Chapter 10
Sinusoidal Steady-State Analysis

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Sinusoidal Sources

- Sinusoidal function

$$v(t) = A\sin(\omega t)\ \mathrm{V}$$

- Sinusoid is periodic function having period of T.

$$v(t+T) = v(t)$$

- Frequency f and angular frequency ω is defined as below

$$f = \frac{1}{T}\ [\boldsymbol{Hz}] \qquad \omega = 2\pi f = \frac{2\pi}{T}\ [\boldsymbol{rad/s}]$$

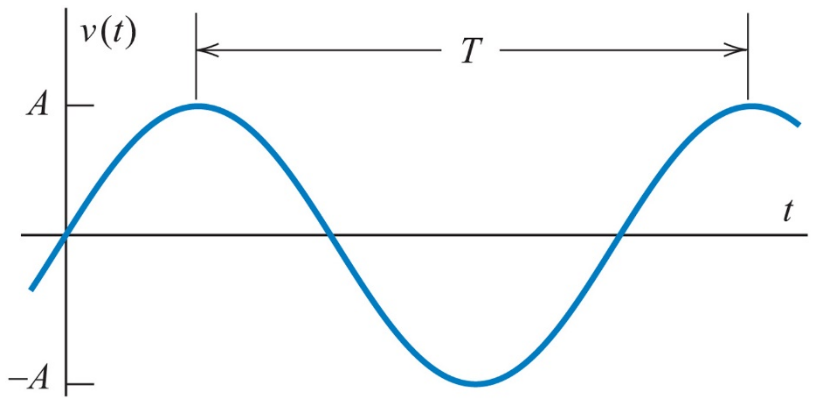


FIGURE 10.2−1

What is Phase?

- Consider the effect of replacing t by $t + t_a$ where t_a is some arbitrary time.

- We have

  $$v(t + t_a) = A\sin(\omega(t + t_a)) = A\sin(\omega t + \omega t_a) = A\sin(\omega t + \theta)\ V$$

  where $$\theta = \omega t_a = \frac{2\pi}{T} t_a = 2\pi \frac{t_a}{T}\ [\boldsymbol{rad}]$$

  θ is called the **phase angle** of the sinusoid $A\sin(\omega t + \theta)$.

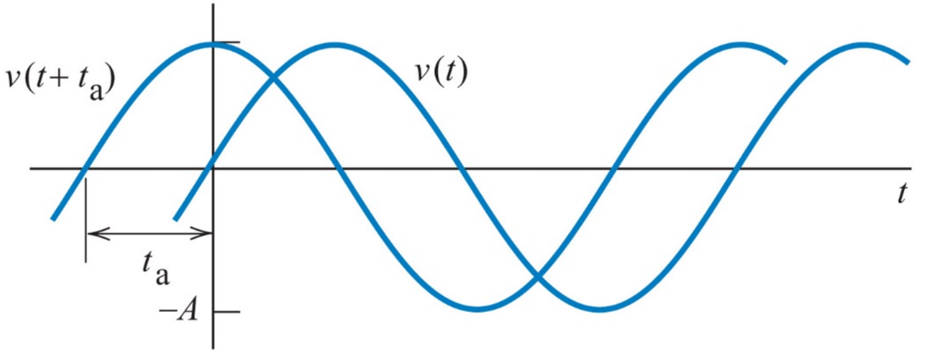


FIGURE 10.2−3

Example 10.2-1 Phase Shift and Delay

- Consider the sinusoids

$$v_1(t) = 10\ \cos(200t + 45°)\ \text{V}\ \text{ and }\ v_2(t) = 8\ \sin(200t + 15°)\ \text{V}$$

Determine the time by which $v_2(t)$ is advanced or delayed with respect to $v_1(t)$.

Solution (1/2)

$v_1(t)$ and $v_2(t)$ have the same frequency but different amplitudes

Therefore, period of both sinusoids are given by

$$\mathbf{200 = \frac{2\pi}{T} \quad \Rightarrow \quad T = \frac{\pi}{100} = 0.0314159 = 31.4159\, ms}$$

To compare the phase angle of $v_1(t)$ and $v_2(t)$, represent $v_2(t)$ as

$$\boldsymbol{v_2(t) = 8\, sin(200t + 15°) = 8\, cos(200t - 75°)\; V}$$

Then difference of phase angles of $v_1(t)$ and $v_2(t)$ is given by

$$\boldsymbol{\theta_2 - \theta_1 = -75° - 45° = -120° = -\frac{2\pi}{3}[rad]}$$

The minus sign indicates a delay rather than an advance. Convert this angle to time.

$$\boldsymbol{\theta_2 - \theta_1 = 2\pi\frac{t_d}{T} \quad \Rightarrow \quad t_d = \frac{(\theta_2 - \theta_1)T}{2\pi} = -10.47ms}$$

Finally, $v_2(t)$ is delayed by about 10.5 ms with respect to $v_1(t)$

Solution (2/2)

Also MATLAB plot shows.

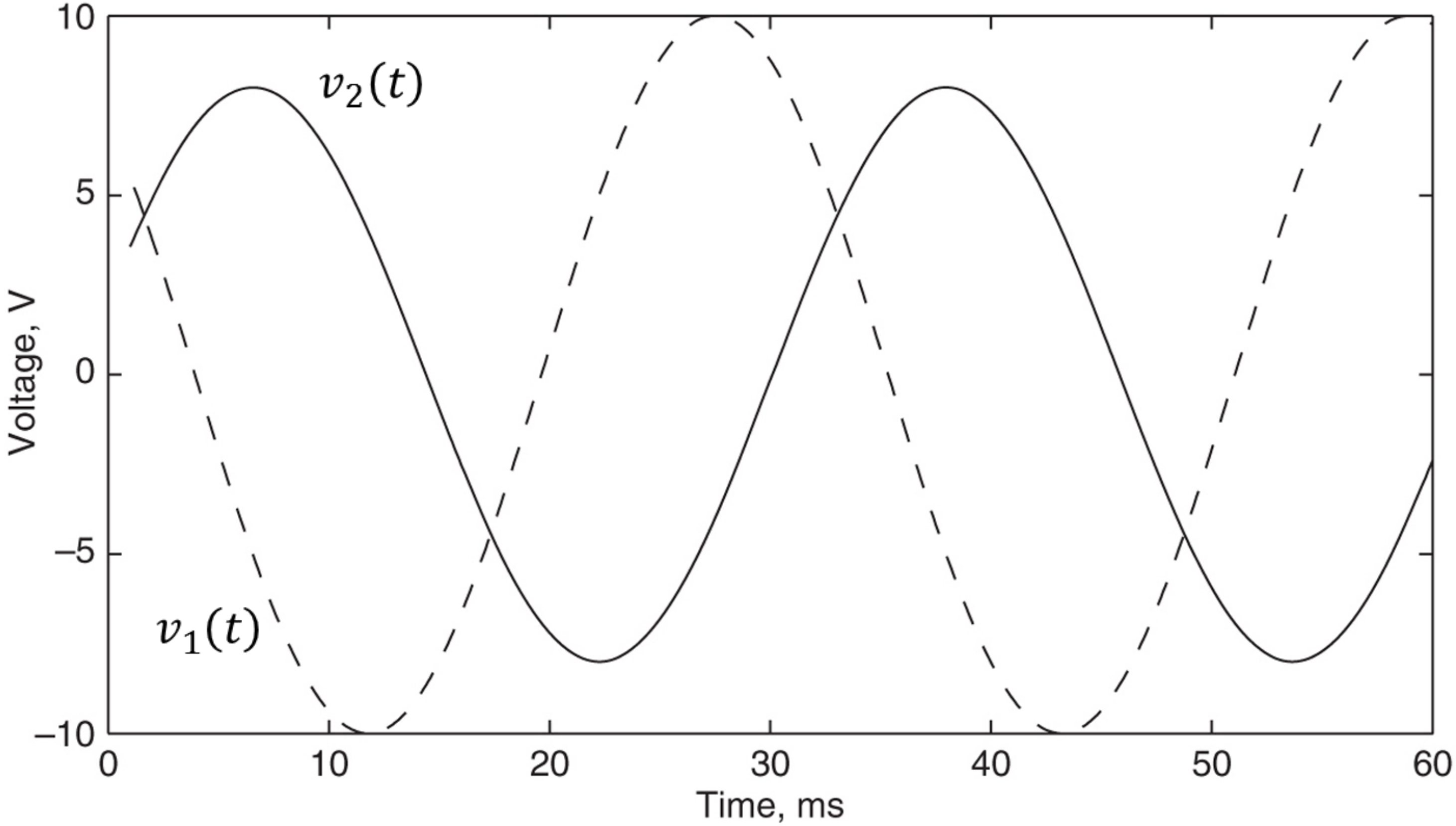
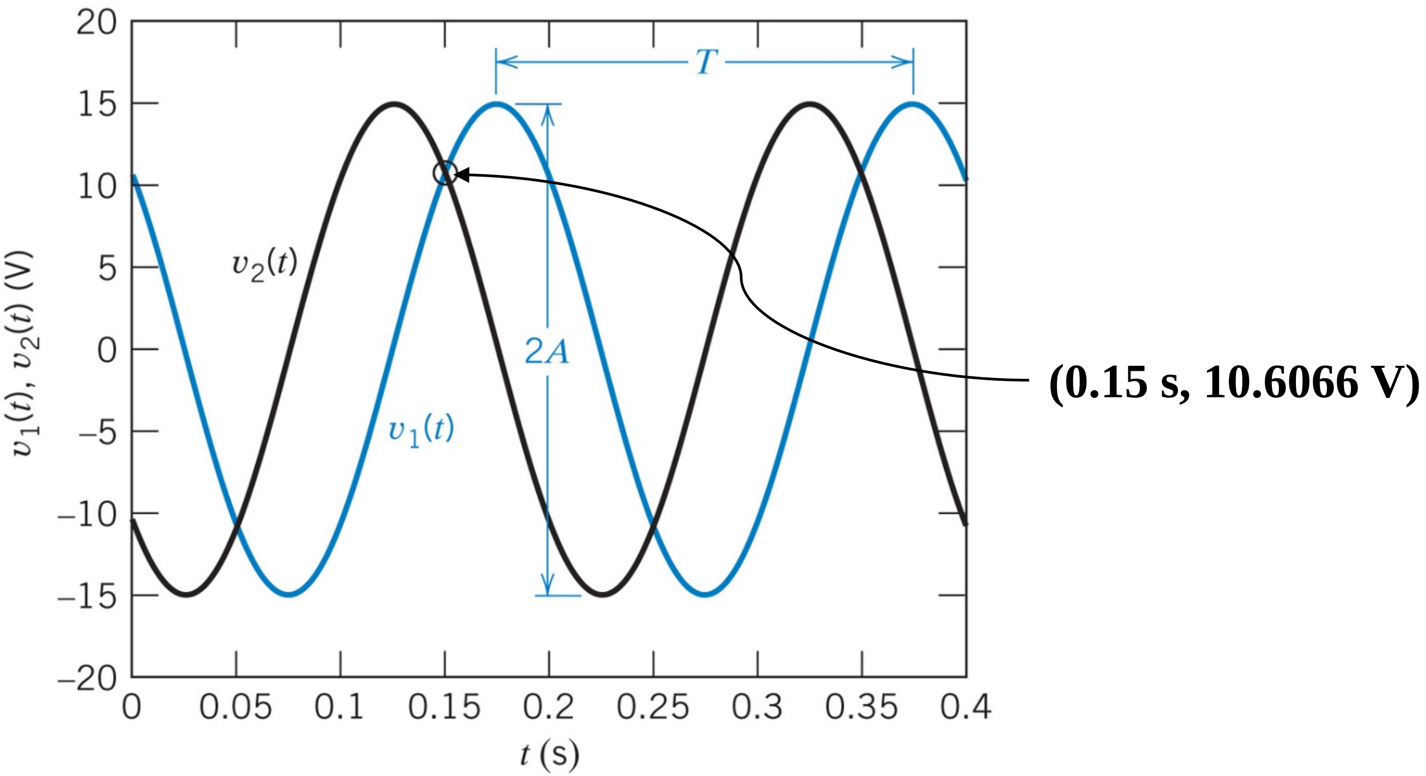


FIGURE 10.2-4

Example 10.2-2 Graphical and Analytic Representation of Sinusoids

- Determine the analytic representations of the sinusoidal volatges $v_1(t)$ and $v_2(t)$ shown in Figure 10.2-5



Solution

$v_1(t)$ and $v_2(t)$ have the same amplitude and period :

$$2A = 30 \;\Rightarrow\; A = 15\,V \text{ and } T = 0.2\,s \;\Rightarrow\; \omega = \frac{2\pi}{0.2} = 10\pi\ rad/s$$

Because $v_1(t_1)$ and $v_2(t_1) = 10.6066\,V$ at $t_1 = 0.15\,s$, and $v_1(t)$ is increasing, phase angle θ_1 is calculated as

$$\theta_1 = -cos^{-1}\left(\frac{v_1(t_1)}{A}\right) - \omega t_1 = -5.498\,rad = -315^\circ = 45^\circ$$

Then $v_1(t)$ is represented as

$$v_1(t) = 15\cos(10\pi t + 45^\circ)\quad V$$

Because $v_2(t)$ is decreasing at time t_1, the phase angle θ_2 of $v_2(t)$ is calculated as

$$\theta_2 = cos^{-1}\left(\frac{v_2(t_1)}{A}\right) - \omega t_1 = -3.927\,rad = -225^\circ = 135^\circ$$

Then $v_1(t)$ is represented as

$$v_2(t) = 15\cos(10\pi t + 135^\circ)\quad V$$

Another Representation : Phasor

- Characteristics of sinusoids with same period are determined by the amplitude and phase angle.

- A phasor is a complex number that is used to represent the amplitude and phase angle of a sinusoid. The relationship between the sinusoid and the phasor is described by

$$A\cos(\omega t + \theta) \quad \leftrightarrow \quad A\angle\theta$$

Exponential Form of Phasor

- Exponential form of phasor

$$A\angle\theta = Ae^{j(\omega t+\theta)}$$

- From Euler's formular,

$$Ae^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + j\,A\sin(\omega t+\theta)$$

Example 10.3-1 Phasor and Sinusoids

- Determine the phasors corresponding to the sinusoids

$$i_1(t) = 120\cos(400t + 60°)\ mA \text{ and } i_2(t) = 100\sin(400t - 75°)\ mA$$

Solution

Using the relationship between the sinusoid and the phasor, we have

$$\mathbf{I}_1(\omega) = 120\angle 60^\circ\, mA$$

For i_2, we have to express this using the cosine instead of sine

$$i_2(t) = 100\, sin(400t - 75^\circ) = 100\, cos(400t - 165^\circ)\, mA$$

Then, we have

$$\mathbf{I}_2(\omega) = 100\angle -165^\circ\, mA$$

Polar and Rectangular Forms of Phasor

- Since phasor is a complex number, it indicates a point on complex plane as we can see in figure 10.3-1.

- Figure 10.3-1 (a) and (b) are showing same phasor in polar and rectangular forms respectively.

- From figure 10.3-1 (a), we have

$$A = |\mathbf{V}| \text{ and } \theta = \angle \mathbf{V}$$

- And comparing this with (b), we have

$$a = \mathrm{Re}\{\mathbf{V}\} \text{ and } b = \mathrm{Im}\{\mathbf{V}\}$$

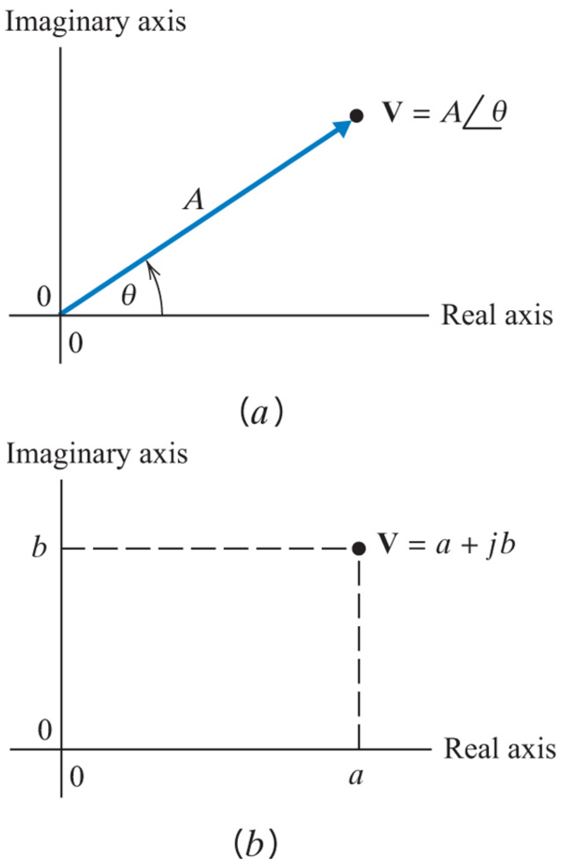


FIGURE 10.3-1

How to Convert Form?

- Since polar and rectangular forms indicate same point,

$$\mathbf{V} = A\angle\theta = a + jb$$

- From figure 10.3-1, we can find

$$a = A\cos(\theta),\ b = A\sin(\theta),\ A = \sqrt{a^2 + b^2}$$

$$\theta = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right) & a > 0 \\ 180^\circ - \tan^{-1}\left(\frac{b}{-a}\right) & a < 0 \end{cases}$$

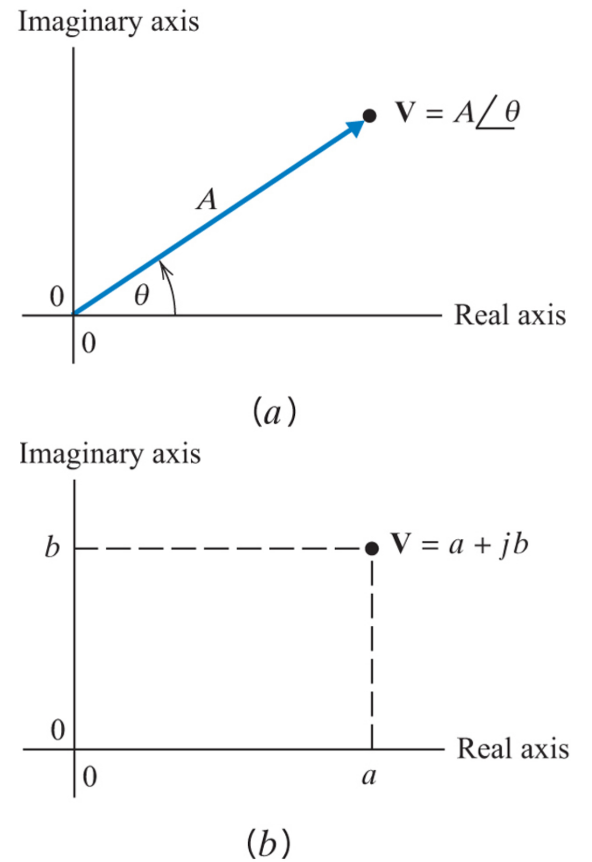


FIGURE 10.3-1

Example 10.3-2 Rectangular and Polar Forms of Phasors

- Consider the phasors

$$\mathbf{V}_1 = 4.25\angle 115° \text{ and } \mathbf{V}_2 = -4 + j3$$

Solution

Using conversion equation

$$\mathbf{V}_1 = Re\{\mathrm{V}_1\} + jIm\{\mathrm{V}_1\} = 4.25\cos(115°) + j4.25\sin(115°) = -1.796 + j3.852$$

$$|\mathbf{V}_2| = |-4 + j3| = \sqrt{(-4)^2 + 3^2} = 5 \qquad \angle\mathbf{V}_2 = 180° - tan^{-1}\left(\frac{3}{-(-4)}\right) = 143°$$

Therefore,

$$\mathbf{V}_1 = -1.796 + j3.852 \;\text{ and }\; \mathbf{V}_2 = 5\angle 143°$$

Arithmetic Operations

- Let's consider doing arithmetic with two arbitrary phasors, $\mathbf{V}_1$ and $\mathbf{V}_2$, each represented in both rectangular and polar forms.

$$\mathbf{V}_1 = a + jb = E\angle\theta \text{ and } \mathbf{V}_2 = c + jd = F\angle\varphi$$

- Then, phasors are added and subtracted using the rectangular forms of the phasors

$$\mathbf{V}_1 + \mathbf{V}_2 = (a + j\,b) + (c + j\,d) = (a + c) + j\,(b + d)$$
$$\mathbf{V}_1 - \mathbf{V}_2 = (a + j\,b) - (c + j\,d) = (a - c) + j\,(b - d)$$

- On the other hands, phasors are multiplied and divided using the polar forms of the phasors

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = \left(E\angle\theta\right)\left(F\angle\phi\right) = EF\angle(\theta + \phi) \text{ and } \frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{A\angle\theta}{B\angle\phi} = \frac{A}{B}\angle(\theta - \phi)$$

- And, the conjugate form of the phasor $\mathbf{V}_1$ is denoted as $\mathbf{V}_1{}^*$ and defined as

$$\mathbf{V}_1{}^* = (a + j\,b)^* = a - j\,b$$
$$= \left(E\angle\theta\right)^* = E\angle -\theta$$

Example 10.3-3 Arithmetic Using Phasors

- Consider the phasors

$$\mathbf{V}_1 = -1.796 + j3.852 = 4.25\angle 115° \text{ and } \mathbf{V}_2 = -4 + j3 = 5\angle 143°$$

Solution

Using rectangular forms of $\mathbf{V}_1$ and $\mathbf{V}_2$,

$$\mathbf{V}_1 - \mathbf{V}_2 = (-1.796 + j3.852) + (-4 + j3) = -5.796 + j6.852$$

Then, with polar forms,

$$\begin{aligned}\mathbf{V}_1 \cdot \mathbf{V}_2 &= (4.25\angle 115°)(5\angle 143°) = (4.25)(5)\angle(115° + 143°) \\ &= 21.25\angle 258° = 21.25\angle -102°\end{aligned}$$

$$\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{4.25\angle 115°}{5\angle 143°} = \frac{4.25}{5}\angle(115° - 143°) = 0.85\angle -28°$$

Euler's Formula

- As we learned at engineering mathematics,
$$\cos\varphi + j\sin\varphi = e^{j\varphi}$$
Consequently, $$A\angle\varphi = A\cos\varphi + j\,A\sin\varphi = Ae^{j\varphi}$$

- $\boldsymbol{Ae^{j\varphi}}$ is called the exponential form of a phasor.
The conversion between the polar and exponential forms is immediate.

- Next, consider
$$A\,e^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + j\,A\sin(\omega t+\theta)$$

Taking the real part of both sides gives,

$$A\cos(\omega t+\theta) = Re\{A\,e^{j(\omega t+\theta)}\} = Re\{A\,e^{j\theta}e^{j\omega t}\}$$

Kirchhoff's Law for AC Circuits

- Then, consider a sinusoid and corresponding phasor,

$$v(t) = A\,cos\,(\omega t + \theta) \quad \text{and} \quad \mathbf{V}(\omega) = A\angle\theta = Ae^{j\theta}$$

Therefore, $$v(t) = \mathrm{Re}\{\,\mathbf{V}(\omega)e^{j\omega t}\,\}$$

- Then consider a KVL from an ac circuit, for example,

$$0 = \sum_i v_i(t) = \sum_i \mathrm{Re}\{\mathbf{V}_i(\omega)e^{j\omega t}\} = Re\left\{e^{j\omega t}\sum_i \mathbf{V}_i(\omega)\right\}$$

- This is required to be true for all values of time t. Let $t = 0$ and $\pi/2$.

$$0 = Re\left\{\sum_i \mathbf{V}_i(\omega)\right\} \quad \text{and} \quad 0 = Re\left\{-j\sum_i \mathbf{V}_i(\omega)\right\} = Im\left\{\sum_i \mathbf{V}_i(\omega)\right\}$$

- Therefore, $$0 = \sum_i \mathbf{V}_i(\omega)$$

Example 10.3-4 Kirchhoff's Laws for AC Circuits

- The input to the circuit shown in Figure 10.3-3 is the voltage source voltage,

$$v_s(t) = 25cos(100t + 15°)\ V$$

$$v_C(t) = 25cos(100t - 22°)\ V$$

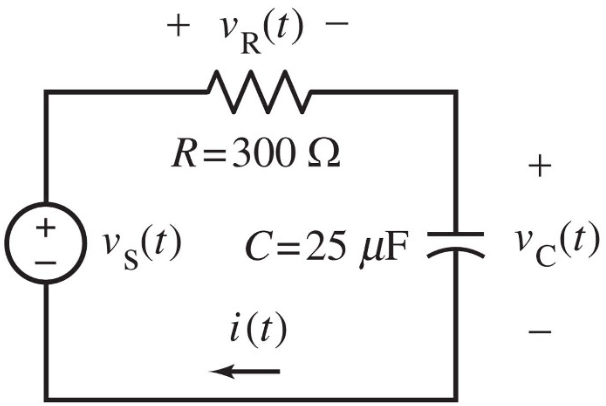


FIGURE 10.3−3

Solution

Apply KVL,

$$v_R(t) = v_s(t) - v_C(t) = 25\cos(100t + 15°) - 20\cos(100t - 22°)$$

Writing this equation using phasor, we have

$$\begin{aligned}\mathbf{V}_R(\omega) = \mathbf{V}_s(\omega) - \mathbf{V}_C(\omega) &= 25\angle 15° - 20\angle -22° \\ &= (24.15 + j6.47) - (18.54 - j7.49) \\ &= 5.61 + j13.96 \\ &= 15\angle 68.1° \quad V\end{aligned}$$

$$\mathbf{V}_R(\omega) = 15\angle 68.1°\ V \quad \leftrightarrow \quad v_R(t) = 15\cos(100t + 68.1°)\ V$$

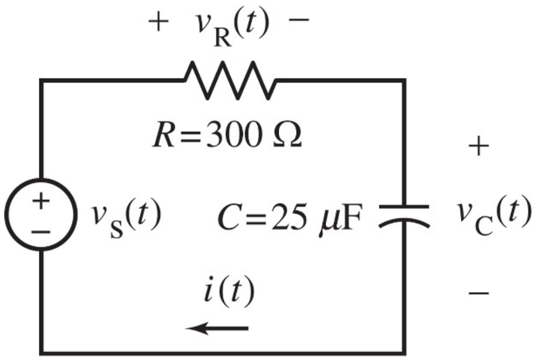


FIGURE 10.3-3

What is Impedance?

- Figure 10.4-1 (a) represents **time domain** and (b) represents **frequency domain**
- Figure 10.4-1(a) shows an element of an ac circuit. We can write

$$v(t) = V_m \cos(\omega t + \theta) \quad \text{and} \quad i(t) = I_m \cos(\omega t + \varphi)$$

The corresponding phasors are

$$\mathbf{V}(\omega) = V_m \angle \theta \quad \text{and} \quad \mathbf{I}(\omega) = I_m \angle \varphi$$

- Then, the impedance is denoted as $\mathbf{Z}(\omega)$ so

$$\mathbf{Z}(\omega) = \frac{\mathbf{V}(\omega)}{\mathbf{I}(\omega)} = \frac{V_m \angle \theta}{I_m \angle \varphi} = \frac{V_m}{I_m} \angle(\theta - \varphi) \ \ [\Omega]$$

- **Ohm's law for ac circuits** satisfy

$$\mathbf{V}(\omega) = \mathbf{Z}(\omega)\mathbf{I}(\omega)$$

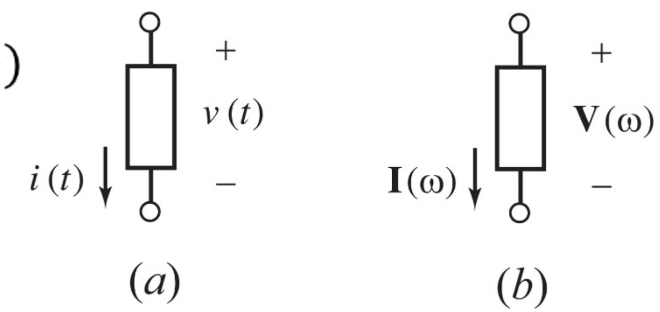


FIGURE 10.4-1

cf) Admittance

- For dc circuits, we used inverse of the resistance as the conductance of an element.
- Of course, there is the ac version of the conductance, which is called admittance.
- Consider same element with previous slide

$v(t) = V_m \cos(\omega t + \theta)$ and $i(t) = I_m \cos(\omega t + \varphi)$

The corresponding phasors are

$$\mathbf{V}(\omega) = V_m \angle\theta \quad \text{and} \quad \mathbf{I}(\omega) = I_m \angle\varphi$$

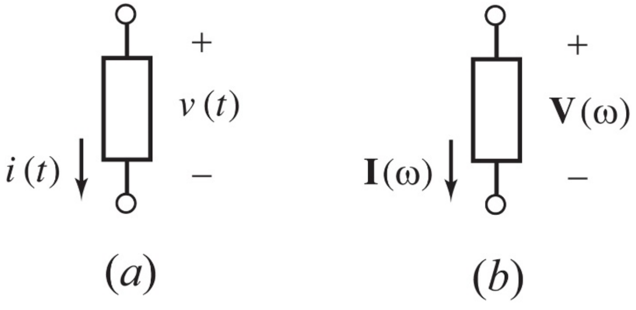


FIGURE 10.4-1

- Then, the admittance is denoted as $\mathbf{Y}(\omega)$ so

$$\mathbf{Y}(\omega) = \frac{\mathbf{I}(\omega)}{\mathbf{V}(\omega)} = \frac{I_m \angle\varphi}{V_m \angle\theta} = \frac{I_m}{V_m} \angle(\varphi - \theta) \quad [S]$$

- Obviously, the admittance is inversion of the impedance

$$\mathbf{Y}(\omega) = \frac{1}{\mathbf{Z}(\omega)}$$

Impedance of Capacitor

- Figure 10.4-2 shows an capacitor of an ac circuit. For $v_c(t) = \mathrm{A}\cos(\omega t + \theta)$, as we studied at chapter 7,

$$i_c(t) = C\frac{d}{dt}v_c(t) = -\omega CA\sin(\omega t + \theta)) = \omega CA\cos(\omega t + \theta + 90°)$$

The corresponding phasors are

$$\mathbf{V}_c(\omega) = A\angle\theta \text{ and } \mathbf{I}_C(\omega) = \omega CA\angle(\theta + 90°) = (\omega C\angle 90°)(A\angle\theta) = j\omega CA\angle\theta$$

Therefore,

$$\boldsymbol{Z}_c(\omega) = \frac{\mathbf{V}_C(\omega)}{\mathbf{I}_C(\omega)} = \frac{A\angle\theta}{j\omega CA\angle\theta} = \frac{1}{j\omega C}$$

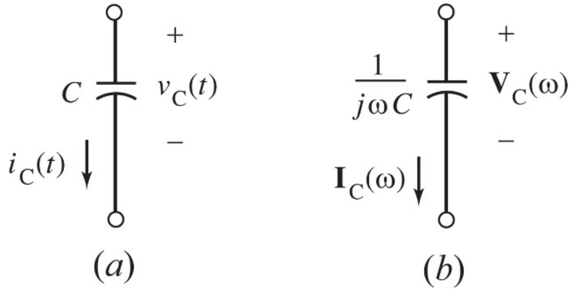


FIGURE 10.4−2

Impedance of Inductor

- Figure 10.4-3 shows an inductor of an ac circuit. For $i_L(t) = \text{A}\cos(\omega t + \theta)$, as we studied at chapter 7,

$$v_L(t) = L\frac{d}{dt}i_L(t) = -\omega LA\sin(\omega t + \theta)) = \omega LA\cos(\omega t + \theta + 90°)$$

The corresponding phasors are

$$\mathbf{I}_L(\omega) = A\angle\theta \text{ and } \mathbf{V}_L(\omega) = \omega LA\angle(\theta + 90°) = (\omega L\angle 90°)(A\angle\theta) = j\omega LA\angle\theta$$

Therefore,

$$\boldsymbol{Z}_L(\omega) = \frac{\mathbf{V}_L(\omega)}{\mathbf{I}_L(\omega)} = \frac{j\omega LA\angle\theta}{A\angle\theta} = j\omega L$$

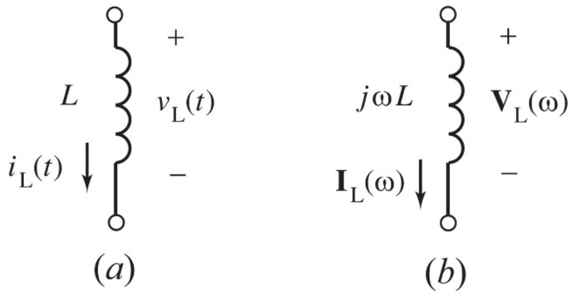


FIGURE 10.4−3

Impedance of Elements

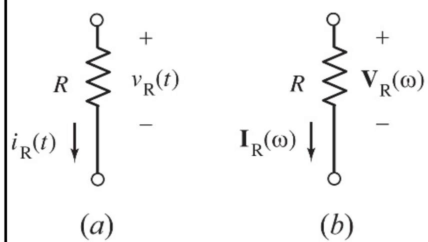
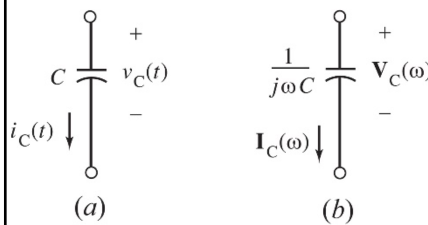
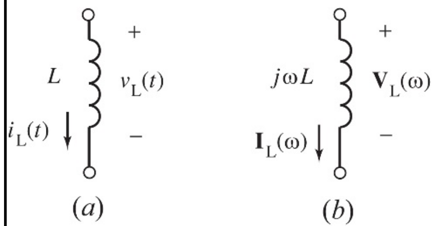
Element	Resistor	Capacitor	Inductor
	 FIGURE 10.4−4	 FIGURE 10.4−2	 FIGURE 10.4−3
Impedance [Ω]	R	$\dfrac{1}{j\omega C}$	$j\omega L$

CHART 10.4−1

Example 10.4-1 Impedances

- The input to the ac circuit shown in Figure 10.4-5 is the source voltage

$$v_s(t) = 12cos(1000t + 15°)\ V$$

Determine (a) the impedances of the capacitor, inductor, and resistance and (b) the current $i(t)$

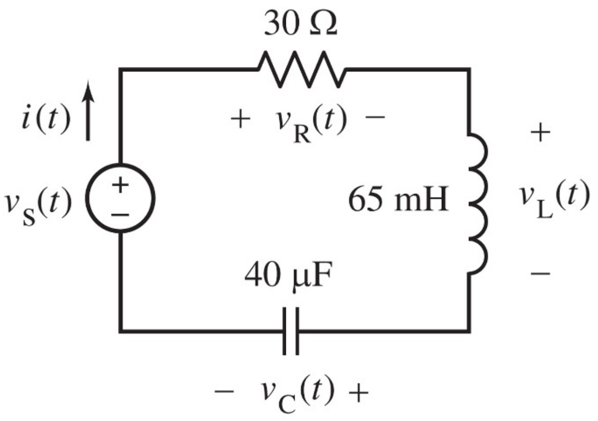


FIGURE 10.4−5

Solution

(a) input frequency is $\omega = 1000\ rad/s$. From the chart 10.4-1, impedances of each elements are

$$\boldsymbol{Z}_R(\omega) = 30\ \Omega,\ \boldsymbol{Z}_L(\omega) = j\omega L = j65\ \Omega,\ \text{and}\ \boldsymbol{Z}_C(\omega) = \frac{1}{j\omega C} = -j25\ \Omega$$

(b) applying KVL using phasor,

$$\mathbf{V}_s(\omega) = \mathbf{V}_R(\omega) + \mathbf{V}_L(\omega) + \mathbf{V}_C(\omega) = \mathbf{I}(\omega)\cdot\boldsymbol{Z}_R(\omega) + \mathbf{I}(\omega)\cdot\boldsymbol{Z}_L(\omega) + \mathbf{I}(\omega)\cdot\boldsymbol{Z}_C(\omega)$$

$$= \mathbf{I}(\omega)\cdot\{\boldsymbol{Z}_R(\omega) + \boldsymbol{Z}_L(\omega) + \boldsymbol{Z}_C(\omega)\}$$

$$12\angle 15° = \mathbf{I}(\omega)\cdot\{30\angle 0° + 65\angle 90° + 25\angle -90°$$

$$= \mathbf{I}(\omega)\cdot 50\angle 53.13°$$

$$\mathbf{I}(\omega) = \frac{12\angle 15°}{50\angle 53.13°} = 0.24\angle -38.13°$$

$$i(t) = 0.24\cos(1000t - 38.13°)$$

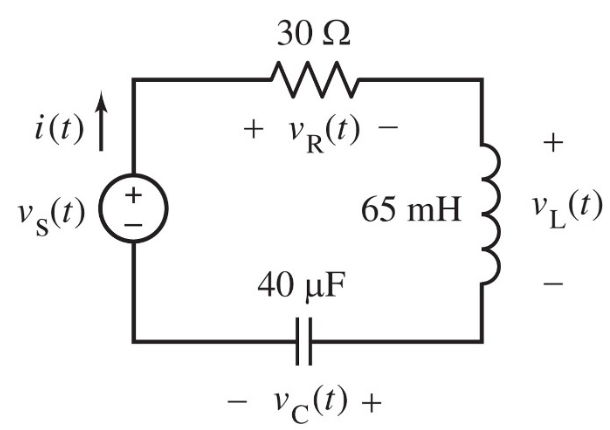


FIGURE 10.4−5

Example 10.4-2 AC Circuits in the Frequency Domain

- The input to the ac circuit shown in Figure 10.4-7 is the source voltage

$$v_s(t) = 48\cos(500t + 75°)\ V$$

Determine the voltage $v(t)$.

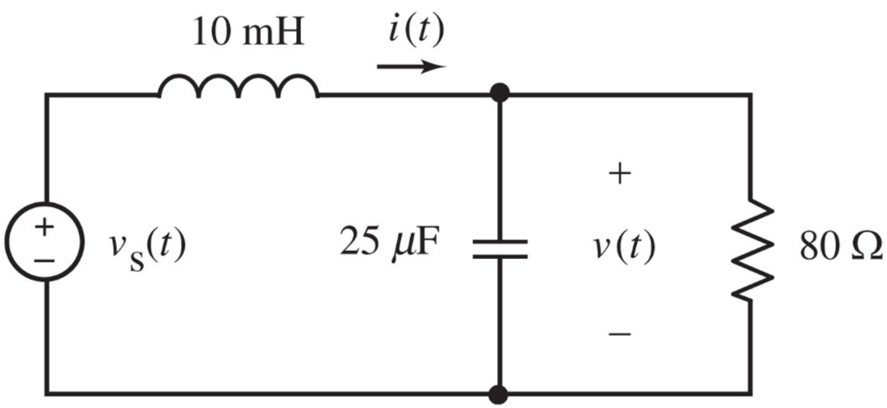


FIGURE 10.4−7

Solution (1/2)

The input frequency is $\omega = 500\ rad/s$. From the chart 10.4-1, impedances of each elements are

$$\boldsymbol{Z}_R(\omega) = 80\ \Omega,\ \boldsymbol{Z}_L(\omega) = j\omega L = j5\ \Omega,\ \text{and}\ \boldsymbol{Z}_C(\omega) = \frac{1}{j\omega C} = -j80\ \Omega$$

Figure 10.4-8 represents same circuit in frequency domain. By applying Ohm's law to each of the impedances,

$$\mathbf{V}_L(\omega) = \mathbf{I}(\omega)\cdot 5\angle 90° \text{ and } \mathbf{V}(\omega) = \mathbf{I}_C(\omega)\cdot 80\angle -90° = \mathbf{I}_R(\omega)\cdot 80\angle 0°$$

By applying KCL at node a, we have

$$\mathbf{I}(\omega) = \mathbf{I}_C(\omega) + \mathbf{I}_R(\omega)$$

$$\Rightarrow \frac{48\angle 75° - \mathbf{V}(\omega)}{5\angle 90°} = \frac{\mathbf{V}(\omega)}{80\angle -90°} + \frac{\mathbf{V}(\omega)}{80\angle 0°}$$

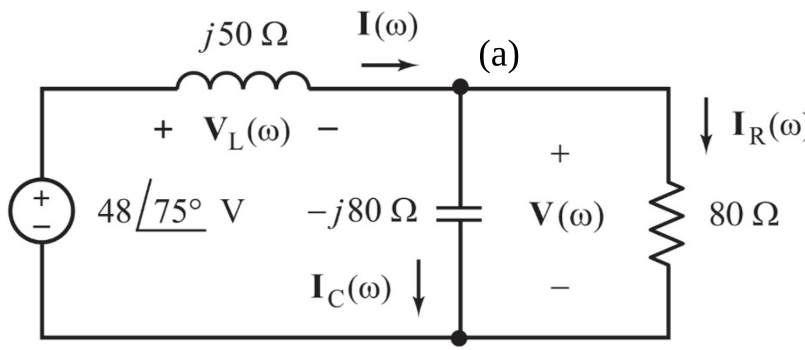


FIGURE 10.4−8

Solution (2/2)

From the equation, we have

$$\frac{48\angle 75°}{5\angle 90°} = \mathbf{V}(\omega)\left\{\frac{1}{5\angle 90°} + \frac{1}{80\angle -90°} + \frac{1}{80\angle 0°}\right\}$$

$$\Rightarrow 9.6\angle 15° = \mathbf{V}(\omega)\left\{\frac{1}{j5} + \frac{1}{-j80} + \frac{1}{80}\right\} = \mathbf{V}(\omega)\cdot\frac{-16j + j + 1}{80} = \frac{1 - j15}{80}\cdot\mathbf{V}(\omega)$$

$$\Rightarrow \mathbf{V}(\omega) = 80\cdot\frac{9.6\angle 15°}{1 - j15} = 51.1\angle 101.2°$$

Therefore,

$$v(t) = 51.1\cos(500t + 101.2°)\ \ V$$

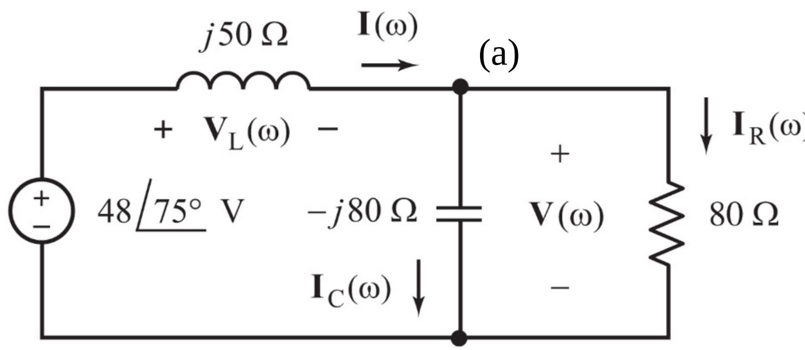


FIGURE 10.4−8

Example 10.4-3 AC Circuits Containing a Dependent Source

- The input to the ac circuit shown in Figure 10.4-10 is the source voltage

$$v_s(t) = 12\cos(1000t + 45°)\ V$$

Determine the voltage $v_o(t)$.

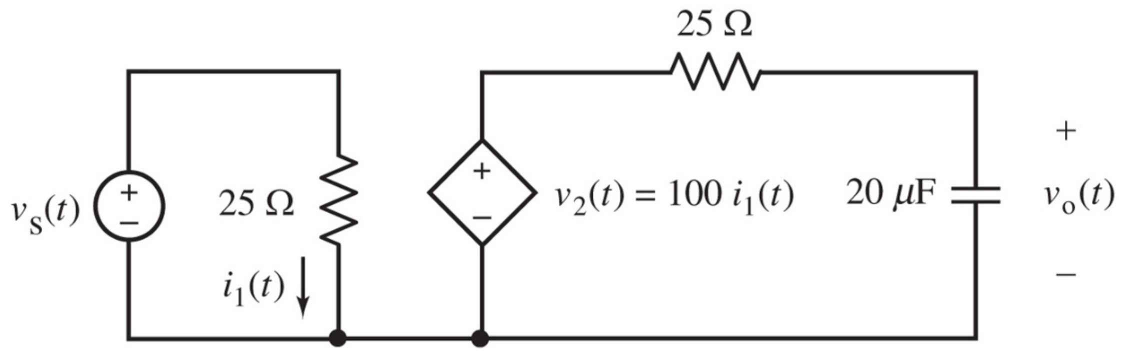


FIGURE 10.4–10

Solution

The input frequency is $\omega = 1000\ rad/s$. Impedance of the capacitor is,

$$\boldsymbol{Z}_C(\omega) = \frac{1}{j\omega C} = -j50\ \Omega$$

Figure 10.4-11 represents same circuit in frequency domain. By applying Ohm's law to each of the impedances, we can find $\mathbf{V}_2(\omega)$ from $\mathbf{I}_1(\omega)$.

$$\mathbf{V}_2(\omega) = 100\mathbf{I}_1(\omega) = 100 \cdot \frac{12\angle 45°}{25} = 48\angle 45°$$

By applying KVL at right mesh, we have

$$\mathbf{V}_2(\omega) - 25\mathbf{I}_2(\omega) = \mathbf{V}_o(\omega) \qquad \Rightarrow 48\angle 45° - 25\frac{\mathbf{V}_o(\omega)}{50\angle -90°} = \mathbf{V}_o(\omega)$$

$$\Rightarrow 48\angle 45° = (1 + j0.5)\mathbf{V}_o(\omega) \qquad \Rightarrow \mathbf{V}_o\ (\omega) = 42.9\angle 18.4°$$

Therefore,

$$v(t) = 42.9\cos(1000t + 18.4°)\ \ V$$

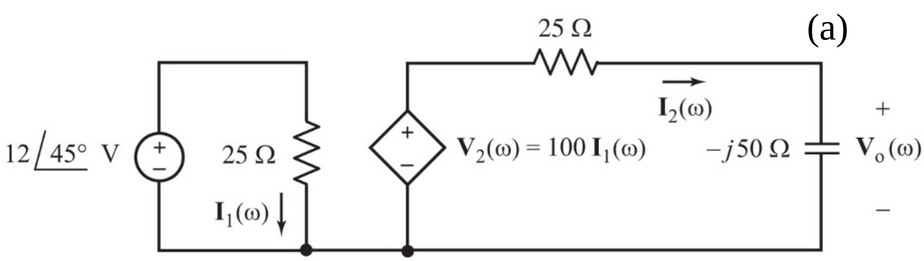


FIGURE 10.4−11

Series Impedances

- Figure 10.5-1 shows two series elements connected to "Circuit A".

  We have $\mathbf{I} = \mathbf{I}_1 = \mathbf{I}_2$ and $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = \mathbf{Z}_1\mathbf{I}_1 + \mathbf{Z}_2\mathbf{I}_2 = (\mathbf{Z}_1 + \mathbf{Z}_2)\,\mathbf{I}$.

- Therefore, $\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2$
- This generalizes to the case of n series impedances

$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n$$

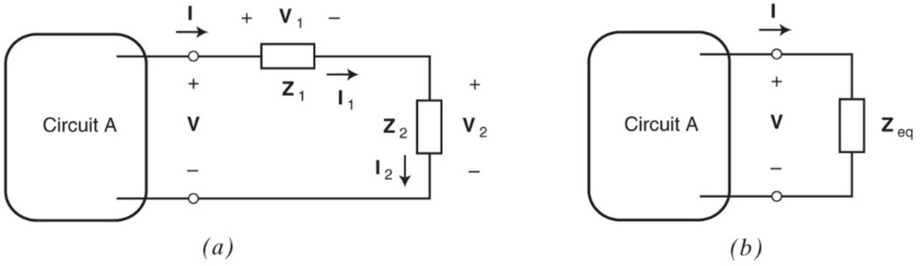


FIGURE 10.5-1

Parallel Impedances

- Figure 10.5-2 shows two parallel elements connected to "Circuit A".

  We have $\mathbf{V} = \mathbf{V}_1 = \mathbf{V}_2$ and $\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = \frac{\mathbf{V}_1}{\mathbf{Z}_1} + \frac{\mathbf{V}_2}{\mathbf{Z}_2} = \left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}\right) \mathbf{I}.$
- Therefore,

$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}}$$

- This generalizes to the case of n series impedances

$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_n}}$$

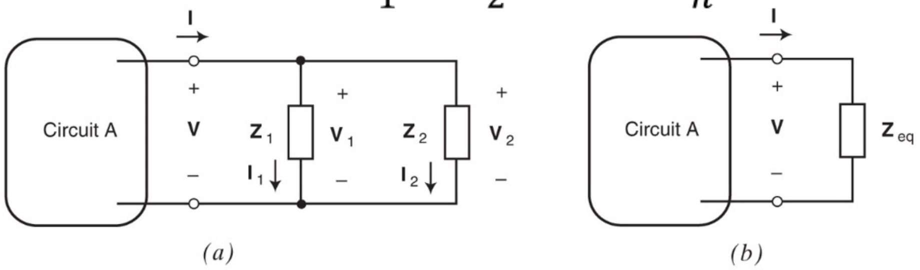


FIGURE 10.5−2

Example 10.5-1 Analysis of AC Circuits Using Impedances

- Determine the steady state current $i(t)$ in the RLC circuit shown in Figure 10.5-3a, using phasors and impedances.

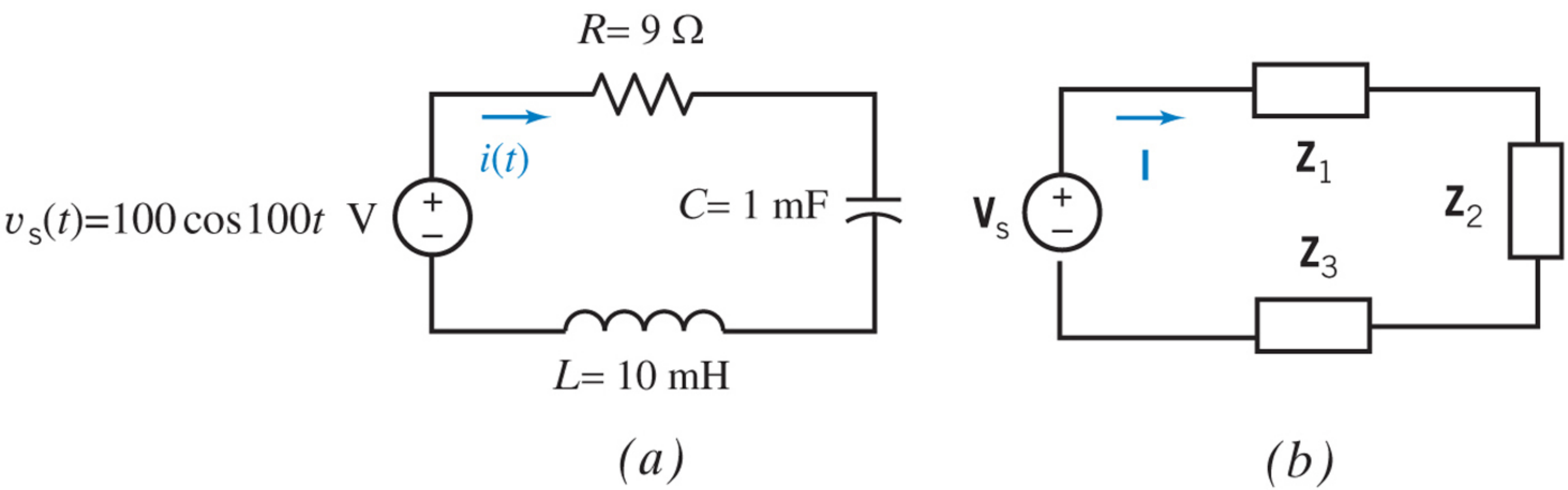


FIGURE 10.5−3

Solution

The input frequency is $\omega = 100\ rad/s$. Let's represent the circuit in using phasors and impedances as shown in (b)

$$\mathbf{Z}_1(\omega) = 9\ \Omega,\ \mathbf{Z}_2(\omega) = \frac{1}{j\omega C} = -j10\ \Omega,\ \text{and}\ \mathbf{Z}_3(\omega) = j\omega L = j1\ \Omega$$

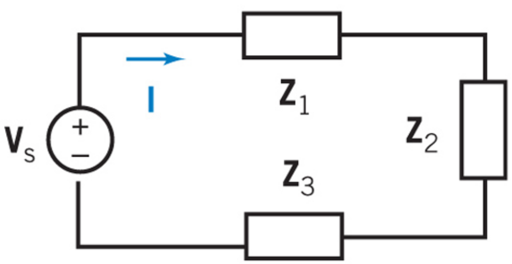
The input phasoir is $\mathbf{V}_s = 100\angle 0°$

Using KVL,

$$\mathbf{V}_s = \mathbf{I}\cdot\left(\mathbf{Z}_1(\omega) + \mathbf{Z}_2(\omega) + \mathbf{Z}_3(\omega)\right)$$
$$\Rightarrow \mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}_1(\omega) + \mathbf{Z}_2(\omega) + \mathbf{Z}_3(\omega)}$$
$$= \frac{100\angle 0°}{9 - j10 + j1} = \frac{100\angle 0°}{9\sqrt{2}\angle -45°}$$
$$= 7.86\angle 45°\ A$$

Therefore,

$$i(t) = 7.86\cos(100t + 45°)\ \ A$$



(b)

FIGURE 10.5−3(b)

Example 10.5-2 Voltage Division Using Impedances

- Consider the circuit shown in Figure 10.5-4a. The input to the circuit is the voltage of the voltage source,

$$v_s(t) = 7.28\cos(4t + 77°)\ V$$

The output is the voltage across the inductor $v_o(t)$. Determine the steady-state output voltage $v_o(t)$.

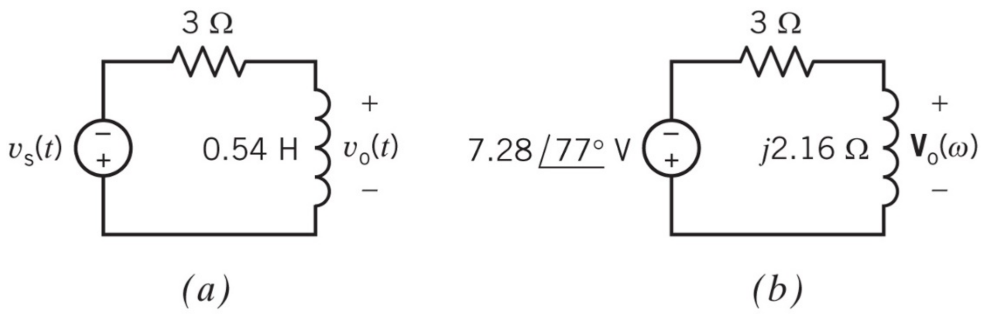


FIGURE 10.5−4

Solution

The input and output are sinusoids of the same frequency. After the circuit has reached steady state, voltage divider is also satisfied at ac circuit.

$$\mathbf{V}_s = 7.28\angle 77°, \quad \omega = 4$$

$$\mathbf{V}_o(\omega) = \frac{\mathbf{Z}_L}{\mathbf{Z}_R + \mathbf{Z}_L}\mathbf{V}_s = \frac{2.16\angle 90°}{3\angle 0° + 2.16\angle 90°}(-7.28\angle 77°) = 4.25\angle 131.25°$$

Therefore,

$$v_o(t) = 4.25\cos(4t + 131.25°) \;\; V$$

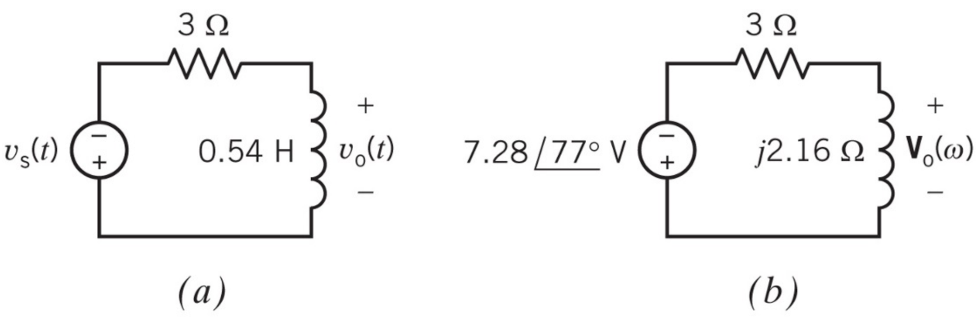


FIGURE 10.5-4

Example 10.5-3 AC Circuit Analysis

- Consider the circuit shown in Figure 10.5-5a. The input to the circuit is the voltage of the voltage source,

$$v_s(t) = 7.68\cos(2t + 47°)\ V$$

The output is the voltage across the resistor,

$$v_o(t) = 1.59\cos(2t + 125°)\ V$$

Determine capacitance C of the capacitor.

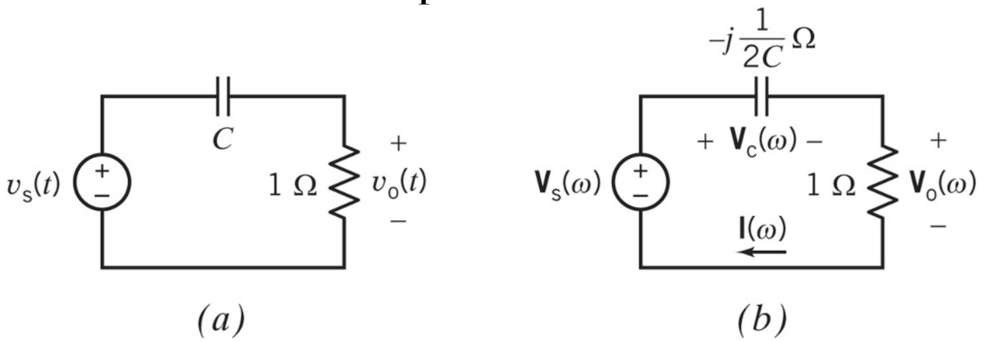


FIGURE 10.5-5

Solution

The input and output are sinusoids of the same frequency. After the circuit has reached steady state, voltage divider is also satisfied at ac circuit.

$$\mathbf{V}_s = 7.68\angle 47°, \mathbf{V}_o = 1.59\angle 125°, \quad \omega = 2$$

$$\mathbf{V}_o = \frac{\mathbf{Z}_R}{\mathbf{Z}_R + \mathbf{Z}_C}\mathbf{V}_s \Rightarrow \mathbf{Z}_C = \frac{\mathbf{V}_s}{\mathbf{V}_o}\mathbf{Z}_R - \mathbf{Z}_R = \left(\frac{\mathbf{V}_s}{\mathbf{V}_o} - 1\right)\mathbf{Z}_R$$

$$\mathbf{Z}_C = \left(\frac{7.68\angle 47°}{1.59\angle 125°} - 1\right)1\angle 0° = 4.27\angle -90° = \frac{1}{j2C} = \left(\frac{1}{C}\angle 0°\right)(0.5\angle -90°)$$

Therefore,

$$C = 0.106 \;\; F$$

Example 10.5-4 AC Circuit Analysis

- Consider the circuit shown in Figure 10.5-6a. The input to the circuit is the voltage of the voltage source $v_s(t)$, and the output is the voltage across the 4-Ω resistor, $v_o(t)$. When the input is $v_s(t) = 8.93\cos(2t + 54°)\ V$, the corresponding output is $v_o(t) = 3.83\cos(2t + 83°)\ V$. Determine the voltage across the 9-Ω resistor $v_a(t)$ and the value of the capacitance C of the capacitor.

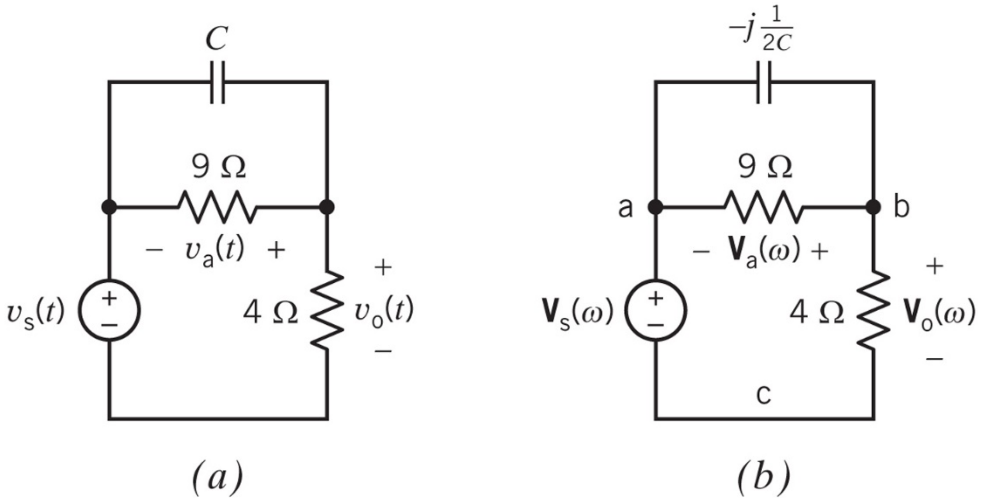


FIGURE 10.5−6

Solution (1/2)

The input and output are sinusoids of the same frequency. After the circuit has reached steady state, voltage divider is also satisfied at ac circuit.

$$\mathbf{V}_s(\omega) = 8.93\angle 54°, \mathbf{V}_o(\omega) = 3.83\angle 83°, \quad \omega = 2$$

$$\mathbf{V}_a(\omega) = \mathbf{V}_o(\omega) - \mathbf{V}_s(\omega) = 3.83\angle 83° - 8.93\angle 54° = (0.47 + j3.80) - (5.25 + j7.22)$$
$$= -4.78 - j3.42 = 5.88\angle 216°$$

Therefore, the voltage across the 9-Ω resistor $v_a(t)$ is given

$$v_a(t) = 5.88\cos(2t + 216°) \quad V$$

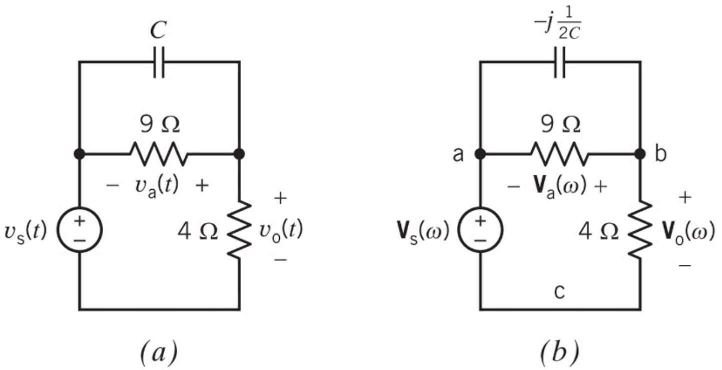


FIGURE 10.5−6

Solution (2/2)

Then, applying KCL at node b, we have

$$\mathbf{I}_{9-\Omega} + \mathbf{I}_C = \mathbf{I}_{4-\Omega} \Rightarrow \frac{-\mathbf{V}_a(\omega)}{\mathbf{Z}_{9-\Omega}} + \frac{-\mathbf{V}_a(\omega)}{\mathbf{Z}_C} = \frac{\mathbf{V}_o(\omega)}{\mathbf{Z}_{4-\Omega}} \Rightarrow -\mathbf{V}_a(\omega)\left\{\frac{1}{\mathbf{Z}_{9-\Omega}} + \frac{1}{\mathbf{Z}_C}\right\} = \frac{\mathbf{V}_o(\omega)}{\mathbf{Z}_{4-\Omega}}$$

$$-5.88\angle 216°\left\{\frac{1}{9\angle 0°} + \frac{1}{-j\,\frac{1}{2C}}\right\} = 5.88\angle 36°\left(\frac{1}{9} + j2C\right) = \frac{3.83\angle 83°}{4\angle 0°}$$

$$\Rightarrow \frac{1}{9} + 2Cj = \frac{0.958\angle 83°}{5.88\angle 36°} = 0.163\angle 47°$$

$$\Rightarrow C = \frac{0.163\angle 47° - \frac{1}{9}}{2j} = 0.06 \;\; F$$

Example 10.5-5 Equivalent Impedance

- Determine the equivalent impedance of circuit shown in Figure 10.5-7a at the frequency $\omega = 1000\ rad/s$.

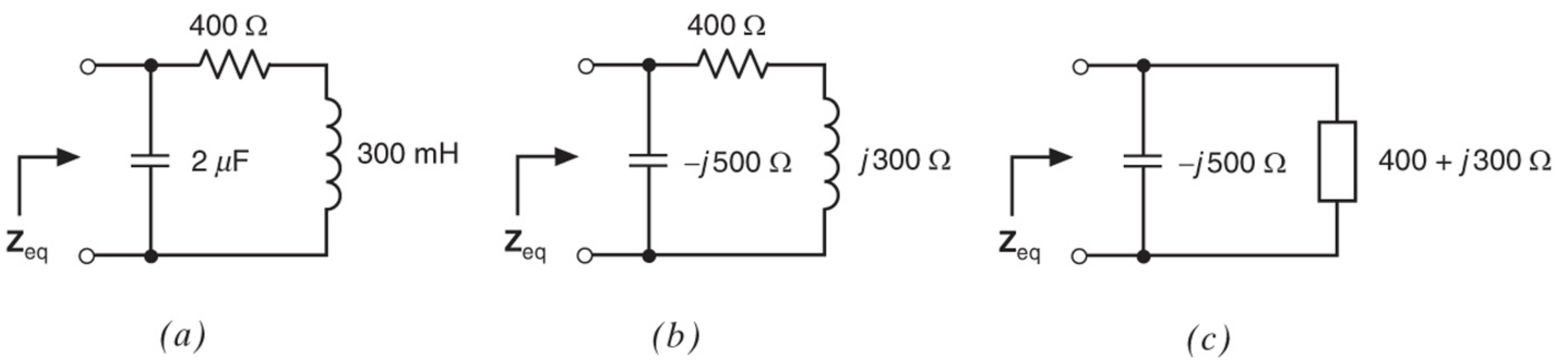


FIGURE 10.5−7

Solution

Represent the circuit in the frequency domain shown in Figure 10.5-7(b). Then replace series impedance by an equivalent impedance, we have the circuit shown in Figure 10.5-7(c). Then find the equivalent impedance of parallel impedances.

$$\mathbf{Z}_{eq} = \mathbf{Z}_C || (\mathbf{Z}_R + \mathbf{Z}_L) = \frac{\mathbf{Z}_C(\mathbf{Z}_R + \mathbf{Z}_L)}{\mathbf{Z}_C + (\mathbf{Z}_R + \mathbf{Z}_L)}$$

Since $\omega = 1000\, rad/s$,

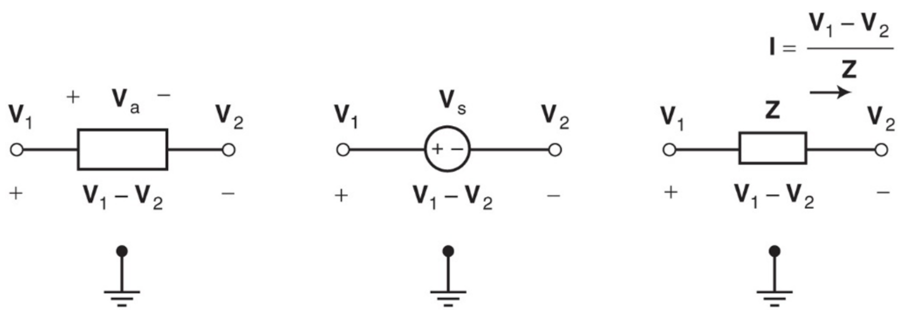
$$\mathbf{Z}_R = 400\ \ \Omega, \mathbf{Z}_L = j\omega L = j300\ \ \Omega, \text{and } \mathbf{Z}_C = \frac{1}{j\omega C} = -j500\ \ \Omega$$

Therefore,

$$\mathbf{Z}_{eq} = \frac{-j500(400 + j300)}{-j500 + (400 + j300)} = 559.0\angle - 26.5°\Omega$$

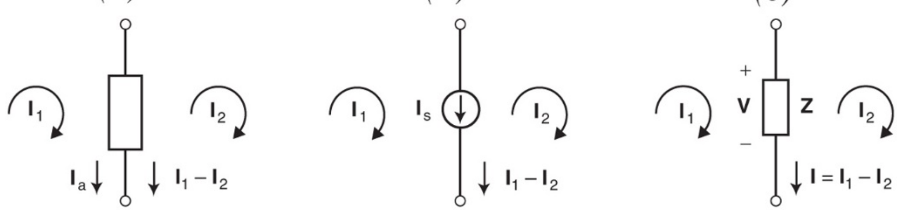
Mesh and Node Equations

- Ohm's and Kirchhoff's laws can be used to an ac circuits as well as dc circuits.
- Therefore, mesh and node equations are available with ac circuits as following convention



(a) (b) (c)

$$\mathbf{I} = \frac{\mathbf{V}_1 - \mathbf{V}_2}{\mathbf{Z}}$$



(a) (b) (c)

$$\mathbf{V} = \mathbf{Z}(\mathbf{I}_1 - \mathbf{I}_2)$$

Example 10.6-2 Mesh Equations for AC Circuits

- Determine the mesh currents for the circuit below.

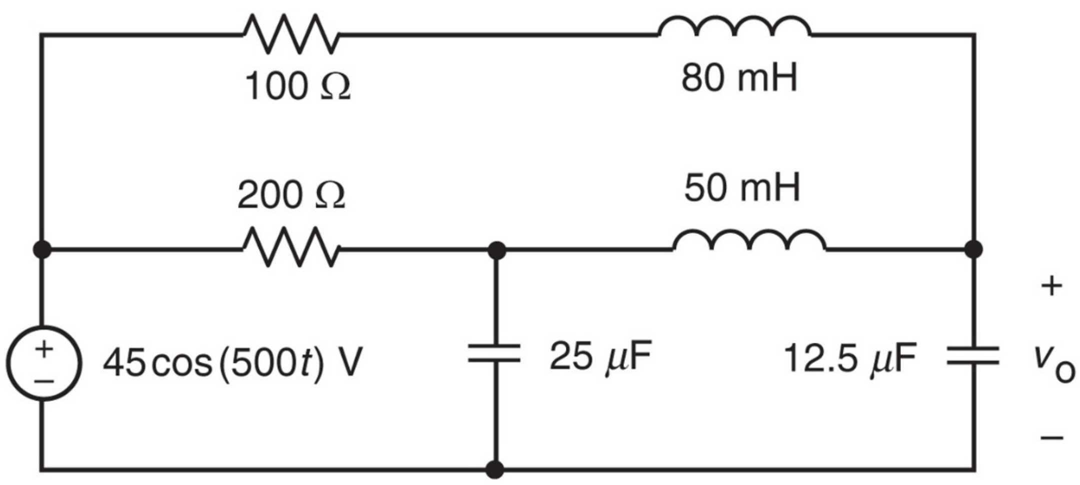


FIGURE 10.6–7

Solution

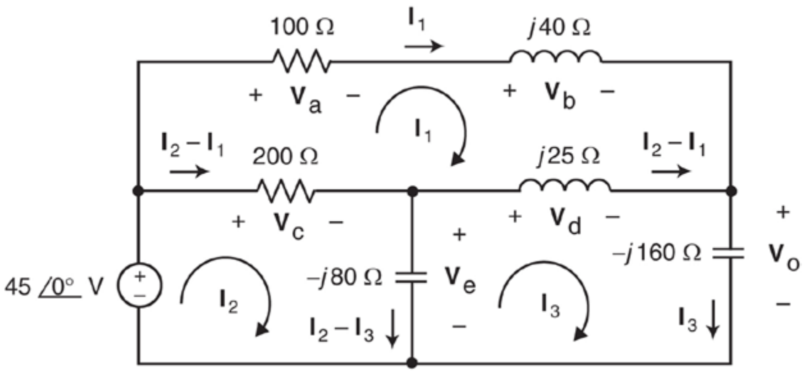
- Represent the circuit in frequency domain.

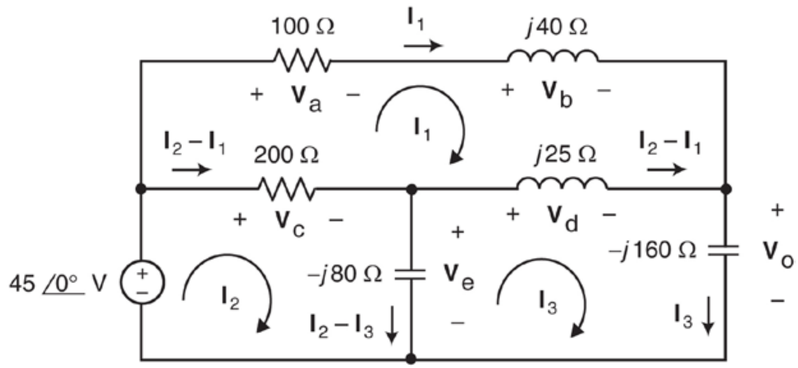
$$\boldsymbol{Z_C} = \frac{1}{j\omega C} \qquad \boldsymbol{Z_L} = j\omega L$$

- Apply KVL to each mesh.

$$100\boldsymbol{I_1} + j40\boldsymbol{I_1} - j25(\boldsymbol{I_2} - \boldsymbol{I_1}) = 0$$
$$200(\boldsymbol{I_2} - \boldsymbol{I_1}) - j80(\boldsymbol{I_2} - \boldsymbol{I_3}) - 45\angle 0 = 0$$
$$-j25(\boldsymbol{I_2} - \boldsymbol{I_1}) - j160\boldsymbol{I_3} + j80(\boldsymbol{I_1} - \boldsymbol{I_3}) = 0$$

$$\boldsymbol{I_1} = 0.374\angle 115\ [A]$$
$$\boldsymbol{I_2} = 0.575\angle 25\ [A]$$
$$\boldsymbol{I_3} = 0.171\angle 28\ [A]$$





Example 10.6-3 Node Equations for AC Circuits with a Supernode

- The input to the ac circuit shown in Figure 10.6-10 is the voltage source voltage

$$v_s(t) = 10\cos(10t) \;\; V$$

The output is the current $i(t)$ in resistor R_1. Determine $i(t)$.

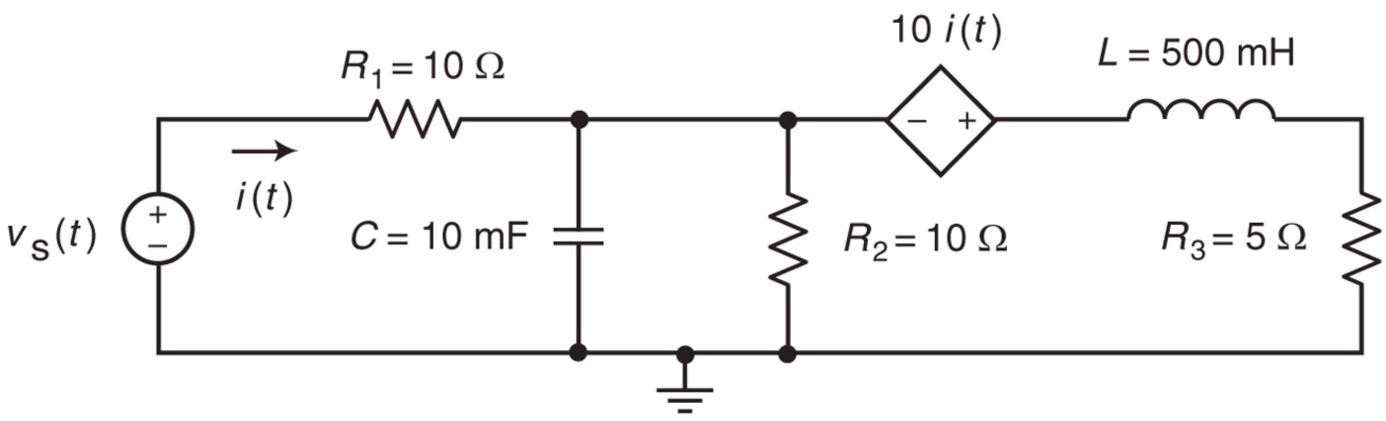


FIGURE 10.6−10

Solution (1/3)

First, we will represent the circuit in the frequency domain using phasors and impedances. The impedances of the capacitor and inductor are

$$\mathbf{Z}_C = -j\frac{1}{10(0.010)} = -j10\ \ \Omega \text{ and } \mathbf{Z}_L = j10(0.5) = j5\ \ \Omega$$

The frequency domain representation of the circuit is shown in Figure 10.6-11

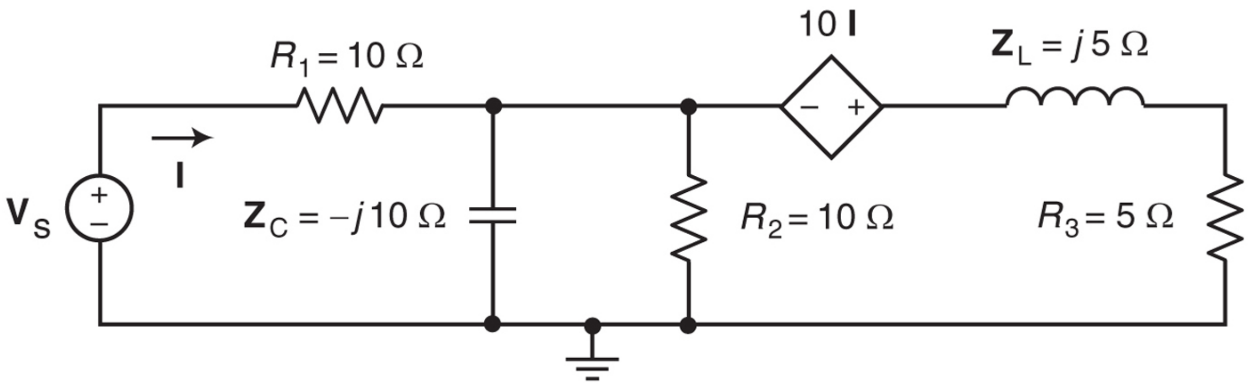


FIGURE 10.6-11

Solution (2/3)

We can analyze this circuit by node equations. To simplify this process, replace parallel and series impedances by equivalent impedances as shown in Figure 10.6-12.

$$\mathbf{Z}_C = -j\frac{1}{10(0.010)} = -j10\ \ \Omega \text{ and } \mathbf{Z}_L = j10(0.5) = j5\ \ \Omega$$

The frequency domain representation of the circuit is shown in Figure 10.6-11

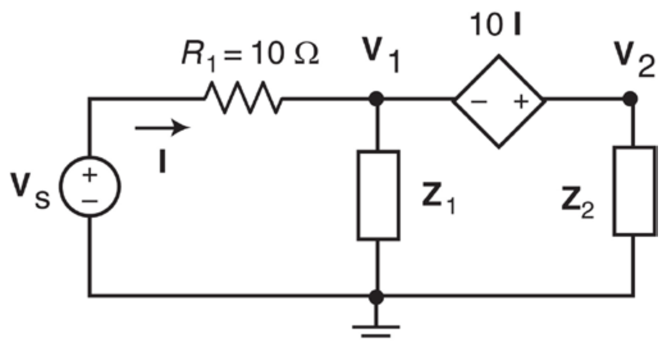


FIGURE 10.6-12

$$\mathbf{Z}_1 = 10||(-j10) = \frac{-j100}{10-j10} = 5 - j5\ \ \Omega \text{ and } \mathbf{Z}_2 = 5 + j5\ \ \Omega$$

Solution (3/3)

Next, consider the dependent source in Figure 10.6-12. We can find **I** and also express the dependent source voltage as

$$\mathbf{I} = \frac{\mathbf{V}_s - \mathbf{V}_1}{R_1} \quad \text{and} \quad 10\mathbf{I} = \mathbf{V}_2 - \mathbf{V}_1$$

Apply KCL to supernode containing dependent source identified in Figure 10.6-12 to get

$$\mathbf{I} = \frac{\mathbf{V}_1}{\mathbf{Z}_1} + \frac{\mathbf{V}_2}{\mathbf{Z}_2} = \frac{\mathbf{V}_1}{\mathbf{Z}_1} + \frac{\mathbf{V}_1 + 10\mathbf{I}}{\mathbf{Z}_2} \quad \Rightarrow \quad (\mathbf{Z}_1 + \mathbf{Z}_2)\mathbf{V}_1 + \mathbf{Z}_1(10 - \mathbf{Z}_2)\mathbf{I} = 0$$

Organizing Eqs. Into matrix form, we get

$$\begin{bmatrix} 1 & R_1 \\ \mathbf{Z}_1 + \mathbf{Z}_2 & \mathbf{Z}_1(10 - \mathbf{Z}_2) \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_s \\ 0 \end{bmatrix}$$

Solving these equations, perhaps using MATLAB, gives

$$\mathbf{V}_1 = 4.4721\angle 63.4^\circ \ V \text{, and } \mathbf{I} = 0.89443\angle -26.6^\circ \ A$$

In the time domain, the output current is

$$i(t) = 0.89443 \cos(10t - 26.6^\circ) \ A$$

Example 10.6-4 AC Circuits Containing OP Amps

- The input to the ac circuit shown in figure below is the voltage source voltage

$$V_s(t) = 125\cos(500t + 15) \quad [mV]$$

Determine the output voltage $v_o(t)$.

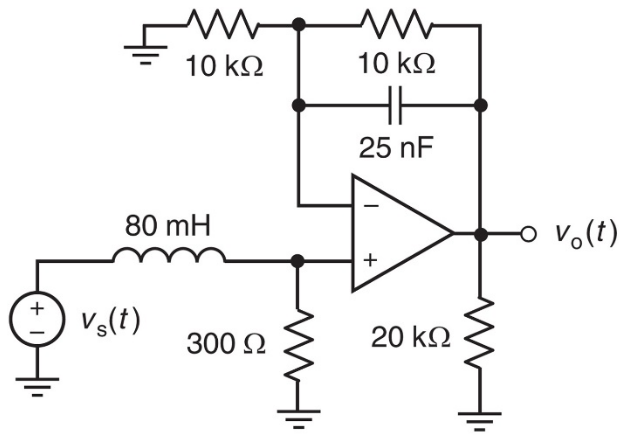
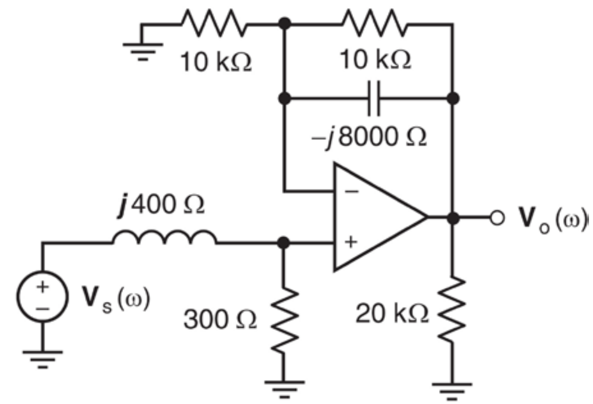


FIGURE 10.6−13

Solution

- Represent the circuit in frequency domain.

$$\mathbf{Z}_C = \frac{1}{j\omega\mathbf{C}} \qquad \mathbf{Z}_L = j\omega\mathbf{L}$$



- Apply KCL at the each node of OP amp

$$\frac{\mathbf{V}_s - \mathbf{V}_a}{j400} = \frac{\mathbf{V}_a}{300} + 0 \;\rightarrow\; \mathbf{V}_s = \mathbf{V_a}\left(1 + \frac{j400}{300}\right) \rightarrow \mathbf{V}_a = \frac{300}{300 + j400}\mathbf{V}_s \cdots (1)$$

$$\frac{\mathbf{V}_a - \mathbf{V}_o}{10000} + \frac{\mathbf{V}_a}{10000} + \frac{\mathbf{V}_a - \mathbf{V}_o}{-j8000} = 0 \cdots (2)$$

- Equation (2) gives

$$\mathbf{V}_o = \left(\frac{16 + j10}{8 + j10}\right)\mathbf{V}_a = \left(\frac{16 + j10}{8 + j10}\right)\left(\frac{300}{300 + j400}\right)\mathbf{V}_s = (0.884\angle - 72.5°)\mathbf{V}_s$$

$$= (0.884\angle - 72.5°)(125\angle 15°) = 111\angle - 57.5°$$

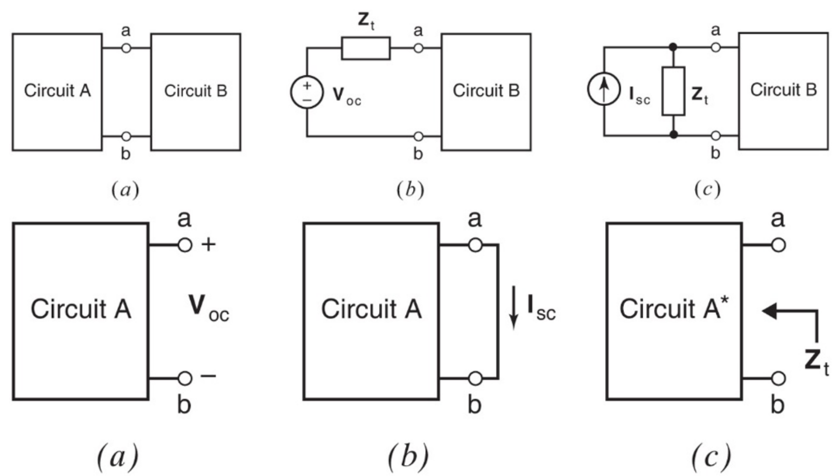
$$v_o(t) = 111cos(500t - 57.5°)[mV]$$

Thévenin and Norton Equivalent Circuit

- Thévenin and Norton equivalent circuits of an ac circuit can be determined by
  the open-circuit voltage $\mathbf{V}_{oc}$,
  the short-circuit current $\mathbf{I}_{sc}$,
  and the Thévenin impedance $\mathbf{Z}_t$.

  They are related by the equation

$$\mathbf{V}_{oc} = \mathbf{Z}_t \mathbf{I}_{sc}$$



Example 10.7-1 Thévenin Equivalent Circuit

- Find the Thévenin equivalent circuit of the ac circuit in Figure shown in Figure 10.7-3.

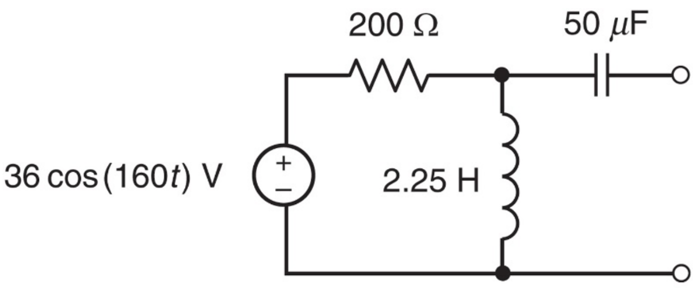


FIGURE 10.7−3

Solution (1/3)

Begin by representing the circuit from Figure 10.7-3 in the frequency domain, using phasors and impedances. As shown in Figure 10.7-4.

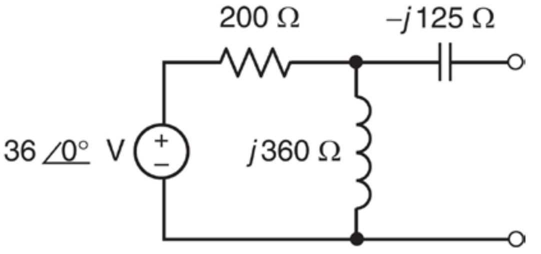


FIGURE 10.7−4

Next, we determine the open-circuit voltage and the Thévenin impedance using circuits shown in Figure 10.7-5.

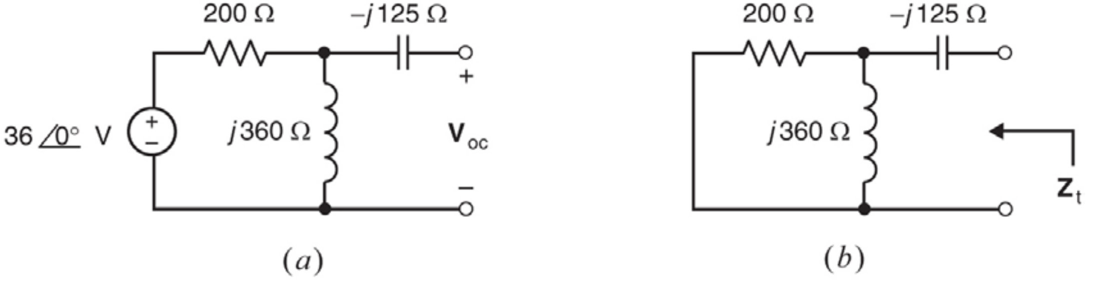


FIGURE 10.7−5

Solution (1/3)

For the open-circuit voltage, using voltage division, we calculate

$$\mathbf{V}_{oc} = \frac{j360}{200 + j360} 36\angle 0° = 31.470\angle 29.1°\ V$$

Next, the Thévenin impedance is determined using circuit shown in Figure 10.7-5 b. Then we calculate

$$\mathbf{Z}_t = -j125 + \frac{200(j360)}{200 + j360} = 152.83 - j40.094 = 158\angle - 14.7°\ \Omega$$

Therefore, the Thévenin equivalent circuit is as below

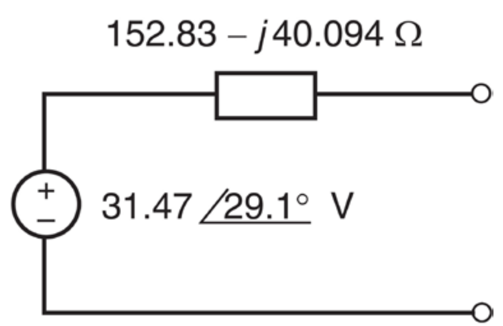


FIGURE 10.7–6

Source Transformation

- Thévenin and Norton equivalent circuits of an ac circuit can be transformed to each other like those of an dc circuit.

$$\mathbf{V}_{\mathrm{oc}} = \mathbf{Z}_{\mathrm{t}} \mathbf{I}_{\mathrm{sc}}$$

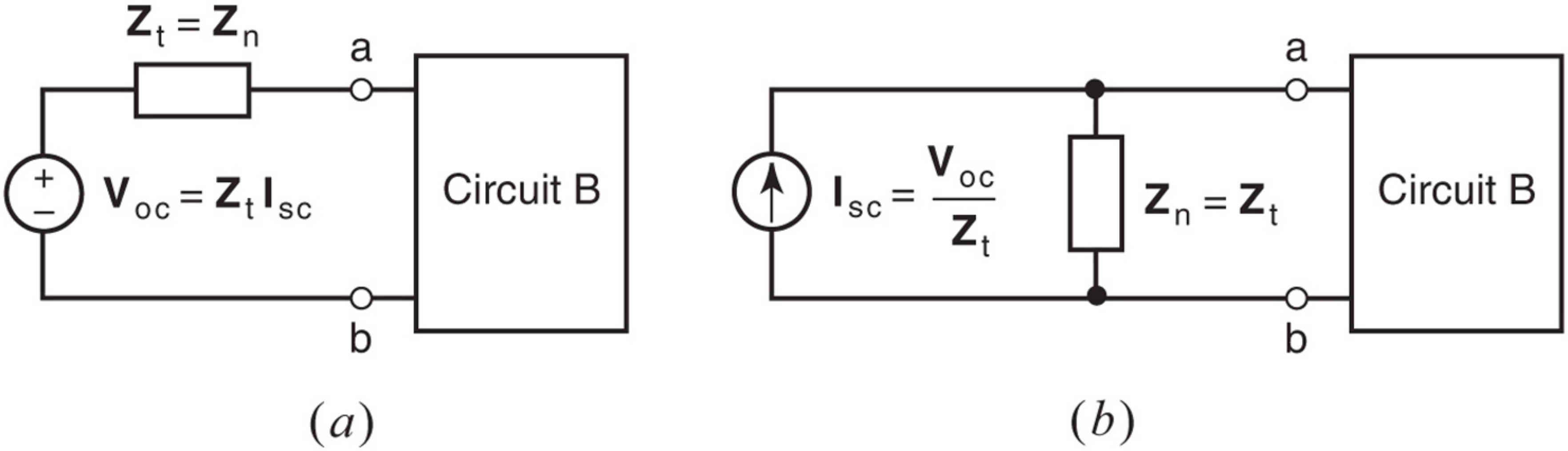


FIGURE 10.7-11

Example 10.7-3 Source Transformation

- Use source transformation and equivalent impedance to find the Thévenin and equivalent circuit of an ac circuit shown in Figure 10.7-3

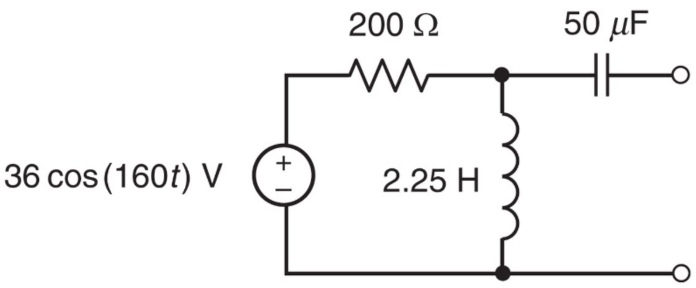
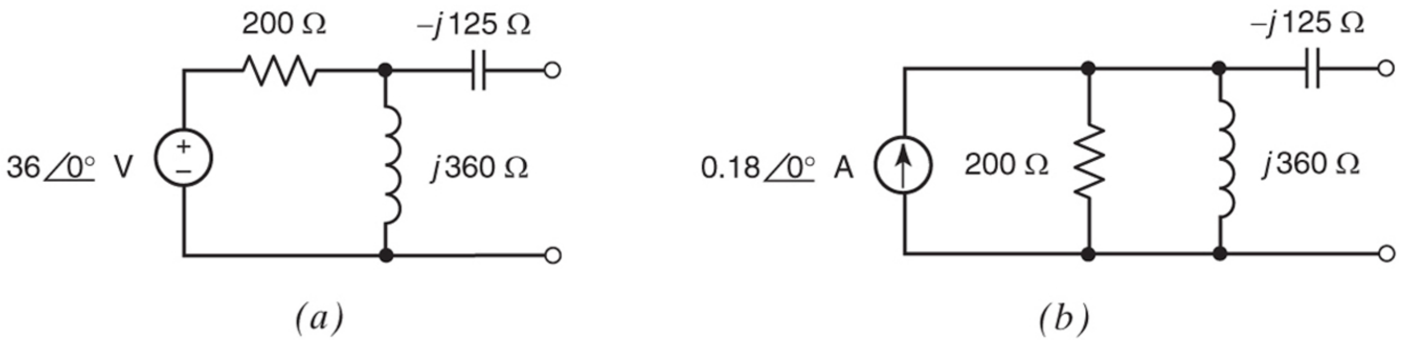


FIGURE 10.7−3

Solution (1/3)

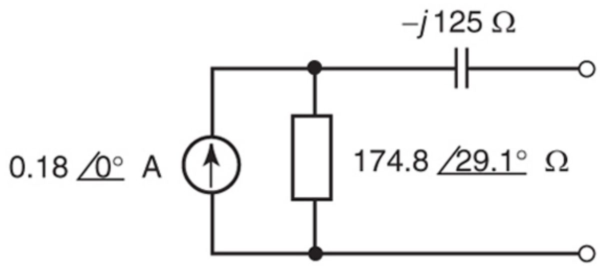
Figure 10.7-12 illustrates the process. Figure 10.7-12a identifies a series combination of a voltage source and impedance. A source transformation replaces this series combination with a parallel combination of a current source and impedance in Figure 10.7-12b.



Then we calculate the equivalent impedance of 200-Ω resistor and j360-Ω inductor to obtain Figure 10.7-12c

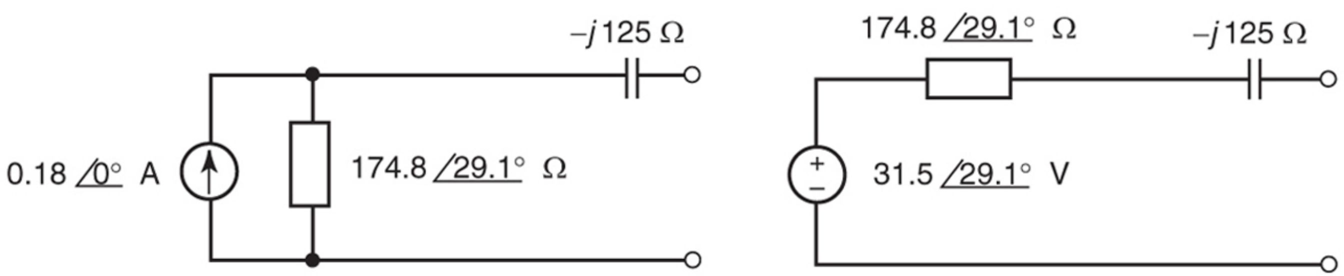
$$200||j360 = \frac{(200)(j360)}{200 + j360} = \frac{72000\angle 90°}{411.8\angle 60.9°} = 174.8\angle 29.1°$$

Solution (2/3)



(c)

Then, Figure 10.7-12 c and d shows a source transformation replacing the parallel combination with a series combination of a voltage source and impedance in Figure 10.7-12d



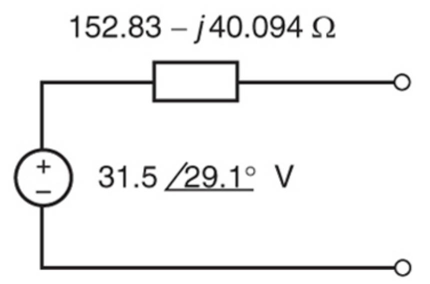
(d) (e)

Solution (3/3)

Then we calculate the equivalent impedance of $174.8\angle 29.1°$-Ω impedance and $-j125$-Ω inductor to obtain Figure 10.7-12f

$$174.8\angle 29.1° - j125 = 152.83 + j84.907 - j125 = 152.83 - j40.094$$

Then, Figure 10.7-12 f shows the Thévenin equivalent circuit of Figure 10.7-3.



(f)

Superposition

- Also, the superposition is available with ac circuits with linear elements.

- As we did with dc circuits, when we set all but one input to zero, the other inputs become 0-V voltage sources(short circuits) and 0-A current sources(open circuits)

Example 10.8-1 Surperposition

- Determine the voltage $v_o(t)$ across the 8-Ω resistor in the circuit shown in Figure 10.8-1.

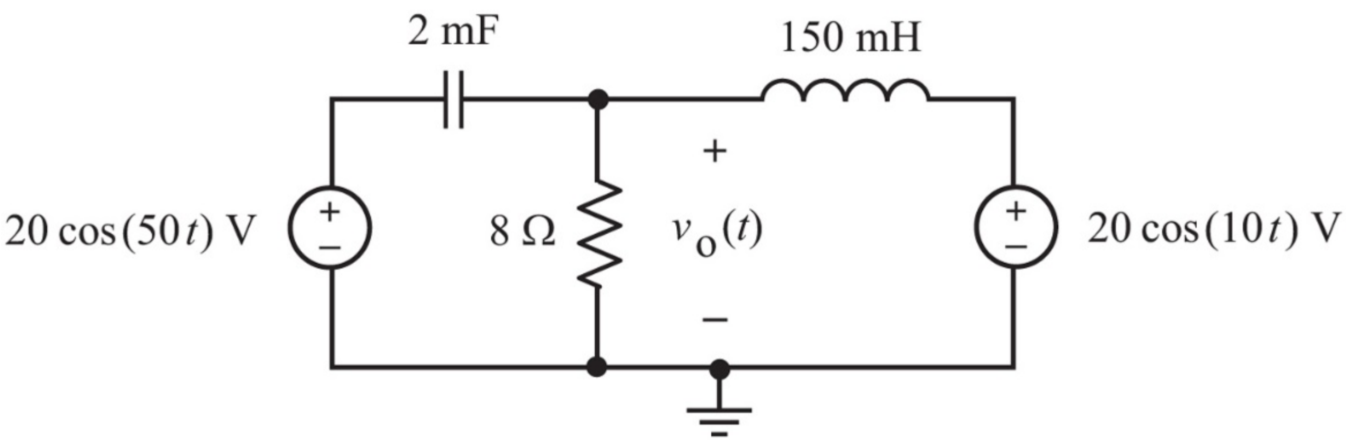


FIGURE 10.8−1

Solution (1/3)

The voltage $v_o(t)$ is caused by two sinusoidal sources, one having a frequency of 50 rad/s and the other having a frequency of 10 rad/s. Let $v_{o1}(t)$ be the part of $v_o(t)$ caused by 50 rad/s source acting alone, and let $v_{o2}(t)$ be the part of $v_o(t)$ by 10 rad/s source acting alone. Figure 10.8-2 shows each circuits used to calculate $v_{o1}(t)$ and $v_{o2}(t)$.

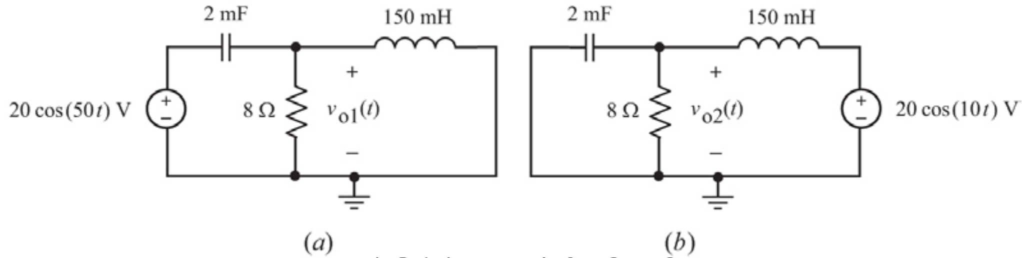


FIGURE 10.8−2

Then the frequency domain representations are shown in Figure 10.8-3.

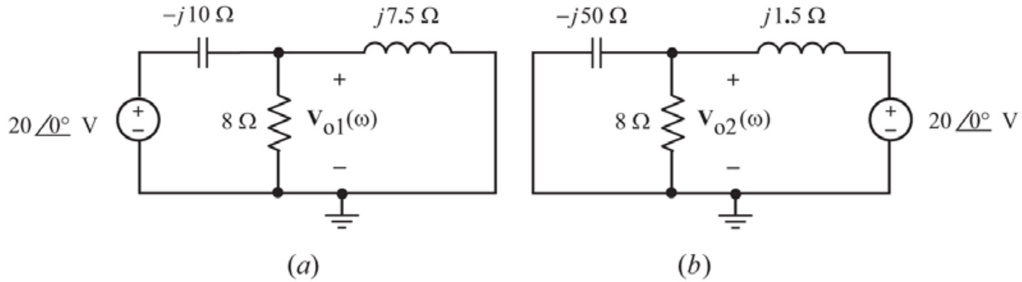


FIGURE 10.8−3

Solution (2/3)

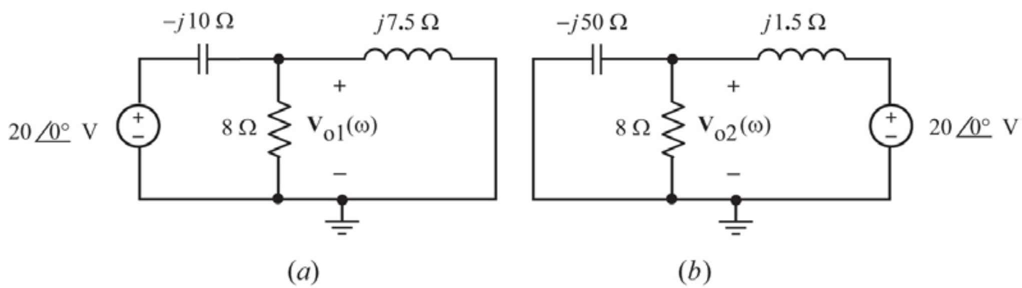


FIGURE 10.8−3

Using equivalent impedance and voltage division in Figure 10.8-3a, we calculate

$$\mathbf{V}_{o1} = \frac{\dfrac{8(j7.5)}{8+j7.5}}{-j10+\dfrac{8(j7.5)}{8+j7.5}}(20\angle 0°) = 15.46\angle 104.9°$$

Similarly in Figure 10.8-3b, we calculate

$$\mathbf{V}_{o2} = \frac{\dfrac{8(-j50)}{8-j50}}{j1.5+\dfrac{8(-j50)}{8-j50}}(20\angle 0°) = 20.24\angle -10.94°$$

The corresponding sinusoids are

$$v_{o1}(t) = 15.46\cos(50t + 104.9°) \text{ and } v_{o2}(t) = 20.24\cos(10t - 10.94°)$$

Solution (3/3)

The response to both sources working together is equal to the sum of the responses to the two sources working separately, therefore

$$v_o(t) = v_{o1}(t) + v_{o2}(t) = 15.46\cos(50t + 104.9°) + 20.24\cos(10t - 10.94°)$$

The output voltage $v_o(t)$ is plotted in Figure 10.8-4. As expected, it is not sinusoid.

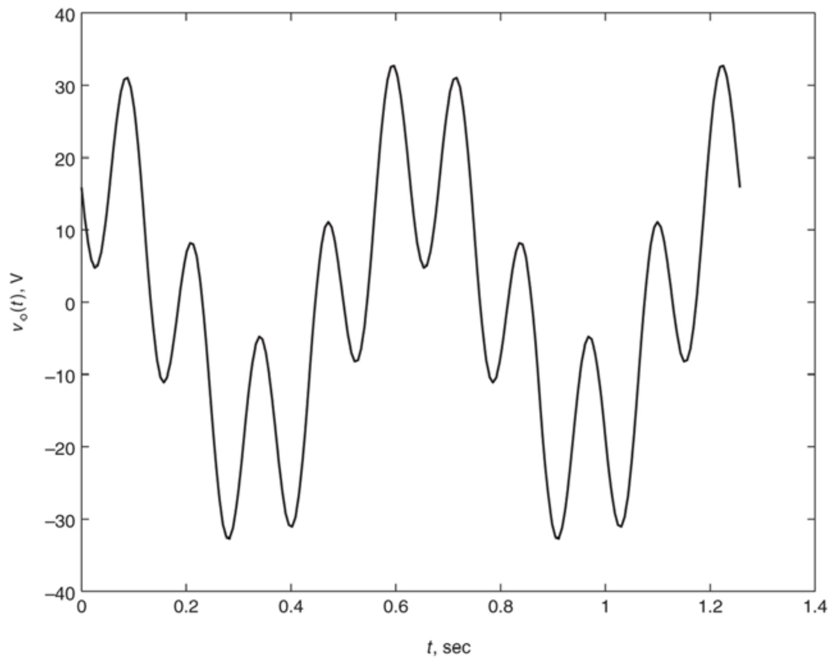


FIGURE 10.8−4

Phase Diagram

- A **phasor diagram** is a graphical representation of phasors and their relationship on complex plane.

- Consider a circuit represented in the time and frequency domain in Figure 10.9-1. The input is the current of the current source, $i(t)$.

  Then phasor voltages across the impedances in Figure 10.9-1(b) are given by

$$\mathbf{V}_R = R(I_m\angle 0°),\ \mathbf{V}_L = j\omega L(I_m\angle 0°) = \omega L I_m\angle 90°, \text{ and } \mathbf{V}_C = \frac{1}{j\omega C}(I_m\angle 0°) = \frac{I_m}{\omega C}\angle -90°$$

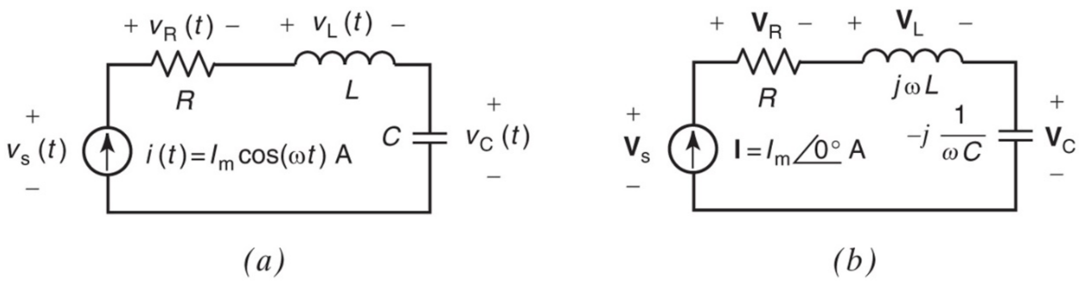


FIGURE 10.9−1

Phase Diagram (cont'd)

- Then the voltage across the input current source $\mathbf{V}_s$ is given by,

$$\mathbf{V}_s = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C = RI_m\angle 0° + \omega L I_m \angle 90° + \frac{I_m}{\omega C}\angle -90°$$
$$= I_m\angle 0°\left(R + j\omega L - j\frac{1}{\omega C}\right) = RI_m + j\left(\omega L - \frac{1}{\omega C}\right)I_m$$

This phasor is shown in complex plane in Figure 10.9-2(c)

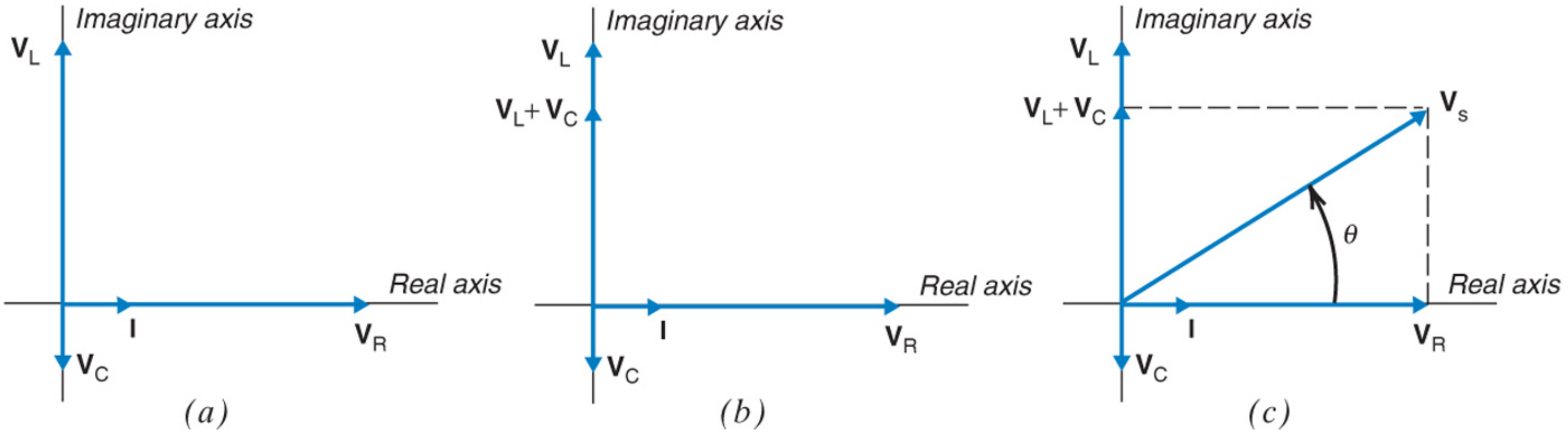


FIGURE 10.9−2

Example 10.9-1 Phasor Diagrams

- Consider the circuit shown in Figure 10.9-1a when $R = 80\ \Omega, L = 8\ H, C = 5\ mF$ and

$$i(t) = 0.25\cos(10t)\ A$$

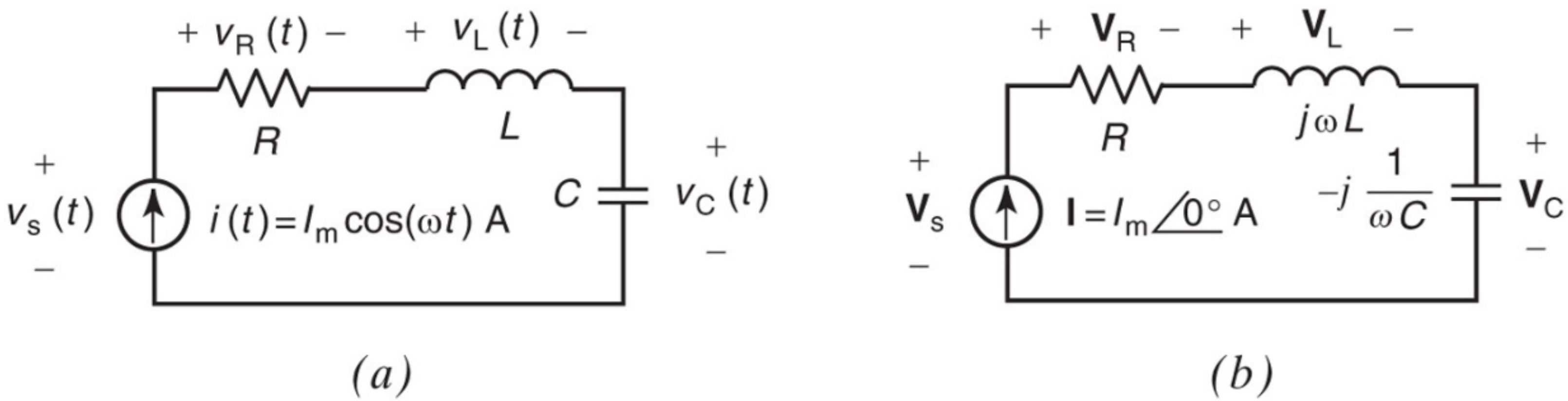


FIGURE 10.9−1

Solution

Noticing that $\omega = 10\ rad/s$, we calculate

$$j\omega L = j80\ \Omega \text{ and } -j\frac{1}{\omega C} = -j\frac{1}{0.05} = -j20\ \Omega$$

The phasor voltages across the impedances in Figure 10.9-1b are

$$\mathbf{V}_R = 80(0.25\angle 0°) = 20\angle 0°\ V, \mathbf{V}_L = j80(0.25\angle 0°) = 20\angle 90°\ V,$$
$$\text{and } \mathbf{V}_C = -j20(0.25\angle 0°) = 5\angle -90°\ V$$

These phasors are drawn in the complex plane in Figure 10.9-2a.

The phasor $\mathbf{V}_L + \mathbf{V}_C$ shown in the complex plane in Figure 10.9-2b is given by

$$\mathbf{V}_L + \mathbf{V}_C = j20 - j5 = 15\angle 90°\ V$$

Using KVL gives

$$\mathbf{V}_S = \mathbf{V}_R + (\mathbf{V}_L + \mathbf{V}_C) = 20 + j15 = 25\angle 36.9°\ V$$

This phasor is shown in the complex plane in Figure 10.9-2c

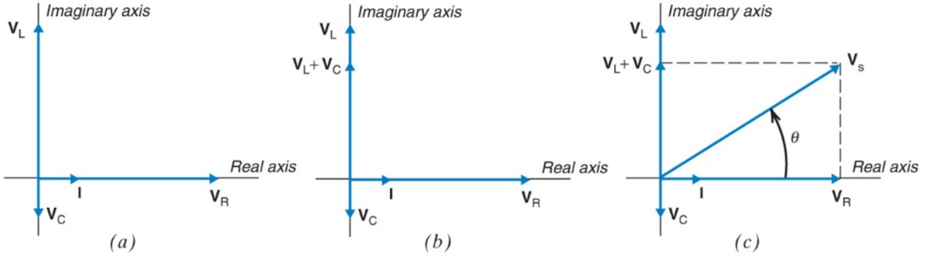


FIGURE 10.9−2

OP-Amps in AC Circuits

- Figure 10.10-1 shows two frequently used op-amp circuits.

  The ratio of output-to-input voltage $\mathbf{V}_o/\mathbf{V}_s$ for inverting amplifier (a) is determined by

$$\frac{\mathbf{V}_o}{\mathbf{V}_s} = -\frac{\boldsymbol{Z}_2}{\boldsymbol{Z}_1}$$

  Then for noninverting amplifier (b) is determined by

$$\frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{\boldsymbol{Z}_1 + \boldsymbol{Z}_2}{\boldsymbol{Z}_1}$$

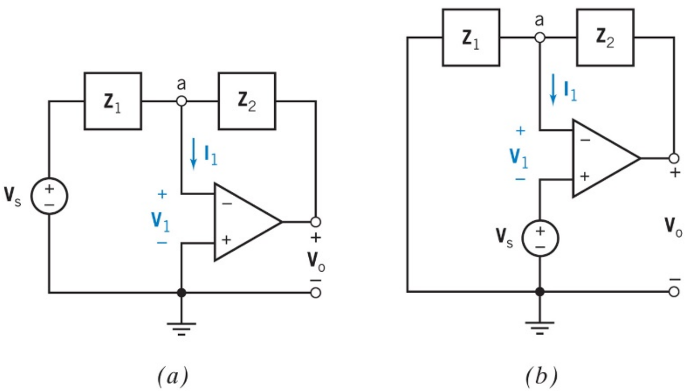


FIGURE 10.10-1

Example 10.10-1 AC Amplifiers

- Find the ratio $\mathbf{V}_o/\mathbf{V}_s$ for the circuit of Figure 10.10-2 when $R_1 = 1\ k\Omega$, $R_2 = 10\ k\Omega$, $C_1 = 0\ F$, and $C_2 = 0.1\ \mu F$ for $\omega = 1000\ rad/s$.

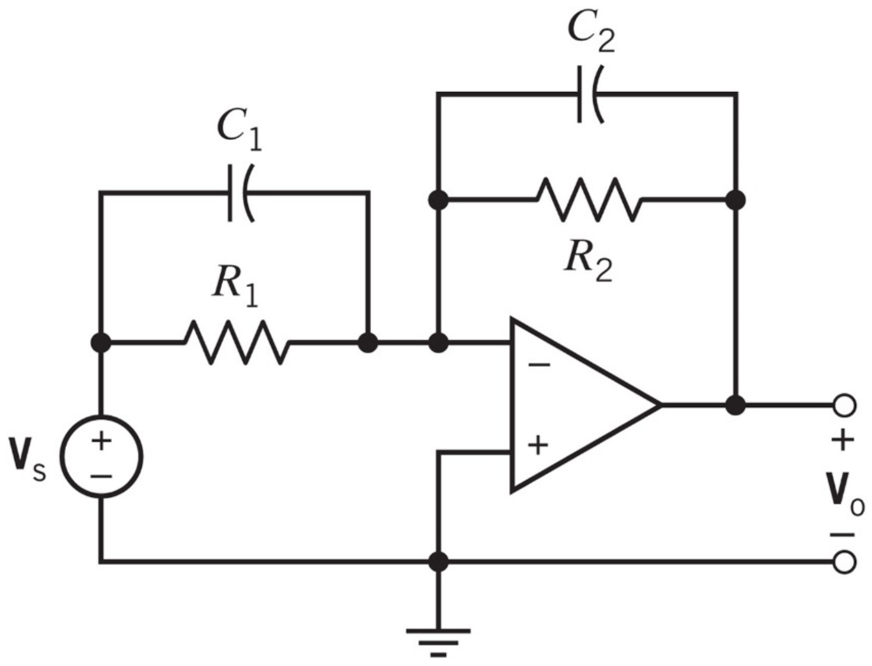


FIGURE 10.10–2

Solution

The circuit of Figure 10.10-2 is an example of the inverting amplifier shown in Figure 10.10-1a. Using equation 10.10-3 and 10.10-6, we obtain

$$\frac{\mathbf{V}_o}{\mathbf{V}_s} = -\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = -\frac{\dfrac{R_2}{1 + j\omega C_2 R_2}}{\dfrac{R_1}{1 + j\omega C_1 R_1}} = -\frac{R_2(1 + j\omega C_1 R_1)}{R_1(1 + j\omega C_2 R_2)}$$

Using the given values of R_1, R_2, C_1, and C_2 gives

$$\frac{\mathbf{V}_o}{\mathbf{V}_s} = -\frac{10^4(1 + j10^3(0)10^3}{10^3(1 + j10^3(0.1 \times 10^{-6})10^4)} = -\frac{10}{1 + j} = 7.07\angle 135°$$

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How to Find Complete Response?

-

- To find the complete response of circuits with sinusoidal inputs that are subject to abrupt changes,

1. Represent the circuit by a differential equations.
2. Find the general solution. This is the natural response $v_n(t)$.
3. Find a particular solution. This is the forced response $v_f(t)$.
4. Represent the response of the circuit as $v(t) = v_n(t) + v_f(t)$.
5. Use the initial conditions to evaluate the unknown constants.

Example 10.11-1 Complete Response

- Determine $v(t)$, the voltage across the capacitor in Figure 10.11-1, both before and after the switch closes.

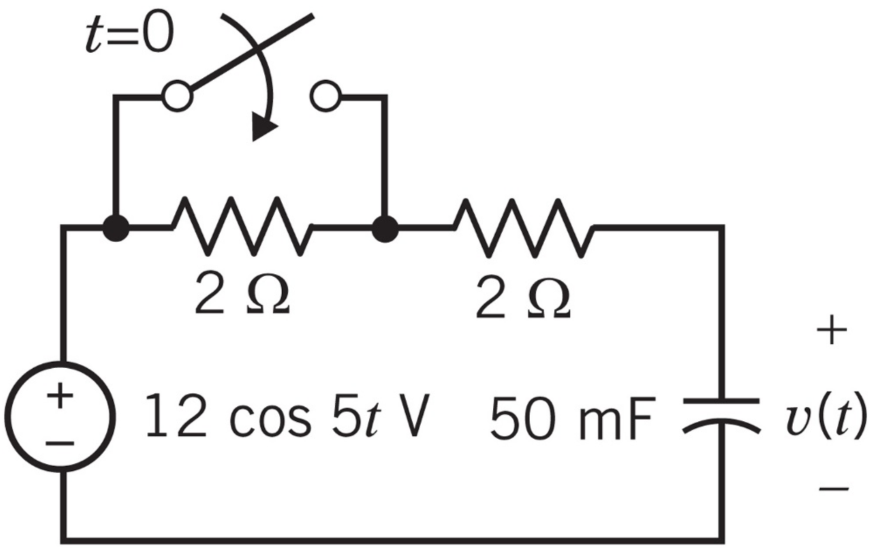


FIGURE 10.11−1

Solution (1/5)

Step 1: For $t < 0$, the switch is open and the circuit is at steady state.
Since the open switch acts like an open circuit, we can represent the circuit with Figure 10.11-2a and its frequency domain representation b.

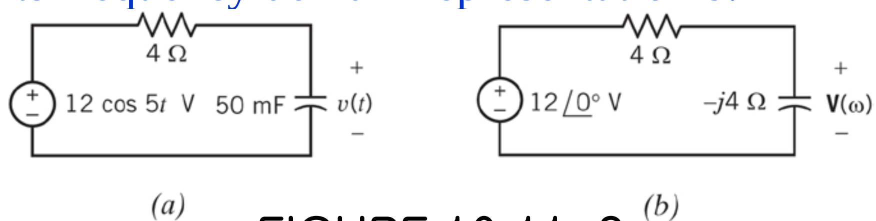


FIGURE 10.11−2

Using voltage division in the frequency domain gives

$$\mathbf{V}(\omega) = \left(\frac{-j4}{4-j4}\right)(12\angle 0^\circ) = \frac{48\angle -90^\circ}{5.66\angle -45^\circ} = 8.485\angle -45^\circ\ V$$

In the time domain,

$$v(t) = 8.485\cos(5t - 45^\circ)\ \ V$$

Therefore immediately before the switch closes, the capacitor voltage is

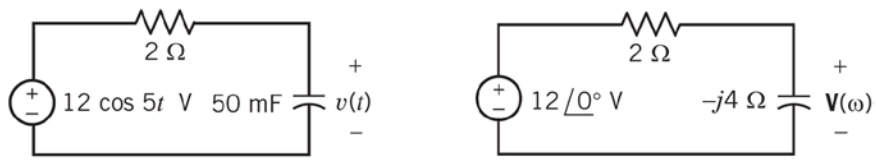
$$v(0-) = \lim_{t\to 0-} v(t) = 8.485\cos(-45^\circ) = 6\ V$$

$$v(0-) = v(0+) = 6\ V$$

Solution (2/5)

Step 2: For $t > 0$, the switch is closed.

Eventually, the circuit will reach a new steady state. Since the closed switch acts like an short circuit, we can represent the circuit with Figure 10.11-3a and its frequency domain representation b.



(a) (b)

FIGURE 10.11−3

Using voltage division in the frequency domain gives

$$V(\omega) = \left(\frac{-j4}{2-j4}\right)(12\angle 0°) = \frac{48\angle -90°}{4.47\angle -63.4°} = 10.74\angle -26.6°\, V$$

In the time domain,

$$v(t) = 10.74\cos(5t - 26.6°)\ V$$

Solution (3/5)

Step 3: For immediately after $t = 0$, the switch is closed but the circuit is not at steady state. We must find the complete response of a first-order circuit.

In Figure 10.11-3a, capacitor is connected to a series voltage source and resistor, that is, a Thévenin equivalent circuit. We can identify R_t and v_{oc} as shown in Figure 10.11-4.

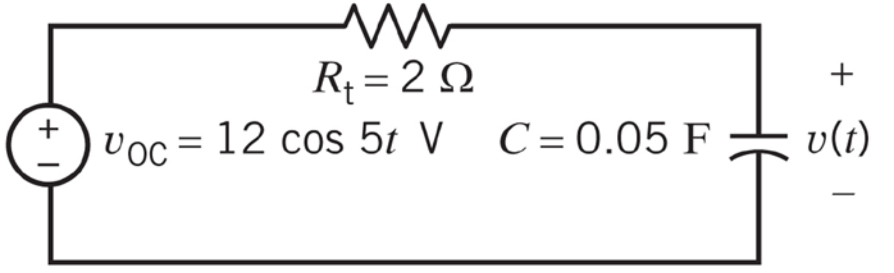


FIGURE 10.11-4

Consequently, the time constant of the circuit is

$$\tau = R_t v_{oc} = 2 \times 0.05 = 0.1\ s$$

Then the natural response of the circuit is

$$v_n(t) = Ke^{-\frac{1}{\tau}t} = Ke^{-10t}\ V$$

Solution (4/5)

The steady state response for $t > 0$ can be used as the forced reponse, so

$$v_f(t) = 10.74\cos(5t - 26.6°)\ V$$

The complete reponse is

$$v(t) = v_n(t) + v_f(t) = Ke^{-10t} + 10.74\cos(5t - 26.6°)\ \ V$$

We can find the constant, K, using the initial capacitor voltage, $v(0+)$:

$$v(0+) = 6 = Ke^{-0} + 10.74\cos(-26.6°) = K + 9.6, \text{thus, } K = -3.6$$

Then,

$$v(t) = -3.6e^{-10t} + 10.74\cos(5t - 26.6°)\ \ V$$

Solution (5/5)

Step 4: Summerize the results.
The capacitor voltage, $v(t)$ is,

$$v(t) = \begin{cases} 8.485\cos(5t - 45°)\ V, & \text{for } t \le 0 \\ -3.6e^{-10t} + 10.74\cos(5t - 26.6°)\ \ V, & \text{for } t \ge 0 \end{cases}$$

And Figure 10.11-5 shows the capacitor voltage as a function of time:

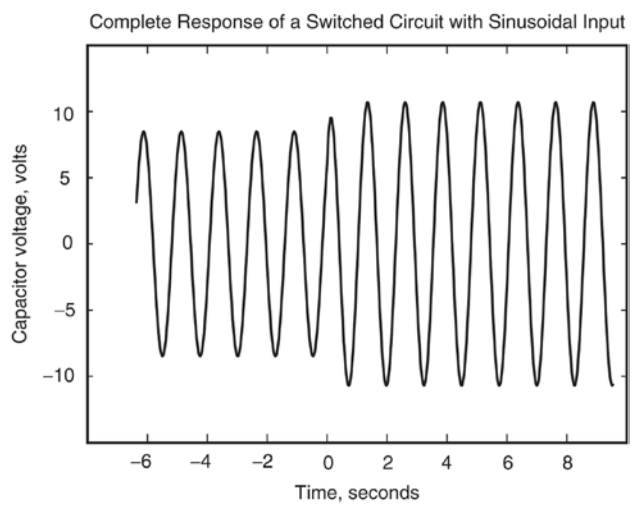


FIGURE 10.11–5

Example 10.11-2 Responses of Various Types of Circuits

- The input to the ac circuit shown in figure below is the voltage source voltage. The output of each circuit is the current *i*(t). Determine the output of each of the circuit.

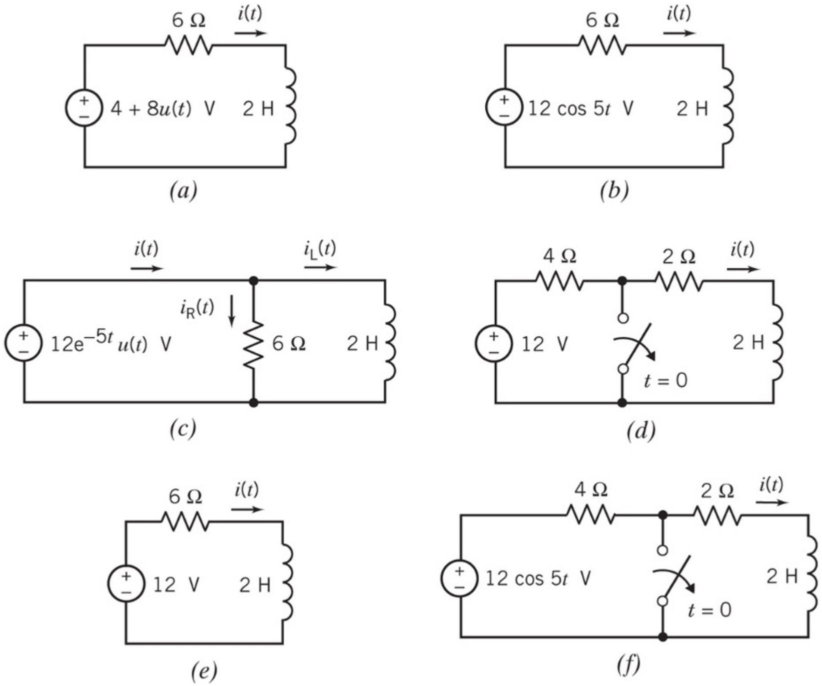
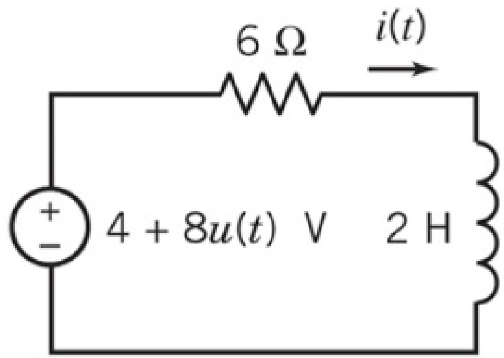


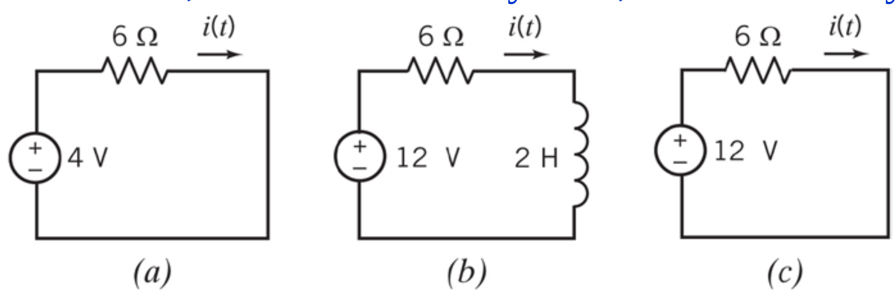
FIGURE 10.11−6

Solution (1/7)

- (a)



  - Draw the circuit when t<0, t>=0 before steady state, t>0 after steady state



  - Initial condition at t=0

$$i(0+) = i(0-) = \frac{4}{6}\ [A]$$

  - Find the Thevenin equivalent circuit for figure (b)

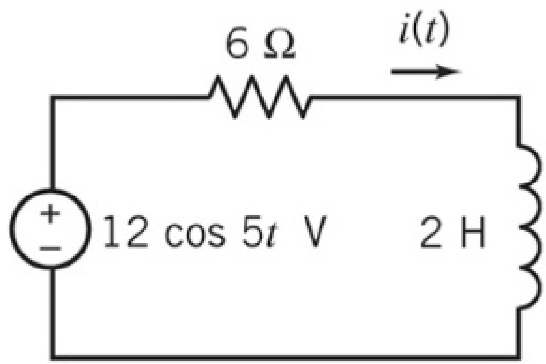
$$R_t = 6\ [\Omega], V_{OC} = 12\ [V] \rightarrow i_{SC} = 2\ [A], \tau = \frac{1}{3}\ [s]$$

  - Determine current

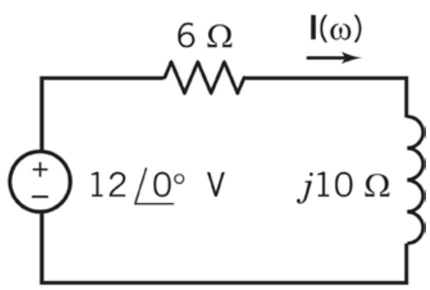
$$i(t) = i_{SC} + (i(0+) - i_{SC})e^{-t/\tau} = 2 - 1.33e^{-3t}\ [A]$$

Solution (2/7)

- (b)



  - There is no switch and input does not change abruptly, so we expect the circuit is at steady state
  - Using Ohm's law,

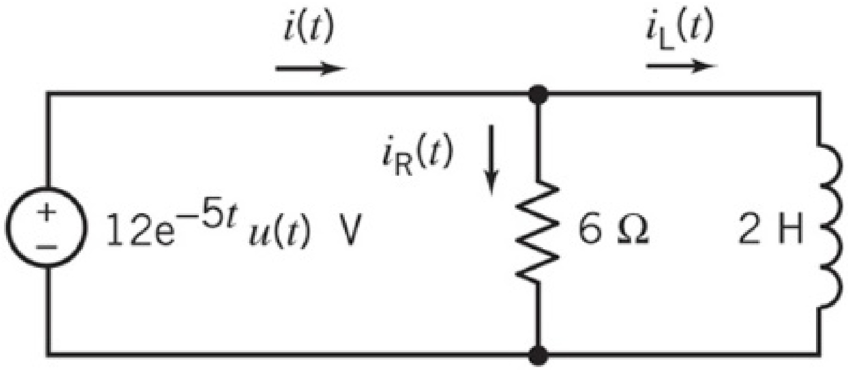


$$\mathbf{I}(\omega) = \frac{12\angle 0}{6 + j10} = 1.03\angle - 59\ [A]$$

$$i(t) = 1.03cos(5t - 59)\ [A]$$

Solution (3/7)

- (c)



  - Apply KCL

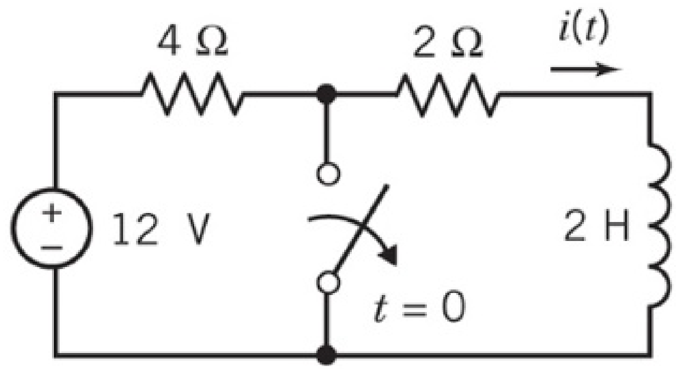
$$i_R(t) = \frac{12e^{-5t}}{6}, \quad i_L(t) = \frac{1}{L}\int_0^t v(t)dt + i_L(0) = 1.2 + i_L(0) - 1.2e^{-5t}$$

$$i(t) = i_R(t) + i_L(t)$$

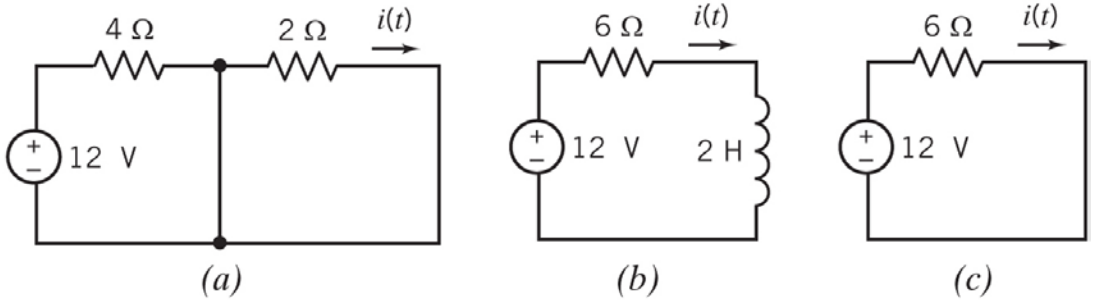
$$i(t) = 1.2 + \ 0.8e^{-5t} \ [\mathrm{A}]$$

Solution (4/7)

- (d)



  - Draw the circuit when t<0, t>=0 before steady state, t>0 after steady state



  - Initial condition at t=0

$$i(0+) = i(0-) = 0\ [A]$$

  - Find the Thevenin equivalent circuit for figure (b)

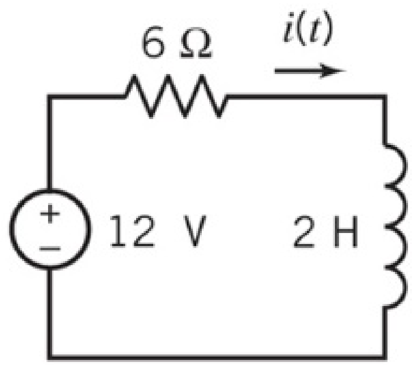
$$R_t = 6\ [\Omega], V_{OC} = 12\ [V] \rightarrow i_{SC} = 2\ [A], \tau = \frac{1}{3}\ [s]$$

  - Determine current

$$i(t) = i_{SC} + (i(0+) - i_{SC})e^{-t/\tau} = 2 - 2e^{-3t}\ [A]$$

Solution (5/7)

- (e)



  - There is no switch and input does not change abruptly, so we expect the circuit is at steady state.
  - Find the Thevenin equivalent circuit for figure (e)

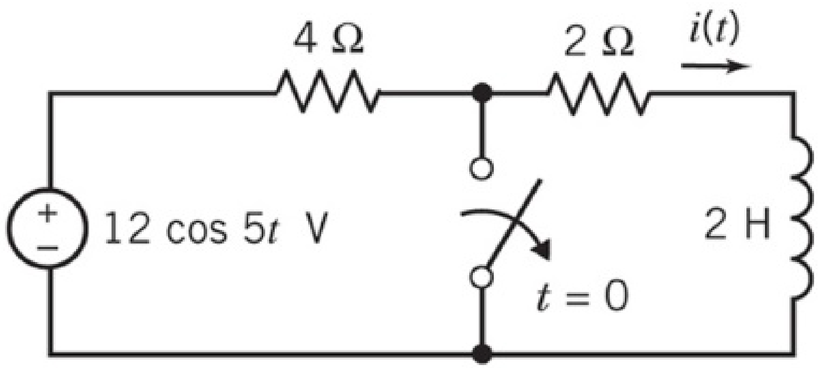
$$R_t = 6\ [\Omega], V_{OC} = 12\ [V] \rightarrow i_{SC} = 2[A], \tau = \frac{1}{3}\ [s]$$

  - The output current is given as

$$i(t) = 2\ [A]$$

Solution (6/7)

- (f)



- Applying KVL for t>0 gives

$$2i(t) + 2\frac{d}{dt}i(t) = 0$$

- Initial condition at t=0

$$i(0+) = i(0-) = 0\ [A]$$

- The Thévenin equivalent resistance for t > 0 and time constant is given,

$$R_t = 6\ [\Omega], \tau = \frac{L}{R_t} = \frac{1}{3}$$

- Then the natural response of the circuit is

$$i_n(t) = Ke^{-3t}\ [A]$$

Solution (7/7)

- From figure 10.11-10, we can find the steady-state response as the forced response from representation in frequency

$$\mathbf{I}(\omega) = \frac{12}{6+j10} = 1.03\angle -59° \; [A]$$

$$i_f(t) = 1.03\cos(5t - 59°)$$

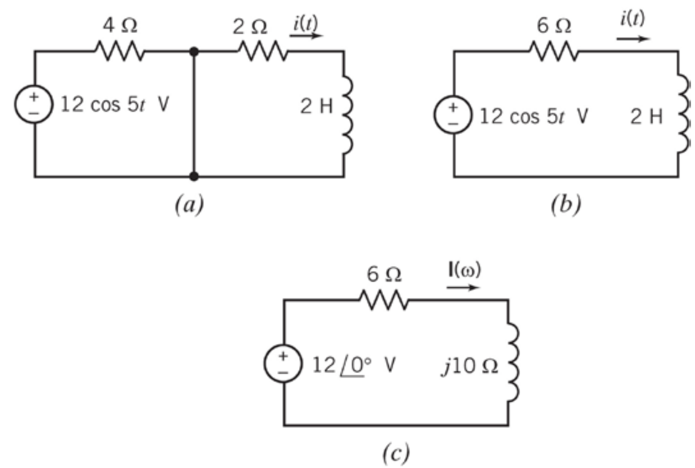


FIGURE 10.11−10

- Then,

$$\begin{aligned} i(t) &= i_n(t) + i_f(t) \\ &= Ke^{-3t} + 1.03\cos(5t - 59°) \; [A] \end{aligned}$$

- At t=0,

$$\begin{aligned} i(0) &= K + 1.03\cos(-59°) \\ &= K + 0.53 = i(0+) = 0 \end{aligned}$$

- Finally,

$$i(t) = -0.53e^{-3t} + 1.03\cos(5t - 59°) \; [A]$$