Chapter 3
Gyroscopic Conservative Systems

According to the definitions given in Section 1.5, in particular in connection with Tables 1.2 and 1.3, systems of the *gyroscopic conservative* type contain nonworking reactions and noncirculatory as well as gyroscopic loads.

3.1. General Aspect

In the linear case, the kinetic energy (1.122) and the potential energy (1.123) of the noncirculatory loads are

$$T = \frac{1}{2}\sum_{i,k=1}^{n} m_{ik}\dot{q}_i\dot{q}_k\,, \qquad V = \frac{1}{2}\sum_{i,k=1}^{n} c_{ik}q_i q_k \tag{3.1}$$

respectively. The generalized gyroscopic forces have the form (1.130)

$$Q_i = -\sum_{i,k=1}^{n} g_{ik}\dot{q}_k \tag{3.2}$$

and the differential equations of motion (1.113) are

$$\sum_{k=1}^{n}(m_{ik}\ddot{q}_k + g_{ik}\dot{q}_k + c_{ik}q_k) = 0 \qquad \text{(i=1,2,...,n)} \quad (3.3)$$

where the matrices (m_{ik}) and (c_{ik}) are constant and symmetric, (m_{ik}) is positive definite, and (g_{ik}) is constant and antimetric.

Setting

$$q_k = A_k e^{it} \qquad \text{(k=1,2,...,n)} \qquad (3.4)$$

we obtain from (3.3) the system (1.118)

$$\sum_{k=1}^{n}(m_{ik}\lambda^2 + g_{ik}\lambda + c_{ik})A_k = 0 \quad \text{(i=1,2,...,n)} \qquad (3.5)$$

Excluding the trivial solution $A_1 = A_2 = ... = A_n = 0,$, which corresponds to the equilibrium state, we finally arrive at the characteristic equation (1.119)

$$\det(m_{ik}\lambda^2 + g_{ik}\lambda + c_{ik}) = 0 \qquad (3.6)$$

The value of a determinant is not altered when its lines and columns are interchanged. Because of the symmetry of $(m_{ik}),(c_{ik})$ and the antimetry of g_{ik}, such an interchange in (3.6) is equivalent to a change in the sign of λ. Thus the roots still appear in pairs, $\lambda_i, -\lambda_i$, one of the

two having a positive real part unless they are purely imaginary or zero.
Table 3.1

According to Lagrange's theorem (Section 1.6) the equilibrium of a conservative system is stable whenever the potential energy of the noncirculatory loads is positive definite. The theorem holds for linear and nonlinear systems with or without gyroscopic forces; it is merely subject to the condition that the total energy be continuous. In Section 2.1 we saw that, in the absence of gyroscopic forces, the system, provided that it is linear, is unstable whenever the potential energy is not positive definite. These statements are illustrated by Table 3.1; they yield:

THEOREM 5. *A conservative linear system cannot be made unstable by gyroscopic forces*.

On the other hand, there is no reason why the system, originally unstable, should remain so when gyroscopic forces are added. As a matter of fact, we are already in possession of a counterexample. In Section 1.3 we treated a simplified model of a shaft having two distinct flexural rigidities. It consisted of a particle (Figure 1.11) attracted by the axes of a rotating coordinate system. It became evident (Figure 1.12) that the equilibrium may be stable although V is not positivie definite. Hence, stabilization by gyroscopic forces is possible. This effect was first studied in detail by Thomson and Tait [65].

If the system is simple in the sense defined in Section 2.1, the potential energy (3.1) has the form (2.10)

$$V = \frac{1}{2}\sum_{i,k=1}^{n}(a_{ik} - Pb_{ik})q_i q_k \tag{3.7}$$

where (a_{ik}) is positive definite and (b_{ik}) is either definite (semidefinite) or indefinite. In problems of this type it often happens that the load parameter P represents a centrifugal force and hence can be interpreted as the square ω^2 of an angular velocity. For small values of P, the expression (3.7) is positive definite and therefore the system is stable.

For sufficiently large values of *P*, *V* is not positive definite. The transition takes place at a certain value $P=P_1$. This value is characterized by the appearance of at least one nontrivial configuration $q_k (k=1.....n)$ for which *V* is stationary. Since the gyroscopic forces are zero when the system is at rest, this is a nontrivial equilibrium configuration, corresponding to a vanishing root λ.

In the λ-plane the situation is at first similar to the one described by Figure 2.1. For small values of *P*, all the roots $\lambda_i, -\lambda_i$ are situated on the imaginary axis. With increasing *P*, at least one pair moves towards the origin and arrives there for $P=P_1$. In contrast to the nongyroscopic case, however, the roots are not tied to the axes and it is necessary that the pair leave the imaginary axis while proceeding away from the origin.

For instance, in the special case $c_1 = c_2 = c$ of the problem illustrated by Figure 1.11 (1.60) takes the form

$$\lambda_{1,2}^{\ 2} = -(\sqrt{\frac{c}{m}} \pm \omega)^2 \qquad (3.8)$$

It follows that $\lambda^2 < 0$ and that λ^2 also moves along the negative real axis, arriving at the origin for $|\omega| = \sqrt{c/m}$ and moving away from it for larger values of $|\omega|$. Thus the four roots of the characteristic equation (3.8) remain on the imaginary axis after passing the origin.

The last result corresponds to the fact that, in general, the system need not be unstable for $P > P_1$. However, for P_1 and possibly for other values $P_1, P_2,,.....$ of the load parameter (in short, whenever a pair of roots pass as through the origin) a static instability occurs, and it is clear that it is unaffected by the presence of gyroscopic forces.

In Figure 3.1 the load *P* is plotted on the vertical axis. The static instabilities at $P_2 > P_1, P_3 > P_2,.....$ are indicated by crosshatching and the uncertain regions between them by diagonal lines.

Refer to Figure 3.1

From the foregoing discussion we obtain :

THEOREM 6. *In linear stability problems of the gyroscopic conservative type, the kinetic method alone supplies all the critical loads. Provided that the system is simple, it is stable for any load parameter* $P < P_1$, *where* P_1 *may be obtained by the energy approach as the smallest value of P for which the potential energy is not positive definite. The equilibrium and the imperfection methods supply* P_1. *along with other possible static instabilities at* $P_2 > P_1, P_3 > P_2, \ldots$ *Any other load* $P > P_1$. *may or may not be critical.*

In problems of this type, it follows that the static approaches may be used to find $P_1, P_2, \ldots$ They do not give any indication, however, concerning stability for other load parameters $P \geq P_1$. Because of the stabilizing effect of the gyroscopic forces, the static instabilities may be the only ones present. This is true, e.g., for the problem of Figure 1.11, provided that $c_1 = c_2$.

In the nonlinear case, Lagrange's theorem still holds. It follows that a gyroscopic conservative system is stable so long as its potential energy is positive definite. For values of *P* under which the potential energy is not positive definite, the behavior of the system is uncertain, even more so than in the linear case, since the static instabilities occurring there may prove harmless in the linear case, since the static instabilities occurring there may prove harmless in the nonlinear case.

3.2. Critical Angular Velocities

It has been mentioned that the particle of Figure 1.11 (Section 1.3) may be considered a model of a disk mounted on a shaft rotating with angular velocity ω, which will be assumed to be positive. The case $c_1 = c_2$ corresponds to a shaft with a single flexural rigidity and is characterized by a single critical angular velocity ω_1, while the case $c_1 < c_2$ represents a shaft with distinct flexural rigidities and is characterized by a critical interval $\omega_1 \le \omega \le \omega_2$.

In this and the following sections the problem will be generalized in different directions. In principle, it can be treated in a coordinate system at rest or in a system rotating with the shaft. In the first case it takes the aspect of a resonance problem of the theory of oscillation; in the second case it is a stability problem of the gyroscopic type.

As a first generalization, let us consider a shaft with a single flexural rigidity, carrying *n* disks which may be represented by the particles $m_1, m_2, \ldots. m_n$, at the sections $x_1, x_2, \ldots. x_n$, while the mass of the shaft will be neglected. Figure 3.2 shows the particle m_i. The deflections of the shaft at

Refer to Figure 3.2

x_i are y_i and z_i. The restoring forces transmitted by the shaft to the particle are denoted by Y_i and Z_i. If the coordinate system rotates with the angular velocity ω of the shaft, the centrifugal force $m_i\omega^2(\dot{y}_i, \dot{z}_i)$ and the Coriolis force $2m_i\omega^2(\dot{z}_i, -\dot{y}_i)$ must be added.

For the calculation of the restoring forces, let us consider the shaft at rest (Figure 3.3) under the influence of the forces Y_k. By means of the influence numbers a_{jk} for the sections j and k, the deflection y_i can be written

$$y_j = \sum_k a_{jk} Y_k \qquad (j=1,2,...,n) \qquad (3.9)$$

Here and in the remainder of this section all sums are to be extended over the sections $x_1, x_2, ...x_n$. Because of Castigliano's theorem, we also have

$$y_j = \frac{\partial V^{(i)}}{\partial Y_j} \qquad (3.10)$$

where $V^{(i)}$ is the energy of deformation, written in terms of the Y_j. Thus, we obtain

$$\frac{\partial y_j}{\partial Y_k} = \frac{\partial y_k}{\partial Y_j} \tag{3.11}$$

Figure 3.3

or, because of (3.9), Maxwell's symmetry relations

$$a_{jk} = a_{kj} \tag{3.12}$$

From (3.9) and (3.10) it follows that

$$V^{(i)} = \frac{1}{2}\sum_{j,k} a_{jk} Y_j Y_k \tag{3.13}$$

Since it must be possible to solve (3.9) for the Y_j, the determinant Δ of the is nonzero. The solution of (3.9) then is

$$Y_j = \sum_k c_{jk} y_k \quad \text{with} \quad c_{jk} = \frac{A_{kj}}{\Delta} \tag{3.14}$$

where A_{kj} is the cofactor of a_{kj} and hence also

$$c_{jk} = c_{kj} \tag{3.15}$$

From (3.13), (3.14), and the identity

$$\sum_j a_{jk} A_{pj} = \Delta \delta_{xp} = \begin{Bmatrix} \Delta (k = p) \\ 0 (k \neq p) \end{Bmatrix} \tag{3.16}$$

we further obtain

$$\left.\begin{aligned} V^{(i)} &= \frac{1}{2}\sum_{j,k,p,q} a_{jk} c_{jp} c_{kq} y_p y_q \\ &= \frac{1}{2}\sum_{j,k,p,q} a_{jk}\frac{A_{pj}}{\Delta} c_{kq} y_p y_q = \frac{1}{2}\sum_{p,q} c_{pq} y_p y_q \end{aligned}\right\} \tag{3.17}$$

Since the trivial equilibrium configuration of the nonrotating shaft is stable, the matrix (c_{pq}) is positive definite. So is the matrix (a_{jk}), as can be readily seen in normal coordinates.

Similar results are obtained for the displacements z_j and the forces Z_k. The corresponding matrices are again (a_{jk}) and (c_{jk}), since the shaft has a single flexural rigidity. The differential equations of motion of the particle m_j therefore are

$$\left.\begin{aligned} m_i\ddot{y}_i &= m_i\omega^2 y_i + 2m_i\omega\dot{z}_i - \sum_k c_{ik} y_k \\ m_i\ddot{z}_i &= m_i\omega^2 z_i - 2m_i\omega\dot{y}_i - \sum_k c_{ik} z_k \end{aligned}\right\} \tag{3.18}$$

In order to find the static instabilities in the rotating coordinate system, it is sufficient to consider one of the two sets of equilibrium conditions obtained from (3.18), for example,

$$\sum_k c_{ik} z_k - m_i \omega^2 z_i = 0 \qquad \text{(i=1,2,...,n)} \qquad (3.19)$$

Nontrivial equilibrium configurations occur whenever

$$\begin{vmatrix} c_{11} - m_1\omega^2 & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} - m_2\omega^2 & & c_{2n} \\ \vdots & & & \vdots \\ c_{n1} & \cdots & \cdots & c_{nn} - m_n\omega^2 \end{vmatrix} = 0 \qquad (3.20)$$

Because of the symmetry of (c_{jk}), the roots $\omega_1^2, \ldots\ldots \omega^2_n$ of (3.20) are real. Since (c_{jk}) is positive definite, they are positive. Hence, there are as many static instabilities as there are disks. Apart from inversions in the sense of rotation, they correspond to so many critical angular velocities $\omega_1, \ldots\ldots \omega_n$. Of course, some of them may coincide.

Let ω_1 be the smallest critical angular velocity. It then follows from Theorem 6(Section 3.1) that the shaft is stable for $0 \le \omega \le \omega_1$ and certainly unstable for $\omega_1, \ldots\ldots \omega_n$. In order to explore the intervals between these critical values, the motion of the system must be investigated. If this is done, for simplicity, in a coordinate system at rest, the differential equations (3.18) reduce to

$$m_i \ddot{y}_i + \sum_k c_{ik} y_k = 0, \quad m_i \ddot{z}_i + \sum_k c_{ik} z_k = 0 \tag{3.21}$$

They represent free oscillations of the system which now is not only conservative but also nongyroscopic. Since such oscillations are harmless, we are tempted to conclude immediately that the intervals between the angular velocities $\omega_1, \ldots\ldots \omega_n$ are stable. However, the differential equations (3.21) do not contain ω and hence do not explain wht the values $\omega_1, \ldots\ldots \omega_n$ are critical. The apparent paradox is readily solved [82] by the observation that, because of imperfections, additional forces with the circular frequency ω are acting on the particle m_i. They give rise to perturbation terms on the right-hand sides of (3.21). Hence, the observer at rest is confronted with a resonance problem. The resonance frequencies correspond to the circular frequencies of the free motion, i.e., to the values $\omega_1, \ldots\ldots \omega_n$, which thus are confirmed as the only critical angular velocities.

By means of (3.14) and (3.9) the equilibrium conditions (3.19) can be written in terms of the forces. We obtain

$$\sum_k a_{ik} Y_k - \frac{1}{m_i \omega^2} Y_i = 0 \qquad \text{(i=1,2,...,n)} \qquad (3.22)$$

If all the Y_k are zero, so are the y_i, according to (3.9). The existence of a nontrivial equilibrium configuration therefore requires that

$$\begin{vmatrix} a_{11} - \frac{1}{m_1\omega^2} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \frac{1}{m_2\omega^2} & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} - \frac{1}{m_n\omega^2} \end{vmatrix} = 0 \qquad (3.23)$$

Along with (3.20), this is a second characteristic equation from which the critical angular velocities may be obtained.

The shaft treated here is an example of the case where the regions between the $P_1 = \omega_i^2$ in Figure 3.1 are stable. If the shaft has two distinct flexural rigidities, each of the principal directions y and z, respectively, has its own set of matrices (c_{jk}), (a_{jk}). This does not necessarily mean that the number of critical angular velocities is simply doubled. In the case of a single disk, e.g., the two matrices (c_{jk}) each reduce to a single element, c_1, c_2 respectively. The differential equations of motion (3.18) become

$$\left.\begin{aligned} m\ddot{y} &= m\omega^2 y + 2m\omega\dot{z} - c_1 y \\ m\ddot{z} &= m\omega^2 z - 2m\omega\dot{y} - c_2 z \end{aligned}\right\} \qquad (3.24)$$

Apart from a slight change in the notation, this is system (1.56). According to Section 1.3, the interval $\omega_1^2 = c_1 / m \leq \omega^2 \leq c_2 / m = \omega_2^2$ is critical, provided $c_1 < c_2$. So this is an example of the case where one of the regions indicated by diagonal lines in Figure 3.1 is unstable.

Problems

1. Verify the critical angular velocities

$$\omega_1 = \left(\frac{48\alpha}{ml^3}\right)^{1/2}, \quad \omega_2 = \left(\frac{384\alpha}{ml^3}\right)^{1/2} \qquad (3.25)$$

of the shaft of Figure 3.4, having a single flexural rigidity α and short bearings at the ends.

Refer to Figure 3.4

2. Solve the analogous problem for long rigid bearings.

3. Drop one of the two disks in Figure 3.4 and confirm the critical interval

$$\frac{256\alpha}{3ml^3} \le \omega^2 \le \frac{256\alpha}{ml^3} \qquad (3.26)$$

under the assumption that the shaft has two distinct flexural rigidities α and 2α.

3.3 Influence of Gyroscopic Moments

In the preceding section the disks were treated as particles. This means that their rotatory inertia was neglected. Actually, the disks do not merely move in translations but they also rotate, and this additional motion is apt to affect the critical angular velocities. In order to show this, let us consider a shaft with a single flexural rigidity, carrying a single disk, and let us treat the problem in a coordinate system at rest.

In Figure 3.5 the centroid of the disk, which is assumed to be exactly centered at the section x_1, is denoted by *S*. The disk has the properties of a symmetric gyro with its figure axis tangential to the deflection curve. In the deformed state this is a space curve with the deflections y, z and the corresponding slopes y', z'. For small slopes, the tangential unit vector in an arbitrary section x is given by $\nu = (1, y', z')$. If ν_1 denotes this vector in the section, a principal system of the disk can be defined by the axes, ξ, η, ζ where ξ has the direction of ν_1, and η, ζ are normal to ξ, the axis η being parallel to the plane x, y and ζ (in our approximation)parallel to x, z.

The configuration of the disk is described by the coordinates $y_1, z_1, y_1^{'}, z_1^{'}$ and the angle φ of rotation in its own plane, measured from the axis η.

The state of motion consists of a translation with the velocities $\dot{y}_1, \dot{z}_1$ superposed by a gyroscopic motion with a spin of angular velocity $\omega_s = \dot{\varphi} = \omega$ and a precession described by the angular velocity $\omega_P = (0, -\dot{z}_1', \dot{y}_1')$. In the absence of a spin the reactions of the disk on the shaft would be the forces

$$Y = -\ \ddot{m}_1,\ Z = -m\ddot{z}_1 \qquad (3.27)$$

and the moments

$$M_y{}' = A\ddot{z}'_1,\ M_z{}' = -A\ddot{y}'_1 \qquad (3.28)$$

where *m* denotes the mass of the disk and *A* its equatorial moment of inertia (taken for an arbitrary diameter in the plane of the disk). According to the theory of gyroscopic the spin gives rise to an additional moment, the so-called gyroscopic moment $M_g = C\omega_s \times \omega_p$, where *C* is the axial moment f inertia (referred to the figure axis ξ). In our approximation its components are

$$M_y{}' = -\ C\omega\dot{\ }\ _1,\ M_z{}'' = -C\omega\dot{z}'_1 \qquad (3.29)$$

The total reaction on the shaft is thus given by the forces (3.27) and the moments

$$M_y = A\ddot{z}'_1 - C\omega\dot{y}'_1 \,,\; M_z = -A\ddot{y}'_1 - C\omega\dot{z}'_1 \qquad (3.30)$$

The influence numbers we need here are the deflection *a* and the inclination *b* at x_1 caused by a unit force acting at x_1, and the deflection b' and the inclination *c* at x_1 caused by a unit moment acting at x_1. It can be shown [4] that the matrix of these influence numbers is symmetric, i.e., that $b' = b$. Moreover, the matrix is positive definite; its determinant Δ therefore is positive. We now have

$$\left.\begin{array}{l} y_1 = a + Y \;{}_z b \\ y_1' = b + Y \;{}_z c \end{array}\right\} \;,\; \left.\begin{array}{l} z_1 = aZ - bM_y \\ z_1' = bY - cM_y \end{array}\right\} \qquad (3.31)$$

and the inversions

$$\left.\begin{array}{l} Y = \frac{1}{\Delta}(cy_1 - by_1') \\ Z = \frac{1}{\Delta}(cz_1 - bz_1') \end{array}\right\} \;,\; \left.\begin{array}{l} M_y = \frac{1}{\Delta}(bz_1 - ay_1') \\ M_z = \frac{1}{\Delta}(-by_1 + ay_1') \end{array}\right\} \qquad (3.32)$$

The differential equations of motion follow from (3.27), (3.30) and (3.32).

They read

$$\left.\begin{aligned} m\ddot{y}_1+\frac{c}{\Delta}y_1-\frac{b}{\Delta}y'_1&=0\\ m\ddot{z}_1+\frac{c}{\Delta}z_1-\frac{b}{\Delta}z'_1&=0\\ A\ddot{y}'_1+C\omega\dot{z}'_1-\frac{b}{\Delta}y_1+\frac{a}{\Delta}y'_1&=0\\ A\ddot{z}'_1+C\omega\dot{y}'_1-\frac{b}{\Delta}z_1+\frac{a}{\Delta}z'_1&=0 \end{aligned}\right\} \qquad (3.33)$$

An inspection of the coefficients confirms that the system is conservative and that, because of the gyroscopic moments, it is gyroscopic even in the coordinate system at rest. In fact, (3.33) may be interpreted as the system of Lagrange's equations obtained from the kinetic energy

$$T=\frac{1}{2}m(\dot{y}_1^2+\dot{z}_1^2)+\frac{1}{2}A(\dot{y}'^2_1+\dot{z}'^2_1) \qquad (3.34)$$

the potential energy

$$V=\frac{1}{2\Delta}[c(y_1^2+z_1^2)-2b(y_1y'_1+z_1z'_1) \qquad (3.35)$$

and the generalized gyroscopic forces

$$-C\omega\dot{z}_1,\ C\omega\dot{y}_1 \qquad (3.36)$$

Since m and A are positive, T is a positive definite quadratic form. So is *V*, because the matrix of the coefficients a,b,c is positive definite. Thus the total energy is continuous and positive definite, irrespective of the value of ω.

It follows from Lagrange's theorem (Section 1.6) that no instabilities of the equilibrium are to be expected. In order to explain the occurrence of critical angular velocities (which are static instabilities for the rotating observer), imperfections in the form of inaccuracies in the centering of the disk must be taken into account. In the coordinate system at rest, they give rise to instationary forces causing resonance.

The imperfections just mentioned can be represented by an additional mass $\mu << m$ which is rigidly connected with the disk and has the coordinates

$$\xi = e_1 \, , \, \eta = e_2 \cos\varphi \, , \, \varsigma = e_2 \sin\varphi \qquad (3.37)$$

in the system of principal axes. The quantities e_1 and e_2 are eccentricities, causing static and kinetic unbalance. In the system at rest, μ has the coordinates

$$x_1 + e_1 \, , \; y_1 + e_2 \cos\varphi \, , \; z_1 + e_2 \sin\varphi \qquad (3.38)$$

If we neglect the influence of the eccentricities on the angular velocity ω, the additional mass μ gives rise to the inertia forces

$$\left.\begin{aligned} T_x &= 0 \\ T_y &= -\mu(\ddot{y}_1 - e_2\omega^2 \cos\omega t) \\ T_z &= -\mu(\ddot{z}_1 - e_2\omega^2 \sin\omega t) \end{aligned}\right\} \qquad (3.39)$$

Taking these forces into consideration in the differential equations of motion and neglecting their moments, which are small of higher order, we obtain, in place of the first two equations (3.33),

$$\left.\begin{aligned} (m+\mu)\ddot{y}_1 + \frac{c}{\Delta} y_1 - \frac{b}{\Delta} y_1' &= \mu e_2 \omega^2 \cos\omega t \\ (m+\mu)\ddot{z}_1 + \frac{c}{\Delta} z_1 - \frac{b}{\Delta} z_1' &= \mu e_2 \omega^2 \sin\omega t \end{aligned}\right\} \qquad (3.40)$$

while the last two equations remain unchanged. The mass μ can be included in *m*. Moreover, we may introduce complex variables

$$r = y_1 + iz_1 \, , \qquad v = y_1' + iz_1' \qquad (3.41)$$

In terms of these, the differential equations of motion can be written in the complex form

$$\left.\begin{aligned} m\ddot{r} + \frac{c}{\Delta}r - \frac{b}{\Delta}v &= \mu e_2 \omega^2 \exp(i\omega t) \\ A\ddot{v} - iC\omega\dot{v} - \frac{b}{\Delta}r + \frac{a}{\Delta}v &= 0 \end{aligned}\right\} \qquad (3.42)$$

It is easy to verify that the general solution of the homogeneous system obtained from (3.42) by setting $e_2 = 0$ is bounded. In order to find a particular solution of (3.42), we set

$$r = A'\exp(i\omega t), \; v = B'\exp(i\omega t) \qquad (3.43)$$

where A' and B' are complex constants. Inserting (3.43) in (3.42), we obtain the system

$$\left.\begin{aligned} (\frac{c}{\Delta} - m\omega^2)A' - \frac{b}{\Delta}B' &= \mu e_2 \omega^2 \\ -\frac{b}{\Delta}A' + [\frac{a}{\Delta} + (C-A)\omega^2]B' &= 0 \end{aligned}\right\} \qquad (3.44)$$

with the determinant

$$f(\omega^2) = -m(C-A)\omega^4 + [\frac{c}{\Delta}(C-A) - \frac{a}{\Delta}m]\omega^2 + \frac{ac - b^2}{\Delta^2} \qquad (3.45)$$

So long as $f(\omega^2) \neq 0$, the particular solution (3.43) is bounded. For $f(\omega^2) \to 0$ the amplitudes A' and B' are apt to increase beyond limit. The roots of $f(\omega^2) = 0$ therefore represent critical angular velocities. The characteristic equation, multiplied by Δ / ω^4, reads

$$g(\omega^2) = \frac{1}{\omega^4} - [am - c(C - A)]\frac{1}{\omega^2} - \Delta m(C - A) = 0 \qquad (3.46)$$

Neglecting the gyroscopic effect by taking $A = C = 0$, we obtain

$$\frac{1}{\omega^4} - \frac{am}{\omega^2} = 0 \qquad (3.47)$$

or

$$\omega_0^2 = \frac{1}{ma} = \frac{c_1}{m} \qquad (3.48)$$

as was to be expected in view of Section 3.2. The same result is obtained for $C = A$, that is, for a rotor which is a spherical gyro. In any event, the critical angular velocities occur in pairs $\omega, -\omega$. Limiting ourselves to positive values, we have

$$\frac{1}{\omega_{1,2}^2} = \frac{1}{2}[am - c(C - A)] \pm \left\{\frac{1}{4}[am - c(C - A)]^2 + \Delta m(C - A)\right\}^{\frac{1}{2}} \qquad (3.49)$$

In practice, most rotors are oblate gyros $(C > A)$. In the case of a flat disk, for example, $C \cong 2A$. The negative sign in (3.49) then yields a complex value for ω_2. It follows that there is a single critical angular velocity ω_1, which, incidentally, is greater than ω_0, since the gyro effect tends to

stiffen the shaft.
The radicand in (3.49) can be written

$$\frac{1}{4}[am + c(C - A)]^2 - b^2 m(C - A) \qquad (3.50)$$

For an elongated rotor $(C < A)$ it is positive but smaller than the square of the first term on the right-hand side of (3.49); thus, there are two critical angular velocities ω_1 and ω_2. It can be shown that $\omega_1 < \omega_0 < \omega_2$.
In some of the literature on critical speeds [62,5] a second set of critical states is discussed and is referred to as the critical angular velocities of counter-rotation. They are explained by equilibrium considerations in a coordinate system rotating with the angular velocity of the shaft, but in the inverse sense. However, such considerations seem to have little to do with the stability of the shaft. Since the effects obtained are not supplied by the kinematic approach, one may safely conclude that they either do not exist or are caused by influences hithero neglected, such as oscillations of the foundations [82]. In any case, there is little experimental evidence for these effects.

Problems

1. Verify the formula

$$\frac{\omega_0^2}{\omega_{1,2}^2} = 1 - \frac{1}{2}\left(1 + \frac{C-A}{3ml^2}\right) \mp \left\{ \frac{1}{4}\left[1 - \frac{1}{3}\frac{(C-A)}{ml^2}\right]^2 + \frac{1}{9}\frac{(C-A)^2}{m^2 l^4} \right\}^{1/2} \qquad (3.51)$$

for the critical angular velocities the shaft of Figure 3.6. Plot $\omega_{1,2}/\omega_0$ as functions of $(C-A)/ml^2$.

Refer to Figure 3.6

2. Verify the value $\omega_1/\omega_0 = 1.32$ for the shaft of Problem 1, carrying a thin circular disk of radius $r = l$.

3.4. Influence of Compression

Critical speeds were first observed in steam turbines. Here, as in other instances, the shaft transmits a torque and an axial force, and these loads affect the critical values of the angular velocity. The influence of the axial force, which alone will be discussed in this section, was first studied by Melan [47].

As an example corresponding to Euler's buckling Case 3, let us consider the shaft of Figure 3.7, equipped with a single disk and a single flexural

Refer to Figure 3.7

rigidity, and let us neglect gyroscopic effects. The problem was treated in Section 1.3 without an axial force. There is a single critical angular velocity, given by (3.48),

$$\omega_1^2 = \frac{c_1}{m} = \frac{1}{m} \qquad (3.52)$$

where c_1 and a are the single elements to which the matrices (c_{jk}) and (a_{jk}), respectively, reduce in this case. More specifically, a is the displacement of the centroid *S* of the disk caused by a unit force orthogonal to the undeformed axis, acting at *S*. The only modification of the problem consists of the addition of the nongyroscopic conservative force *P*. Since, by this addition, the character of the system is not altered, the result will still be given by (3.52). However, in computing the influence number *a*, the force *P* must now be taken into account.

Figure 3.8 shows the shaft loaded by the forces *P* and *Q*. The differential equation of the deflection curve is

$$\alpha y'' = Q(l - x) + P(y_l - y) \tag{3.53}$$

Figure 3.8

or

$$y'' + \kappa y = \frac{Q}{P}\kappa^2 (l - x) + \kappa^2 y_l \quad , \quad \kappa^2 = \frac{P}{\alpha} \tag{3.54}$$

The general solution is

$$y = A\cos\kappa x + B\sin\kappa x + \frac{Q}{P}(l - x) + y_l \tag{3.55}$$

and the boundary conditions,

$$\left.\begin{aligned} y(0) &= A + \frac{Q}{P}l + y_l = 0 \\ y'(0) &= B\kappa - \frac{Q}{P} = 0 \\ y(l) &= y_l = A\cos\kappa l + B\sin\kappa l \end{aligned}\right\} \qquad (3.56)$$

yield

$$B = \frac{Q}{P}\frac{1}{\kappa}, \quad A = -\frac{Q}{P}\frac{1}{\kappa}\tan\kappa l \qquad (3.57)$$

and

$$y_l = \frac{Q}{P\kappa}(\tan\kappa l - \kappa l) \qquad (3.58)$$

It follows from (3.58) that, for $Q = 0$, the displacement y_l is zero unless $\tan\kappa l = \infty$, that is, $\kappa l = \pi/2, 3\pi/2, \ldots$ This confirms Euler's buckling load (1.10) with $k = 1/4$.

Let us assume now that the load P is small compared to the static buckling load P_1. This implies that $\kappa l << \pi/2$, that is, that $\kappa l << 1$. Expanding (3.58) for small values κl, we obtain

$$y_l = \frac{Ql^3}{3\alpha}\left(1 + \frac{2}{5}\frac{Pl^2}{\alpha} + \cdots\right) \qquad (3.59)$$

For $P=0$, this yields the well-known deflection

$$y_l(P=0)=\frac{Ql^3}{3\alpha} \tag{3.60}$$

For $Q=1$, we obtain the influence number

$$a=\frac{l^3}{3\alpha}(1+\frac{2}{5}\frac{Pl^2}{\alpha}+\cdots) \tag{3.61}$$

required in (3.52). Thus, in a first approximation,

$$\omega_1^2=\frac{1}{ma}=\frac{3\alpha}{ml^3}(1-\frac{2}{5}\frac{Pl^2}{\alpha}) \tag{3.62}$$

Comparing this with the value

$$\omega_0^2=\frac{3\alpha}{ml^3} \tag{3.63}$$

valid in the absence of *P*, we finally have

$$\frac{\omega_1}{\omega_0}=1-\frac{1}{5}\frac{Pl^2}{\alpha} \tag{3.64}$$

It should be kept in mind that (3.64) is an approximation, valid for $P << P_1$. The result shows that the critical angular velocity is decreased by compression and increased by tension $(P<0)$. Other cases can be treated in an analogous manner. Some results are given in Table 5.3 of Section 5.5.

The influence of a torque represented by a constant moment vector (Figure 3.9) has been calculated in a similar way. However, we have shown in Section 1.5 that, in general, a constant moment vector is circulatory. Therefore, this problem is not conservative. It will be treated in Section 5.5, and it will be shown there that, in general, it is incorrect to approach it by static means.

Problems

1. Find a first approximation for the critical angular velocity of the shaft illustrated in Figure 3.10, acted upon by compressive forces applied at the ends. Check the result against (5.83) and Table 5.3

Refer to Figure 3.10