Chapter 2

Nongyroscopic Conservative Systems

The majority of stability problems arising in structures are of the nongyroscopic conservative type. Therefore, we will restrict ourselves in this chapter to *nongyroscopic conservative* systems.

2.1. General Aspect

According to the definitions given in connection with Tables 1.2 and 1.3 (Section 1.5), nongyroscopic conservative systems only contain nonworking reactions and noncirculatory loads. In the linear case, the energies (1.122) and (1.123) are

$$T = \frac{1}{2}\sum_{i,k=1}^{n} m_{ik}\dot{q}_i\dot{q}_k\,, \qquad V = \frac{1}{2}\sum_{i,k=1}^{n} c_{ik}q_i q_k \tag{2.1}$$

and the differential equations of motion (1.124) are

$$\sum_{i,k=1}^{n}(m_{ik}\ddot{q}_k + c_{ik}q_k) = 0 \quad \text{(i=1,2,...,n)} \qquad (2.2)$$

where the metrices (m_{ik}) and (c_{ik}) are constant and symmetric, and the first one, in addition, is positive definite.

A well-known algebraic principle (see, e.g., [61]) asserts that it is always possible to find a real linear transformation of the coordinates q_k,

$$\varphi_i = \sum_{k=1}^{n}\alpha\ q_k \text{(i=1,2,...,n)} \qquad (2.3)$$

such that two quadratic forms such as (2.1), one of which is positive definite, assume the so-called *normal forms*

$$T = \frac{1}{2}\sum_{k=1}^{n} m_i\varphi_i^2 \quad , \quad V = \frac{1}{2}\sum_{k=1}^{n} c_i\varphi_i^2 \qquad (2.4)$$

(containing only purely quadratic terms). Here, the m_i are positive. The φ_i are called the normal coordinates of system. Applying Lagrange's equations (1.100) along with (1.101) to the energies in the forms (2.4), we obtain the differential equations of motion in terms of the normal coordinates,

$$m_i \ddot{\varphi}_i + c_i \varphi_i = 0 \qquad \text{(i=1,2,...,n)} \qquad (2.5)$$

Here, the differential equations for the various coordinates are independent. The transition from the original coordinates q_i to the normal coordinates φ_i facilitates the discussion of the stability. Setting

$$\varphi_i = A_i e^{\lambda_i t} \qquad \text{(i=1,2,...,n)} \qquad (2.6)$$

we immediately obtain

$$\lambda_i^2 = -\frac{c_i}{m_i} \qquad \text{(i=1,2,...,n)} \qquad (2.7)$$

It is clear that the discussion can be based on the equal and opposite roots λ_i and $-\lambda_i$ obtained from (2.7). Incidentally, it can be shown that they coincide with the roots of the characteristic equation obtained from (1.124) by means of (1.119). They are positive and negative ,respectively,for $c_i < 0$ and conjugate imaginary for $c_i > 0$. In the last case (which is harmless although critical in the sense of Section 1.7) the corresponding fundamental solutions (2.6) are harmonic oscillations and thus bounded; in the first case they are exponential functions with real arguments, one of them unbounded. If $c_i = 0$, the fundamental solutions of (2.5) are $\varphi_i = A_i$ and $\varphi_i = B_i t$. The first one represents an infinity of nontrivial equilibrium configuration; the second one is unbounded. Since the general solution is obtained by superposition of the fundamental solutions for the various normal coordinates, the situation is as follows; so long as all the c_i are positive (see Figure 2.1, where the small squares indicate the $\lambda^2{}_i$ and the circles the $\lambda_i < 0$), the equilibrium configuration $\varphi_i = 0 (i = 1, ..., n)$ is stable. When at least one c_i is negative, it is unstable. Whenever one of the c_i is zero, the system is kinetically and statically unstable.

Since the inequalities $c_i > 0 (i = 1, ..., n)$ are also the necessary and sufficient conditions for *V* to be positive definite, Lagrange's theorem can be inverted or, in other words, replaced by:

THEOREM 1. *The equilibrium of a nongyroscopic conservative linear system is stable exactly as long as the potential energy is positive* *definite.*

It was pointed out in Section 1.2 that the potential energy is the sum,

$$V = V^{(i)} + V^{(e)} \tag{2.8}$$

of the deformation energy $V^{(i)}$ and the potential energy $V^{(e)}$ of the external loading. With a few exceptions characterized by large displacements - one of them is the problem of buckling of helical springs solved by Haringx [20,21] and others [79] (see also [5]) - it is possible to write (2.8) in the form

$$V = V^{(i)} - PU \tag{2.9}$$

where $V^{(i)}$ is independent of the load and $P \geq 0$ is a parameter indicating the intensity of the loading. Problems of this type will be referred to as *simple stability problems* and the corresponding systems as *simple systems.*

In a stability problem where the unloaded state is stable, $V^{(i)}(q_k)$ is positive definite and $U(q_k)$ is at least capable of assuming positive values. The normal coordinates are dependent on *P*, and, in general, the expression for $V^{(i)}(\varphi_k)$ and $U(\varphi_k)$ will not assume their normal forms. However, in a simple stability problem the potential energy, according to (2.9) and (2.1), may be written

$$V = \frac{1}{2}\sum_{i,k=1}^{n}(a_{ik} - Pb_{ik})q_i q_k \qquad (2.10)$$

where the matrix (a_{ik}) is positive definite and (b_{ik}) either indefinite or positive definite (semidefinite). For $P=0$, *V* is positive definite and hence the system is stable. For sufficiently large values of *P*, *V* is not positive definite and hence the system is unstable. The transition takes place at certain value $P = P_1$. In Figure 2.1, at least one pair of roots λ_i, $-\lambda_i$ lying originally on the imaginary axis, moves towards the origin while *P* increases. For $P = P_1$, the two roots of the pair meet at the origin; $P > P_1$ one of them proceeds along the positive real axis, the other along the negative one. The motion of the roots need not be monotonic in *P*

[87]. However, for certain values $P_2 > P_1, P_3 > P_2 \ldots$ of the load parameter, other pairs of roots may pass through the origin. Since the equilibrium is unstable for any $P \geq P_1$, this is of no consequence.

Refer to Figure 2.2

In Figure 2.2 the load *P* is plotted on the vertical axis. The unstable domain for $P \geq P_1$ is indicated by crosshatching. From Theorem 1 or from Figure 2.2 follow:

THEOREM 2. *In linear stability problems of the nongyroscopic conservative type all the critical loads are supplied not only by the kinetic approach but also by the energy method. Provided that the problem is simple, the system is stable for any load parameter* $P < P_1$ *and unstable for any* $P \geq P_1$, *where* P_1 *is the smallest value of P for which the potential energy is not positive definite. The equilibrium and* *the imperfection approaches also supply* P_1 *along with other values* $P_2 > P_1, P_3 > P_2 \cdots$, *which, however, are insignificant.*

It follows that in problems of this type all the approaches introduced in Section 1.2 can be applied to find P_1, even though it does not follow from the equilibrium or the imperfection approach that any load $P > P_1$ is critical. This and the fact that most classical problems are purely nongyroscopic explains why the static methods have been so successful in the past.
The foregoing discussion is limited to linear systems. The results, however, are readily generalized for nonlinear systems, provided that they are accessible to linearization.
So long as the quadratic approximation (2.10) of the potential energy is positive definite, all the roots λ_i of the characteristic equations are imaginary (Figure 2.1), since the λ_i^2 are negative. Thus, the case is indeed critical in Lyapunov's sense (Section 1.8). However, according

to Theorem A (Section 1.8) the exact expression for $V(q_i)$ is positive definite, and from Lagrange's theorem (Section 1.6) it follows that the equilibrium is stable. When the quadratic approximation admits negative values in an arbitrarily small vicinity of the equilibrium configuration, according to Theorem B (Section 1.8) so does the exact expression. Moreover, there exists at least one positive root of the characteristic equation, and it follows from Theorem D (Section 1.8) that the equilibrium is unstable. Finally, if the quadratic approximation is positive semidefinite, at least one root is zero and thus creates a static instability. In this case, the linear system is endangered. According to what has been said at the end of Section 1.6, however, this does not necessarily imply a danger for the nonlinear system. We thus have:

THEOREM 3. *The equilibrium configuration of a nongyroscopic conservative system is stable so long as the quadratic approximation*

(2.10) of its potential energy is positive definite. It is stable of unstable when the approximation is positive semidefinite, and it is unstable in any other case. Theorem 2, if applied to the linearized system, still supplies the correct stability regions, except that their boundaries may belong to the stable or unstable domains.

In the next sections a few examples of nongyroscopic conservative systems will be considered.

2.2 Influence of Shear on Buckling

According to Table 1.4, Euler buckling is a nongyroscopic conservative problem. If shear is taken into account, the results given by (1.10) in connection with Table 1.1 need some (minor) corrections.
Figure 2.3 shows a section of a prismatic homogeneous beam subjected to

Refer to Figure 2.3

a bending moment M and a shear force Q. The curvature κ_1 of the deflection curve caused by M is

$$\kappa_1 = \frac{M}{\alpha} \qquad (2.11)$$

where α denotes the constant flexural rigidity. If, for sumplicity, the section is treated as an element, the shear force Q causes a shear angle.

$$\gamma = \frac{Q}{\beta} \qquad (2.12)$$

where β denotes the constant shear rigidity. Any variation of γ with x (Figure 2.4) is equivalent to a second curvature

$$\kappa_2 = \frac{Q'}{\beta} \qquad (2.13)$$

Refer to Figure 2.4

so that the differential equation of the deflection curve is

$$y'' = \kappa_1 + \kappa_2 = \frac{M}{\alpha} + \frac{Q'}{\beta} \qquad (2.14)$$

The last term is (2.14), which is usually neglected, gives rise to small corrections even in simple bending problems, e.g., in the case of a cantilever (Figure 2.5) loaded at the free end. Here, $M(x) = P(l - x)$ and $Q(x) = P,$

Figure 2.5

hence

$$y'' = \frac{P}{\alpha}(1 - x) \tag{2.15}$$

Since Q is constant, the curvature of the deflection curve is not modified by the shear. However, the second of the two boundary conditions

$$y(0) = 0\,, \qquad y'(0) = \frac{Q(0)}{\beta} = \frac{P}{\beta} \tag{2.16}$$

depends on Q. The solution of (2.15) and (2.16) is

$$y = \frac{P}{6\alpha}x^3(3l - x) + \frac{P}{\beta}x \tag{2.17}$$

It is obvious that in the approximation considered here the influence of the shear consists of a rotation of the cantilever through the angle $\frac{P}{\beta}$ about the centroid of the section $x = 0.$. In fact, the deflection at the free end is

$$f = \frac{Pl^3}{3\alpha} + \frac{Pl}{\beta} \tag{2.18}$$

If the load, instead of acting at the end, is uniformly distributed over the cantilever (as in Problem 1 at the end of this section), the shear decreases the curvature but increases the deflections.

In order to pass from simple bending to stability problems, let us reconsider Euler's Case 5 in Table 1.1 of Section 1.2, taking shear into account. Since the problem is still nongyroscopic conservative, the equilibrium approach may be used. The bending moment and the shear force (Figure 2.6) are

Refer to Figure 2.6

$M(x) = Py$ and $Q(x) = Py'$, respectively. Thus, the deflection curve is subject to the differential equation

$$y'' = -\frac{P}{\alpha} y + \frac{P}{\beta} y'' \tag{2.19}$$

or

$$y'' + \kappa y = 0, \text{ where } \kappa^2 = \frac{P/\alpha}{1 - P/\beta} \tag{2.20}$$

Apart from the modified significance of κ^2, the differential equation is identical with (1.21). Also, the boundary conditions are still given by (1.16). According to (1.25), the smallest eigenvalue is $\kappa = \pi / l$, written in terms of the load by means of the second relation (2.20),

$$P_1 = \frac{\alpha \kappa_1^2}{1 + (\alpha / \beta)\kappa_1^2} = \frac{\alpha \pi^2}{l^2 + (\alpha / \beta)\pi^2} \tag{2.21}$$

Comparison with (1.20) shows that the smallest critical load (2.21), beyond which the column is unstable, is decreased by the shear.
In Class 1 through 4 of Table 1.1 (Section 1.2) the differential equation of the deflection curve is also unaffected by the shear, apart from the modified significance of κ. In Cases 2 and 3, the treatment is the same as in Case 5, since Q does not appear in the end conditions. In Case 1, some of the end conditions contain Q, which, however, is zero in the first buckling mode. It is obvious that in cases the buckling load

is decreased by the influence of shear. In Case 4, a new calculation is required to obtain P_1.

Problems

1. Verify the curvature

$$y'' = \frac{P}{2l\alpha}(1-x)^2 - \frac{P}{l\beta} \tag{2.22}$$

and the maximal deflection

$$f = \frac{Pl^3}{8\alpha} + \frac{Pl}{2\beta} \tag{2.23}$$

of a uniformly loaded cantilever,

2.Verify the buckling load

$$P_1 = \frac{4\alpha\pi^2 / l^2}{1+4(\alpha/\beta)\pi^2 / l^2} \tag{2.24}$$

for Euler's problem in Case 1, if shear is taken into account.

3.Verify the characteristic equation

$$\tan \kappa l = \frac{\beta + \alpha\kappa^2}{\beta + 2\alpha\kappa^2}\kappa l \tag{2.25}$$

in Euler's Case 4, taking shear into account. Compare it with the characteristic equation in the absence of shear, and show that the influence of shear here too decreases the buckling load.

2.3. Buckling by Tension

It is easy to see that a rod is apt to buckle under tension, provided that the load is applied in a suitable manner. So long as the load is constant in magnitude and direction, the problem is still conservative and nongyroscopic. Let us consider an example, using the equilibrium approach. The influence of shear will again be neglected.

Figure 2.7 shows a rod supported as in Euler's Case 5. The axial load P is applied at the end of a rigid handle of length a which is positioning downwards

Refer to Figure 2.7

and is in alignment with the tangent of the deflection curve at the upper end. Since here a horizontal reaction Q_o is to be expected, the differential equation of the deflection curve is

$$\alpha y'' = P(y + ay') + Q_0(l - x) \tag{2.26}$$

or

$$y'' - \kappa^2 y = \kappa^2 a y'_l + \frac{Q_0}{P}\kappa^2(l - x) \quad , \qquad \kappa^2 = \frac{P}{\alpha} \tag{2.27}$$

The general solution is

$$y = A\cosh\kappa x + B\sinh\kappa x - \frac{Q_0}{P}(l - x) - ay'_l \tag{2.28}$$

The constants A and B are determined by the boundary conditions

$$y(0) = y(l) = 0, \qquad y'(l) = y'_l \tag{2.29}$$

along with the equilibrium condition

$$Pay'_l + Q_0 l = 0 \tag{2.30}$$

for the moments with respect to the lower hinge. In fact, (2.28) through (2.30) yield the linear homogeneous system

$$\left.\begin{aligned}
&A - \frac{Q_0}{P} l - y'_l\, a = 0 \\
&A\cosh\kappa l + B\sinh\kappa l - y'_l\, a = 0 \\
&A\kappa\sinh\kappa l + B\kappa\cosh\kappa l + \frac{Q_0}{P} - y'_l = 0 \\
&\frac{Q_0}{P} l + y'_l\, a = 0
\end{aligned}\right\} \tag{2.31}$$

with the characteristic equation

$$\begin{vmatrix} 1 & 0 & -l & -a \\ \cosh\kappa l & \sinh\kappa l & 0 & -a \\ \kappa\sinh\kappa l & \kappa\cosh\kappa l & 1 & -1 \\ 0 & 0 & l & a \end{vmatrix} = 0 \tag{2.32}$$

Adding the last line to the first and developing with respect to the first line we obtain

$$\begin{vmatrix} \sinh \kappa l & 0 & -a \\ \kappa \cosh \kappa l & 1 & -1 \\ 0 & 1 & a \end{vmatrix} = 0 \tag{2.33}$$

that is,

$$(l+a)\sinh \kappa l - a\kappa l \cosh \kappa l = 0 \tag{2.34}$$

or

$$\tan \kappa l = \frac{\kappa l}{1 + l/a} \tag{2.35}$$

Figure 2.8 illustrates the graphical solution of the transcendental equation (2.35). It supplies a single eigenvalue κ_1. The buckling load depends on $\alpha_1 l$ and also on a. For $a << l$ the coefficient of kl on the right-hand side of

Refer to Figure 2.8

(2.35) is small. In this case, $\tanh \kappa_1 l$ is nearly unity, hence

$$\kappa_1 l \cong 1 + \frac{1}{a} \simeq \frac{1}{a} \quad \text{and} \quad P_1 = \alpha\kappa_1^2 \simeq \frac{\alpha}{a^2} \tag{2.36}$$

Problems of this type have been discussed by Grammel [18]. (See also [5].) In terms of the corresponding Euler problems, Grammel solved Case 5 with handles at both ends, while the solution outlined above corresponds to

Refer to Figure 2.9

Case 5 with a single handle. In Case 1 and 2 there is obviously no buckling; other cases are contained among the problems at the end of this section.

In Figure 2.9 a combination of the problem of Figure 2.7 with the corresponding Euler problem is shown. The total loading amounts to a couple whose moment is proportional to the slope of the deflection curve at the loaded end. The differential equation of the deflection curve is

$$\alpha y'' = Pay'_l + Q_0(l - x) \qquad (2.37)$$

Its general solution,

$$y = -\frac{Q_0}{\alpha}\frac{x^3}{6} + \left(\frac{P}{\alpha}ay'_l + \frac{Q_0}{\alpha}l\right)\frac{x^2}{2} + c_1 x + c_2 \qquad (2.38)$$

subjected to the boundary conditions

$$y(0) = y(l) = 0 \quad , \quad y'(l) = y'_l \qquad (2.39)$$

and to the equilibrium condition

$$Pay'_l + Q_0 l = 0 \qquad (2.40)$$

yields $c_2 = 0$ and the linear homogeneous system

$$\left.\begin{aligned} & y'_l \frac{Pa}{\alpha}\frac{l^2}{2} + \frac{Q_0}{\alpha}\frac{l^3}{3} + c_1 l = 0 \\ & y'_l \left(\frac{Pa}{\alpha}l - 1\right) + \frac{Q_0}{\alpha}\frac{l^2}{2} + c_1 = 0 \\ & y'_l \frac{Pa}{\alpha} + \frac{Q_0}{\alpha}l = 0 \end{aligned}\right\} \qquad (2.41)$$

for y'_l , Q_0/α *and* c_1. The characteristic equation is

$$\begin{vmatrix} \frac{Pa}{\alpha}\frac{l}{2} & \frac{l^2}{3} & 1 \\ \frac{Pa}{\alpha}l-1 & \frac{l^2}{2} & 1 \\ \frac{Pa}{\alpha} & l & 0 \end{vmatrix} = 0 \tag{2.42}$$

It has a single solution,

$$P_1 = \frac{3\alpha}{la} \tag{2.43}$$

representing the buckling load.

This problem, corresponding to Euler's Case 5, has again been solved by Grammel [18], along with numerous similar problems. Again, Cases 1 and 2 are trivial. Herrmann and * [24] have treated a somewhat similar problem, concerned with buckling of a bar by shear forces applied on the surface.

Problems

1. Establish the characteristic equation for buckling of the rod shown in Figure 2.10. Verify the approximation

$$P_1 = \alpha\kappa_1^2 = \frac{\alpha}{a^2} \tag{2.44}$$

valid for $a << l$.

Refer to Figure 2.10

2. Use the result of Problem 1 to find P_1 for the rod of Figure 2.11.

Refer to Figure 2.11

3.Verify the buckling loads

$$P_1 = \frac{\alpha}{la}, \qquad P_1 = \frac{4\alpha}{la}, \qquad P_1 = \frac{2\alpha}{la}, \tag{2.45}$$

in the cases shown in Figure 2.12 through 2.14.

Refer to Figure 2.12
Refer to Figure 2.13
Refer to Figure 2.14

2.4. Buckling of Plates

In order to see how the methods developed above can be applied to more elaborate structures, let us consider a thin rectangular plate.
In the case of a column, the potential energies of the internal and external forces are given by (1.32) and (1.34), respectively. They can be written

$$V^{(i)} = \frac{1}{2}\int_0^l \alpha \left(\frac{d^2 y}{dx^2}\right)^2 dx, \qquad V^{(e)} = \frac{1}{2}\int_0^l N\left(\frac{dy}{dx}\right)^2 dx \tag{2.46}$$

where α is the flexural rigidity and $N = -P$ the normal force. It is easy to verify that these expressions are valid also in cases where α and N are functions of x. For constant values of α and N the energy approach, applied to the sum of the energies (2.46) and to admissible configurations of the deflection curve, supplies the differential equation (1.40) along with the dynamic boundary conditions, which in Case 5 are given by (1.41).

Figure 2.15 shows an element of a thin plate loaded along the edges, by forces lying in the center plane. The state of stress is plane, and the stress

Refer to Figure 2.15

resultant N_x, N_y and N_{xy} are obtained by integration of σ_x, σ_y and τ_{xy} over the thickness h of the plate. When the plate buckles under the influence of the edge loads, let $w(x, y)$ denote the vertical displacement of the center plane. According to Timoshenko and Gere [67, pp. 337

and 340], the deformation energy of the plate is given by the surface intergral

$$V^{(i)}=\frac{1}{2}\iint_F D\left\{(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})^2-2(1-\nu)[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-(\frac{\partial^2 w}{\partial x\partial y})^2]\right\}dxdy \qquad (2.47)$$

Here, ν denotes Poisson's ratio, and the constant

$$D=\frac{Eh^3}{12(1-\nu^2)}=\frac{\alpha}{1-\nu^2} \qquad (2.48)$$

containing Young's modulus *E*, represents the flexural rigidity of the plate per unit length. The potential energy of the external loads is

$$V^{(e)}=\frac{1}{2}\iint_F\left\{N_x(\frac{\partial w}{\partial x})^2+2N_{xy}(\frac{\partial w}{\partial x})(\frac{\partial w}{\partial y})+N_y(\frac{\partial w}{\partial x\partial y})^2\right\}dxdy \qquad (2.49)$$

The analogy between (2.47), (2.49), and the two expressions (2.46) is obvious. With (2.47) and (2.49), the energy approach supplies the partial differential equation

$$D(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4})=N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_y\frac{\partial^2 w}{\partial y^2} \qquad (2.50)$$

for the center surface of the plate of constant thickness along with the dynamic boundary conditions. For a built-in edge $x=0$ the boundary conditions are

$$w=0 \qquad \frac{\partial w}{\partial x}=0 \qquad (x = \;\; 0 \tag{2.51}$$

If the edge is simply supported, we have instead

$$w=0 \qquad \frac{\partial^2 w}{\partial x^2}=0 \qquad (x = \;\; 0 \tag{2.52}$$

For a free edge one obtains

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0\,, \quad \frac{\partial^3 w}{\partial x^3}+(2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}=0\,, \quad (x=0) \tag{2.53}$$

In the special case of a rectangular plate simply supported along its four edges and uniformly loaded along the edges b, the coordinate system can be chosen as shown in Figure 2.16. The differential equation (2.50) then reduces to

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + P\frac{\partial^2 w}{\partial x^2} = 0 \tag{2.54}$$

Figure 2.16

and the boundary conditions become

$$\begin{aligned} &w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad (x = 0, x = \\ &w = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad (y = 0, y = b) \end{aligned} \tag{2.55}$$

A typical solution of the boundary conditions is

$$w(x, y) = A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{2.56}$$

where m and n are entire numbers. According to the theory of Fourier series, the most general admissible configuration of the plate is supplied by summation of (2.56) over all positive integers *m* and *n*. Inserting (2.56) in (2.54), we obtain

$$\pi^2 D\left(\frac{m^4}{a^4} + 2\frac{m^2 n^2}{a^2 b^2} + \frac{n^4}{b^4}\right) - P\frac{m^2}{a^2} = 0 \qquad (2.57)$$

If *P* is arbitrary, this equations will, in general, not be satisfied by positive integers *m* and *n*. it follows that nontrivial equilibrium configurations only exist for loads of the type

$$P_{mn} = \pi^2 D\frac{a^2}{m^2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \quad \text{(m,n positive integers)} \quad (2.58)$$

These are the eigenvalues of the problem; the corresponding expressions (2.56) are the eigenfunctions. According to the equilibrium approach, the buckling load is represented by the smallest eigenvalue. For a given *m*, the value $n = 1$ supplies the smallest eigenvalue

$$P_{m1} = \frac{\pi^2 D}{a^2}\left(1 + \frac{1}{m}\frac{a^2}{b^2}\right)^2 \qquad (2.59)$$

According to (2.56), the sections $x = const$ take the shapes of single semiwaves when buckling sets in. For $b -> \infty$ the loads (2.59) tend to the buckling loads $m^2\pi^2 D / a^2$ of a strip which is free along the edges *a*, to be obtained from the corresponding Euler loads (1.26) by substituting *D* for α (and *a* for *l*). It follows that the term containing *b* represents the stiffening effect of the supports *a*.
Let us write (2.59) in the form

$$P_{m1} = \frac{\pi^2 D}{b^2} k_m \tag{2.60}$$

where

$$k_m = (\frac{\mu}{m} + \frac{m}{\mu})^2 \quad \text{and} \quad \mu = \frac{a}{b} \tag{2.61}$$

In Figure 2.17 the functions $k_m(\mu)$ are plotted. For a given ratio μ the lowest of these curves determines the buckling load. Every one of the functions $k_m(\mu)$ has a minimum $k_m = 4$ for $\mu = m$. The curves for *m* and *m*+1

Refer to Figure 2.17

intersect for $\mu = \sqrt{m(m+1)}$. Up to $\mu = \sqrt{2}$, that is, for broad, square and not too narrow plates, the buckling load is P_{11}. Here, buckling occurs with a semiwave also in the longitudinal direction. For $\mu > \sqrt{2}$, that is, for narrow plates, the buckling load is successively given by $P_{21}, P_{31}, \ldots$ and buckling takes place with an increasing number of semiwaves. For the square plate $(b = a$ and hence $\mu = 1$ the buckling load is $P_{11} = 4\pi^2 D/a^2 = 39.4784\ldots D/a^2$.

Problems

1.Verify expressions (2.46) for the case where α and N are functions of x.

2.Dicuss buckling of a square plate simply supported along its four edges and uniformly compressed by P in the direction P and by Q in the direction y.

2.5. Rayleigh's Principle

The examples treated in the preceding sections have been chosen for their simplicity. Actually, there are many problems whose solution is considerably more difficult. In such cases one will look for approximate solutions. The most efficient methods to obtain such approximations are based on the energy approach. In order to explain them and to compare their results with exact solutions, let us once more consider the simply supported rectangular plate treated in Section 2.4.

The potential energy of the loaded plate has the form of (2.9),

$$V = V^{(i)} - PU \tag{2.62}$$

where $V^{(i)}$ is the deformation energy (2.47). $P > 0$ is the total load, and *U*, according to (2.49), has the form

$$U = \frac{1}{2}\iint_F \left(\frac{\partial w}{\partial x}\right)^2 dxdy \tag{2.63}$$

The problem is simple in the sense of Section 2.1. The energy approach is based on a comparison of the potential energies $V = V^{(i)} - PU$ $V[(w(x, y)]$ for all admissible configurations, i.e., for all continuous functions $(w(x, y)$ having continuous first partial derivatives with respect to x and y and satisfying the kinematic boundary conditions

$$w = 0 \qquad (x = 0,\ x = a,\ y = 0) \tag{2.64}$$

Since $V^{(i)}$ and *U* are positive definite, *V* has the same property for sufficiently small values of *P* but ceases to be positive definite when *P* is sufficiently large. The buckling load makes the transition : for $P = P_1$ there is a nontrivial configuration $w_1(x, y)$ for which $V = 0$, but still no configuration $w\,(x, y)$ for which $V < 0$. It follows that V is stationary for $w_1(x, y)$, that is, that P_1 is the smallest values of P for which the variational problem

$$\delta(V^{(i)} - PU) = 0 \tag{2.65}$$

restricted to fixed values of P and to admissible functions $w\,(x, y)$, has nontrivial solution $w_1(x, y)$.

Moreover,

$$V^{(i)}[w_1] - P_1 U[w_1] = 0 \tag{2.66}$$

Equations (2.65) and (2.66) may be reinterpreted. Provided that we ask for the minimum of $V^{(i)}$ subject to the side condition $U = 1$, we again obtain (2.65) with P as a Lagrangean multiplier. The solution is the function $w_1(x,y)$, and because of (2.66) and the side condition, the corresponding value P_1 of the multiplier is the minimum of V. A function $w(x,y)$ remains admissible if it is multiplied by a constant λ. At the same time the expressions $V^{(i)}[w]$ and $U[w]$ are increased in the ratio λ^2. It is therefore irrelevant whether one asks for the minimum of $V^{(i)}$ subject to the side condition $U=1$, or simply for the minimum of $V^{(i)}/U$.

This ratio,

$$R[w]=\frac{V^{(i)}[w]}{U[w]} \tag{2.67}$$

is referred to as *Rayleigh's quotient* [55]. It follows from (2.66) that

$$P_1=R[w_1] \tag{2.68}$$

that is, that the buckling load can be obtained as the minimum of Rayleigh's quotient for admissible functions. This result will be referred to as *Rayleigh's principle*.

It was mentioned in Section 2.4 that solving the variational problem in the usual way one obtains the differential equation of the center surface along with dynamic boundary conditions. This confirms that in problems of the type considered here the equilibrium approach is legitimate. It is true that, by proceeding from here in a purely formal manner, one gets all the eigenvalues and and the corresponding eigenfunctions. However, if the basic question concerning the minimum of $V^{(i)}$ or R is kept in mind, the equilibrium and the energy approaches are completely equivalent.

It is clear that the argumentation presented above is not restricted to rectangular plates, although this case has been chosen as an example. Actually, Rayleigh's principle is valid for any nongyroscopic conservative system(and, in a proper formulation, even for more general systems accessible to the energy approach). The quotient (2.67), calculated for an arbitrary admissible function w, represents an upper bound

$$R[w] \geq P_1 \tag{2.69}$$

for the buckling load. It can be shown that, in general, this upper bound is also a good approximation for P_1, provided that one has succeeded in choosing w so that it does not differ too much from the eigenfunction w_1 .

For the square plate $a=b$ considered in Section 2.4, with simply supported edges and a uniform load parallel to one pair of edges, the function

$$w(x,y) = cx(a-x)y(a-y) \tag{2.70}$$

where c is a constant, is admissible and may be considered an approximation of the first eigenfunction. The corresponding functions $V^{(i)}$ and U are

$$V^{(i)} = \frac{Dc^2}{2}\iint_F \left\{ \begin{aligned} &4[y^2(a-y)^2 + 2xy(a-x)(a-y) + x^2(a-x)^2]^2 \\ &-2(1-v)[4xy(a-x)(a-y) - (a-2x)^2(a-2y)^2]\end{aligned} \right\} dxdy \tag{2.71}$$

and

$$U = \frac{c^2}{2}\iint_F (a-2x)^2 y^2 (a-y)^2\, dxdy \qquad (2.72)$$

Integration yields

$$V^{(i)} = \frac{11}{45} Dc^2 a^6, \qquad U = \frac{1}{180} c^2 a^8 \qquad (2.73)$$

Thus,

$$R = \frac{V^{(i)}}{U} = 44\frac{D}{a^2} \qquad (2.74)$$

is an upper bound for P_{11}. Compared with the exact value $P_{11} = 4\pi^2 D/a^2 = 39.4784...D/a^2$, the error is approximately 11.5 percent.

Problem

1. Check (2.71) through (2.73) by carrying out the calculations.

2.6. The Methods of Ritz and Galerkin

If Rayleigh's principle is used as in the preceding section, the quality of the approximation obtained for the first eigenvalue is a matter of skill, since it depends entirely on the choice of a function which is at the same time simple and sufficeintly close to the first eigenfunction. An improved technique has been proposed by Ritz [56]. It is based on the following idea: instead of forming Rayleigh's quotient with a single function conveniently chosen, one may apply a linear combination

$$w = \sum c_k w_k \qquad (2.75)$$

of any number of admissible functions w_k. Any such combination is itself and admissible function and yields an upper bound for the first eigenvalue. By a suitable choice of the coefficients, however, Rayleigh's quotient, i.e., the upper bound, can be minimized.

In order to demonstrate this process, let us consider Euler's column in

Case 3 (Table 1.1, Section 1.2) instead of the plate, since here the ideas are not obscured by the calculations. The potential energy $V = V^{(i)} - PU$ is given by (1.35). The kinematic boundary conditions are

$$y(0) = y'(0) = 0 \tag{2.76}$$

and the variation of $V[y(x)]$ has the form already given by (1.36). Partial integration yields

$$\int_0^l (\alpha y'''' + Py'')\eta dx + \alpha y''\eta\Big|^l - (\alpha y''' + Py')\eta\Big|^l = 0 \tag{2.77}$$

where $\eta(x) = \delta y(x)$, and use has been made of (2.76). Since $\eta(x)$ is an arbitrary admissible function, (2.77) supplies the differential equation

$$\alpha y'''' + Py'' = 0 \tag{2.78}$$

along with the dynamic end conditions

$$y''(l) = \alpha y'''(l) + Py'(l) = 0 \tag{2.79}$$

indicating that at the free end the bending moment vanishes and the shear force is given by the projection of P into the end section. The buckling load is the smallest solution of the eigenvalue problem defined by (2.78) in conjunction with (2.76) and (2.79). Its exact value is

$$P_1 = \frac{\pi^2 \alpha}{2l^2} = 2.4674 \cdots \frac{\alpha}{l^2} \tag{2.80}$$

Let us now forget the exact solution and try to obtain approximations for P_1. In a first step, we calculate Rayleigh's quotient

$$R(y) = \frac{V^{(i)}[y]}{U[y]} \tag{2.81}$$

for a simple admissible function, e.g., for

$$y(x) = cx^2 \tag{2.82}$$

The integrals $V^{(i)}$ and U become

$$V^{(i)} = 2\alpha c^2 l \qquad U = \frac{2}{3} c^2 l^3 \tag{2.83}$$

and (2.81) supplies the upper bound

$$R = 3\frac{\alpha}{l^2} \tag{2.84}$$

for (2.80), which exceeds the true value by 21.6 per cent,
A more accurate result is obtained if the method of Ritz is applied to the linear combination

$$y(x) = c_1 y^{(1)}(x) + c_2 y^{(2)}(x) \tag{2.85}$$

of the two admissible functions

$$y^{(1)}(x) = x^2 \qquad y^{(2)}(x) = x \tag{2.86}$$

To minimize $R[C_1 y^{(1)} + C_2 y^{(2)}]$, one might set $\frac{\partial R}{\partial C_i} = 0$. One may as well return to the original problem (2.65),requiring that

$$\frac{\partial}{\partial c_1}(V^{(i)} - PU) = 0, \quad \frac{\partial}{\partial c_2}(V^{(i)} - PU) = 0 \tag{2.87}$$

where $V^{(i)}$ and U are calculated from (2.85). The meaning of (2.87) is obvious; the variational problem (2.65), open to all admissible functions, is restricted to trial functions of the type (2.85) and thus becomes an ordinary extremum problem. As the class of functions taken into consideration is restricted, the minimum we are looking for can only be raised. This confirms that the smallest value of P to be obtained from (2.87) is an upper bound for the buckling load P_1.

Inserting $V = V^{(i)} - PU$ from (1.35) and the trial function y from (2.85) in (2.87) we obtain

$$\left.\begin{aligned}\int_0^l (\alpha y'' y^{(1)}{}'' - P y' y^{(1)}{}')dx = 0\\ \int_0^l (\alpha y'' y^{(2)}{}'' - P y' y^{(2)}{}')dx = 0\end{aligned}\right\} \qquad (2.88)$$

where y is still given by (2.85). The integrals are readily calculated for (2.86); they yield

$$\left.\begin{aligned}(4\alpha - \frac{4}{3}Pl^2)c^2 + (6\alpha - \frac{3}{2}Pl^2)lc_2 = 0\\ (6\alpha - \frac{3}{2}Pl^2)c^2 + (12\alpha - \frac{9}{5}Pl^2)lc_2 = 0\end{aligned}\right\} \qquad (2.89)$$

This is a homogenenous linear system for C_1 and C_2. Since we ar not interested in the trivial solution, we require that the determinant

$$(4\alpha - \frac{4}{3}Pl^2)(12\alpha - \frac{9}{5}Pl^2) - (6\alpha - \frac{3}{2}Pl^2)^2 = 0 \qquad (2.90)$$

be zero. Thus we obtain the quadratic equation

$$\frac{P^2 l^4}{\alpha^2} - \frac{104}{3}\frac{Pl^2}{\alpha} + 80 = 0 \qquad (2.91)$$

with the roots

$$P^{(1)} = 2.486\frac{\alpha}{l^2}, \; P^{(2)} = 32.181\frac{\alpha}{l^2} \qquad (2.92)$$

The first of these roots is the approximation we are looking for. It is, in fact, an upper bound, exceeding the true value by only 0.77 per cent. It can be shown that $P^{(a)}$ (another upper bound for P_1) is an approximation of the second eigenvalue P_2. Moreover, these results can be improved by extending (2.85), including more admissible functions.

By means of a partial integration of (2.88), analogous to the one leading from (1.36) to (2.77), we obtain

$$\int_0^l (\alpha y'''' + Py'')y^{(1)}dx + \alpha y'' y^{(1)\,\prime}\Big|^l - (\alpha y''' + Py')\,y^{(1)}\Big|^l = 0 \qquad (2.93)$$

along with a similar relation, containing $y^{(2)}$ in place of $y^{(1)}$. If we assume that the functions $y^{(1)}$ and $y^{(2)}$ satisfy not only the kinematic boundary conditions (2.76) but also the dynamic end conditions (2.79), the last two terms vanish and we have

$$\int_0^l (\alpha y'''' + Py'')y^{(1)}dx = 0\,, \quad \int_0^l (\alpha y'''' + Py'')y^{(2)}dx = 0 \qquad (2.94)$$

with y still given by (2.85).

The last relations suggest a modification of the method of Ritz. In order to explain this modified approach, proposed by Galerkin [17], let us continue our concern with the present example and again approximate the first eigenfunction by a linear combination of type (2.85), composed of functions $y^{(1)},....,$, which, in contrast to the approach of Ritz, now are required to satisfy all the boundary conditions of (2.76) and (2.79). In general, such a combination will not satisfy the differential equation (2.78) of the problem. However, multiplying the left-hand side of the differential equation, formulated for the function (2.85), in turn by $y^{(1)}$ and integrating the products over the interval $0,....l$, we obtain expressions which can be made zero by an appropriate choice of the coefficients c_1 . The significance of this process is clear: the left-hand sides of (2.94) may be interpreted as weighted means of the differential expression $\alpha y'''' + Py''$,formed with the weights $y^{(1)}$ and the process implies that, in place of this expression, at least the weighted means are made zero so as to satisfy the differential equation in the average.

It has been shown by Leipholz [41] that, from a more general point of view, the approach of Ritz may be considered a special case of the Galerkin method. For the special case considered here this is shown above; a general proof has been given by Grammel [5]. With skillfully chosen trial functions, the method of Ritz generally supplies excellent approximations of the critical load. The reason is obvious from Rayleigh's principle: since Rayleigh's quotient is stationary for the first eigenfunction, this function may be considerably modified without an appreciable change in quotient.

The formalism involved in Galerkin's approach is simpler than that of the Ritz method. On the other hand, the method of Galerkin requires more elaborate trial functions.

Problems

1. Apply the method of Ritz to Euler's problem in Case 5, using the trial function (2.85) with $y^{(1)} = x(l-x)$ and $y^{(2)} = x^2(l-x)^2$. Compare the approximation obtained with the exact value of the buckling load and show that the error is 0.51 per cent.

2. Use the Ritz approach to confirm the upper bound 7.889 α/W for the square of the critical length of a (prismatic and homogenenous) vertical column builit in at the lower end and loaded by its own weight *W*. A more accurate value [67, p.103] is $7.837\alpha/W$.

2.7. The Mass Distribution

The techniques discussed in the last sections are based on the energy approach. There are other methods by which approximate solutions can be obtained. One of them exploits the possibility of approximating the given system by one with a smaller degree of freedom. An elastic column, e.g., may be replaced by a chain of rigid elements (Figure 2.18) connected by hinges with elastic restoring moments. Such a simplified model may be treated by any one of the methods introduced in Section 1.2, provided that the problem is conservative and nongyroscopic.

Refer to Figure 2.18

There is a fundamental difference between the kinetic approach and the static method: the motion of the system depends on the mass distribution, whereas the mass does not appear in static considerations. However, because of Theorems 2 and 3 (Section 2.1) the static approaches are legitimate in nongyroscopic conservative systems; they must therefore supply the correct results. In order words, we have:

THEOREM 4. *In a nongyroscopic conservative system, whether linear* *or merely accessible to linearization, the critical loads are independent of the mass distribution.* Despite this theorem, the mass distribution deserves particular care in cases where the real system is replaced by a simplified model, at least if the kinetic approach is to be used. It has happened that the model, although predicting the smallest critical load with sufficient accuracy, supplied an incorrect over-all picture of the instability for loads $P > P_1$. Theorem 4 rests on the assumption that the kinetic energy is positive definite. This condition, always satisfied in real systems, may be inadvertently violated if the simplified model is not properly chosen.

The model of Figure 2.18 represents a column in Euler's Case 3. Let l be the common length of the rigid elements, and let the concentrated masses m_1, m_2 and their locations a_1, a_2 on the single members be arbitrary. If the influence of gravity and all terms of order higher than two are neglected, the energies of the system are

$$T = \frac{1}{2}[m_1 a_1{}^2 \dot{\vartheta}_1{}^2 + m_2 (l\dot{\vartheta}_1 + a_2 \dot{\vartheta}_2)^2] \qquad (2.95)$$

and

$$V = \frac{c}{2}[\vartheta_1^2 + (\vartheta_2 - \vartheta_1)^2] - \frac{Pl}{2}(\vartheta_1^2 + \vartheta_2^2) \qquad (2.96)$$

Using Lagrange's approach in the form (1.100), we obtain the differential equations of motion

$$\left.\begin{aligned} (m_1 a_1{}^2 + m_1 l^2)\ddot{\vartheta}_1 + m_2 l a_2 \ddot{\vartheta}_2 + (2c - Pl)\vartheta_1 - c\vartheta_2 &= 0 \\ m_2 l a_2 \ddot{\vartheta}_1 + m_2 a_2{}^2 \ddot{\vartheta}_2 - c\vartheta_1 + (c - Pl)\vartheta_2 &= 0 \end{aligned}\right\} \qquad (2.97)$$

Setting

$$\vartheta_1 = Ae^{\lambda t}, \qquad \vartheta_2 = Be^{\lambda t} \qquad (2.98)$$

we obtain

$$\left.\begin{aligned} [(m_1 a_1{}^2 + m_1 l^2)\lambda^2 + 2c - Pl]A + (m_2 l a_2 \lambda^2 - c)B &= 0 \\ (m_2 l a_2 \lambda^2 - c)A + (m_2 a_2{}^2 \lambda^2 + c - Pl)B &= 0 \end{aligned}\right\} \qquad (2.99)$$

The characteristic equation,

$$p_0 \lambda^4 + p_2 \lambda^2 + p_4 = 0 \qquad (2.100)$$

is quadratic in λ^2. Its coefficients are

$$\left.\begin{aligned} p_0 &= m_1 m_2 a_1{}^2 a_2{}^2 \\ p_2 &= [m_1 a_1{}^2 + m_2 a_2{}^2 + m_2 (l + a_2)^2] c - [m_1 a_1{}^2 + m_2 (l^2 + a_2{}^2) \\ p_4 &= c^2 - 3cPl + P^2 l^2 \end{aligned}\right\} \tag{2.101}$$

The choice of the same spring constant c for both hinges implies that we are thinking of a column of constant flexural rigidity. In order to approximate the case of a uniform mass distribution, we can either set $m/2, a_2 = a_2 = l/2$ or $m_1 = m/2, m_2 = m/4, a_2 = a_2 = l$. In the first case the total mass m appears concentrated in the centers of the rigid memeber; in the second case it is concentrated at their ends. As is to be expected, the smallest critical load P_1 turns out to be the same in the two cases. It is an approximation of Euler's buckling load.

Another choice is $m_1 = \varepsilon m, m_2 = m, a_2 = a_2 = l$ where $\varepsilon > 0$. It corresponds to a total mass $(1+\varepsilon)m$. However, since the mass distribution is irrelevant, the same value for P_1 is to be expected as in the two cases discussed above. Moreover, P_1 must be independent of ε, even when, with $\varepsilon -> 0$, the total mass appears more and more concentrated at the free end.

The coefficient (2.101) are

$$\left.\begin{aligned} p_0 &= m^2 l^4 \varepsilon \\ p_2 &= m l^2 [(5+\varepsilon)c - (2+\varepsilon)Pl] \\ p_4 &= c^2 - 3cPl + P^2 l^2 \end{aligned}\right\} \qquad (2.102)$$

Let us consider the case in which ε is small. Linearizing the discriminant $\Delta = p_o^2 - 4 p_o p_4$ of (2.100) with respect to ε , we obtain

$$\Delta = p_o^2 - 4 p_o p_4 \qquad (2.103)$$

The coefficient p_0 is always positive, as are p_o, p_4 and Δ for sufficiently small values of *P*. The roots λ_1^2 and λ_2^2 then are negative and the system is stable. In our approximation the zeros of the discriminant are

$$Pl = \left(\frac{10+\varepsilon}{4} \pm i \frac{\sqrt{\varepsilon}}{2}\right)c \qquad (\Delta = 0 \qquad (2.104)$$

Since the right-hand side is complex, there is no real P satisfying (2.104). It follows that λ_1^2 and λ_2^2 are always distinct and real. In Figure 2.19, p_2 and p_4 are plotted against Pl/c. The zero of p_2 is

$$Pl = \frac{5+\varepsilon}{2+\varepsilon}c \qquad (p_2 = 0 \tag{2.105}$$

Figure 2.19

the zeros of p_4 are

$$Pl = \frac{1}{2}(3 \pm \sqrt{5})c \qquad (p_4 = 0 \tag{2.106}$$

The zero of p_2 lies between the zeros of p_4. Hence, for

$$P < P_1 = (3-\sqrt{5})\frac{c}{2l} \tag{2.107}$$

p_2 and p_4 are positive: $\lambda_1^2 < 0, \lambda_2^2 < 0$, and the trivial equilibrium configuration is thus stable. If, on the other hand, $P \geq P_1$ we either have $p_4 \leq 0$ and thus $\lambda_1^2 \leq 0, \lambda_2^2 \geq 0$, or $p_4 > 0, p_2 < 0$, and hence $\lambda_1^2, \lambda_2^2 > 0$. In either case the trivial equilibrium configuration is unstable, and it turns out, therefore, that, in accordance with Theorem 2 (Section 2.1) there is a single stable domain $P < P_1$ and a single unstable domain $P \geq P_1$ for arbitrarily small positive values of ε.

Instead of letting ε tend toward zero, we might have assumed $\varepsilon = 0$ from the beginning, concentrating, in this way, the entire mass at the upper end of the system. In this case (2.102) reduces to

$$\left.\begin{aligned} p_0 &= 0 \\ p_2 &= ml^2(5c - 2Pl) \\ p_4 &= c^2 - 3cPl + P^2l^2 \end{aligned}\right\} \tag{2.108}$$

Now the characteristic equation (2.100) is merely of the first degree in λ^2. This implies that two of the four fundamental solutions are lost and that

$$\lambda^2 = -\frac{p_4}{p_2} \tag{2.109}$$

where the right-hand side is always real. According to Figure 2.19, the root λ^2 is negative for $P < P_1$ and again between the zero of p_2 and the second zero of p_4. From (2.105) and (2.106) it follows that now there are two stable domains,

$$P < P_1 = (3-\sqrt{5})\frac{c}{2l}, \qquad \frac{5c}{2l} < P < (3+\sqrt{5})\frac{c}{2l} \tag{2.110}$$

and hence also two unstable ones,

$$P_1 \le P \le \frac{5c}{2l}, \; P \ge (3+\sqrt{5})\frac{c}{2l} \tag{2.111}$$

The system considered here is simple. The last results - or, more precisely, the existence of the second stable domain (2.11)- are inconsistent with Theorem 2. The inconsistency is explained by the fact that, by setting $\varepsilon = 0$, two fundamental solutions, one of which is unstable in the second interval (2.110), are lost. The reduction in degree of the characteristic equation has another aspect [87] which is readily confirmed either by means of (2.99) or by considering the possible motions of the system illustrated in Figure 2.18: with $\varepsilon = 0$

the matrix (m_{ik}) loses its positive definite character and becomes positive semidefinite. However, the theorems of Section 2.1 have been established under the assumption that (m_{ik}) be positive definite.
It follows from this example that, whenever a system is replaced by a simpler model, care must be taken that the mass concentration of the model preserves the definiteness of the matrix (m_{ik}).

Problem

1. Verify the smallest critical load

$$P = (3-\sqrt{5})\frac{c}{2l} \tag{2.112}$$

of the system illustrated in Figure 2.18 for the mass distributions

$$\left.\begin{array}{l}(a) m_1 = m_2 = m/2, a_1 = a_2 = l/2 \\ (b) m_1 = m/2, m_2 = m/4, a_1 = a_2 = l\end{array}\right\} \tag{2.113}$$

Compare the results with Euler's buckling load in Case 3, associating the spring constant *c* of the model with the flexural rigidity * of the column in such a manner that the maximal deflection caused by a couple acting at the free end is the same for the column and the model.