Game Theory

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Definition

- Game theory provides a mathematical tool for the analysis of interactive decision-making process
- Distinction between a game and an optimization problem
  - A game should involve multiple decision makers that can be influenced by others' behavior.
  - An optimization problem involves only a single decision maker.
- Game theory can be a design tool to find solutions of decentralized problems

Classification (1)

- Non-cooperative vs. Cooperative
  - Non-cooperative: players make decisions independently
  - Cooperative: groups of players ("coalitions") may enforce cooperative behavior
    - a competition between *coalitions* of players, rather than between individual players
- Static vs. Dynamic
  - Static: all players make decisions simultaneously, without knowledge of other players' strategies (one stage)
  - Dynamic: when players interact by playing a similar stage game numerous times, (multiple stages)

Classification (2)

- Complete information vs. Incomplete information
  - Complete information: Every player knows the payoffs and strategies available to other players but the players may not see all of the moves made by other players
- Perfect information vs. Imperfect information
  - An example of perfect information game: chess
    each player can see all of the pieces on the board at all times
  - An example of imperfect information game: card game
    each player's cards are hidden from other players

Static Games

1. The players simultaneously choose their actions; and then
2. The players receive their own payoff that depends on the combination of actions just chosen by all players

One stage

Example: Prisoner’s dilemma

- Two men, charged with a joint violation of law, are held separately by the police. Each is told that

1) If one confesses and the other does not, the former will be given a reward of $100 and the latter will be fined by $200.

2) If both confesses, each will be fined by $100.

3) If neither confesses, both will go clear.

How can we describe the prisoner’s dilemma mathematically

Strategic-form Representation

- A static game can be mathematically described by the strategic-form representation

$G = [K, \{A_k\}, \{u_k(a)\}],$ where

- $K = \{1, 2, \cdots, N\}$ is the finite set of players.
- A_k is the set of strategies (actions) available to the player k.
- $u_k(a)$ is the utility (payoff) for the player k.

Example: Prisoner’s dilemma

- Strategic-form representation for prisoner’s dilemma

$$G = [K, \{A_k\}, \{u_k(a)\}]$$

- $K = \{\text{prisoner1}, \text{prisoner 2}\}$, $A_k = \{\text{confess}, \text{not confess}\}$
- $u_k(a)$:

		prisoner2	
		confess	not confess
prisoner1	confess	(-1 , -1)	(1 , -2)
	not confess	(-2 , 1)	(0 , 0)

  - A finite game because the A_k’s are countable
  - A complete information game because the $u_k(a)$’s are common knowledge among the players

Dominated Strategy & Its Iterative Deletion

- Prisoner's Dilemma

		prisoner2	
		confess	not confess
prisoner1	confess	(-1 , -1)	(1 , -2)
	not confess	(-2 , 1)	(0 , 0)

dominated strategy

$$\Rightarrow u_1(C, X) > u_1(NC, X)$$

	prisoner2	
	confess	not confess
prisoner1 confess	(-1 , -1)	(1 , -2)

$$\Rightarrow u_2(C, C) > u_2(C, NC)$$

  - The outcome: (confess, confess)

Iterative Deletion of Dominated Strategies

- Left/Middle/Right Game

player 2

Player 1

	L	R
L	(1 , 1)	(0.5, 1.5)
M	(2 , 0)	(1, 0.5)
R	(0 , 3)	(0 , 2)

dominated

$$\Rightarrow \quad u_1(M, X) > u_1(L, X)$$
$$u_1(M, X) > u_1(R, X)$$

player 2

player1	L	R
M	(2 , 0)	(1 , 0.5)

$$\Rightarrow \quad u_2(M, R) > u_2(M, L)$$

  - The outcome: (M, R)

Nash Equilibrium

- Non-cooperative game where the players compete through self-optimization
- A joint strategy which no player can increase its utility by *unilaterally* deviating from.

Strategy $\mathbf{a}^* \in \mathbf{A}$ is a NE *if* $u_k(\mathbf{a}^*) \geq u_k(\hat{a}_k, \mathbf{a}^*_{-k}) \ \ \forall k, \forall \hat{a}_k \in A_k$

- Prisoner’s Dilemma: (confess, confess)
- Left/Middle/Right: (Middle, Right)

Multiple NEs

- Battle of Roses Game

		husband	
		football	ballet
wife	football	(1 , 2)	(0 , 0)
	ballet	(0 , 0)	(2 , 1)

  - Two NEs: (football, football), (ballet, ballet)

Mixed Strategy (1)

- Matching pennies game

  Each of two players has a penny and must choose either head or tail facing up. If two pennies match, the player 1 wins; otherwise, the player 2 wins

		player2	
		head	tail
Player1	head	(1 , -1)	(-1 , 1)
	tail	(-1 , 1)	(1 , -1)

pure strategy

  - There is no Nash equilibrium

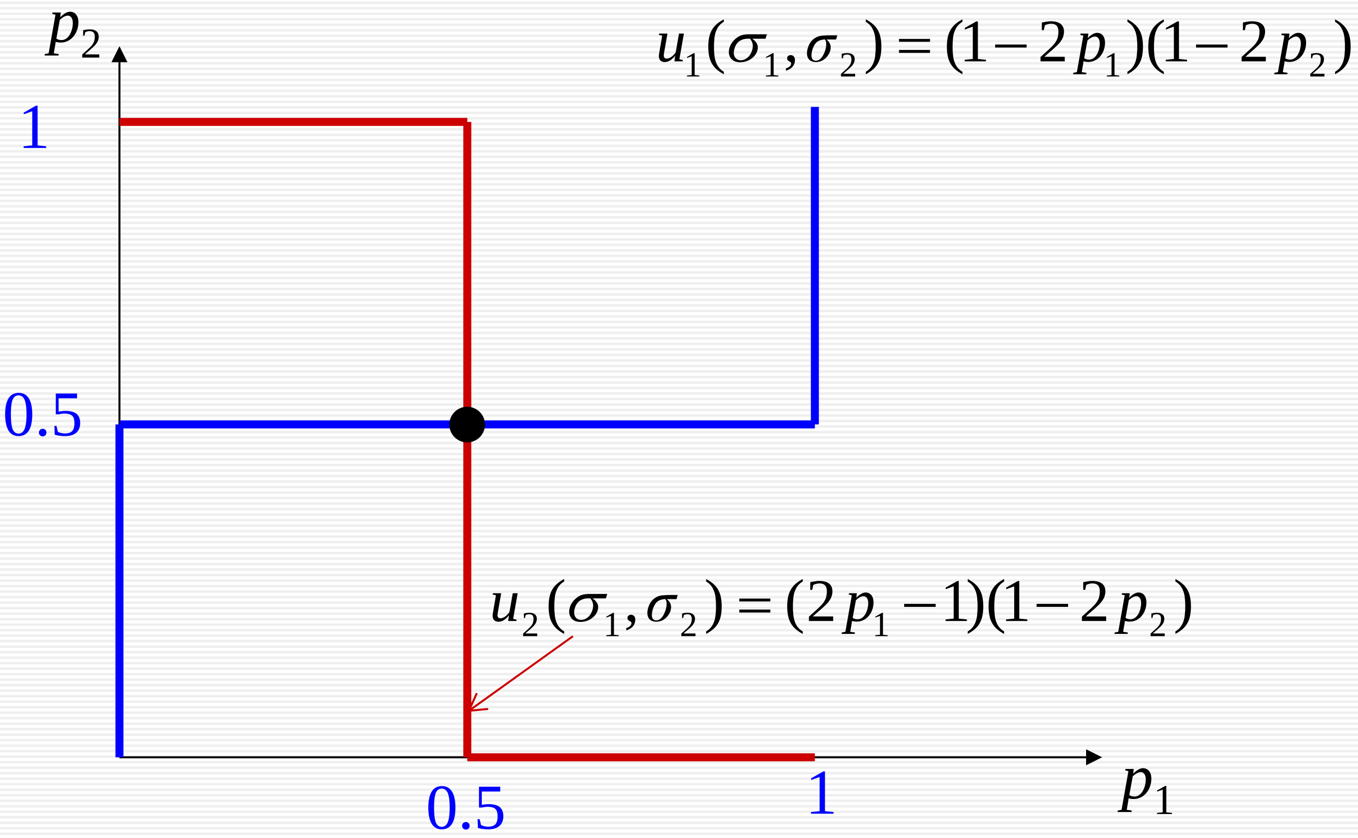
Mixed Strategy (2)

- Matching Pennies Game
  - A probability distribution over the strategy set
  - The maxed strategy
    - Player1: $\sigma_1 = (p_1, 1 - p_1)$
    - Player2: $\sigma_2 = (p_2, 1 - p_2)$

    where p_i is the probability that player i chooses head.
  - $$u_1(\sigma_1, \sigma_2) = p_1 p_2 - p_1(1 - p_2) - (1 - p_1) p_2 + (1 - p_1)(1 - p_2)$$
    $$= (1 - 2p_1)(1 - 2p_2)$$
    $$u_2(\sigma_1, \sigma_2) = -p_1 p_2 + p_1(1 - p_2) + (1 - p_1) p_2 - (1 - p_1)(1 - p_2)$$
    $$= -(1 - 2p_1)(1 - 2p_2)$$

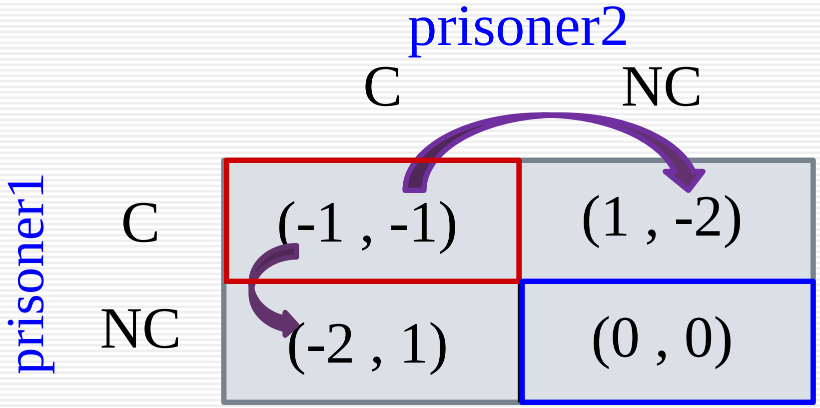
Mixed Strategy (3)

- Matching Pennies Game



Pareto Optimality (1)

- Prisoner's Dilemma



		prisoner2	
		C	NC
prisoner1	C	(-1 , -1)	(1 , -2)
	NC	(-2 , 1)	(0 , 0)

  - Neither prisoner 1 nor prisoner 2 gets any incentive to deviate from (C,C) unilaterally.
  - But, if both prisoners could jointly change their strategies, they can be willing to play (NC, NC)

Pareto Optimality (2)

- A strategy profile is Pareto-optimal *if we cannot increase the payoff without decreasing that of at least one other player*
- A strategy profile is Pareto dominated if some other strategy would make at least one player better off without hurting any other player
- A NE can be Pareto-dominated by a Pareto-optimal strategy
- Any Pareto-optimal strategy does not Pareto-dominate another Pareto-optimal strategy

Pareto Optimality (3)

- Prisoner’s Dilemma

prisoner2

		C	NC
prisoner1	C	(-1 , -1)	(1 , -2)
	NC	(-2 , 1)	(0 , 0)

  - (C, C) : a NE, but not Pareto-optimal
  - (NC,C), (C, NC), (NC,NC) : Pareto-optimal but not NE
  - (C, C) is Pareto-dominated by (NC,NC)

Sequential Game

- A class of a dynamic game
- a game where one player chooses his action before the others choose theirs. Importantly, the later players must have some information of the first's choice.
- Extensive-form representation

Extensive-form Representation

- Static Game

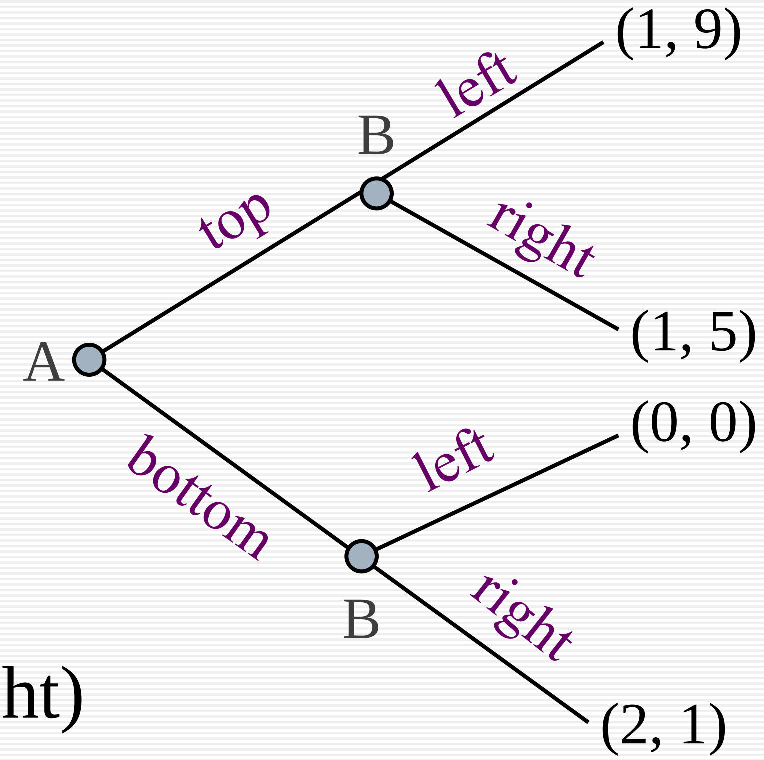
All players act simultaneously

		Player B	
		left	right
Player A	top	(1, 9)	(1 , 5)
	bottom	(0, 0)	(2 , 1)

NE: (top,left), (bottom, right)

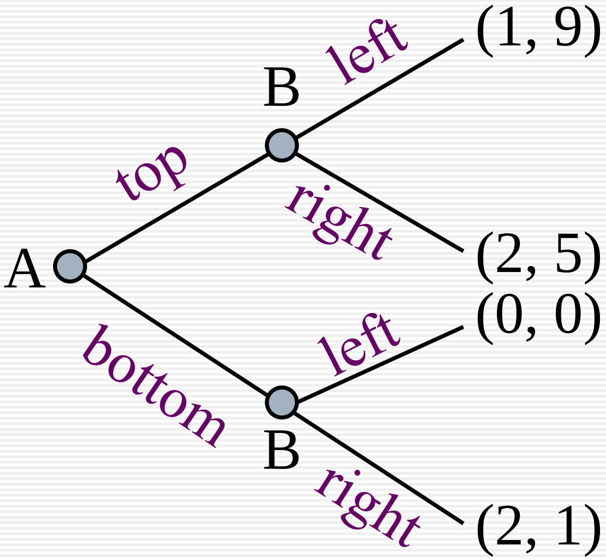
- Sequential Game

Extensive–form representation



Sequential Game Representation

- Extensive–form



- Strategic-form

		B			
		LL	LR	RL	RR
A	top	(1,9)	(1,9)	(2,5)	(2,5)
	bottom	(0,0)	(2,1)	(0,0)	(2,1)

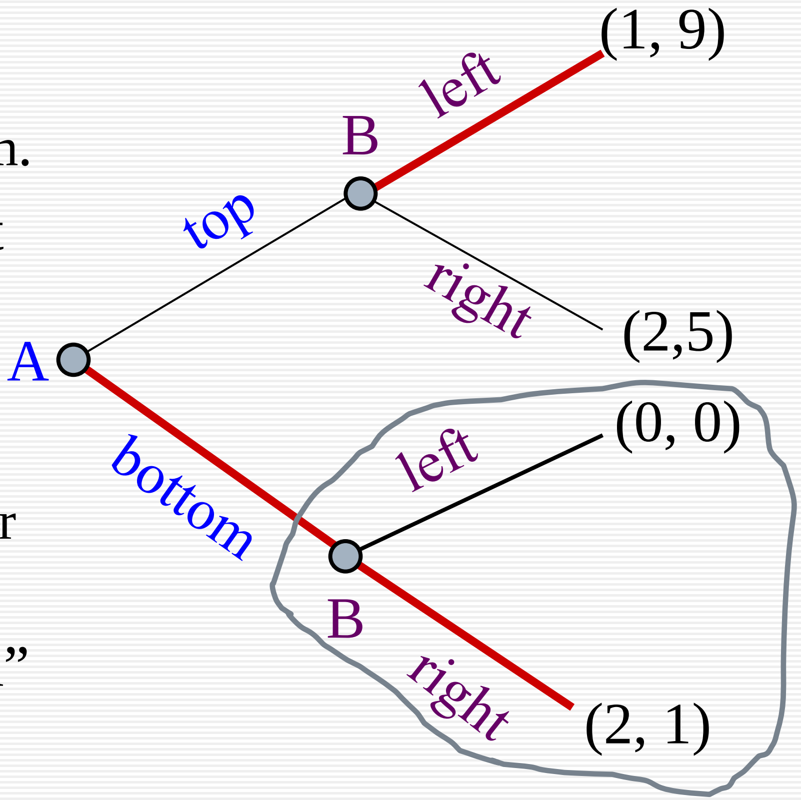
NE: (top, LL)

(bottom, LR),

(bottom, RR)

Sequential game: Backward Induction

- Would Player A ever choose Top?
  - Only if he believes that Player B will play Left after he plays Bottom.
- Player B could threaten to play Left if Player A goes for Bottom.
  - However, this is not a credible threat
  - When A has chosen Bottom, Player B will prefer to play Right.
- So, “ I'll play Left if you go Bottom” is an empty threat from B.



- (Top, Left) is not credible because it is based on an empty threat.
- (Bottom, Right) is the only credible outcome (subgame-perfect NE)

Example: Resource Allocation in OFDMA Systems

- *M* subchannels, *K* users in *N* Cells
- Power: $\mathrm{p}^n = (p_1^n, \cdots, p_M^n)$ $\mathrm{P} = [\mathrm{p}^1\ \mathrm{p}^2 \cdots \mathrm{p}^N]$
- Subchannel Allocation of BS *n*: $\mathrm{A}^n = [a_{m,k}^n]_{M \times K}$
- SINR of user *k* in cell *n* for given P:

$$\gamma_{m,k}^n = \frac{G_{m,k}^n p_m^n}{\sum_{l=1, l\neq n}^N G_{m,k}^l p_m^l + \sigma^2}$$

- Achievable date rate of user *k*

$$R_{m,k}^n(\mathrm{P}) = W \log_2\left(1 - \frac{1.5}{\ln(5\mathrm{BER})}\gamma_{m,k}^n\right)$$

$$R_k(\mathrm{P}, \mathrm{A}^n) = \sum_{m=1}^M a_{m,k}^n R_{m,k}^n(\mathrm{P})$$

Downlink Resource Allocation Game (1)

- Noncooperative RA game

$$G = [\mathcal{N}, \{\mathcal{P}^n \times \mathcal{A}^n\}, \{u_n\}]$$

- $\mathcal{N} = \{1, 2, \cdots, N\}$
- $\mathcal{P}^n = \{\mathrm{p}^n \mid 0 \le \sum_{m=1}^{M} p_m^n \le P_{\max}\}$
- $\mathcal{A}^n = \{\mathrm{A}^n \mid a_{m,k}^n \in \{0,1\}\ \forall m,k \text{ and } \sum_{k \in U_n} a_{m,k}^n = 1\}$
- $u_n(\mathrm{P}, \mathrm{A}^n) = \sum_{k \in U_n} \mu_k R_k(\mathrm{P}, \mathrm{A}^n) - c \sum_{m=1}^{M} p_m^n$

$$\text{NRAG}: \quad \max_{\mathrm{P}^n \in \mathcal{P}^n, \mathrm{A}^n \in \mathcal{A}^n} u_n(\mathrm{p}^n, \mathrm{P}^{-n}, \mathrm{A}^n)$$

Downlink Resource Allocation Game (2)

- Problem of optimizing the subchannel allocation for a given network power vector, $\mathrm{P_o}$

$$\boxed{\max_{\mathrm{A}^n \in \mathcal{A}^n} \sum\nolimits_{k \in U_n} \mu_k R_k(\mathrm{P_o}, \mathrm{A}^n)}$$

$$\Rightarrow \max_{\mathrm{A}^n \in \mathcal{A}^n} \sum\nolimits_{k \in U_n} \sum\nolimits_{m=1}^{M} a_{m,k}^n \mu_k R_{m,k}^n(\mathrm{P_o})$$

- Optimal subchannel allocation for given $\mathrm{P_o}$

$$a^{*n}_{m,k} = \begin{cases} 1 & \text{if } k = \arg\max_{k \in U_n} \mu_k R_{m,k}^n(\mathrm{P}_o) \\ 0 & \text{otherwise} \end{cases}$$

Downlink Resource Allocation Game (3)

- Optimal subchannel assignment matrix A*(P) can be determined once P is determined.

$$\max_{\mathrm{A}^n \in \mathcal{A}^n} \sum_{k \in U_n} \sum_{m=1}^{M} a_{m,k}^n \mu_k R_{m,k}^n(\mathrm{P})$$

$$= \sum_{m=1}^{M} \max_{k \in U_n} \left(\mu_k R_{m,k}^n(\mathrm{P}) \right)$$

- Our RA game becomes Power Allocation game
- Noncooperative Power Allocation game

$$\mathrm{NPAG}: \max_{\mathrm{P}^n \in \mathcal{P}^n} \; u_n(\mathrm{p}^n, \mathrm{P}^{-n}, A^{*n}(\mathrm{P}))$$

$$\Rightarrow \max_{\mathrm{P}^n \in \mathcal{P}^n} \sum_{m=1}^{M} \left(\max_{k \in U_n} \left(\mu_k R_{m,k}^n(\mathrm{P}) \right) - c p_m^n \right) \qquad \forall n \in \mathcal{N}$$