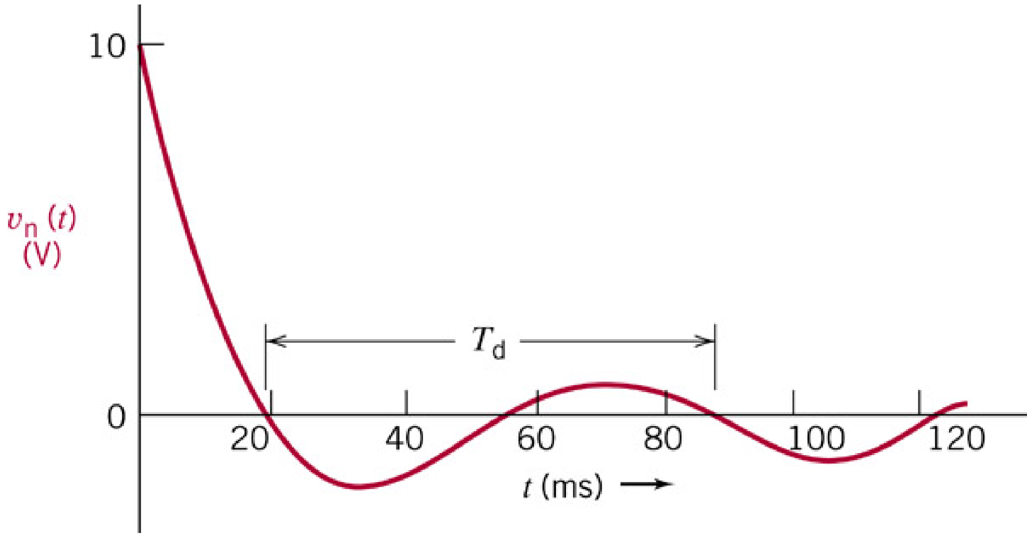
$$B_1 = v(0) = 10$$

We can use Eq. 9.7-8 to obtain

$$B_2 = \frac{\alpha}{\omega_d} B_1 - \frac{v(0)}{\omega_d RC} - \frac{i(0)}{\omega_d C}$$

$$= \frac{60 \times 10}{80} - \frac{10}{80 \times 25/3000} - \frac{-0.6}{80 \times 10^{-3}}$$

$$= 7.5 - 15.0 + 7.5 = 0$$



- Therefore, the natural response is

$$v_n(t) = 10e^{-60t} \cos 80t \text{ V}$$

- A sketch of this response is shown in Figure 9.6-1.

Natural Response of an Underdamped Unforced Parallel RLC Circuit

- The ***period of the oscillation*** is the time interval, denoted as T_{d}. expressed as

$$\boxed{T_d = \frac{2\pi}{\omega_d}\ \mathrm{s}}$$

  However, the natural response of an underdamped circuit is not a pure oscillatory response. Thus we may approximate T_{d} by the period between the first and third zero crossings, as shown in Figure 9.6-1.

- The frequency in hertz is

$$f_d = \frac{1}{T_d}\ \mathrm{Hz}$$

- The period of the oscillation of the circuit of example 9.6-1 is

$$T_d = \frac{2\pi}{80} = 79\ \mathrm{ms}$$

Forced Response of an RLC Circuit

- The forced response of an RLC circuit described by a second-order differential equation must satisfy the differential equation and no arbitrary constants. The response to a forcing function will often be of the same form as the forcing function.
- We consider the differential equation for the second-order circuit as

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t) \qquad (9.7\text{-}1)$$

The forced response x_f must satisfy Eq. 9.7-1.

$$a_2 \frac{d^2x_f}{dt^2} + a_1 \frac{dx_f}{dt} + a_0 x_f = f(t)$$

- If the forcing function is a constant, we expect the forced response also to be a constant since the derivatives of a constant are zero.
- If the forcing function is of the form $f(t) = Be^{-at}$, we expect

$$x_f = De^{-at}$$

Forced Response of an RLC Circuit

- If the forcing function is a sinusoidal function, we can expect the forced response to b a sinusoidal function. If $f(t) = A\sin\omega_0 t$, we will try

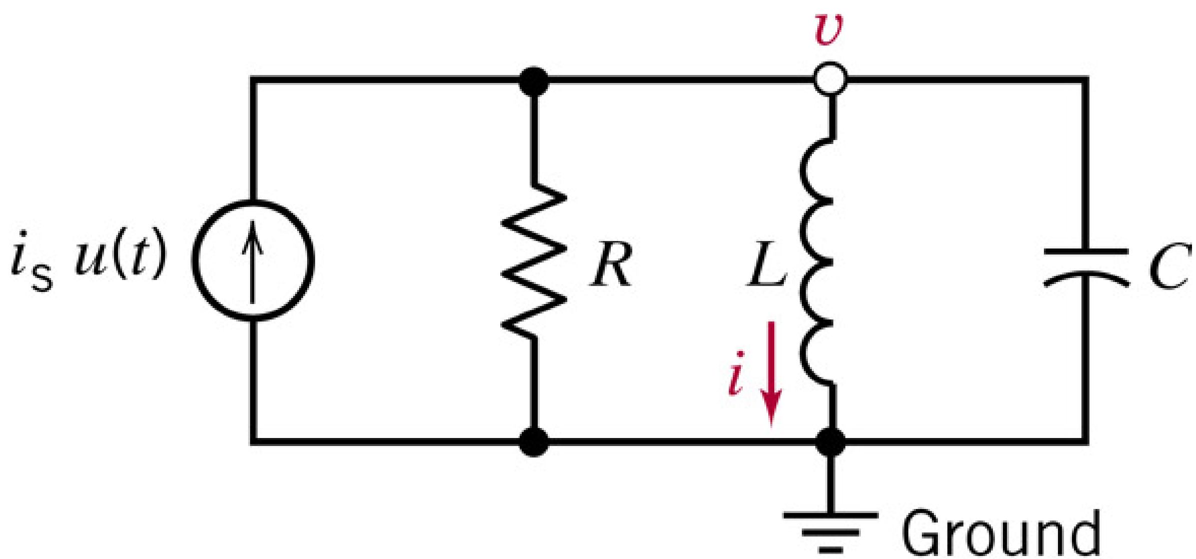
$$x_{\mathrm{f}} = M\sin\omega_0 t + N\cos\omega_0 t = Q\sin(\omega_0 t + \theta)$$

- Table 9.7-1 summarizes selected forcing functions and their associated assumed solutions.

Forcing Function	Assumed solution
K	A
Kt	$At + B$
Kt^2	$At^2 + Bt + C$
$K\sin\omega t$	$A\sin\omega t + B\cos\omega t$
Ke^{-at}	Ae^{-at}

Example 9.7-1 **Forced Response to an Exponential Input**

- Find the forced response for the inductor current i_f for the parallel RCL circuit shown in Figure 9.7-1 when $i_s=8e^{-2t}$A. Let R=6Ω, L=7H, and C=1/42F.



Solution

- The source current is applied at t=0 as indicated by the unit step function $u(t)$. The KCL equation at the upper node is

$$i + \frac{v}{R} + C\frac{dv}{dt} = i_s$$

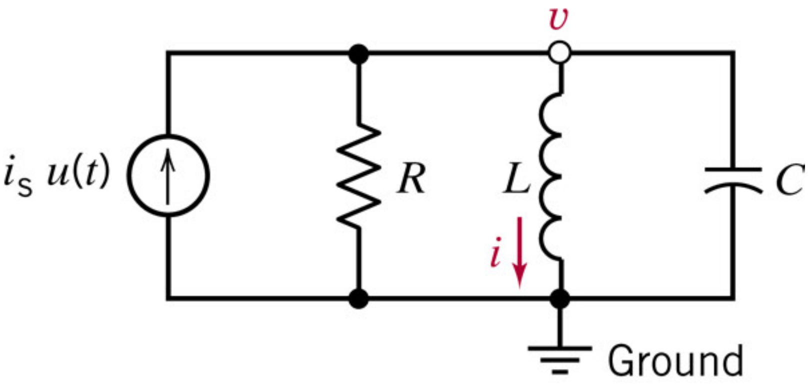
We wish to obtain the second-order differential equation in terms of i.

$$v = L\frac{di}{dt} \quad \text{and} \quad \frac{dv}{dt} = L\frac{d^2 i}{dt^2}$$

Substituting the component values and the source i_s, we obtain

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{LC}i_s$$

$$\frac{d^2 i}{dt^2} + 7\frac{di}{dt} + 6i = 48e^{-2t}$$



Solution

- We wish to obtain the forced response, so we assume that the response will be

$$i_{\mathrm{f}} = Be^{-2t}$$

where B is to be determined. Substituting the assumed solution into the differential equation, we have

$$4Be^{-2t} + 7(-2Be^{-2t}) + 6Be^{-2t} = 48e^{-2t}$$

or

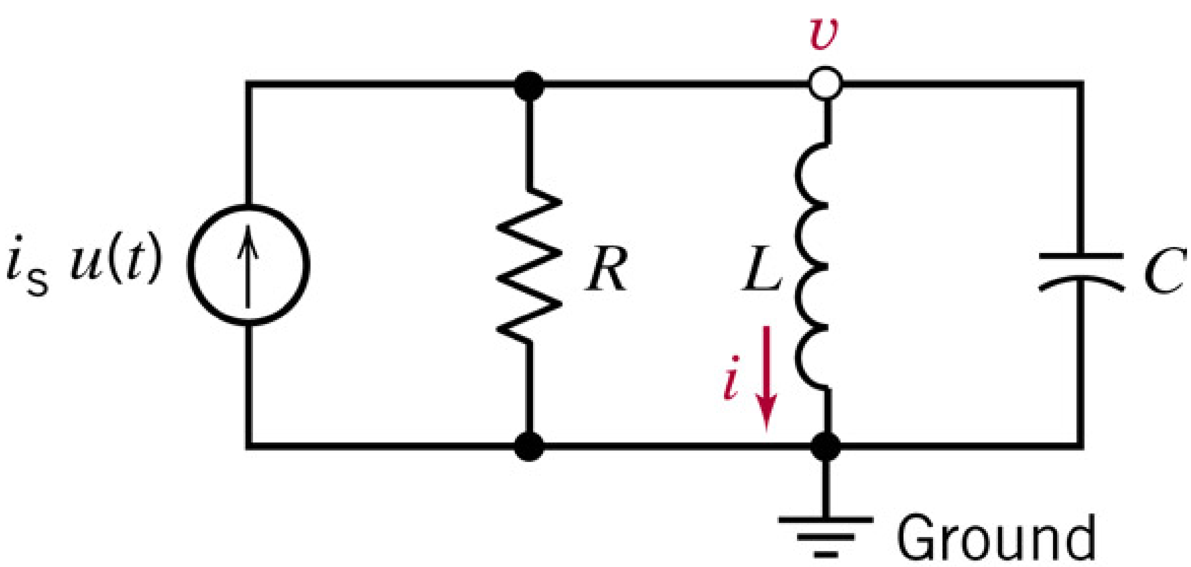
$$(4 - 14 + 6)Be^{-2t} = 48e^{-2t}$$

Therefore, B=-12 and

$$i_{\mathrm{f}} = -12e^{-2t}$$

Example 9.7-2 **Forced Response to a Constant Input**

- Find the forced response i_f of the circuit of Example 9.7-1 when $i_s = I_0$, where I_0 is a constant.



Solution

- Since the source is a constant applied at t=0, we expect the forced response to be a constant also.
  As a 1st method, we will use the differential equation to find the forced response.
  2nd method, we will demonstrate the alternative method that uses the steady-state behavior of the circuit to find i_f.

- The differential equation with the constant source is obtained,

$$\frac{d^2 i}{dt^2} + 7\frac{di}{dt} + 6i = 6I_0$$

  Again, we assume that the forced response is i_f=D, a constant,. Since the first and second derivatives of the assumed forced response are zero. We have

$$6D=6I_0$$

  or

$$D=I_0$$

  Therefore,

$$i_f=I_0$$

Solution

- Another approach is to determine the steady-state response if of the circuit of Figure 9.7-1 by drawing the steady-state circuit model.

$$i_f = I_0$$

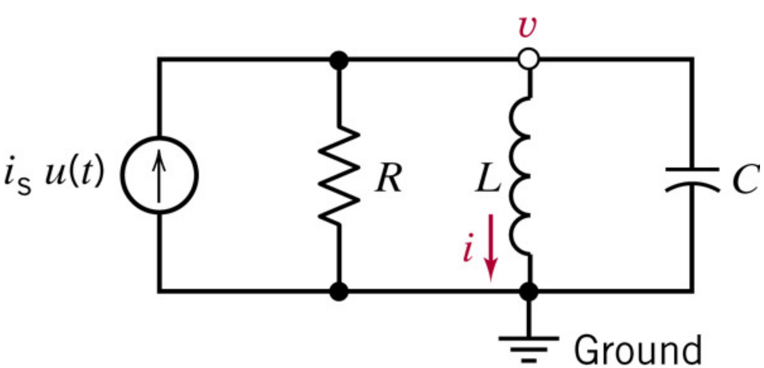


Fig 9.7-1

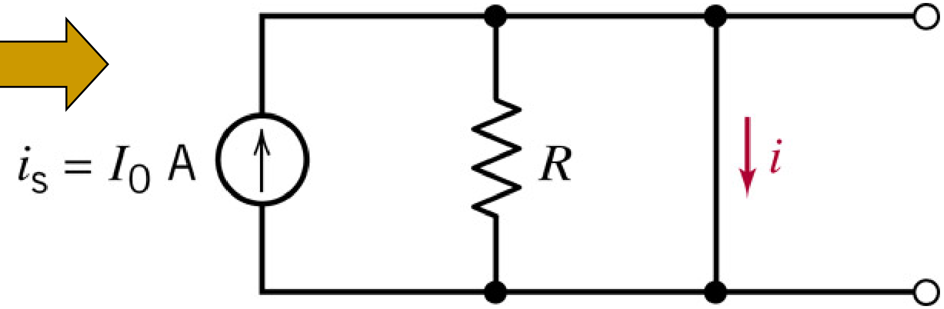


Fig 9.7-2

Forced Response of an RLC Circuit

- Again, consider the circuit of Example 9.7-1 and 9.7-2 (Figure 9.7-1) when the differential equation is

$$\frac{d^2 i}{dt^2} + 7\frac{di}{dt} + 6i = 6i_S \qquad (9.7\text{-}9)$$

The characteristic equation of the current is

$$s^2 + 7s + 6 = 0$$

or

$$(s+1)(s+6) = 0$$

Thus, the natural response is

$$i_n = A_1 e^{-t} + A_2 e^{-6t} \qquad (9.7\text{-}10)$$

Forced Response of an RLC Circuit

- Consider the special case where $i_S = 3e^{-6t}$.

  Then we at first expect the forced response to be

$$i_f = Be^{-6t} \qquad (9.7\text{-}11)$$

  However, the forced response and one component of the natural response would then both have the form De^{-6t}.

  Let's try substituting Eq. 9.7-11 into the differential equation (9.7-9). We then obtain

$$36Be^{-6t} - 42Be^{-6t} + 6Be^{-6t} \neq 18e^{-6t}$$

  or
$$0 \neq 18e^{-6t}$$

  which is an impossible solution. Therefore, we need another form of the forced response when one of the natural response terms has the same form as the forcing function.

Forced Response of an RLC Circuit

- Let us try the forced response

$$i_{\mathrm{f}} = Bte^{-6t} \qquad (9.7\text{-}12)$$

- Then, substituting Eq. 9.7-12 into Eq. 9.7-9, we have

$$B(-6g - 6g + 36tg) + 7B(g - 6tg) + 6Btg = 18g$$

where $g = g(t) = e^{-6t}$. We have

$$B = -\frac{18}{5} \quad \text{and} \quad i_{\mathrm{f}} = -\frac{18}{5} te^{-6t}$$

In general, if the forcing function is of the same form as one of the components of the natural response, x_{n1}, we will use

$$\boxed{x_{\mathrm{f}} = t^{p} x_{\mathrm{n1}}}$$

where the integer p is selected so that the x_{f} is not duplicated in the natural response. Use the lowest power, p, of t that is not duplicated in the natural response.

Complete Response of an RLC Circuit

- The *complete response* is the sum of the natural response and the forced response; thus

$$x = x_n + x_f$$

- Let us consider the series RLC circuit of Figure 9.2-2 with a differential equation (9.2-8) as

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_S$$

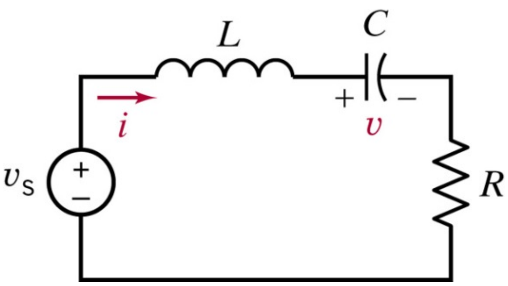


Fig. 9.2−2

When $L = 1\ \text{H}, \quad C = 1/6\ \text{F}, \quad \text{and} \quad R = 5\ \Omega$, we obtain

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 6v_S \qquad (9.8\text{-}1)$$

We let $\quad v_S = \dfrac{2e^{-t}}{3}\ \text{V}, \ v(0) = 10\ \text{V}, \ \text{and} \ \dfrac{dv(0)}{dt} = -2\ \text{V/s}$

Complete Response of an RLC Circuit

- To obtain natural response, we write the characteristic equation using operators as

$$s^2 + 5s + 6 = 0$$

or

$$(s+2)(s+3) = 0$$

Therefore the natural response is

$$v_n = A_1 e^{-2t} + A_2 e^{-3t}$$

- The forcing response is obtained by examining the forcing function and noting that its exponential response has a different time constant than the natural response,

$$v_f = Be^{-t} \qquad (9.8\text{-}2)$$

We can determine B by substituting Eq. 9.8-2 into Eq. 9.8-1. Then we have

$$Be^{-t} + 5\left(-Be^{-t}\right) + 6\left(Be^{-t}\right) = 4e^{-t}$$

or

$$B = 2$$

Complete Response of an RLC Circuit

- The complete response is then

$$v = v_{\mathrm{n}} + v_{\mathrm{f}} = A_1 e^{-2t} + A_2 e^{-3t} + 2e^{-t}$$

In order to find A_1 and A_2 we use the initial conditions.
At t=0 we have v(0)=10, so we obtain

$$10 = A_1 + A_2 + 2 \qquad (9.8\text{-}3)$$

From the fact that dv/dt=-2 at t=0, we have

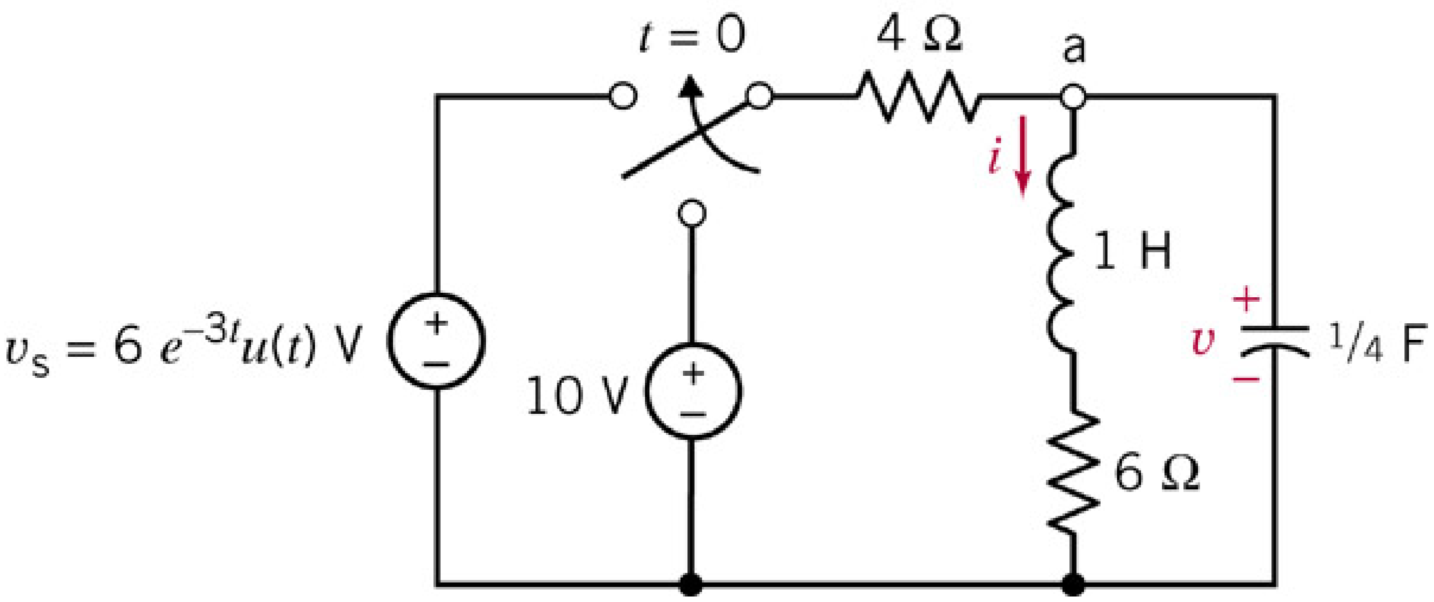
$$-2A_1 - 3A_2 - 2 = -2 \qquad (9.8\text{-}4)$$

Solving the Eqs. 9.8-3 and 9.8-4 by Cramer's rule,
We have A_1=24 and A_2=-16. Therefore,

$$v = v_{\mathrm{n}} + v_{\mathrm{f}} = 24e^{-2t} - 16e^{-3t} + 2e^{-t} \text{ V}$$

Example 9.8-1 **Complete Response of a Second-Order Circuit**

- Find the complete response v(t) for t>0 for the circuit of Figure 9.8-1. Assume the circuit is at steady state at t=0⁻



Solution

- First, we determine the initial conditions of the circuit. At t=0- we have the circuit model shown in Figure 9.8-2, where we replace the capacitor with an open circuit and the inductor with a short circuit. Then the voltage and inductor current is

$$V(0^-)=6V,\ i(0-)=1A$$

After the switch is thrown, we can write KVL for the right-hand mesh of Figure 9.8-1 to obtain

$$-v+\frac{di}{dt}+6i=0 \qquad (9.8-5)$$

The KCL equation at node a will provide a second equation in terms of v and i as

$$\frac{v-v_s}{4}+i+\frac{1}{4}\frac{dv}{dt}=0 \qquad (9.8-6)$$

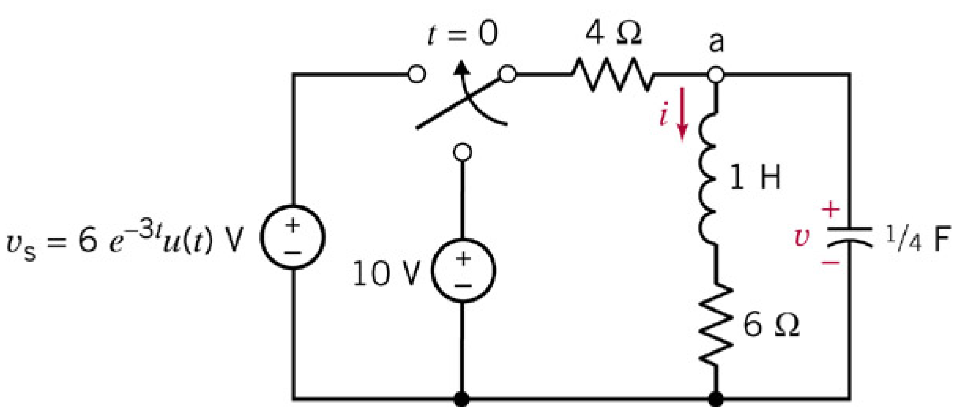


Fig 9.8-1

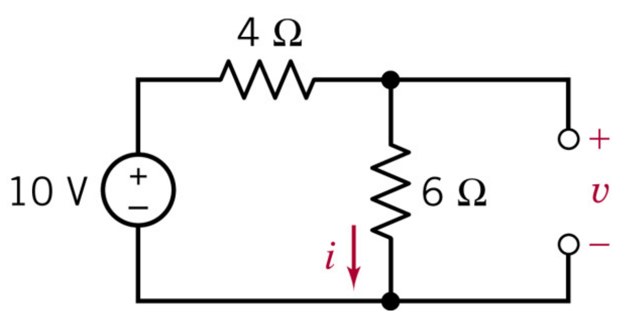


Fig 9.8-2

Solution

- Equations 9.8-5 and 9.8-6 may be rearranged as

$$\left(\frac{di}{dt}+6i\right)-v=0$$

$$i+\left(\frac{v}{4}+\frac{1}{4}\frac{dv}{dt}\right)=\frac{v_s}{4}$$

We will use operators so that s=d/dt, $s^2=d^2/dt^2$, and 1/s=∫dt.

$$(s+6)i-v=0 \qquad (9.8\text{−}9)$$

$$i+\frac{1}{4}(s+1)v=\frac{v_s}{4} \qquad (9.8\text{−}10)$$

The characteristic equation is obtained from Cramer's rule as the determinant Δ

$$\Delta=\frac{1}{4}(s+6)(s+1)+1$$

Set the determinant to zero to obtain $(s+6)(s+1)+4=0$

Therefore, the roots of the characteristic equation are

$$s_1=-2 \quad \text{and} \quad s_2=-5$$

Solution

- To find the second-order differential equation describing the circuit, we use Cramer's rule for Eqs. 9.8-9 and 9.8-10 to solve for v in order to obtain

$$v = \frac{(s+6)(v_s/4)}{\Delta} = \frac{(s+6)v_s}{s^2+7s+10}$$

Of course, this equation can be rewritten as

$$(s^2+7s+10)v = (s+6)v_s$$

and hence the second-order differential equation is

$$\frac{d^2v}{dt^2} + 7\frac{dv}{dt} + 10v = \frac{dv_s}{dt} + 6v_s \qquad (9.8\text{–}11)$$

The natural response v_n is

$$v_n = A_1 e^{-2t} + A_2 e^{-5t}$$

The forced response is assumed to be of the form

$$v_f = Be^{-3t} \qquad (9.8\text{–}12)$$

Substituting v_f into the differential equation, we have

$$9Be^{-3t} - 21Be^{-3t} + 10Be^{-3t} = -18e^{-3t} + 36e^{-3t}$$

Therefore,

$$B = -9$$

$$v_f = -9e^{-3t}$$

Solution

- The complete response is then

$$v = v_n + v_f = A_1 e^{-2t} + A_2 e^{-5t} - 9e^{-3t} \qquad (9.8\text{–}13)$$

Since v(0)=6, we have

$$v(0) = 6 = A_1 + A_2 - 9$$

or

$$A_1 + A_2 = 15 \qquad (9.8\text{–}14)$$

We also know that i(0)=1A. We can use Eq. 9.8-8 to determine dv(0)/dt and then evaluate the derivative of Eq. 9.8-13 at t=0. Eq. 9.8-8 states that

$$\frac{dv}{dt} = -4i - v + v_s$$

At t=0 we have

$$\frac{dv(0)}{dt} = -4i(0) - v(0) + v_s(0) = -4 - 6 + 6 = -4$$

Solution

- Let us take the derivative of Eq. 9.8-13 to obtain

$$\frac{dv}{dt} = -2A_1e^{-2t} - 5A_2e^{-5t} + 27e^{-3t}$$

At t=0 we obtain

$$\frac{dv(0)}{dt} = -2A_1 - 5A_2 + 27$$

Since dv(0)/dt=-4, we have

$$2A_1 + 5A_2 = 31 \qquad (9.8\text{–}15)$$

Solving Eqs. 9.8-15 and 9.8-14 simultaneously, we obtain

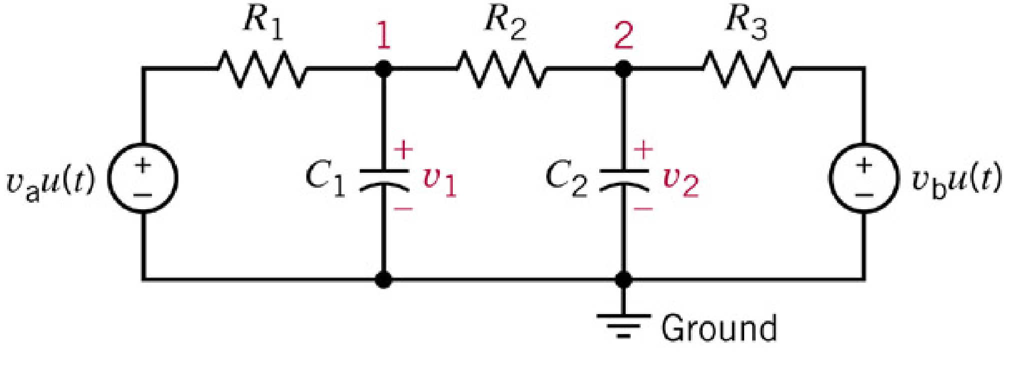
$$A_1 = \frac{44}{3} \quad \text{and} \quad A_2 = \frac{1}{3}$$

Therefore,

$$v = \frac{44}{3}e^{-2t} + \frac{1}{3}e^{-5t} - 9e^{-3t}\,\mathrm{V}$$

State Variable Approach to Circuit Analysis

- The *state variables* of a circuit are a set of variables associated with the energy of the energy storage elements of the circuit.
  Thus, they describe the complete response of a circuit to a forcing function and the circuit's initial conditions.
  We will choose as the state variables those variables that describe the energy storage of the circuit
  Thus, we will use the independent capacitor voltages and the independent inductor currents.
- Consider the circuit shown in Figure 9.9-1. The state variables are v1 and v2.



State Variable Approach to Circuit Analysis

- Writing the KCL at nodes 1 and 2, we have

node 1: $$C_1 \frac{dv_1}{dt} = \frac{v_a - v_1}{R_1} + \frac{v_2 - v_1}{R_2} \qquad (9.9\text{-}1)$$

node 2: $$C_2 \frac{dv_2}{dt} = \frac{v_b - v_2}{R_3} + \frac{v_1 - v_2}{R_2} \qquad (9.9\text{-}2)$$

Equations9.9-1 and 9.9-2 can be rewritten as

$$\frac{dv_1}{dt} + \frac{v_1}{C_1R_1} + \frac{v_1}{C_1R_2} - \frac{v_2}{C_1R_2} = \frac{v_a}{C_1R_1} \qquad (9.9\text{-}3)$$

$$\frac{dv_2}{dt} + \frac{v_2}{C_2R_3} + \frac{v_2}{C_2R_2} - \frac{v_1}{C_2R_2} = \frac{v_b}{C_2R_3} \qquad (9.9\text{-}4)$$

State Variable Approach to Circuit Analysis

- Assume that $C_1R_1 = 1,\ C_1R_2 = 1,\ C_2R_3 = 1,$ and $C_2R_2 = 1/2$

  Then we have

$$\frac{dv_1}{dt} + 2v_1 - v_2 = v_a \qquad (9.9\text{-}5)$$

$$-2v_2 + \frac{dv_2}{dt} + 3v_2 = v_b \qquad (9.9\text{-}6)$$

Using operator, we have

$$(s+2)v_1 - v_2 = v_a$$

$$-2v_2 + (s+3)v_2 = v_b$$

If we wish to solve for v_1, we use Crammer's rule to obtain

$$v_1 = \frac{(s+3)v_a + v_b}{(s+2)(s+3) - 2} \qquad (9.9\text{-}7)$$

State Variable Approach to Circuit Analysis

- The characteristic equation is obtained from the denominator and has the form

$$s^2 + 5s + 4 = 0$$

The characteristic roots are s=-4 and s=-1. The second-order differential equation can be obtained by rewriting Eq. 9.9-7 as

$$\left(s^2 + 5s + 4\right)v_1 = \left(s + 3\right)v_a + v_b$$

Then the differential equation for v_1 is

$$\frac{d^2 v_1}{dt^2} + 5\frac{dv_1}{dt} + 4v_1 = \frac{dv_a}{dt} + 3v_a + v_b \qquad (9.9\text{-}8)$$

State Variable Approach to Circuit Analysis

- We now proceed to obtain the natural response

$$v_{1n} = A_1 e^{-t} + A_2 e^{-4t}$$

and forced response, which depends on the form of the forcing function. If v_a=10 V and v_b=6 V, v_{1f} will be a constant. (see Table 9.7-1)

We obtain v_{1f} by substituting v_a and v_b into Eq. 9.9-8, obtaining

$$4v_{1f} = 3v_a + v_b$$

or

$$4v_{1f} = 30 + 6 = 36$$

Therefore,

$$v_{1f} = 9$$

Then

$$v_1 = v_{1n} + v_{1f} = A_1 e^{-t} + A_2 e^{-4t} + 9 \qquad (9.9\text{-}9)$$

State Variable Approach to Circuit Analysis

- If we know that $v_1(0)$=5 V and $v_2(0)$=10 V, we first use $v_1(0)$=5 along with Eq. 9.9-9 to obtain

$$v_1(0) = A_1 + A_2 + 9$$

and, Therefore, $$A_1 + A_2 = -4 \qquad (9.9\text{-}10)$$

Now we need the value of $dv1/dt$ at t=0. Referring back to Eq. 9.9-5, we have

$$\frac{dv_1}{dt} = v_a + v_2 - 2v_1$$

Therefore, at t=0 we have

$$\frac{dv_1(0)}{dt} = v_a(0) + v_2(0) - 2v_1(0) = 10 + 10 - 2(5) = 10$$

The derivative of the complete solution at t=0 is

$$\frac{dv_1(0)}{dt} = -A_1 - 4A_2$$

Therefore,

$$A_1 + 4A_2 = -10 \qquad (9.9\text{-}11)$$

State Variable Approach to Circuit Analysis

- Solving Eqs. 9.9-10 and 9.9-11, we have

$$A_1 = -2 \quad \text{and} \quad A_2 = -2$$

Therefore,

$$v_1(t) = -2e^{-t} - 2e^{-4t} + 9 \text{ V}$$

As you encounter circuits with two or more energy storage elements, you should consider using the state variable method of describing a set of first-order differential equations.

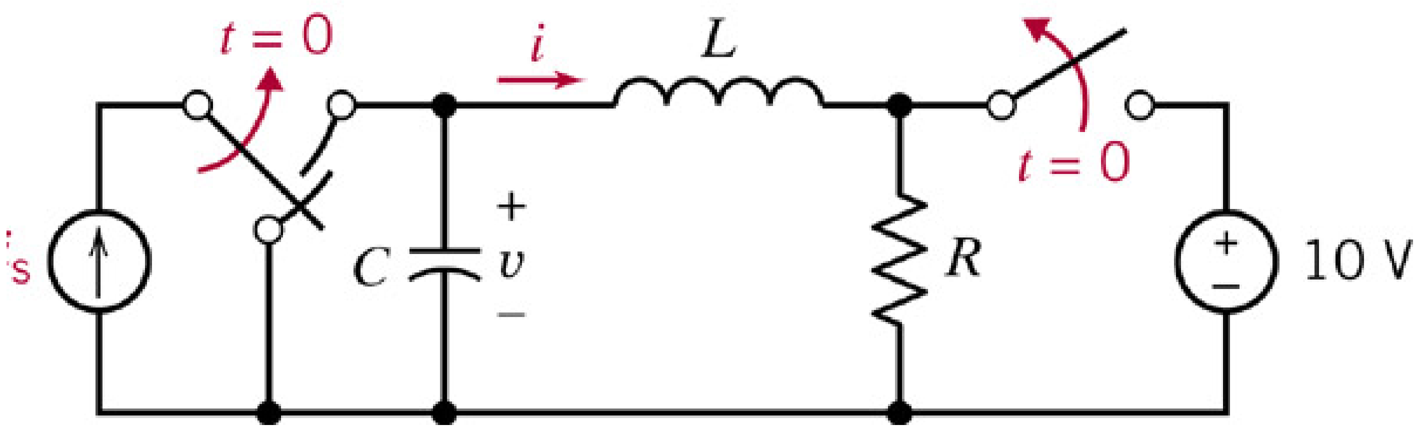
- The **state variable method** uses a first-order differential equation for each state variables to determine the complete response of a circuit.

summary table

Table 9.9-1 **State Variable Method of Circuit Analysis**
1. Identify the state variables as the independent capacitor voltage and inductor currents.
2. Determine the initial conditions at t=0 for the capacitor voltages and the inductor currents
3. Obtain a first-order differential equation for each state variable using KCL or KVL.
4. Use the operator s to substitute for d/dt
5. Obtain the characteristic equation of the circuit by noting that it can be obtained by setting the determinant of Cramer's rule equal to zero.
6. Determine the roots of the characteristic equation, which then determine the form of the natural response.
7. Obtain the second-order (or higher-order) differential equation for the selected variable x by Cramer's rule.
8. Determine the forced response xf by assuming an appropriate form of x_f and determining
9. Obtain the complete solution $x= x_n + x_f$
10. Use the initial conditions on the state variables along with the set of first-order differential equations (step3) to obtain dx(0)dt
11. Using x(0) and dx(0)/dt for each state variable, find the arbitrary constants A_1, A_2, … A_n to obtain the complete solution x(t)

Example 9.9-1 **Complete Response of a Second-Order Circuit**

- Find i(t) for t>0 for the circuit shown in Figure 9.9-2 when R=3Ω, L=1H, C=1/2F, and is=$2e^{-3t}$A. Assume steady state at t=0^-.



Solution

- First, we identify the state variables as i and v. The initial conditions at t=0 are obtained by considering the circuit with the 10-V source connected for a long time at t=0-. Then v(0)=10V and i(0)=0A. At t=0, the voltage source is disconnected and the current source is connected.

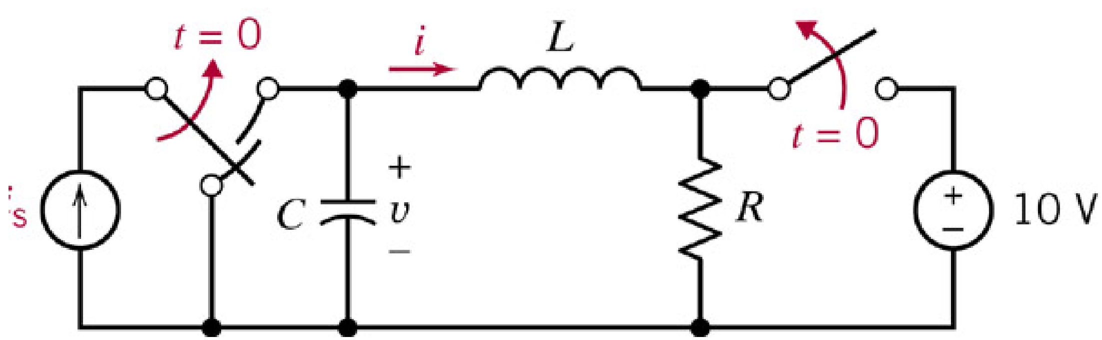
- The first differential equation is obtained by using KVL around the RLC mesh

$$L\frac{di}{dt}+Ri=v$$

The second differential equation is obtained by using KCL at the node at the top of the capacitor

$$C\frac{dv}{dt}+i=i_s$$

Substituting the component values, we have $\dfrac{di}{dt}+3i-v=0$ and $\dfrac{dv}{dt}+2i=2i_s$



Solution

- Using the operator s=d/dt, we have

$$(s+3)i - v = 0 \qquad (9.9\text{–}14)$$

$$2i + sv = 2i_s \qquad (9.9\text{–}15)$$

Therefore the characteristic equation obtained from the determinant is

$$(s+3)s + 2 = 0$$

Thus, the roots of the characteristic equation are

$$s_1 = -2 \quad \text{and} \quad s_2 = -1$$

Since we wish to solve for i(t) for t>0, we use Cramer's rule to solve Eqs. 9.9-14 and 9.9-15 for i, obtaining

$$i = \frac{2i_s}{s^2 + 3s + 2}$$

Therefore, the differential equation is

$$\frac{d^2 i}{dt^2} + 3\frac{di}{dt} + 2i = 2i_s \qquad (9.9\text{–}16)$$

The natural response is $i_n = A_1 e^{-t} + A_2 e^{-2t}$

Solution

- We assume the forced response is of the form

$$i_f = Be^{-3t}$$

Substituting if into Eq. 9.9-16, we have

$$(9Be^{-3t}) + 3(-3Be^{-3t}) + 2Be^{-3t} = 2(2Be^{-3t})$$

Therefore, B=2 and

$$i_f = 2e^{-3t}$$

The complete response is $i = A_1e^{-t} + A_2e^{-2t} + 2e^{-3t}$

Since i(0)=0, $0 = A_1 + A_2 + 2$ (9.9−17)

We need to obtain di(0)/dt from Eq. 9.9-12, which we repeat here as

$$\frac{di}{dt} + 3i - v = 0$$

Therefore, at t=0 we have

$$\frac{dv(0)}{dt} = -3i(0) + v(0) = 10$$

V should be i

Solution

- The derivative fo the complete response at t=0 is

$$\frac{dv(0)}{dt} = -A_1 - 2A_2 - 6$$

V should be i

Since di(0)/dt=10, we have

$$-A_1 - 2A_2 = 16$$

and repeating Eq. 9.9=17, we have

$$A_1 + A_2 = -2$$

Adding these two equations, we determine that A1=12 and A2=-14. Then we have the complete solution for i as

$$i = 12e^{-t} - 14e^{-2t} + 2e^{-3t}\,(\mathrm{A})$$

Roots in the Complex Plane

- Let us consider the roots of a parallel RLC circuit. The characteristic equation (9.4-3) is

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

and the roots are $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where $\alpha = 1/(2RC)$ and $\omega_0^2 = 1/(LC)$

When, $\omega_0^2 > \alpha^2$ the roots are complex and

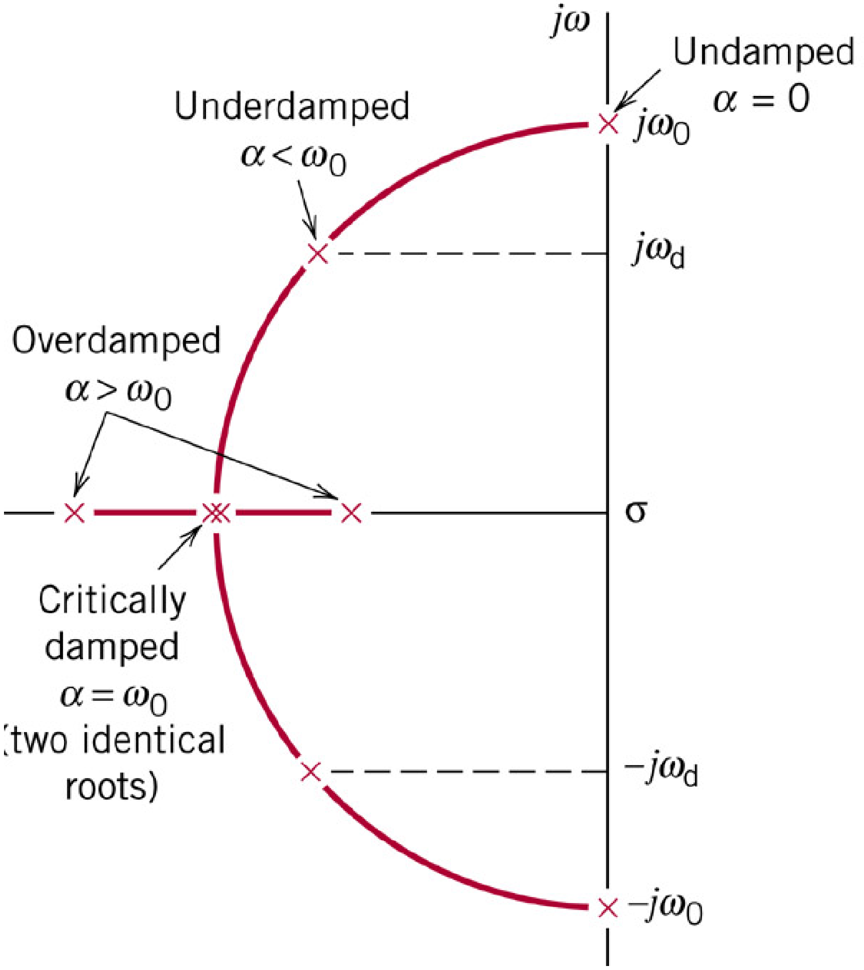
$$s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

- In general, roots are located in the complex plane, the location being defined by coordinates measured along the real of σ-axis and the imaginary or *j*ω-axis. This is referred to as the *s*-plane of as the ***complex frequency plane***.

Roots in the Complex Plane

- Figure 9.10-1
  The complete s-plane showing the locations of the two roots, s1 and s2, of the characteristic equation in the left-hand portion of the s-plane. The roots are designated by the x simble

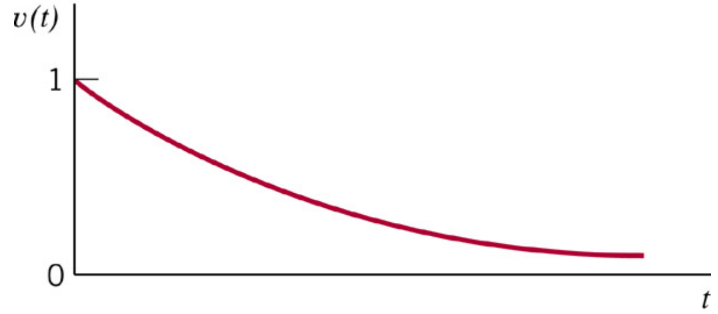
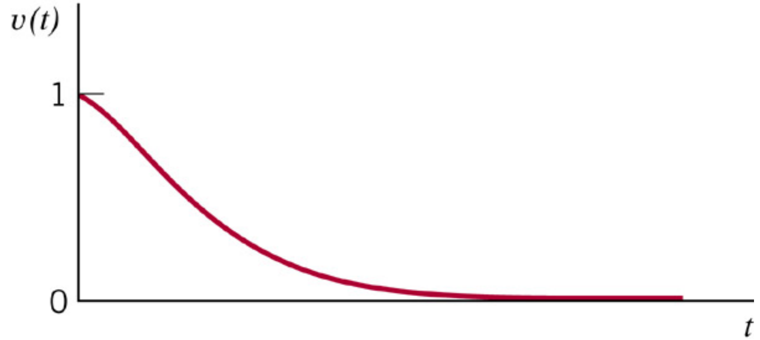
  for the four conditions:
  (1) undamped, α=0
  (2) underdamped, $\alpha^2 < \omega_0^2$
  (3) critically damped, $\alpha^2 = \omega_0^2$
  (4) overdamped,. $\alpha^2 > \omega_0^2$



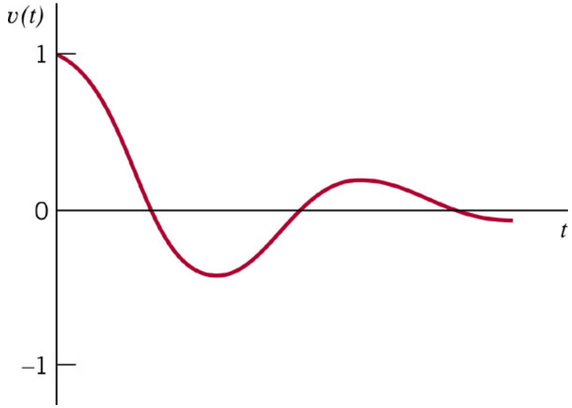
Roots in the Complex Plane

- A summary of the root locations, the type of response, and the form of the response for $v(0)=1$ and $i(0)=0$ is presented in Table 9.10-1

Table 9.10-1 **The Natural Response of a Parallel RLC Circuit**

Root Location	Type of Response	Form of Response for v(0)=1V and i(0)=0
$s=-r_1, -r_2$ two real roots	Overdamped	
$s=-r_1, -r_2$ two equal roots	Critically damped	

Roots in the Complex Plane

Table 9.10-1 The Natural Response of a Parallel RLC Circuit		
Root Location	Type of Response	Form of Response for v(0)=1V and i(0)=0
$s=-\alpha\pm j\omega_d$	Underdamped	
$s=\pm j\omega_0$	Undamped