Preliminary Study

Probability Distribution

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Probability (1)

- Sample space
  - A mathematical abstraction of the collection of all possible experimental outcomes
  - Example
    - If the experiment consists of flipping two coins, the sample space denoted by S is as

      S= { (H, H), (H, T), (T, H), (T, T)}
- Sample point
  - A possible outcome of a real world experiment

Probability (2)

- Event
  - A set of sample points
  - A subset of the sample space
  - Example
    - In the experiment of flipping two coins, the event E that at least one tail occurs is as

      $E = \{ (H, T), (T, H)\}$
  - A simple event: an event with only single sample point

Probability (3)

- Probability
  - Assignment of a real number P(E) to each event E of the sample space S, satisfying the following three axioms
    1. $0 \leq P(E) \leq 1$
    2. $P(S) = 1$
    3. For any sequence of events E_1, E_2, … that are mutually exclusive,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

  - We refer to P(E) as the probability of the event E

Probability (4)

- Joint probability
  - A probability for the outcomes of combined experiments
  - The joint probability of events A and B: $P(A,B)$
- Conditional probability
  - $P(B|A) = P(A,B)/P(A)$
- Statistical independence
  - $P(A,B) = P(A)\,P(B)$ or $P(B|A) = P(B)$

Random Variable (1)

- Random variable
  - A real valued function defined on the sample space
  - Because the value of a random variable is determined by the outcome of the experiment, we may assign a probability to a possible value of the random variable
  - A random variable is characterized by a probability distribution function

Random Variable (2)

1. Discrete r.v.
   – Countable number of possible values
2. Continuous r.v.
   – Uncountable number of possible values

Discrete Random Variable (1)

- Discrete random variable, *X*
  - Probability distribution

$$P_X\left(x\right) = \Pr\{X = x\}$$

  - Cumulative distribution function (CDF)

$$F_X\left(x\right) = \sum_{x_i \le x} P_X(x_i)$$

  - Statistically independent random variables, *X* and *Y*

$$P_{XY}\left(x_i, y_i\right) = P_X(x_i)P_Y(y_i) \quad \text{for all values } (x_i, y_i)$$

$$F_{XY}\left(x, y\right) = \sum_{x_i \le x}\sum_{y_i \le y} P_X(x_i)P_Y(y_i)$$

Discrete Random Variable (2)

- Mean and variance of a discrete r. v., *X*
  - Mean

$$E[X] = \sum_{\text{all } k} x_k P_X(x_k)$$

  - *n*-th moment

$$E[X^n] = \sum_{\text{all } k} (x_k)^n P_X(x_k)$$

  - Variance

$$Var(X) = E[X^2] - (E[X])^2$$

Discrete Random Variable (3)

- Examples
  - Uniform (n_1, n_2)

$$\Pr\{X = k\} = \frac{1}{n_2 - n_1 + 1}, \qquad k = n_1, n_1 + 1, \ldots, n_2 \quad (n_2 > n_1)$$

  - Poisson (λ)

$$\Pr\{X = n\} = \frac{e^{-\lambda}\lambda^n}{n!}, \qquad n = 0, 1, 2 \ldots$$

  - Bernoulli (p)

$$\Pr\{X = n\} = p^n (1-p)^{1-n}, \qquad n = 0, 1$$

Discrete Random Variable (4)

- Examples
  - Binomial (n, p)

$$\Pr\{X = k\} = \binom{n}{k} p^n (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n$$

  - Geometric (p)

$$\Pr\{X = k\} = (1-p)^{k-1} p, \qquad k = 1, 2, \ldots$$

  - Negative binomial (k, p)

$$\Pr\{X = n\} = \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} p, \qquad n = k, k+1, \ldots$$

Continuous Random Variable (1)

- An uncountable number of possible values
- Probability density function: $f_X(x)$
- Cumulative distribution function: $F_X(x)$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

$$f_X(\mathrm{x}) = \frac{dF}{dx}$$

$$\Pr\{a \le X \le b\} = \int_a^b f_X(x)dx$$

Continuous Random Variable (2)

- Examples
  - Uniform (a, b)

$$f_X(x) = \frac{1}{b-a}, \qquad a \le x \le b$$

  - Exponential (λ)

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \ge 0$$

  - Normal (μ, σ^2)

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2 / 2\sigma^2}, \qquad -\infty < x < \infty$$

Continuous Random Variable (3)

- Examples
  - Erlang (n, λ) : n-stage

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, \qquad x > 0, n \geq 2$$

    - When n=1, exponential
  - Gamma(α, λ)

$$f_X(\mathrm{x}) = \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \qquad x > 0$$

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}$$

    - When α is a natural number, Erlang

Continuous Random Variable (4)

- Mean and variance of a continuous r. v., X
  - Mean:

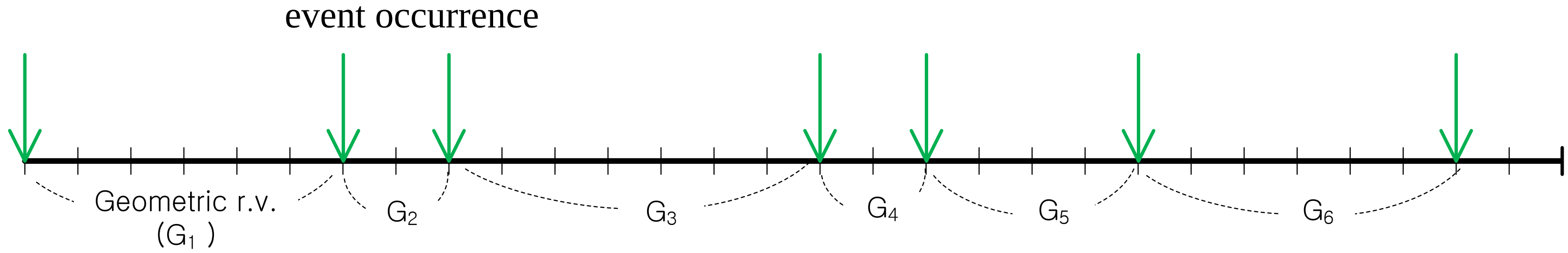
$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) dx$$

  - The nth moment :

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

Geometric Distribution

- Bernoulli r.v., B
  - Success or Failure
  - $\Pr\{B = 0\} = 1 - p$
  - $\Pr\{B = 1\} = p$
- Geometric r.v., G
  - the number of Bernoulli trials until success (event occur)
  - $\Pr\{G = k\} = (1 - p)^{k-1} p$



✓ In discrete time domain, Geometric r.v. represents interevent time

Binomial Distribution

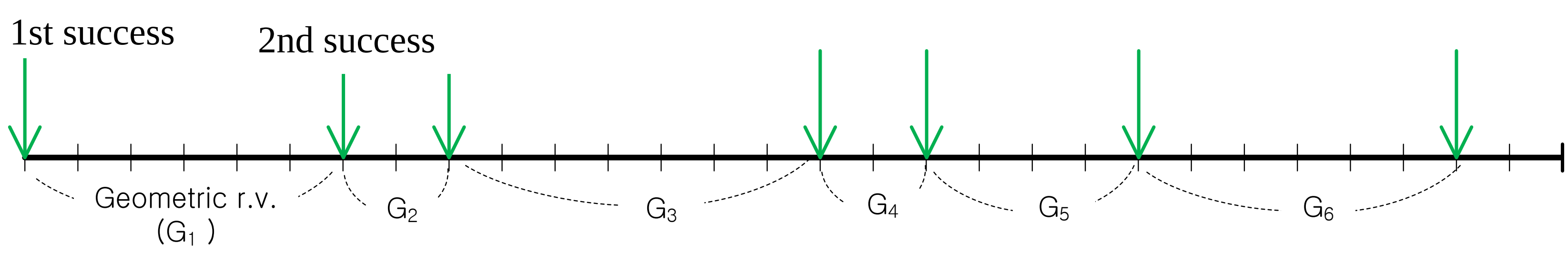
- Binomial r.v., X
  - the number of successes in n Bernoulli trials

$$\Pr\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

- $X = B_1 + B_2 + \cdots + B_n$ (B_i: Bernoulli r.v. ; 0 or 1)

Negative Binomial Distribution

- Negative Binomial r.v., X
  - the number of Bernoulli trials until the k-th success
  - $\Pr\{X = k\} = \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} p$
  - $X = G_1 + G_2 \cdots + G_k$ (G_i: Geometric r.v.)



Exponential Distribution

- Exponential r.v., Y
  - $\Pr\{Y > t\} = e^{-\lambda t}$
  - Cumulative distribution function (CDF): $F_Y(t) = \Pr\{Y \leq t\}$

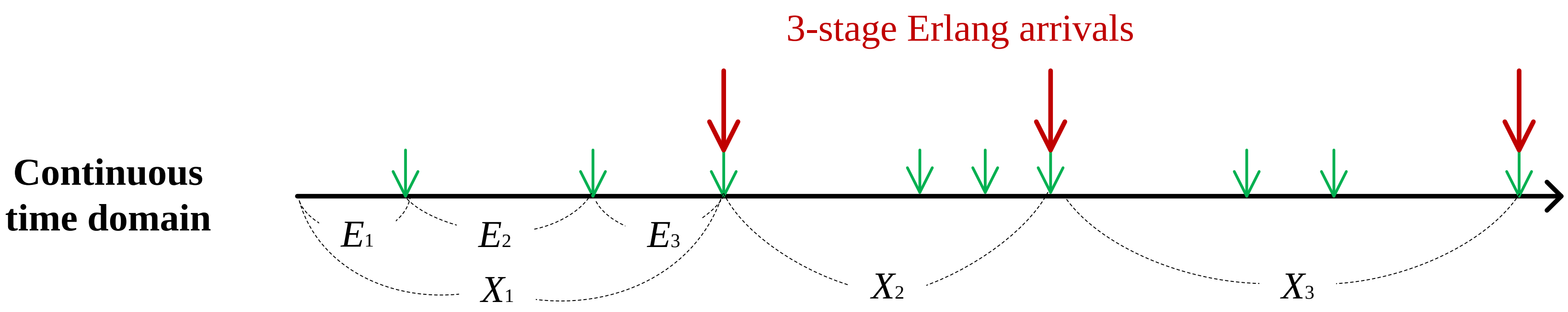
    $$F_Y(t) = 1 - e^{-\lambda t}$$

  - Probability distribution function (PDF): $f_Y(t) = \frac{d}{dt} F_Y(t)$

    $$f_Y(t) = \lambda e^{-\lambda t}$$

k-stage Erlang Distribution (1)

- k-Erlang r.v., X
  - $X = E_1 + E_2 + \cdots + E_k$ (E : exponential r.v.)
    - k-fold convolution of exponential distribution
  - Hypo-exponential
  - An example of 3-stage Erlang distribution



  - $\Pr\{X > t\} = \sum_{j=0}^{k-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!}$: the probability of events less than k during t

k-stage Erlang Distribution (2)

- k-stage Erlang r.v.
  - Cumulative distribution function (CDF): $F_Y(t) = \Pr\{X \leq t\}$

$$F_Y(t) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!}$$

$$= 1 - \left(e^{-\lambda t} + \sum_{j=1}^{k-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!} \right)$$

  - Probability distribution function (PDF): $f_X(t) = \frac{d}{dt} F_X(t)$

$$f_X(t) = \lambda e^{-\lambda t} - \sum_{j=1}^{k-1} \left\{ \frac{\lambda (\lambda t)^{j-1} e^{-\lambda t}}{(j-1)!} - \frac{\lambda (\lambda t)^j e^{-\lambda t}}{j!} \right\}$$

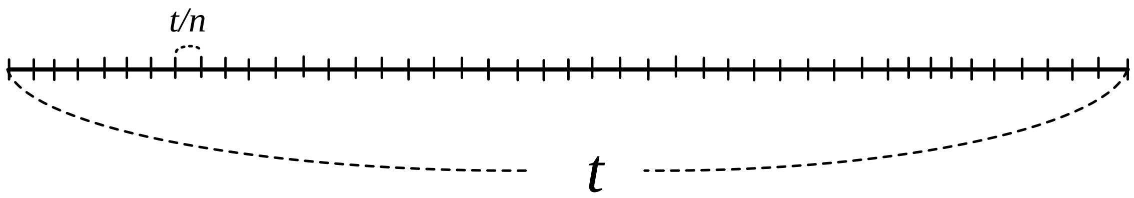
$$= \frac{\lambda (\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}$$

Relationship between random variables

Discrete Random Variable	Continuous Random Variable
Geometric	Exponential
Binomial	Poisson
Negative binomial	Erlang

Binomial $\rightarrow$ Poisson (1)

- Binomial r.v. in the discrete time domain can be represented as Poisson r.v. in the continuous time domain
- Proof



  - An interval of duration t is divided into n sub-intervals, each of which has the length of t/n
  - Bernoulli trial at each sub-interval
    - p : the success probability of a Bernoulli trial
  - Average number of successes (events) during t
    - λt: Poisson ; np : Bernoulli
    - $\lambda t = np$

Binomial → Poisson (2)

- When r.v. X represents the number of success events of n trials,
  $P_k = \Pr\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}$

- $P_0 = (1-p)^n = \left(1 - \frac{\lambda t}{n}\right)^n \xrightarrow{n \to \infty} \lim_{n\to\infty}\left(1 - \frac{\lambda t}{n}\right)^n = e^{-\lambda t} \quad \cdots \; (1)$

- $$\frac{P_{k+1}}{P_k} = \frac{\binom{n}{k+1}(1-p)^{n-k-1}p^{k+1}}{\binom{n}{k}(1-p)^{n-k}p^k} = \frac{(n-k)p}{(k+1)(1-p)}$$

$$= \frac{np\left(1-\frac{k}{n}\right)}{(k+1)(1-p)} = \frac{\lambda t\left(1-\frac{k}{n}\right)}{(k+1)\left(1-\frac{\lambda t}{n}\right)}$$

$$\xrightarrow{n \to \infty} \quad \lim_{n\to\infty}\frac{P_{k+1}}{P_k} = \frac{\lambda t}{(k+1)} \quad \cdots \; (2)$$

Binomial → Poisson (3)

- When $n \to \infty$, from (1)and (2)

$$P_k = \frac{\lambda t}{k} P_{k-1} = \frac{\lambda t}{k} \frac{\lambda t}{k-1} P_{k-2} = \cdots = \frac{(\lambda t)^k}{k!} P_0$$

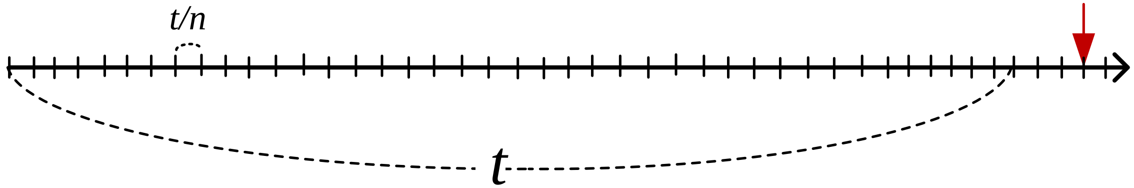
$$= \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

: Poisson distribution

- When $n \to \infty$, since the discrete time domain become continuous time domain, Binomial r.v. in the discrete time domain can be represented as Poisson r.v. in the continuous time domain

Geometric → Exponential (1)

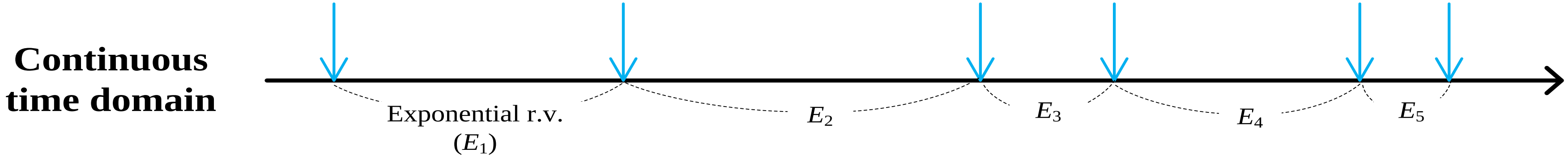
- Exponential r.v. can be obtained as a limiting form of Geometric r.v.
  - λ : the occurrence rate of events in continuous time domain
  - Y: exponential r.v.



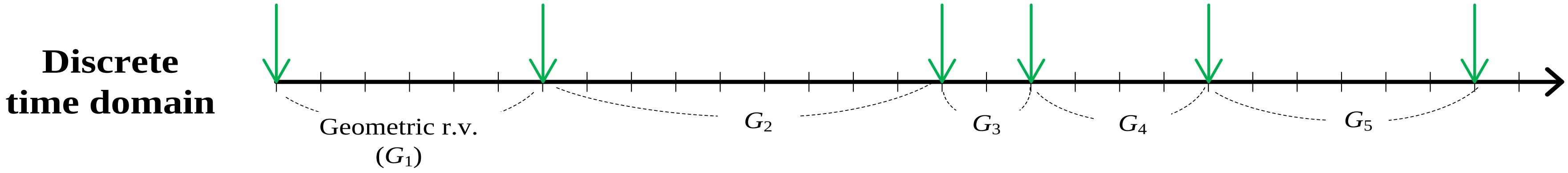
  - Let X be a geometric r.v.: $\Pr\{X > n\} = (1-p)^n$
  - An average number of successes (events) for n trials: $np = \lambda t$
  - $\Pr\{Y > t\} = \lim_{n\to\infty} \Pr\{X > n\} = \lim_{n\to\infty} (1-p)^n$
    $$= \lim_{n\to\infty} \left(1 - \frac{\lambda t}{n}\right)^n = e^{-\lambda t}$$
  - $\Pr\{Y \le t\} = 1 - e^{-\lambda t}$: exponential distribution

Geometric → Exponential (2)

- Arrival events occur according to Poisson process



- Arrival events occur according to Bernoulli process



Memoryless Property (1)

- Memoryless property of geometric r.v. X
  - A geometric r.v. X has memoryless property if for all nonnegative integers n, m

$$\Pr\{X = n + m | X > n\} = \Pr\{X = m\}$$

< ***Proof*** >

- $$\Pr\{X = n + m | X > n\} = \frac{\Pr\{X=n+m\}}{\Pr\{X>n\}} = \frac{(1-p)^{n+m-1}p}{(1-p)^n} = (1-p)^{m-1}p = \Pr\{X = m\}$$

$\therefore$ Geometric r.v. X has memoryless property

Memoryless Property (2)

- Memoryless property of exponential r.v. X
  - A exponential r.v. X has memoryless property if for all nonnegative t, x

$$\Pr\{X \le t + x | X > t\} = \Pr\{X \le x\}$$

< ***Proof*** >

- $\Pr\{X \le t + x | X > t\} = \frac{\Pr\{x < X \le t + x\}}{\Pr\{X > t\}}$

$$= \frac{\Pr\{X \le t + x\} - \Pr\{X \le t\}}{\Pr\{X > t\}}$$

$$= \frac{1 - e^{-\lambda(t+x)} - (1 - e^{-\lambda t})}{e^{-\lambda t}}$$

$$= 1 - e^{-\lambda x}$$

$$= \Pr\{X \le x\}$$

$\therefore$ Exponential r.v. X has memoryless property.