Chapter 1

Introduction

To establish the characteristic features which distinguish stability problems from ordinary problems, let us consider a simple example.

1.1 A simple Problem

Figure 1.1 shows a column built in at one end and loaded eccentrically at the other. Incidentally, in dealing with columns, rods, shafts, etc., we shall always assume that they are prismatic (cylindric),homogeneous, elastic, and obey

Hooke's law, unless a statement of the contrary is made. Let l be the length of the column, $\alpha = \mathrm{EI}$ its flexural rigidity, e the eccentricity of the load, and f the deflection at the free end. If the coordinate system of Figure 1.1 is

Refer to Figure 1.1

used and the slope of the deflection curve is assumed to be small compared with unity, the bending moment in section x is $\mathrm{M} = \mathrm{P}(e + f - y)$, and hence the (linearized) differential equation of the deflection curve is

$$\alpha \mathrm{y}^{\mathrm{n}} = \mathrm{P}(e + f - y) \qquad (1.1)$$

With the notation

$$\frac{P}{\alpha} = \kappa^2 > 0, \qquad (1.2)$$

the differential equation (1.1) becomes

$$y'' + \kappa^2 y = \kappa^2 (e + f). \qquad (1.3)$$

Since the lower end is bulit in, and since f is the (unknown) deflection at the upper end, the boundary conditions are

$$y(0) = y'(0) = 0, \qquad y(l) = f \qquad (1.4)$$

The general solution of (1.3),

$$y = A\cos\kappa x + B\sin\kappa x + e + f, \tag{1.5}$$

contains three unknown constants, A,B, and f. They can be determined by means of the boundary conditions (1.4). In this way we obtain the solution

$$y = \frac{1-\cos\kappa x}{\cos\kappa l} = e \tag{1.6}$$

and the end deflection

$$f = \left(\frac{1}{\cos\kappa l} - 1\right)e. \tag{1.7}$$

Equation (1.6) already exhibits one of the typical features of stability problems. In contrast to the results obtained in ordinary situations of the theory of structures, the deflections are not proportional to the load. This is illustrated by Figure 1.2, where i denotes the radius of inertia of the cross section and f/i is plotted versus $kl = l\sqrt{P/\alpha}$ for various values of e/i. The deviation from proportionality occurs in spite of the fact that the relations used have been linearized on the assumption that the slope of the deflection curve remains small. The deviation is explained by the fact that it has been necessary to account for the deformation while setting up the expression for the bending moment. In simple bending problems this is not necessary; actually, most of the content of the theory of structures and the theory of elasticity are based on the assumption that the equilibrium conditions are satisfied by the forces acting on the undeformed system.

In stability problems this assumption, which is essential for Kirchhoff's general uniqueness theorem[30], must be dropped, and, in fact, we will see that in many situations the deformation of a structure will not be uniquely determined by the loading.

The curves in Figure 1.2 have a common vertical asymptote at $kl = \pi / 2$, that is, for

$$P_1 = \alpha \kappa^2 = \frac{\pi^2 \alpha}{4l^2}. \tag{1.8}$$

Refer to Figure 1.2

This implies that, no matter how small the eccentricity *, the deflections become infinite for the critical value (1.8) of the load. Moreover, the result remains true for apparently centrically loaded columns, since small eccentricities can never completely be avoided.

Actually, the linearized differential equation loses its validity long before the deflection f becomes comparable with the length l. Thus the curves of Figure 1.2 are reliable solely in the vicinity of the horizontal axis. Moreover, most columns will leave the range of validity of Hooke's law and will even break long before the load reaches the critical value(1.8).

The load for which the column fails approaches P_1 with decreasing eccentricity e. If it were possible to make e exactly zero, the section $0 \le \kappa l \le \pi / 2$ of the axis κl and the asymptote would take the place of the curves in Figure

1.2. The deflection then would be zero for any load $P \prec P_1$ and become arbitrary for the critical load (1.8). The uniqueness of the solution would become lost, and the column would be apt to buckle, at least under any attempt to further increase the load.

A more elaborate investigation of the linearized problem along the lines to be developed in Section 1.2 shows that, for $e = 0$ and $P < P_1$ (Figure 1.3), the straight shape of the column represents a stable equilibrium configuration in the sense that small perturbations result in oscillations confined to the immediate vicinity of the equilibrium position. For $P = P_1$ and infinite number of additional equilibrium configuration appear (as is the case in neutral equilibrium) in which the column is bent. For $P \succ P_1$ the straight shape is again the only equilibrium configuration, but it has become unstable; and arbitrary small perturbation is followed by a rapid increase of the deflections and by the destruction of the column. Thus it becomes clear that buckling is a stability problem.

Buckling of columns and rods was first investigated by Euler [16]. He showed that the vertical line at ** has to be replaced by a curve as indicated in Figure 1.3 if the analysis is based on the exact differential equation of the deflection curve. For $P < P_1$ this curve represents stable equilibrium, while the points $P > P_1$ on the P - axis correspond to unstable equilibrium positions. In the case of a slender rod, the deflections corresponding to stable equilibrium may become considerable. However, columns in most structures fail while the deflections are still very small ; in such cases P_1 may still be considered the critical load, also referred to as the buckling load.

Problem

1. In Figure 1.4 an instability problem of a different type is shown. The structure consists of two bars hinged with a initial slope $\alpha = \arcsin h/l$ and loaded subsequently. Verify the relation

$$P = 2EF\frac{h-f}{l}[l(l^2 - 2hf + f^2)^{-1/2} - 1] \qquad (1.9)$$

between the deflection f of the middle hinge and the load P. Plot f/l versus $P/(2EF)$ for $h/l = 1/10$, and discuss the stability of the system.

Refer to Figure 1.4

1.2. Classical Approaches

If the end conditions are varied, the buckling problem of Section 1.1 appears in five different versions as indicated in

Table 1.1. Euler has shown that the buckling load is

$$P_1 = k\pi^2 \frac{\alpha}{l^2}, \qquad (1.10)$$

where the notations defined in Section 1.1 are used and k is a numerical factor varying from case to case as shown at the bottom of Table 1.1. This formula is

Refer to Table 1.1

based on the assumption that the load P, acting at the upper end of the column, remains constant with respect to magnitude and direction in the process of buckling. The problem treated in Section 1.1 reappears as Case 3 in Table 1.1. The value $k = \frac{1}{4}$ is confirmed by a comparison of (1.8) and (1.10)

There exist several approaches to stability problems. We will presently discuss some of these approaches by applying them to Euler's buckling Case 5 in its linearized form. To exhibit the underlying ideas, let us compare the actual problem with one of a single degree of freedom: a particle * (Figure1.5) moving without friction along a parabola

$$y = \frac{1}{2} a x^2 \tag{1.11}$$

under the influence of its own weight mg. According to the preliminary definitions given in Section 1.1, the only equilibrium position O is stable as

Refer to Figure 1.5

long as $a > 0$. For $a = 0$ it becomes neutral, and for $a < 0$ it is unstable.

Starting from the observation that mechanical systems are never perfect, we may assume that the parabola is slightly rotated (Figure 1.6) about the origin and hence has the (approximate) equation

$$y = \frac{1}{2}ax^2 - bx, \qquad |b| << 1. \tag{1.12}$$

Refer to Figure 1.6

The equilibrium position is shifted to the lowest point of the rotated parabola.
Since

$$\frac{dy}{dx} = ax - b \tag{1.13}$$

the abscissa of this point is

$$x_0 = \frac{b}{a}. \tag{1.14}$$

For $a \rightarrow 0$ this value tends to infinity, however small $b \neq 0$ is chosen, and this is symptomatic for the loss of stability of the equilibrium position O of the perfect system (Figure

1.5).

It is the approach just described which has been used in the determination of the buckling load (1.8). In order to treat Case 5 in an analogous way, let us introduce, as an imperfection, an eccentricity at the upper end of the

Refer to Figure 1.7

column. Equilibrium requires the presence of the reactions P and Pe/l as indicated in Figure 1.7. The deflection curve is subject to the differential equation

$$\alpha y'' = -P(y + \frac{e}{l}x) \tag{1.15}$$

and to the boundary conditions

$$y(0) = y'(0) = 0, \qquad (1.16)$$

If the notation of (1.2) is used, (1.15) takes the form

$$y'' + \kappa^2 y = -\kappa^2 \frac{e}{l} x. \qquad (1.17)$$

The general solution

$$y = A\cos\kappa x + B\sin\kappa x - \frac{e}{l} x, \qquad (1.18)$$

subjected to the boundary conditions (1.16), becomes

$$y = \left(\frac{\sin\kappa x}{\sin\kappa l} - \frac{x}{l}\right)e \qquad (1.19)$$

Since $y \to \infty (0 < x < l)$ for $kl \to \pi$, the buckling load is

$$P_1 = \alpha\kappa^2 = \frac{\pi^2\alpha}{l^2} \qquad (1.20)$$

Thus the value $k = 1$ in Table 1.1 is confirmed.

The approach just described is based on the presence of imperfections and on the observation that, for a certain value of the load, the equilibrium configuration becomes so far removed from the one of the unloaded system that the structure is endangered. It is obvious that this idea can be applied to more complicated systems. It may be referred to as the *imperfection method* and is characterized by the question: What is the value of the load for which the static displacements of the imperfect system become excessive or even infinite(as in the linear cases treated so far)?

Another approach is concerned with the equilibrium configurations of the perfect system. In the case of the parabola (1.11) of Figure 1.5, the transition between stability and instability of the isolated *equilibrium position* $x=0$, in which we are interested and which will henceforth be denoted as *trivial*, takes place when the parameter a vanishes. For $a = 0$ the parabola coincides with the $x - axis$, and any point $x \neq 0$ on it represents what is called a *nontrivial* equilibrium position. Thus the loss of stability of the trivial equilibrium position is indicated by the appearance of nontrivial equilibrium positions in its vicinity.

In order to apply this approach to Euler's problem in Case 5, we set the eccentricity e in Figure 1.7 equal to zero. The load P then acts directly on the upper hinge, and the reactions Pe/l disappear. The differential equation of the deflection curve becomes

$$y'' + \kappa y = 0 \qquad (1.21)$$

The boundary conditions are still given by (1.16). The general solution of (1.21) is

$$y = A\cos\kappa x + B\sin\kappa x \qquad (1.22)$$

The first boundary condition (1.16) requires that

$$A = 0; \qquad (1.23)$$

from the second one we obtain

$$B\sin\kappa l = 0 \qquad (1.24)$$

For arbitrary values of k (and hence of P) this equation requires that $B = 0$ and hence $y \equiv 0$: the only equilibrium configuration is the trivial one. For

$$\kappa_m l = m\pi \quad \text{(m=1,2,...)} \qquad (1.25)$$

(the values $m = -1, -2...$ do not furnish anything else, and $m = 0$ corresponds to the unloaded column), condition (1.24) is satisfied with arbitrary values of B and thus supplies an infinity of nontrivial equilibrium configurations.

Problems of this type are called *eigenvalue problems*. The ** or the corresponding loads

$$P_m = \alpha \kappa_m{}^2 = \frac{m^2 \pi^2 \alpha}{l^2} \tag{1.26}$$

are called the eigenvalues of the problem, and the corresponding solutions

$$y_m(x) = B_m \sin \frac{m \pi x}{l} \tag{1.27}$$

are the eigenfunctions. The values P_m for which nontrivial equilibrium configurations exist are also referred to as Euler's loads of order m. The present approach does not single out any one of these loads. However, it follows from

the previous approach and from the methods to be discussed presently that the actual buckling load is P_1 it coincides with (1.20).
The approach just discussed is based on the observation that the transition from stability to instability of a (trivial) equilibrium configurations in the vicinity of the trivial one. This approach will be referred to as the *equilibrium method*; it is characterized by the question : what are the values of the load for which the perfect system admits nontrivial equilibrium configurations ?
In order to explain a third approach, based on the potential energy of the system, we need a few definitions. Let $V(x_1, x_2 ,....x_n)$ be a function vanishing for the values $x_1 = x_2 ,= .... = x_n = 0$ of its arguments. If the variables

$x_m (m = 1,\ldots n)$ are interpreted as Cartesian coordinates in a space of n dimensions, V is zero at the origin. The function V is called positive (negative) *definite* if it is positive (negative) for any other set of argument $x_1, x_1, \ldots x_n$ within a sufficiently small vicinity

$$\left| x_m \right| \le g > 0 \tag{1.28}$$

of the origin. If V is of one sign in (1.28) but assumes the value zero at points in (1.28) other than the origin, it is called (positive or negative) *semidefinite*. If V assumes values of either sign within an arbitrarily small domain (1.28), it is referred to as *indefinite*.

The particle of Figure 1.5 has the potential energy

$$V = mgy = \frac{1}{2} mgax^2 \tag{1.29}$$

provided that we take $V(x=0)=0$. The function $V(x)$ is positive definite if and only if $a>0$. Thus the trivial equilibrium position is stable exactly as long as the potential energy is positive definite.

In exploring the vicinity (1.28) of the equilibrium configuration it is essential that only admissible configurations are considered, that is, configurations satisfying all the kinematic constraints. In Figure 1.5, for example, V is positive definite on the parabola, but not if all points is the vicinity of 0 are considered. In the case of a column (Table 1.1), admissible configurations are represented by continuous functions $y(x)$ having continuous first derivatives and satisfying the kinematic (or geometric) end conditions. The requirements concerning continuity are necessary since we want to exclude fracture of the column. (In the case of a string, continuity of the first derivative would not be required.) In Case 3(Table 1.1)the *kinematic end conditions* are $y(0)=0, y'(0)=0$. They represent kinematic (geometric) constraints, while the condition $y''(l)=0$, which states that the bending moment vanishes at the free end, is merely concerned with the forces and is therefore called a *dynamic end condition.*

In order to apply the concept of definiteness to a continuum (e.g., to a column) it must be slightly generalized. Here, the potential energy depends on one or more functions of one or more variables, representing the displacement field. In Euler's problem for instance, V is a so-called *functional*(i.e., a function of a function) of the form $V[y(x)]$, where *y(x)* is an arbitrary admissible function. To the origin of Figure 1.5 there corresponds the trivial equilibrium configuration $y \equiv 0$ of the column (the origin in function space); to any other point on the parabola there corresponds a deflected shape *y(x)* (a point in function space outside the origin). The potential energy is positive definite if

$$V[y \equiv 0] = 0, \qquad V[\,y(x) \not\equiv 0\,] > 0 \tag{1.30}$$

where *y(x)* is an arbitrary admissible function sufficiently close to the trivial function $y \equiv 0$. In a similar way, negative, semi-, and indefinite functions are generalized.

In an elastic system the potential energy is the sum

$$V = V^{(i)} + V^{(e)} \tag{1.31}$$

of the energy $V^{(i)}$ of the internal forces, also referred to as the deformation energy, and the energy $V^{(e)}$ of the external loads. In the case of a column, it follows from the differential equation of the deflection curve that

$$V^{(i)} = \int_0^l \frac{M^2}{2\alpha} dx = \frac{\alpha}{2} \int_0^l y''^2 \, dx. \qquad (1.32)$$

As a consequence of the deflection the ends of the column approach each other by

$$\int_0^l (ds - dx) = \int_0^l (\sqrt{1 + y'^2} - 1) dx = \frac{1}{2} \int_0^l y'^2 \, dx \qquad (1.33)$$

where only terms up to the second degree in y' are retained. Thus the potential energy of the constant load P is

$$V^{(e)} = -\frac{P}{2} \int_0^l y'^2 \, dx \qquad (1.34)$$

and the total potential energy (1.31) of the system becomes

$$V = \frac{\alpha}{2}\int_0^l y''^2\,dx - \frac{P}{2}\int_0^l y'^2\,dx \tag{1.35}$$

It is so normalized that it vanishes in the trivial equilibrium configuration $y \equiv 0$. For $P \leq 0$ and, in fact, also for sufficiently small positive values of P the functional V is positive in any nontrivial admissible configuration $y(x)$; thus, by analogy with the particle of Figure 1.5, the trivial equilibrium configuration is stable. For sufficiently large values of P, however, V will be negative, a least for certain admissible configuration $y(x)$: now V is not positive definite; the trivial equilibrium configuration therefore is unstable. The transition is characterized by the existence of at least one nontrivial admissible configuration $y_1(x)$ in which $V = 0$, while there is still no admissible configuration with $V < 0$.

It is clear that, under appropriate continuity conditions, $V[y_1(x)]$ is stationary. Thus, our problem is to find the smallest value P_1 of P for which V is stationary in a nontrivial admissible configuration $y_1(x)$. In other words, we have to find the smallest value P_1 of P for which the *variational problem*

$$\delta V = \alpha \int_0^l y''\delta y'' dx - P\int_0^l y'\delta y' dx = 0, \tag{1.36}$$

restricted to admissible variations $\delta y(x)$ of $y(x)$, has a nontrivial solution. A function $\eta(x) = \delta y(x)$ is called an *admissible variation* of $y(x)$ if it results in a new admissible configuration $y(x) + \eta(x)$ in the vicinity of $y(x)$. It conforms to the kinematic constraints and thus satisfies the kinematic boundary conditions.

Apart from the sign, δV may be interpreted as the virtual work done by the internal and external forces in an admissible virtual displacement δy. Our problem therefore is equivalent to the one of finding the smallest load for which a nontrivial equilibrium configuration exists. It follows that for problems of the type considered here the energy approach is equivalent to the equilibrium method. In fact, it is easy to see that

$$\delta y' = (\delta y)' = \eta', \qquad \delta y' = \delta y(= \eta) \tag{1.37}$$

Thus, (1.36) takes the form

$$\delta V = \alpha \int_0^l y'' \eta'' dx - P \int_0^l y' \eta' dx = 0, \tag{1.38}$$

In Euler's Case 5 the kinematic boundary conditions (1.16) imply $\eta(0)=\eta(l)=0$. By partial integration we obtain, in place of (1.38),

$$\delta V=\int_0^l(\alpha y''''+Py'')\eta dx+\alpha y''\eta'\Big|_0^l=0. \tag{1.39}$$

Since $\eta(x)$ is an arbitrary admissible function, (1.39) yields the differential equation

$$\alpha y''''+Py''=0 \tag{1.40}$$

and the dynamic boundary conditions

$$y''(0)=y''(l)=0. \tag{1.41}$$

Integrating (1.40) subject to the boundary conditions (1.41) and (1.16), we obtain the differential equation (1.21). It has been shown that the smallest value P_1 of P for which a nontrivial solution of (1.21) and (1.16) exists is given by

(1.20). Thus the result is the same as in the previous approaches.

The approach discussed here is based on the observation that the transition from stability to instability may be indicated by the fact that V ceases to be positive definite. The approach is called the *energy method*; it is characterized by the question: What is the value of the load for which the potential energy of the perfect system ceases to be positive definite?

While the methods considered so far are of a purely static nature, a last approach to be discussed is kinetic. It is concerned with the motion of the system in the vicinity of the equilibrium configuration.

In the case of the particle of Figure 1.5 the differential equation of motion is

$$m\ddot{s} = -\frac{dV}{ds} \tag{1.42}$$

Linearization and use (1.29) yield

$$m\ddot{x} = -\frac{dV}{dx} = -mgax \tag{1.43}$$

or

$$\ddot{x} + agx = 0 \tag{1.44}$$

According to the preliminary definitions given in Section 1.1, the equilibrium position $x = 0$ is stable exactly as long as the general solution of (1.44) is bounded. Thus, we once more obtain the stability condition $a > 0$.

In a similar way the vibrations of the column (Figure 1.8) under the influence of the load P can be investigated. If μ

denotes the (constant) mass per unit length, the inertia forces are

$$dT = \mu \ddot{y}(\zeta, t) d\zeta \tag{1.45}$$

Besides, the reactions P, Q_1, Q_2 must be introduced. According to d'Alembert's principle, the deflection curve is determined by

$$\alpha y''(x,t) = -Py(x,t) + Q_1 x - \mu \int_0^x \ddot{y}(\zeta, t)(x - \zeta) d\zeta \tag{1.46}$$

and

$$y(0,t) = y(l,t) = 0 \tag{1.47}$$

Instead of a differential equation we have obtained an integro-differential equation. It can be simplified by differentiation with respect to *x*. In a first step we have

$$\alpha y'''(x,t) = -Py'(x,t) + Q_1 - \mu \int_0^x \ddot{y}(\zeta,t)\,d\zeta \qquad (1.48)$$

Differentiating once more, we obtain

$$\alpha y''''(x,t) = -Py''(x,t) - \mu \ddot{y}(x,t) \qquad (1.49)$$

or simply

$$\alpha y'''' + Py'' + \mu \ddot{y} = 0 \qquad (1.50)$$

This differential equation is of the fourth order and requires two more boundary conditions,

$$y''(0,t) = y''(l,t) = 0 \qquad (1.51)$$

besides (1.47), stating that the bending moment is zero at either end. They formally follow from the equilibrium of the whole column in the sense of d'Alembert together with the requirement that Equation (1.46), which has been abandoned in favor of the weaker statement (1.50), is at least satisfied for $x=0$ and $x=l$.

In the theory of oscillations, it is shown that the general solution of (1.50),(1.47), and (1.51) is obtained by superposition of an infinite number of so-called *natural vibrations*.

$$y_m(x,t)=\sin\frac{m\pi x}{l}(A_m\cos\omega_m t+B_m\sin\omega_m t)\ (m=1,2,...) \qquad (1.52)$$

The question now is whether or not all these natural vibrations remain arbitrarily small for sufficiently small initial perturbations, i.e., for sufficiently small values of the A_m and B_m. Inserting (1.52) in (1.50), we obtain the equations

$$\mu \omega_m{}^2 = \frac{m^2\pi^2}{l^2}\left(\frac{m^2\pi^2}{l^2}\alpha - p\right) \quad (m=1,2,...) \quad (1.53)$$

for the circular frequencies ω_m of the natural vibrations. As long as all the terms in parentheses are positive, the expressions (1.52) are harmonic oscillations with amplitudes determined by the initial conditions. If one of the terms in parentheses is negative, the corresponding ω_m is imaginary and the oscillation (1.52) is unbounded for arbitrary small perturbations, since

$$\cos i\sigma t = \cosh \sigma t, \qquad \sin i\sigma t = i \sinh \sigma t \qquad (1.54)$$

where $i = \sqrt{(-1)}$. A similar result is obtained in the case where one of the ω_m is zero. Thus the straight column is stable exactly as long as P is smaller than any one of the terms $m^2\pi^2\alpha / l^2$, that is, for $P < P_1$, where P_1 is again the buckling load

(1.20).

This last approach is based on the observation that small perturbations of the equilibrium result in motions which are apt to become dangerous. The approach is called the *vibration method*: it is characterized by the question: What is the value of the load for which the most general free motion of the perfect system in the vicinity of the equilibrium position ceases to be bounded?

Problems

1. Verify the buckling load (Table 1.1) in Euler's Case 1 by means of the imperfection method, based on an inaccurate alignment of the ends (Figure 1.9).

2. Solve Euler's problem in Case 4 by means of the equilibrium approach.

3. Verify the buckling load (Table 1.1) in Euler's Case 2 by means of the energy method.

4. The system of Figure 1.10 consists of two rigid rods of mass m and length l, hinged without friction and loaded by P and their own weights. Neglect the weight of the body carrying the bottom hinge. Apply the vibration approach to verify the value $P_1 = mg$ of the load at which buckling occurs.

5. Show that for $\omega_m = 0$ the corresponding natural vibration of the column in Figure 1.8 is unbounded.

1.3. Critical Review

In Section 1.2 we have applied four different approaches to the solution of one and the same problem. Three of these approaches to the solution of one and the same problem. Three of these approaches (the imperfection, equilibrium, and energy methods) are based on static concepts, while the fourth (the vibration method) is a kinetic approach. Each one of the four methods is characterized by a specific question. Although three seems to be little connection between these questions, the result, as far as the buckling load P_1 of the column in Case 5 is concerned, is the same.

A more careful appreciation of the results, however, reveals certain differences. Some of the approaches lead directly to the buckling load P_1 ; in other cases an infinity of critical loads P_m is obtained, containing P_1 as their smallest value.

The result depends on the precise formulation of the characteristic question. A slight modification of the imperfection approach, for example, would be sufficient to obtain the critical loads of higher orders. Moreover, it often happens in the practical application of the energy method that the characteristic question is forgotten in the course of the calculations and that the variational problem which is finally obtained is solved in a purely formal way, supplying Euler's loads of all orders.

It is not difficult to formulate the characteristic questions in the four approaches in such a way that in the case of the column considered in Section 1.2 and in similar situations the results coincide, supplying the one and only buckling load. However, the fact remains that we are confronted with four entirely different questions. It is by no means evident that the answers to these four questions should always be the same. In fact, even in Euler's problem, when the questions have been reformulated so as to supply solely the critical load of the first order, some differences remain. According to the kinetic approach, for instance, there is no doubt that the column is unstable under any load $P > P_1$. One is tempted to draw the same conclusion from the energy method, but not the slightest information of this kind is supplied by either the imperfection or the equilibrium approach.

The situation is still less satisfactory if a wider class of problems is taken into consideration. There are numerous instances where the results of the various approaches are inconsistent. Some of these cases will be discussed in the remainder of this section.

An important branch of the theory of stability is concerned with rotating systems such as shafts. For an observer taking part in the rotation, at least part of the loading consists in the centrifugal and Coriolis forces. It follows that any critical state will be associated with an angular velocity, and it is, in fact, rather the critical angular velocities than the critical loads which have to be determined.

If the problem of the critical speed of a shaft of noncircular cross section is simplified and linearized the situation of Figure 1.11 is obtained. Here

Refer to Figure 1.11

the particle m, representing the concentrated mass of a disk fixed on the shaft, is supposed to be attracted by the axes of a coordinate system rotating uniformly with the angular velocity ω. The restoring forces, assumed to be linear and determined by the elastic constants c_1 and c_2, represent the rigidities of the shaft with respect to its two principal directions. It is obvious that the origin O is the only equilibrium position, and that it corresponds to a straight shaft with its axis coinciding with the z-axis. We are interested in its stability.

Since the restitutive forces are defined in a coordinate system rotating with the shaft, it is convenient to consider the problem as one of *relative* equilibrium or motion, introducing, besides $-c_1 x$, $-c_2 y$, the centrifugal force $m\omega^2 \bullet (x, y)$

and the Coriolis forces $2m\omega\bullet(\dot{y},-\dot{x})$. The Coriolis force does no work in an actual motion. Thus the total potential energy is

$$V=\frac{1}{2}[(c_1-m\omega^2)x^2+(c_2-m\omega^2)y^2)] \qquad (1.55)$$

Formulating Newton's law, we obtain the differential equations

$$\left.\begin{aligned}\ddot{x}-2\omega\dot{y}+(\frac{c_1}{m}-\omega^2)x=0\\ \ddot{y}-2\omega\dot{x}+(\frac{c_2}{m}-\omega^2)y=0\end{aligned}\right\} \qquad (1.56)$$

for the motion in the vicinity of the equilibrium position. The differential equations (1.56) are linear and homogenous and hence are solved by setting

$$x=Ae^{it},y=Be^{it}. \qquad (1.57)$$

Inserting this in (1.56), we obtain the homogeneous linear system of equations

$$\left.\begin{aligned}(\lambda^2+\frac{c_1}{m}-\omega^2)A-2\omega\lambda B=0\\ 2\omega\lambda A-(\lambda^2+\frac{c_2}{m}-\omega^2)B=0\end{aligned}\right\} \qquad (1.58)$$

for A and B. The trivial solution $A=B=0$ represents equilibrium at O. Nontrivial solutions, corresponding to motions in the vicinity of O, require that the determinant of (1.58) vanish, i.e., that the so-called *characteristic equation*

$$\Delta(\lambda)=\lambda^4+(\frac{c_1}{m}+\frac{c_2}{m}+2\omega^2)\lambda^2+(\frac{c_1}{m}-\omega^2)(\frac{c_2}{m}-\omega^2)=0 \qquad (1.59)$$

be satisfied. The roots of (1.59) are given by

$$\lambda_{1,2}{}^2=-\frac{1}{2}\frac{c_1+c_2}{m}-\omega^2\mp[\frac{1}{4}(\frac{c_2-c_1}{m})^2+2\frac{c_1+c_2}{m}\omega_2{}^2]^{1/2} \qquad (1.60)$$

Provided $\omega\neq 0$ and c_1 and c_2 are positive, the values λ_1^2 and λ_2^2 are real and have distinct absolute values. Thus the four roots of (1.59) occur in pairs: $\lambda_1, -\lambda_1$ and $\lambda_2, -\lambda_2$. They are either real or purely imaginary, and the four of them are distinct as long as none is zero. The general solution is obtained by superposition and is limited as long as all the roots are imaginary, i.e., so long as $\lambda_1^2 < 0$ and $\lambda_2^2 < 0$. In fact, if, for example, $\lambda_1^2 < 0$, the corresponding fundamental solutions (1.57) can be written in terms of $\cos\omega t$ and $\sin\omega t$ with $\omega = i\lambda_1$. If, on the other hand, $\lambda_1^2 > 0$, they may be replaced by $\cosh\lambda_1 t$ and $\sinh\lambda_1 t$. If, finally, $\lambda_1^2 = 0$ then λ_1 is a double root, and (1.57) has to be replaced by a slightly more complicated solution which can be shown to be unbounded.

In the special case when $c_1 \neq c_2$, let us assume that $c_1 < c_2$. Inspecting the coefficients in (1.59), we readily find, by means of Vieta's relations, that $\lambda_1^2 < 0$ and

$$\left.\begin{array}{l}\lambda_2{}^2 < 0, (\omega^2 < \dfrac{c_1}{m}, \omega^2 > \dfrac{c_2}{m}) \\ \lambda_2{}^2 > 0, (\dfrac{c_1}{m} < \omega^2 < \dfrac{c_2}{m})\end{array}\right\} \qquad (1.61)$$

It follows that, according to the kinetic approach, there is a single and finite domain $c_1/m \leq \omega^2 \leq c_2/m$ of critical angular velocities (Figure 1.12). On

Refer to Figure 1.12

the other hand, we conclude from (1.55) that V is positive definite only for $\omega^2 < c_2/m$. On the strength of the energy method, therefore, the infinite domain $\omega^2 \geq c_1/m$ ought to be critical. We finally see that nontrivial equilibrium positions, i.e., constant nonvanishing solutions of (1.56), exist if and

only if $\lambda = 0$ is a root of the characteristic equation (1.59). It follows that the equilibrium approach supplies the two limits, $\omega_1^2 = c_1 / m$ and $\omega^2{}_2 = c_2 / m$ of the unstable domain, but gives no indication concerning the region between them. The same result would be furnished by the imperfection method.
Experiments confirm the prediction of the kinetic approach. In fact, it will become evident that the equilibrium at O is stabilized, for $\omega^2 > c_2 / m$, by the Coriolis force, which does not appear in any one of the static methods. We learn from this example that there are simple problems of practical importance where the kinetic approach alone supplies a correct answer.
The case $c_1 = c_2 = c$, hitherto excluded, correspond to the more common situation of a shaft with a single flexural rigidity

(e.g., a shaft with circular cross section). Physically, the problem is still the same, but from a methodological point of view it is entirely changed. The limits of the unstable domain (Figure 1.12) now conicide. Thus, $\omega_1 = \omega_2 = \sqrt{c/m}$ is the only angular velocity that is possibly critical. However, for $\omega^2 = \omega^2_1 = c/m$, the differential equations of motion become

$$\ddot{x} - 2\omega_1\dot{y} = 0, \quad \ddot{y} - 2\omega_1\dot{x} = 0, \qquad (1.62)$$

and their general solution,

$$\begin{aligned} x &= A\cos 2\omega_1 t + B\sin 2\omega_1 t + C \\ y &= -A\sin 2\omega_1 t + B\cos 2\omega_1 t + D \end{aligned} \qquad (1.63)$$

is bounded. Here the kinetic approach fails to supply a critical value. In accordance with the energy method any $\omega > \omega_1$ ought to be critical, and the remaining approaches yield ω_1 as the only critical angular velocity.

Experimentally, ω_1 is confirmed as the only critical speed. It follows that there are problems where even the kinetic approach fails if imperfections are disregarded. This is confirmed when the problem when the problem is treated in a coordinate system at rest. Here it presents itself as a vibration problem, and there is no perturbation and hence no resonance so long as the system is perfect.

Another example is shown in Figure 1.13 The particle m moves in the fixed plane x, y under the influence of the attractions $-c_1 x, -c_2 y$ from the

Refer to Figure 1.13

coordinate axes and a force μr perpendicular to the radius vector and proportional to the distance from the origin O, which point is again the only equilibrium position. The force μr is not conservative. The potential of the remaining forces is

$$V = \frac{1}{2}(c_1 x^2 + c_2 y^2) \tag{1.64}$$

The differential equations of motion

$$\ddot{x} + \frac{c_1}{m}x + \frac{\mu}{m}y = 0, \qquad \ddot{y} - \frac{\mu}{m}x + \frac{c_2}{m}y = 0, \tag{1.65}$$

are again solved by (1.57). The equations obtained for A and B are

$$\left.\begin{aligned} (\lambda^2 + \frac{c_1}{m})A + \frac{\mu}{m}B = 0 \\ -\frac{\mu}{m}A + (\lambda^2 + \frac{c_2}{m})B = 0 \end{aligned}\right\} \tag{1.66}$$

and the characteristic equation (the condition for the existence of a nontrivial solution) is

$$\Delta(\lambda) = \lambda^4 + \frac{c_1 + c_2}{m}\lambda^2 + \frac{c_1 c_2 + \mu^2}{m^2} = 0 \qquad (1.67)$$

Its roots are given by

$$\lambda^2 = -\frac{1}{2m}\{c_1 + c_2 \pm [(c_1 - c_2)^2 - 4\mu^2]^{1/2}\} \qquad (1.68)$$

It is sufficient for our purpose to consider two special cases. If $c_1 = c_2 = c > 0$, then (1.68) reduces to

$$\lambda^2 = -\frac{1}{m}(c \pm i\mu) \qquad (1.69)$$

It is not difficult to see that, for arbitrary values $\mu \neq 0$, two of the roots, which now consist of two conjugate pairs, have positive real parts. The corresponding fundamental solutions can be written in terms of $e^{\gamma t}\cos\omega t$ and $e^{\gamma t}\sin\omega t$, where ω is real and $\gamma > 0$. It follows that, according to the vibration approach, the origin is unstable even though by (1.64) the potential energy is positive definite. Moreover, because of (1.67), none of the roots is zero, and this implies that the instability is not indicated by the presence of nontrivial equilibrium positions.

For $c_1 = 2\mu > 0$ and $c_2 = -\mu/3$, Equation (1.68) becomes

$$\lambda^2 = -\frac{1}{6}(5 \pm \sqrt{13})\frac{\mu}{m} \tag{1.70}$$

Since now both values of λ^2 are negative, the roots are purely imaginary, and the system is stable although V is not definite.

The cases considered so far were concerned with particles and hence were especially simple. They revealed shortcomings of the classical approaches which are fundamental and may be found also in more complicated systems.

As an example, let us consider a column loaded tangentially (Figure 1.14). This is only a slight modification of Euler's problem in Case 3: the load P, still of constant magnitude, is assumed to be tangential to the deflection curve in the process of buckling instead of retaining of retaining its original direction.

Refer to Figure 1.14

In an assumed nontrivial equilibrium position (Figure 1.14), the sign of the bending moment caused by the tangential load does not agree with the sense of curvature. Accordingly, there is no nontrivial equilibrium position, and the equilibrium approach does not indicate any danger of buckling. In order to confirm this result, we note that (for small inclinations) the differential equation of the deflection curve is

$$\alpha y'' = P(y_l - y) - Py'_l(l - x) \tag{1.71}$$

and that the end conditions are

$$y(0) = y'(0) = 0, \quad y(l) = y_l\,, \quad y'(l) = y'_l \tag{1.72}$$

The general solution of (1.71) is

$$y = A\cos\kappa x + B\sin\kappa x - y'_l(l - x) + y_l \tag{1.73}$$

where is κ is given by (1.2). The end conditions yield the linear and homogenous system

$$\left.\begin{aligned} A - y'_l\, l + y_l &= 0 \\ B\kappa + y'_l &= 0 \\ A\cos\kappa l + B\sin\kappa l &= 0 \\ -A\kappa\sin kl + B\cos\kappa l &= 0 \end{aligned}\right\} \tag{1.74}$$

for A, B, y'_l, y_l . A nontrivial equilibrium configuration corresponds to a nontrivial solution of this system. Equating the determinant to zero, we obtain

$$\begin{vmatrix} 1 & 0 & -l & 1 \\ 0 & \kappa & 1 & 0 \\ \cos\kappa l & \sin\kappa l & 0 & 0 \\ -\kappa\sin\kappa l & \kappa\cos\kappa l & 0 & 0 \end{vmatrix} = 0 \tag{1.75}$$

If we develop this with respect to the last column, we are left with a three-row determinant. Developing once more with respect to the last column, we find

$$\kappa(\cos^2 \kappa l + \sin^2 \kappa l) = \kappa = 0 \tag{1.76}$$

This confirms the fact that there is no nontrivial equilibrium configuration for $P > 0$. It seems rather improbable, however, that the column should not buckle. It will be shown in Section 5.2 that the static approaches are illegitimate in this problem and the column actually buckles under a critical load obtained by means of the vibration method.

Another example, where it is clear a priori that any static approach must fail, is of a column loaded by a pulsating compression, for example, by

$$P = Q + S\cos\omega t\;,\;(Q>0, S>0) \tag{1.77}$$

Case 5(Table 1.1) is particularly simple. The differential equation of motion and the end conditions have been established in Section 1.2. They are given by (1.50), (1.47), and (1.51), where now P is no longer constant but has to be inserted from (1.77). The solution will be given in Section 6.2.

We conclude from there examples that even in a linear theory there are numerous cases where one or more of the approaches discussed in Section 1.2 fail, or where the results obtained by different methods are inconsistent. Additional complications are to be expected in nonlinear problems. Our next aim is to establish a method which is valid without restriction, and it is obvious that such a method must be based on a concept of stability defined more rigorously than has so far been done.

Problems

1. Show by means of the kinetic approach that in Figure 1.11 the limits of the unstable domain $c_1 / m < \omega^2 < c_2 / m$ are themselves unstable.

2. Consider the particle m, attracted by the axes of a rotating coordinate system $x, y,$ as a model of a disk on a shaft with two distinct flexural rigidities. Introduce an

Refer to Figure 1.15

imperfection (Figure 1.15) by assuming that the restoring forces act at $x-a$ and $y-b$, where a and b are constants. Show that the result is the one obtained by the equilibrium approach.

3. Treat the problem of Figure 1.11 in a coordinate system at rest and show that, for $c_1 = c_2 = c$, the equilibrium at O is stable as long as there are no imperfections.

4. Treat the tangentially loaded column of Figure 1.14 by means of the imperfection approach, assuming the presence of a small eccentricity.

5. Show some stages of the deformation for the column of Problem 5 above by plotting the deflection curve for $\kappa l = n\pi / 2 (n = 1, \ldots, 6)$.

1.4. The Stability Concept

The approaches considered so far are concerned with (1) the deflection of the imperfect system, (2) the nontrivial equilibrium configurations of the perfect system, (3) the potential energy, and (4) the motion after the equilibrium has been momentarily disturbed. Actually, we are not interested in nontrivial equilibrium configurations or in the potential energy. The only things we are concerned with are large displacements, measured from the equilibrium configuration in the unloaded state, since they are apt to endanger the system.

The loading of a given system may consist of a number of forces. If we assume that they are increased proportionally from zero, the instantaneous state of loading may be described by a parameter P which will be briefly referred to as the *load*. While P is increased, there occur two possible reasons for large displacement. In the first place, the equilibrium configuration of the system is altered by the application of the load. The corresponding displacements are often small and proportional to the loading. This is the case in the situation which is commonly treated in the linear theory of elasticity. If the increase of the displacements is faster and if, for a certain value of the load, the displacements become excessive or even infinite (as in the case of Figure 1.1), we talk of a *latent instability of the system*, since this effect is usually caused by imperfections and is overlooked if imperfections are not taken into account. The term *kinetic instability* describes the same effect. The system will be referred to as *unstable* and its load as *critical* whenever a static or a kinetic instability occurs. The smallest critical load will be denoted by P_1.

The two types of instability are different. It is true that in certain cases, as in Euler's problem (Section 1.2), they occur under the same load. However, in many other problems (e.g., in some of the cases treated in Section 1.2) they are independent and must be considered separately. It is clear that instabilities of the equilibrium are always obtained by the kinetic approach. In order to take care of latent instabilities, however, the possibility of imperfections has to

be kept in mind.
To make our definitions more precise, let us consider a system with a finite degree of freedom, n, subjected to constant loads. Let the coordinates $\bar{q}_k (k = 1 \ldots n)$ be measured from the equilibrium configuration $\bar{q}_k = 0 (k = 1 \ldots n)$ of the unloaded system. Then the equilibrium of the loaded system will be described by the coordinates

$$\bar{q}_k = a_k(p) \qquad (k = 1, 2 \ldots n) \tag{1.78}$$

where the $a_k(p)$ are functions of the loading. A *static instability* is described by excessive values of at least one $|a_k(p)|$. What values are to be considered as excessive depends on the system and on its allowable stresses. In many cases (as in the problem of Figure 1.1) the decision is simplified by the fact that the $|a_k(p)|$ become infinite under a certain finite loading.

For the study of the motion caused by perturbations, let us use a new set of coordinates,

$$q_k = \bar{q}_k - a_k \quad (k = 1, 2, \ldots n) \tag{1.79}$$

measured from the equilibrium configuration of the loaded system. The corresponding generalized velocities are

$$\dot{q}_k = \dot{\bar{q}}_k \qquad (k = 1, 2, \ldots n) \tag{1.80}$$

We confine our attention to free motions,

$$q_k = q_k(t) \qquad (k = 1 \,,\, n2 \tag{1.81}$$

caused by single perturbations of the type that can be described by a set of initial conditions

$$q_k(0) = q_{k0} \quad , \dot{q}_k(0) = \dot{q}_{k0} \, , \qquad (k = 1, 2, ...n) \qquad (1.82)$$

With these preparations we might define *kinetic stability* in the loaded state by the requirement that all the functions $|q_k(t)|$ remain arbitrarily small for any time $t > 0$ provided the initial values $|q_{k0}|$ and $|\dot{q}_{k0}|$ have been chosen sufficiently small. This definition, however, is asymmetric; although the $|\dot{q}_{k0}|$, in addition to the $|q_{k0}|$, are required to be sufficiently small, no condition resembling the one for $|q_k(t)|$ is formulated for the $|\dot{q}_k(t)|$. It is convenient to use a *symmetric definition* and to define stable equilibrium by the conditions that the $|q_k(t)|$ and the $|\dot{q}_k(t)|$ remain arbitrarily small for any $t > 0$, provided the $|q_{k0}|$ and the $|\dot{q}_{k0}|$ are chosen sufficiently small.

There is a simple way of interpreting this definition in geometrical terms. In a euclidean space of 2*n* dimensions, the so-called *phase space* with cartesian coordinates $q_{k0}, \dot{q}_{k0} (k = 1,...n)$, the configuration and the state of motion of the system are represented by a *phase point* $P(q_k, \dot{q}_k)$. The motion of P in phase space describes the motion of the system in physical space. If $\vec{\mathbf{q}} = (q_k, \dot{q}_k)$ denotes the radius vector of P in phase space, the equation

$$\mathbf{q}^2 = q_1^{\ 2} + \cdots + q_n^{\ 2} + \dot{q}_1^{\ 2} + \cdots + \dot{q}_n^{\ 2} = \eta^2 \qquad (1.83)$$

where η is a constant, represents a hypersphere of radius * and center *O*. According to the definition given above, the equilibrium, corresponding to the origin in phase space, is stable if and only if the phase point remains within a hypersphere (1.83) of arbitrarily small radius, provided its initial position is within a hypersphere of sufficiently small radius and center *O*.

These definitions refer to systems with a finite degree of freedom *n*. Many systems of practical interest, and in particular structures(e.g., Euler's column), are continuous and hence have an infinite degree of freedom. Strictly speaking, one is not allowed to generalize the results obtained for finite *n* and to use them in cases where *n* is infinite. As a matter of fact, Shield and Green [58] have pointed out some difficulties inherent in this generalization. The problem is still being discussed. Recent work by Shield [59] and by Koiter [31,32] indicates that a satisfactory solution will eventually be found. In any case there is not much choice at the present time. It seems that the generalization is reliable in most cases if carried out with due caution. A case which is critical in a certain respect will be discussed in Section 2.7.

From the definitions given above it follows that the stability of the equilibrium should be investigated by means of the kinetic method. In order to establish the path of a phase point released sufficiently close to the origin, the differential equations of motion must be integrated. At the same time the possibility of latent instabilities, caused by imperfections of the system, should be kept in mind.

In most cases of practical interest it is cumbersome to carry out this program. The static approaches are considerably simpler. Although their connectior with our definitions is not obvious, it is a fact that they have been applied to structures, until very recently, with amazing success. Thus the question arises in what cases and why the kinetic method, which is the direct one, may be replaced by the simpler static approaches. The answer to this question calls for a classification of mechanical systems based on the forces to which they are subjected.

Problem

1. Interpret the motion of a harmonic oscillator, having a single degree of freedom and the circular frequency κ, in the phase plane.

1.5. Forces and Systems

Figure 1.16 shows a force $\vec{F}$ whose point of application P

is deplaced from P_1 to P_2 along the curve C. The work of this force

$$W = \oint_{p_1}^{p_2} \mathbf{F} \bullet d\mathbf{r} \tag{1.84}$$

Refer to Figure 1.16

is usually introduced as a line integral, under the tacit assumption that $\vec{F}$ is a stationary field force $\vec{F}(\vec{r})$, defined without reference to the body on which it acts.
Many forces(e.g., pulsating loads) also depend on time. They are functions of the form $\vec{F}(\vec{r},t)$ and are referred to as instationary. Other forces (as the drag of a projectile) depend on the velocity $\vec{v} = \dot{\vec{r}}$ of their point of application and hence

have the form $\vec{F}(\vec{r},\vec{v})$. In such cases the work (1.84) is not a mere line integral: it cannot be computed unless the motion $\vec{r}(t)$ of the point of application is known. If this is the case, the work becomes a time integral

$$W = \int_{t_1}^{t_2} \mathbf{F}\bullet\mathbf{v}\,dt = \int_{t_1}^{t_2} L\,dt \tag{1.85}$$

where the times t_1 and t_2 correspond to the positions P_1 and P_2 on C and

$$L = \mathbf{F}\bullet\mathbf{v} \tag{1.86}$$

is the power or rate of work of $\mathbf{F}$.

Again, many forces, particularly all kinds of reactions, cannot be defined without reference to the body on which they act. If this body is finite, a question arises in connection with the point of application. In the case of a rolling wheel

(Figure 1.17) there are three points A which might be considered as possible points of application of the normal pressure N and the static friction F : the material points (on the wheel and on the guiding body) in which constant takes place, and the instantanuous center of the wheel. The first two points are instantaneously at rest; the third one moves with the velocity v. A similar problem arises in connection with the string force S in Figure 1.18.

Refer to Figure 1.17

It is easy to check that in all applications of the concept of work(energy principle, principle of virtual work, equations of Lagrange, etc.) it is always the displacement of the material point of the moving body which is used in the proofs. Thus (1.84) and (1.85) have to be supplemented by the statement that $d\vec{r}$ or $\vec{v}$ refers to the material point of application on the body in consideration.

Refer to Figure 1.18

Certain forces are usually referred to as conservative because they are compatible with the notion of conservation of energy in a purely mechanical sense. The current definition, requiring that the work (1.84) be independent of the curve *C* (Figure 1.16) connecting P_1 with P_2, is meaningful only for stationary field forces defined without reference to the body on which they act. There are not many forces of this kind and hardly any systems containing no other types of forces. In order to take care of real situations, let us define a *conservative force* by the condition that its work W in any admissible displacement of the system on which it acts depends solely on the initial and final configurations of the system.

Let the coordinates of a system be denoted by $q_k (k = 1 \ldots n)$. Unless a statement to the contrary is made, we shall restrict ourselves to *holonomic* and *scleronomic systems* [86,pp. 43 and 45]. Because of the first limitation, the most general admissible displacement is described by an arbitrary set of increments dq_k of the coordinates. Similarly, any set of generalized velocities $\dot{q}_k$ represents an admissible state of motion. On account of the second restriction, the radius vector of an arbitrary material element of the system has the form $\vec{r}(q_k)$, that is, it does not depend explicitly on the time. It follows that

$$d\mathbf{r} = \sum_{k=1}^{n} \frac{\partial \mathbf{r}}{\partial q_k} dq_k \qquad (1.87)$$

Hence, the elementary work in a real displacement of a force acting on the system is

$$dW = \mathbf{F} \cdot d\mathbf{r} = \sum_{k=1}^{n} P_k dq_k \qquad (1.88)$$

where

$$P_k = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_k} \qquad \text{(k=1,2,.....,n)} \qquad (1.89)$$

Similarly, the rate of work in a real motion is

$$L = \sum_{k=1}^{n} P_k \dot{q}_k \qquad (1.90)$$

If F is conservative, and $V(q_k)(k=1...n)$ denotes its potential, the elementary work is also given by

$$dW = -dV = -\sum_{k=1}^{n} \frac{\partial V}{\partial q_k} dq_k \qquad (1.91)$$

Comparing (1.88) with (1.91), we find

$$P_k = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_k} = -\frac{\partial V}{\partial q_k} \qquad \text{(k=1,2,....n) (1.92)}$$

Now $\vec{r}$ and V are independent of the generalized velocities $\dot{q}_k$ and the time t. It follows from (1.92) that, in general, only forces that depend exclusively on the configuration of the system can be conservative.

It will become evident in the following chapters that, in connection with instability problems, it is important to know whether or not a given force is conservative. With this purpose in mind, let us set up a list of the types of forces we must expect. In the first place we distinguish between *active forces* or *loads* and *reactive forces* or *reactions*. The first are given a priori as functions of the q_k, $\dot{q}_k$ and t. The latter are known a priori and have to be obtained along with the motion, by integrating the differential equations of motion.

In a scleronomic system the work of the *reactions* is never positive. Reactions can therefore be classified (Table 1.2) as either *nonworking* (e.g., normal pressure and static dry friction) or *dissipative* (doing negative work, e.g., kinetic dry friction). According to our definition the nonworking reactions have to be considered as conservative. They represent a first exception from the rule given above since they cannot be derived from a potential energy.

Refer to Table 1.2

Table 1.3 gives a similar classification of *loads*. Those that depend explicitly on the time (e.g., pulsating loads) are called *instationary* and never conservative. Stationary loads may be classified as either velocity-dependent or velocity-independent. In the first case the dependence on the velocity may

Refer to Table 1.3

be such that the work in a real motion (as is the case with

Coriolis forces, Lorentz forces, and gyroscopic moments) is always zero. Such loads are called *gyroscopic*; they are obviously conservative, and they represent a second exception from the rule given above. Velocity-dependent loads doing positive work are unimportant in the present context; those doing negative work (e.g., fluid friction or air drag) are referred to as *dissipative*. Among the velocity-independent loads there are those which can be derived from a stationary, single-valued potential(e.g., gravitational forces). Borrowing an expression from hydrodynamics, we will call them *noncirculatory*. Loads of this type are conservative. All other velocity-independent loads will be referred to as *circulatory*.

Circulatory loads are more common than is usually supposed.

They are doing work on bodies moving cyclically through the same positions and hence play an important role in power-transmitting devices such as cranks,

Refer to Figure 1.19

shafts, pulleys, etc. The force $\vec{F}$ shown in Figure 1.19 is assumed to be rigidly connected with a rigid body and hence is constant for an observer moving with the body. In the translation $A->B$ it does a certain amount of work. If, however, the translation is preceded and followed by rotations through $\pi/2$ and $-\pi/2$, respectively, the same final position is attained without work; thus, $\vec{F}$ is circulatory.

A couple of constant moment vector $\vec{M}$, as shown in Figure 1.20, is likewise circulatory. In fact, in a rotation through * about the

Refer to Figure 1.20

axis z the work of $\vec{M}$ is positive. In two successive rotations through π about the axes x and y the same final position is attained, but now the work of $\vec{M}$ is zero.
If all the forces (loads and reactions) acting on the system are conservative, we talk of a *conservative system*. (In Tables 1.2 and 1.3 the conservative forces are underscored.) A conservative system contains no other forces than nonworking reactions and loads of the noncirculatory and

gyroscopic types. The systems in which we are interested always contain nonworking reactions and noncirculatory loads. In Euler's problem, for example, all reactions are nonworking, and the external load *P* as well as the elastic stresses (the internal loads) are noncirculatoty. If, in addition, gyroscopic loads are present, we shall talk of a *gyroscopic* conservative system; if not, we will refer to it as *nongyroscopic*.

On the other hand, a system is *nonconservative* if it contains at least one nonconservative force, i.e., a dissipative reaction or a load of the dissipative, circulatory, or instationary types. According to which of these is the case, we shall refer to the system as *dissipative*, *circulatory*, or *instationary*. Instationary systems are also called *heteronomous*, whereas

the term *autonomous* is used for systems free of instationary forces.
In the most general case, forces of all kinds will be represent. If, however, besides conservative forces (which are practically always present) only dissipative forces occur, we shall call the system *purely* dissipatvie. In a similar sense we may refer to a system as purely circulatory or purely instationary.
If the system is conservative, the total work of the internal and external forces can be represented by a potential energies of all loads. By analogy to (1.88) and (1.92), the elementary work in a real motion then is given by

$$dW = \sum_{k=1}^{n} P_k dq_k \qquad (1.93)$$

where

$$P_k = -\frac{\partial V}{\partial q_k} \qquad (k=1,2,...,n) \qquad (1.94)$$

Moreover, the principle of virtual work, applied to the actual motion, supplies the energy theorem in the form

$$\frac{dT}{dt} = L \qquad (1.95)$$

where T is the kinetic energy of the system and L is the total rate of work. Integrating (1.95) we finally obtain the theorem of conservation energy

$$T + V = E \qquad (1.96)$$

where E is the constant total energy of the system. This theorem is valid for any conservative system, whether gyroscopic or not.

As a first example, Figure 1.21 shows a particle m moving in a uniformly rotating coordinate system under the action of its weight. Obviously, the weight $\vec{W} = (0,0,-mg)$ and the centrifugal force $\vec{Z} = m\omega^2 \times (x, y, 0)$ are noncirculatory, and the Coriolis force $\vec{C} = 2m\omega \times (\dot{y}, -\dot{x})$ is gyroscopic. Thus the system, i.e., the particle, is conservative; if $\omega \neq 0$, the particle is also gyroscopic. If $\omega = \omega(t)$, $\vec{Z}$ and $\vec{C}$ become instationary and hence the particle is heteronomous. If air drag is taken into account, the particle is dissipative.

The double pendulum of Figure 1.22, equipped with restoring moments acting in the hinges, may be considered a simplified model of an elastic rod

Refer to Figure 1.21

built in at the upper end. So long as the force $\vec{F}$ is absent, the system is conservative and nongyroscopic. With $\vec{F}$ acting in the axis of the second pendulum, it becomes purely circulatory, since $\vec{F}$ is a force of the type shown in Figure 1.19. The spherical pendulum (Figure 1.23), if acted upon by a

Refer to Figure 1.22

couple of constant moment vector (as discussed in connection with Figure 1.20), is likewise circulatory.

In Table 1.4 we once more find the classification of mechanical systems as either conservative system is either nongyroscopic or gyroscopic. In the second class only the most important subclasses are listed: those of the purely dissipative, purely circulatory, and purely instationary types. The examples added in each column show that in all five subclasses practically significant stability problems arise. Incidentally, the problem of critical speed, shown in column two, is gyroscopic only if treated in a coordinate frame rotating with the shaft. Treatment of the buckling problem under the influence of

Refer to Figure 1.23

damping (column three) seems uncommon, but is perfect order once the kinetic nature of the stability concept (Section

1.4) has been recognized.

Refer to Table 1.4

Problem

1. An automobile is driven around a corner with increasing speed. On the floor of the trunk a loose nut is rolling. Introduce all forces acting on the nut (from the viewpoint of the driver) and classify them.

1.6. Lagrange's Stability Theorem

In a holonomic system the increment δq_i of a single coordinate δq_i represents an admissible displacement.

Applying the principle of virtual work in turn to these elementary displacements, one obtains Lagrange's equations [86, p.180]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i \qquad \text{(i=1,2,...,n)(1.97)}$$

where the generalized forces Q_i are the coefficients of the * in the expression of the virtual work

$$\delta W = \sum_{k=1}^{n} Q_i \delta q_i \qquad (1.98)$$

In the case of a conservative system, most texts claim that the generalized forces can be expressed by

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad (1.99)$$

in terms of the potential energy. In this way, the special form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad (i=1,2,...,n) \quad (1.100)$$

of Lagrange's equations is obtained, where

$$L = T - V \qquad (1.101)$$

is the so-called kinetic potential of the system.

Actually, Equations (1.99) and therefore (1.100) only apply if the conservative system is nongyroscopic. It has been shown in (1.88) and (1.92) that the actual work in a real motion of a conservative system is

$$dW = \sum_{i=1}^{n} P_i dp \quad \text{w h e} \; P_i = -\frac{\partial V}{\partial q_i} \qquad (1.102)$$

Here, the forces and displacements belong to the same motion, namely, the actual one. On the other hand, the principle of virtual work is based on the expression (1.98) of the virtual work, where the Q_i are obtained from the real forces (i.e., the ones acting in the real motion), while the * represent the virtual displacement (which is admissible but arbitrary). Here, the forces and the displacements belong to different motions. Hence, the second equation (1.102) does not necessarily imply (1.99). It is true that the Q_i coincide with the P_i so long as the forces acting in the system only depend on the q_i and on t. However, they are different as soon as forces dependent on the $\dot{q}_i$ are present. The only such forces admitted in a conservative system are gyroscopic. Their work in an actual displacement is always zero; in a virtual displacement, the work vanishes only exceptionally. Since they do not occur in the potential energy, these forces would get lost if Lagrange's equations were applied in the form of Equations (1.100) and (1.101)

Let

$$Q_i = P_i + G_i \qquad (i=1,2,3,...,n) \qquad (1.103)$$

where the P_i are defined by (1.102), whereas the generalized gyroscopic forces G_i follow from their virtual work

$$\delta W_g = \sum_{k=1}^{n} G_i \delta q_i \qquad (1.104)$$

It then follows from (1.97) that, for a gyroscopic conservative system, Lagrange's equations can be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = G_i \qquad (i=1,2,...,n)(1.105)$$

As an example, consider the particle of Figure 1.21 moving under its own weight in a uniformly rotating coordinate frame. The kinetic energy is

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad (1.106)$$

The potential energy of the weight and the centrifugal force is

$$V = -\frac{1}{2} m\omega^2 (x^2 + y^2) + mgz \qquad (1.107)$$

Thus the kinetic potential becomes

$$L = T - V = \frac{1}{2} m[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \omega^2 (x^2 + y^2) - 2gz] \qquad (1.108)$$

The Coriolis force is conservative but gyroscopic. From its virtual work

$$\delta W_g = 2m\omega(\dot{y}\delta x - \dot{x}\delta y) \qquad (1.109)$$

we obtain the generalized gyroscopic forces

$$G_x = 2m\omega\dot{y} \qquad G_y = -2m\omega\dot{x} \qquad G_z = 0 \quad (1.110)$$

If (1.108) and (1.110) are inserted, Lagrange's equations (1.105) supply the differential equations of motion,

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x = 0, \quad \ddot{y} - 2\omega\dot{x} - \omega^2 y = 0, \quad \ddot{z} + g = 0 \quad (1.111)$$

They include the gyroscopic terms and are easily checked by means of Newton's law.

Lagrange's equations are a powerful tool whenever the stability of an equilibrium configuration is to be investigated by means of the kinetic method. It has been stated at the end of Section 1.4 that this is not always necessary. In fact, a first shortcut in the form of a general theorem has been provided by Lagrange. This theorem, usually formulated for nongyroscopic conservative systems (i.e., for systems belonging to the first subclass in Table 1.4.), is actually more general [84]; it applies to all systems containing only (nongyroscopic or gyroscopic) conservative and dissipative forces, i.e., to systems of the first three subclasses in Table 1.4.

Let the total energy (1.96) of the system be a continuous function of the coordinates q_k and the generalized velocities $\dot{q}_k$. Equation (1.83) represents a hypersphere of radius about the origin *O* in phase space. According to a well-known theorem of Weierstrass, the energy *E* assumes a minimum value ε on this sphere. In scleronomic systems, the kinetic energy *T* is a positive definite function of the $\dot{q}_k$, dependent also the q_k. If *V* is a positive definite function of the q_k, then $E(q_k, \dot{q}_k)$ is positive definite in phase space. It follows that $\varepsilon > 0$ provided η is sufficiently small. Since E is a continuous function, the initial values q_{k0} and $\dot{q}_{k0}$ can always be chosen so as to make $E < \varepsilon$. In a system containing only conservative and dissipative forces, E does not increase. It follows that $E \le E_o \le \varepsilon$. The phase point remains inside the hypersphere of radius η, and hence the equilibrium is stable. We thus have proved the:

THEOREM OF LAGRANGE. *Provided the total energy is continuous, the equilibrium of a system containing only conservative and dissipative forces is stable whenever the potential energy is positive definite.*

Note that the inversion of this theorem is not true. The particle shown in Figure 1.11 is conservative. Its potential energy (1.55) is positive definite so long as $\omega^2 < c_1 / m$, and it has been shown (Figure 1.12)that, in this range, the equilibrium is, in fact, stable. For $\omega^2 > c_2 / m$, however, the equilibrium again becomes stable although *V* is negative definite.

Lagrange's theorem is the only general principle supplying immediate results for an extensive class of systems, whether or not their differential equations are linear. In order to obtain more results of a fairly general nature, it is convenient to linearize the differential equations of motion. We have repeatedly made use of this possibility, e.g., in connection with the particle of Figure 1.5. Even though such a linearization is not always legitimate (and, in certain cases, may prove impossible), it provides a good starting point for any investigation of a more refined nature. It is true that, with respect to static instabilities, defined by large rather than small values of the displacements, a given system and its linearized version may behave differently. Figure 1.3 shows that a static instability may be less detrimental than it appears from a study of the linear case. There are also examples where the opposite is true. On the other hand, it is usually possible generalize the results obtained with respect to the kinetic instability of a given linear system, since here the definition is based on small values of the $|q_k|$ and $|\dot{q}_k|$. In fact, Lyapunov [44], Chetayev [9]. and others (see, for example [45,10]) have established a number of stability theorems for nonlinear systems, which, although not quite as general as Lagrange's principle, will allow us to transfer most of our essential results from linear to nonlinear systems. Therefore, it seems that a reasonable plan to be adopted in any stability investigation and, in particular, in the structure of the following chapters consists of first considering the linear system and then proceeding to the nonlinear one. Of course, this implies that we restrict ourselves to systems that can be linearized. Moreover, we have to keep in mind that, with respect to static instabilities, a generalization usually is impossible.

Problems

1. Check the differential equations (1.65) of the particle shown in Figure 1.13 by means of Lagrange's equations.
2. Show that the particle in Figure 1.24, moving between prestrained springs, does not obey a single linear differential equation of motion.

Refer to Figure 1.24

1.7. Linear Systems

Whenever the motion of a system is governed by a single set of linear differential equations, containing an equation for each degree of freedom, the system will be referred to as *linear*. If we denote the coordinates by $\bar{q}_k$ and exclude the unimportant (and practically nonexistent) case where the forces depend on $\dddot{\bar{q}}_k, \ddddot{\bar{q}}_k, \ldots$ the differential equations of motion have the form

$$\sum_{k=1}^{n}(m_{ik}\ddot{\bar{q}}_k + g_{ik}\dot{\bar{q}}_k + c_{ik}\bar{q}_k) + h_i = 0 \qquad (i=1,2,...,n) \qquad (1.112)$$

where the m_{ik}, g_{ik}, c_{ik} and h_i depend on the properties of the system and on its loading. For the examples treated in the foregoing sections this is readily verified. The various types of systems listed in Table 1.4 can be distinguished in (1.112) by means of the properties of the coefficients.

If at least one of the coefficients or one of the quantities h_i is a function of time, the system is obviously *heteronomous*.

In order to obtain the various classes of *autonomous* systems, we restrict ourselves from now on to constant coefficients and constant values of h_i. The general solution of the inhomogeneous system (1.112) is obtained by superposition of a particular integral and the general solution of the reduced (homogeneous) system

$$\sum_{k=1}^{n}(m_{ik}\ddot{q}_k + g_{ik}\dot{q}_k + c_{ik}q_k) = 0 \qquad \text{(i=1,2,...,n)} \quad (1.113)$$

An especially simple particular solution of (1.112) is given by

$$\bar{q}_i = a_i \qquad \text{(i=1,2,...,n)} \quad (1.114)$$

where the a_i are constants satisfying the linear equations

$$\sum_{k=1}^{n} c_{ik}a_k = -h_i \qquad \text{(i=1,2,...,n)} \quad (1.115)$$

Exactly the same relations (1.114) and (1.115) are obtained if we inquire about the equilibrium configuration corresponds to a particular integral of the type (1.114). Moreover, if a new set of coordinates,

$$q_i = \bar{q}_i - a_i \qquad \text{(i=1,2,...,n)} \qquad (1.116)$$

is introduced, measured from the equilibrium configuration a_i, the system (1.112) reduces to (1.113). Thus, the general solution of the reduced system (1.113) represents the motion in the vicinity of the equilibrium configuration a_i.

It is obvious that the distinction between the two solutions just discussed corresponds to the distinction, introduced in Section 1.4, between latent instabilies of the system (static instabilities) and instabilities of the equilibrium (kinetic instabilities). It is always possible to introduce the coordinates $\bar{q}_k$ in such a way that the h_i in (1.112) are zero so long as the system is unloaded. If the h_i are nonzero in loaded state, they represent constant generalized forces in the sense of Lagrange. They determine, in conjunction with the c_{ik}, the equilibrium configurations of the loaded system. If the h_i remain zero in the process of loading, the system may be referred to as *perfect*; if at least one h_i becomes nonzero, the system is *imperfect*.

According to (1.115), the perfect system always has the trivial equilibrium configuration $a_i = 0 (i = 1, \ldots, n)$, whereas any equilibrium configuration of the loaded imperfect system is nontrivial . So long as the determinant of the matrix (c_{ik}) is different from zero, the perfect system has only the trivial equilibrium position; also the equilibrium configuration of the imperfect system is unique and finite; actually, the displacements from the trivial equilibrium configuration are small provided the imperfections are small.

When the determinant of the c_{ik} becomes zero, the perfect system admits nontrivial equilibrium configurations. At the same time the a_i of the imperfect system (or, at least, some of them) can become infinite. Thus, a *static instability* arises and is indicated by the appearance of nontrivial equilibrium configurations of the perfect system, i.e., by the equilibrium method.

The motion of the system in the vicinity of the equilibrium configuration is described by the homogenous linear system (1.113). According to the definition given in Section 1.4, the equilibrium is stable so long as the $|q_i(t)|$ and $|\dot{q}_i(t)|$ remain arbitrarily small, provided the $|q_{i0}|$ and $|\dot{q}_{i0}|$ are chosen sufficiently small. Now, any solution of (1.113) remains a resolution when all the $|q_i(t)|$ are multiplied by an arbitrary constant. Thus, the equilibrium is stable if and only if the $|q_i(t)|$ and $|\dot{q}_i(t)|$ remain bounded, provided the $|q_{i0}|$ and $|\dot{q}_{i0}|$ are finite.

In order to solve (1.113), we set

$$q_k = A_k e^{\lambda t} \qquad \text{(i=1,2,...,n)} \qquad (1.117)$$

Inserting this in (1.113), we obtain the homogenous linear system

$$\sum_{k=1}^{n} (m_{ik}\lambda^2 + g_{ik}\lambda + c_{ik})A_k = 0 \qquad \text{(i=1,2,...,n)} \qquad (1.118)$$

for the A_k. The trivial solution $A_k = 0 (k = 1,\dots,n)$ corresponds to the equilibrium state. Nontrivial solutions exist if and only if the determinant vanishes, i.e., if the so-called *characteristic* *equation*

$$\det(m_{ik}\lambda^2 + g_{ik}\lambda + c_{ik}) = 0 \tag{1.119}$$

is satisfied. Provided the $2n$ roots λ_j of (1.119) are distinct and real, they supply $2n$ real fundamental solutions of the form (1.117). By superposition, the general solution, containing $2n$ constants, is obtained; it can be adapted to arbitrary initial conditions by a suitable choice of the constants. If the roots are distinct but (at least, in part) complex, they occur in conjugate complex pairs $\lambda = \lambda' \pm i\lambda''$. The corresponding A_k, obtained from (1.118), are also conjugate complex, i.e., of the form $A_k' \pm iA_k''$, and the corresponding pairs of fundamental solutions (1.117) appear in the form

$$q_k = e^{\lambda' t}[(A'_k \pm iA''_k)\cos\lambda'' t - (A''_k \mp iA'_k)\sin\lambda'' t] \quad (1.120)$$

Since their real as well as their imaginary parts must satisfy (1.113), they may be replaced by the real and independent fundamental solutions

$$B_k \cos\lambda'' t \qquad \text{and} \qquad C_k e^{\lambda' t}\sin\lambda'' t \qquad (1.121)$$

A *kinetic instability* arises if there is at least one set of initial conditions resulting in an unbounded motion, i.e., if at least one of the fundamental solutions is unbounded. The kinetic approach thus finally consists of an investigation of the roots of the characteristic equation. According to (1.117) and (1.121), a positive real part in any root implies an instability.

If the characteristic equation has multiple roots, Equation (1.117) does not always supply a complete set of fundamental solutions. A discussion of this case has been given by Routh [57]. The additional fundamental solutions then contain powers of t multiplied by the exponential functions themselves, except when λ is purely imaginary or zero. Thus, any root with a zero real part may imply an instability.

It is worth noting that the kinetic approach in the form just described also supplies the static instabilities and thus renders an additional investigation by means of the imperfection approach unnecessary. It has been pointed out above that a static instability is always indicated by the appearance of a nontrivial equilibrium configuration in the perfect system, i.e., by a constant solution of (1.113). Such a solution is of the form of (1.117) with $\lambda = 0$. Thus, static instabilites correspond to vanishing roots of the characteristic equation.

It follows from the foregoing discussion that the linear system is stable so long as all the roots of the characteristic equation (1.119) have negative real parts. If at least one of the roots has a positive real part, the system is unstable. A vanishing root indicates a static and possibly also a kinetic instability. If there no roots with positive real parts but at least one pair of purely imaginary roots, we are confronted with a *critical case*: the system may be stable or unstable.

The roots of the characteristic equation (1.119) are determined by the matrices $(m_{ik}),(q_{ik})$, and (c_{ik}) of the coefficients occuring in (1.113). It has been noted that these matrices reflect the properties of the system with which we are concerned. It is therefore to be expected that the various types of systems discussed in Section 1.5 distinguish themselves by their stability properties.

In order to establish the connections between the various systems and the matrices in (1.113), let us suppose that these differential equations have been obtained by formulating Lagrange's equations (1.97). The acceleration terms $m_{ik}\ddot{q}_k$ stem from the kinetic energy, which, in the linear case, is a positive definite quadratic form of the generalized velocities,

$$T = \frac{1}{2}\sum_{i,k=1}^{n} m_{ik}\dot{q}_i\dot{q}_k \qquad (1.122)$$

The matrix (m_{ik}) is constant. It can always be symmetrized and actually must be assumed to be symmetric ; otherwise the acceleration terms would not have the simple form indicated above. Finally, the matrix (m_{ik}) is positive definite, i.e., it has certain properties which ensure the definiteness of T but need not be discussed here in detail.

Let us now assume that the system is *conserative* and *nongyroscopic*. Then the generalized forces are represented by a potential energy, which, in the linear case, is a quadratic form of the coordinates,

$$V = \frac{1}{2}\sum_{i,k=1}^{n} c_{ik} q_i q_k \qquad (1.123)$$

The matrix (c_{ik}) is constant and symmetric by the same reasons as (m_{ik}), but is not necessarily positive definite. The differential equations (1.113) reduce to

$$\sum_{k=1}^{n}(m_{ik}\ddot{\bar{q}}_k + c_{ik}\bar{q}_k) = 0 \qquad \text{(i=1,2,...,n)} \quad (1.124)$$

If the condition that (c_{ik}) be symmetric is dropped, (1.124) also describes the motion of a purely *circulatory* system. In fact, linear generalized forces dependent only on the configuration have the form

$$Q_i = -\sum_{i,k=1}^{n} c_{ik} q_k \qquad \text{(i=1,2,...,n)} \quad (1.125)$$

where the matrix (c_{ik}) is constant. If (c_{ik}) is asymmetric, it can be split up, by

$$(c_{ik}) = (c'_{ik}) + (c''_{ik}) \qquad (1.126)$$

into its symmetric and antimetric parts, with components

$$c'_{ik} = \frac{1}{2}(c_{ik} + c_{ki}) \qquad c''_{ik} = \frac{1}{2}(c_{ik} - c_{ki}) \qquad (1.127)$$

The work of the Q_i is

$$dW = \sum_{k=1}^{n} Q_i dq_i = -\sum_{i,k=1}^{n} c_{ik} q_k dq_i \qquad (1.128)$$

This is an exact differential if and only if

$$\frac{\partial Q_k}{\partial q_i} - \frac{\partial Q_i}{\partial q_k} = c_{ik} - c_{ki} = 2c''_{ik} = 0 \qquad \text{(i=1,2,...,n)} \quad (1.129)$$

It follows that the matrix (c''_{ik}) represents the circulatory forces, whereas the noncirculatory forces are given by (c'_{ik}).
Linear generalized forces dependent on the state of motion have the form

$$Q_i = -\sum_{i,k=1}^{n} g_{ik}\dot{q}_k \qquad (i=1,2,...,n) \qquad (1.130)$$

Thus, the differential equations of a gyroscopic or a dissipative autonomous system are (1.113) where the matrix (g_{ik}) is constant, and (m_{ik}) and (c_{ik}) have the properties discussed above, (c_{ik}) being symmetric if circulatory forces are absent.
By means of relations analogous to (1.126) and (1.127), the matrix (g_{ik}) can be decomposed into its symmetric and antimetric parts, respectively. The rate of work of (1.130) is

$$L=\sum_{i=1}^{n} Q_i \dot{q}_i = -\sum_{i,k=1}^{n} g_{ik} \dot{q}_i \dot{q}_k = -\sum_{i,k=1}^{n} g'_{ik} \dot{q}_i \dot{q}_k \qquad (1.131)$$

since the contributions of (g''_{ik}) cancel. It follows immediately that the matrix (g'_{ik}) is positive definite and represents the dissipatie forces, while the gyroscopic forces, which do no work in real motions, are represented by (g''_{ik}). Thus, (g_{ik}) characterizes a *gyroscopic* or a *dissipative* system according to whether or not it is antimetric.

1.8. Nonlinear Systems

It was mentioned at the end of Section 1.6 that the essential results to be established for the kinetic stability of linear systems can be generalized for nonlinear systems if the

differential equations are accessible to linearization. To facilitate this generalization, let us state a few theorems.
In a system that can be linearized, the quadratic form (1.123) may be considered as a first approximation of the potential energy, obtained by dropping terms of third and higher degrees. It is not difficult to see that the following theorem holds for V (and, incidentally, for any function of this type):

THEOREM A. *If the approximation (1.123) is definite (indefinite),so is the exact function* $V(q_k)$.

For a proof, see Malkin [45, p. 17]. There is no similar theorem for the semidefinite case; hence, Theorem A cannot be inverted. It is possible that the approximation (1.123) is semidefinite, whereas the higher order terms render the exact expression $V(q_k)$ definite or indefinite. However, if the approximation (1.123) is semidefinite with a certain sign, it is not possible that $V(q_k)$ is semidefinite with the opposite sign. In fact, the proof given by Malkin for Theorem A also proves the following extension:

THEOREM B. *If the approximation (1.123) assumes positive (negative) values within an arbitrarily small vicinity of the origin, so does the exact function* $V(q_k)$.

Starting from consideration of this type, Lyapunov [44] has proved a series of stability and instability theorems. The body of these theorems represents a powerful means for dealing with nonlinear problems. The few results we will need in the following chapters are:

THEOREM C. *If, in the linearized system, all the roots of the characteristic equation (1.119) have negative real parts, the equilibrium of the actual (nonlinear) system is stable.*

THEOREM D. *If, in the linearized system, at least one root of the characteristic equation has a positive real part, the equilibrium of the actual system is unstable.*

For proofs of these theorems see, e.g., [45, p.53]. Again, a similar theorem does not exist for the so-called *critical case*, where there are roots with vanishing real parts but no roots with positive real parts. Here, the higher order terms may render the equilibrium stable or unstable.