Poisson Process

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Definition

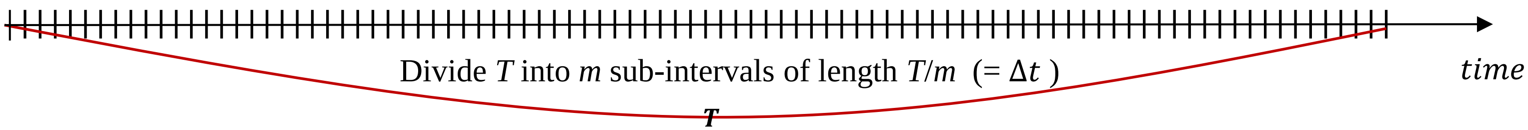
- (Homogeneous) Poisson Process
  - The counting process $\{N(t), t \geq 0\}$ is Poisson process having rate λ ($\lambda > 0$) if
    1) $N(0) = 0$
    2) The events in $(t, t + \Delta t]$ is independent of t
    3) The probability of an event in $(t, t + \Delta t]$ is $\lambda\Delta t + o(\Delta t)$
    4) The probability of no event in $(t, t + \Delta t]$ is 1 - $\lambda\Delta t + o(\Delta t)$

    where $o(\Delta t)$ is any function such that $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$
  - When $\Delta t \to 0$, it implies that
    - Probability of an event within Δt : $\lambda\Delta t$
    - Probability of no event within Δt : $1 - \lambda\Delta t$
    - Probability of more than one event within $\Delta t \approx 0$

  ⇒This can be seen as Bernoulli process

Probability distribution

- Representation with Bernoulli process



  - When $\Delta t \to 0,\ m \to \infty$
  - X: random variable representing the number of events during T
- Since an event occurs at each subinterval according to Bernoulli process with probability $\lambda\Delta t$,

$$\Pr\{X = n\} = \lim_{m\to\infty} \binom{m}{n} (\lambda\Delta t)^n (1 - \lambda\Delta t)^{m-n}$$

$$= \lim_{m\to\infty} \frac{m!}{n!\,(m-n)!} \frac{(\lambda T)^n}{m^n} \frac{\left(1 - \lambda\frac{T}{m}\right)^m}{\left(1 - \lambda\frac{T}{m}\right)^n}$$

$$= \frac{(\lambda T)^n}{n!} \lim_{m\to\infty} \frac{m!}{(m-n)!\,m^n} \frac{\left(1 - \lambda\frac{T}{m}\right)^m}{\left(1 - \lambda\frac{T}{m}\right)^n}$$

$$= \frac{(\lambda T)^n}{n!} \lim_{m\to\infty} \left(1 - \lambda\frac{T}{m}\right)^m = \frac{(\lambda T)^n}{n!} e^{-\lambda T}$$

$$\Rightarrow \quad \Pr\{X = n\} = \frac{(\lambda T)^n}{n!} e^{-\lambda T}$$

Exponential Distribution Property (1)

1. The inter-event time is exponentially distributed

< *Proof* >

- Let $A_c(t)$ denote the probability of no event in $(t_0, t_0 + t]$
- $A_c(t + \Delta t) = Pr\{\text{no event in } (t_0, t_0 + t]\} \times Pr\{\text{no event in } (t_0 + t, t_0 + t + \Delta t]\}$

$$A_c(t + \Delta t) = A_c(t)(1 - \lambda\Delta t + o(\Delta t))$$

$$A_c(t + \Delta t) - A_c(t) = -\lambda\Delta t\, A_c(t) + o(\Delta t)A_c(t)$$

$$\lim_{\Delta t \to 0} \frac{A_c(t+\Delta t) - A_c(t)}{\Delta t} = \lim_{\Delta t \to 0} \left(-\lambda A_c(t) + \frac{o(\Delta t)A_c(t)}{\Delta t}\right)$$

$$A_c'(t) = -\lambda A_c(t)$$

$$\frac{A_c'(t)}{A_c(t)} = -\lambda \qquad \ldots.(1)$$

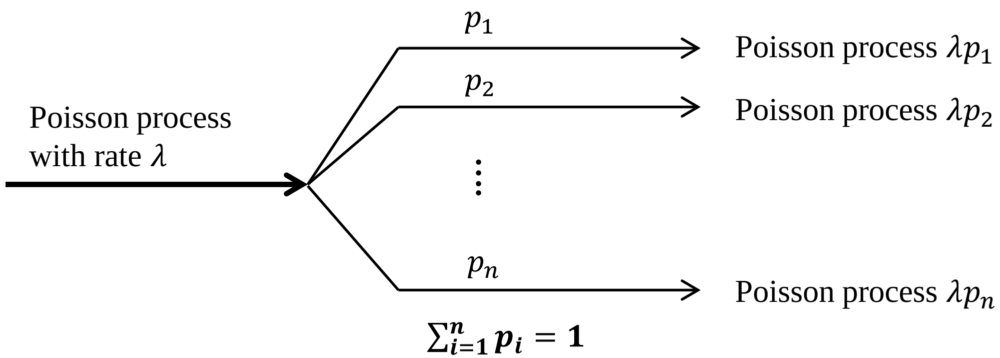
Exponential Distribution Property (2)

- From (1)
  - $\int \frac{A_c'(t)}{A_c(t)} = \ln A_c(t) = -\lambda t + k$
- Since $A_c(0) = 1$
  - $\ln A_c(0) = k = 0$
- Thus, $\ln A_c(t) = -\lambda t$
  - $A_c(t) = e^{-\lambda t}$
- If X is a random variable of inter-event time
  - $A_c(t) = Pr\{X > t\} = e^{-\lambda t}$
  - $Pr\{X \leq t\} = 1 - e^{-\lambda t}$

The last term implies that X is exponentially distributed.

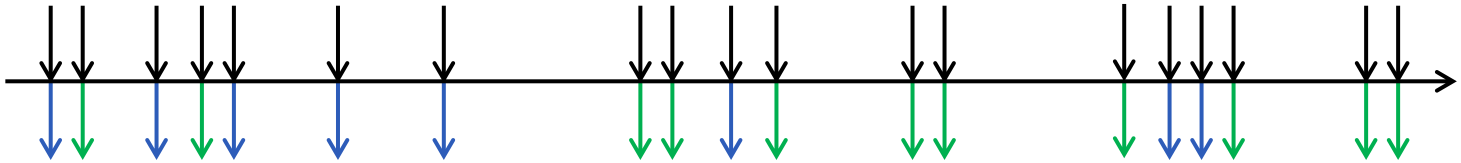
Decomposition Property (1)

2. Decomposition



– Example

- Event arrival follows Poisson process (black arrow)
- Each event is classified with probability
  - ✓ blue arrow with p_1 and green arrow with $(1 - p_1)$
- According to decomposition property, these are also respectively Poisson processes



Decomposition Property (2)

2. Decomposition

< *Proof* >

- Consider a Poisson process with rate λ
- Each event is categorized into two types
  - Type I with probability p
  - Type II with probability (1 - p)
- Let $N_1(t)$ be the number of type I events in [0,t]

  Let $N_2(t)$ be the number of type II events in [0,t]

  Let $N(t)$ be the total number of events in [0,t]
  - $N(t) = N_1(t) + N_2(t)$

Decomposition Property (3)

< *Proof* (cont.) >

$$Pr\{N_1(t)=n, N_2(t)=m\}$$

$$= Pr\{N(t)=n+m\}\, Pr\{N_1(t)=n, N_2(t)=m | N(t)=n+m\}$$

$$= \frac{(\lambda t)^{m+n}}{(m+n)!} e^{-\lambda t} \times \binom{m+n}{n} p^n (1-p)^m$$

$$= \frac{(\lambda p t)^n}{n!} \frac{(\lambda(1-p)t)^m}{m!} e^{-\lambda t}$$

$$P_r\{N_1(t)=n\} = \sum_{m=0}^{\infty} P_r\{N_1(t)=n, N_2(t)=m\}$$

$$= \sum_{m=0}^{\infty} \frac{(\lambda p t)^n}{n!} \frac{(\lambda(1-p)t)^m}{m!} e^{-\lambda t} = \frac{(\lambda p t)^n}{n!} e^{-\lambda t} \sum_{m=0}^{\infty} \frac{(\lambda(1-p)t)^m}{m!}$$

$$= \frac{(\lambda p t)^n}{n!} e^{-\lambda t} e^{\lambda(1-p)t}$$

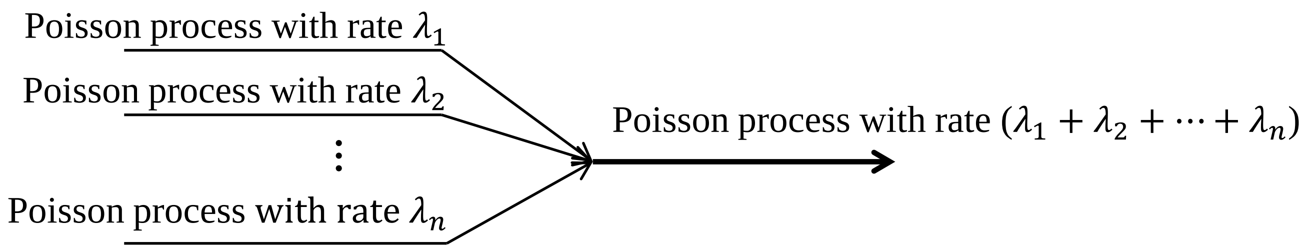
Thus, $Pr\{N_1(t)=n\} = \frac{(\lambda p t)^n}{n!} e^{-\lambda p t}$

Similarly, $Pr\{N_2(t)=m\} = \frac{(\lambda(1-p)t)^m}{m!} e^{-\lambda(1-p)t}$

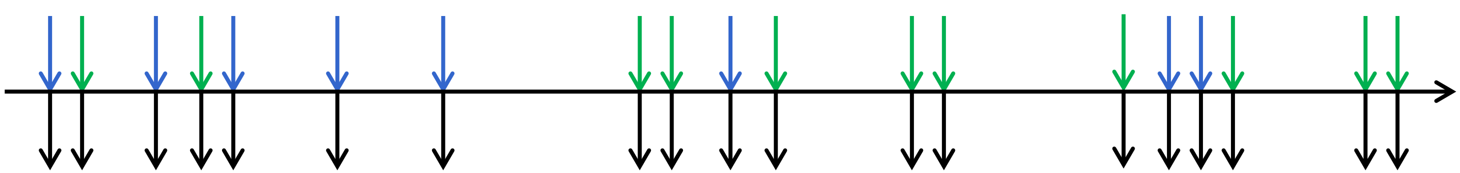
} Poisson process

Superposition Property (1)

3. Superposition



- Example
  - There are two types of events (blue and green), which respectively arrive according to Poisson process
  - The combination of those events (black arrows) also follows Poisson process



Superposition Property (2)

$< Proof >$

– Consider two Poisson processes with rate λ_1 and λ_2

$$Pr\{N(t) = n\} = \sum_{k=0}^{n} Pr\{N_1(t) = k\} \times Pr\{N(t) \ = n | N_1(t) = k\}$$

$$= \sum_{k=0}^{n} \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_1 t} \times \frac{(\lambda_2 t)^{n-k}}{(n-k)!} e^{-\lambda_2 t}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} (\lambda_1 t)^k (\lambda_2 t)^{n-k}$$

$$= \frac{((\lambda_1 + \lambda_2)t)^n}{n!} e^{-(\lambda_1+\lambda_2)t}$$

– Therefore, $\{N(t), t \geq 0\}$ is a Poisson process with rate $(\lambda_1 + \lambda_2)$

Non-Homogeneous Poisson Process (NHPP)

- Definition
  - The counting process $\{N(t), t \geq 0\}$ is NHPP with rate $\lambda(t), t \geq 0$ if
    - $N(0) = 0$
    - $\Pr\{N(t + \Delta t) - N(t) = 0\} = 1 - \lambda(t)\Delta t + o(\Delta t)$
    - $\Pr\{N(t + \Delta t) - N(t) = 1\} = \lambda(t)\Delta t + o(\Delta t)$
    - $\Pr\{N(t + \Delta t) - N(t) \geq 2\} = o(\Delta t)$

- Derivation of pdf

$$\Pr\{N(t + \Delta t) = 0\}$$
$$= \Pr\{N(t) = 0\} \times \Pr\{N(t + \Delta t) - N(t) = 0\}$$

Let $P_n(t) \coloneqq \Pr\{N(t) = n\}$. Then

$$P_0(t + \Delta t) = P_0(t)(1 - \lambda(t)\Delta t + o(\Delta t)) \quad \dots \quad (1)$$

$$P_n(t + \Delta t) = \sum_{k=0}^{n} P_{n-k}(t) \Pr\{N(t + \Delta t) - N(t) = k\} \quad \dots \quad (2)$$

NHPP (2)

- From (1)

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -P_0(t)\lambda(t) + \frac{P_0(t)o(\Delta t)}{\Delta t}$$

$$\lim_{\Delta t\to 0}\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -P_0(t)\lambda(t) + \lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}P_0(t)$$

$$\Rightarrow \qquad \frac{dP_0(t)}{\mathrm{d}t} = -P_0(t)\lambda(t) \qquad \ldots (3)$$

- From (2)

$$P_n(t+\Delta t) = P_n(t)\Pr\{N(t+\Delta t)-N(t)=0\} + P_{n-1}(t)\Pr\{N(t+\Delta t)-N(t)=1\} + \sum_{k=2}^{n} P_{n-k}(t)\Pr\{N(t+\Delta t)-N(t)=k\}$$

$$= P_n(t)(1-\lambda(t)\Delta t + o(\Delta t)) + P_{n-1}(t)(\lambda(t)\Delta t + o(\Delta t)) + o(\Delta t)\sum_{k=2}^{n} P_{n-k}(t)$$

$$\lim_{\Delta t\to 0}\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = -\lambda(t)P_n(t) + \lambda(t)P_{n-1}(t) + \lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}\sum_{k=0}^{n} P_{n-k}(t)$$

$$\Rightarrow \qquad \frac{dP_n(t)}{\mathrm{d}t} = -\lambda(t)P_n(t) + \lambda(t)P_{n-1}(t) \qquad \ldots (4)$$

NHPP (3)

- Define PGF for random variable $N(t)$: $\mathbb{P}(z,t) := \sum_{n=0}^{\infty} P_n(t) z^n$
- Differentiate: $$\frac{d\mathbb{P}(Z,t)}{dt} = \frac{dP_0(t)}{dt} + z\frac{dP_1(t)}{dt} + z^2\frac{dP_2(t)}{dt} + \cdots$$
- From (3), (4)

$$\begin{aligned}
\frac{d\mathbb{P}(Z,t)}{dt} &= -\lambda(t)P_0(t) \\
&\quad - z\lambda(t)\,P_1(t) \;\; + z\lambda(t)P_0(t) \\
&\quad - z^2\lambda(t)\,P_2(t) + z^2\lambda(t)P_1(t) \\
&\qquad \cdots\cdot \\
&= -\lambda(t)(P_0(t) + zP_1(t) \; + z^2\,P_2(t) + \;\ldots) \\
&\quad + z\lambda(t)\;(P_0(t) + zP_1(t) \; + z^2\,P_2(t) + \;\ldots) \\
&= -\lambda(t)\mathbb{P}(z,t) + z\lambda(t)\mathbb{P}(z,t)
\end{aligned}$$

$$\frac{d\mathbb{P}(z,t)}{dt} = -\lambda(t)(1-z)\mathbb{P}(z,t)$$

- $$\left.\frac{d\mathbb{P}(z,t)}{dt}\right/_{\mathbb{P}(z,t)} = -\lambda(t)(1-z) \qquad \ldots (5)$$

NHPP (4)

– Integrating (5)

$$\int_0^t \frac{d\mathbb{P}(z,u)}{du} \Big/ \mathbb{P}(z,u)\, du = \int_0^t -\lambda(u)(1-z)\, du + K(z)$$

$$\ln \mathbb{P}(z,t) = \int_0^t -\lambda(u)(1-z)\, du + K(z)$$

– Since $N(0) = 0,\ \ P_0(0) = 1,\ P_n(0) = 0$ for $n \geq 1 \Rightarrow \mathbb{P}(z,0) = 1$

$$\ln \mathbb{P}(z,0) = \int_0^0 -\lambda(u)(1-z)\, du + K(z) = 0 \quad \Rightarrow \quad K(z) = 0$$

– $$\ln \mathbb{P}(z,t) = -\int_0^t \lambda(u)\, du + z\int_0^t \lambda(u)\, du$$

$$\mathbb{P}(z,t) = e^{-\int_0^t \lambda(u)du} e^{z\int_0^t \lambda(u)du}$$

NHPP (5)

- Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\mathbb{P}(z,t) = e^{-\int_0^t \lambda(u)du} e^{z\int_0^t \lambda(u)du}$

$$\sum_{n=0}^{\infty} P_n(t) z^n = e^{-\int_0^t \lambda(u)du} \sum_{n=0}^{\infty} \frac{\left(z\int_0^t \lambda(u)du\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\int_0^t \lambda(u)du\right)^n}{n!} e^{-\int_0^t \lambda(u)du} z^n$$

- Therefore,

$$P_n(t) = \frac{\left(\int_0^t \lambda(u)\,du\right)^n}{n!} e^{-\int_0^t \lambda(u)du}$$