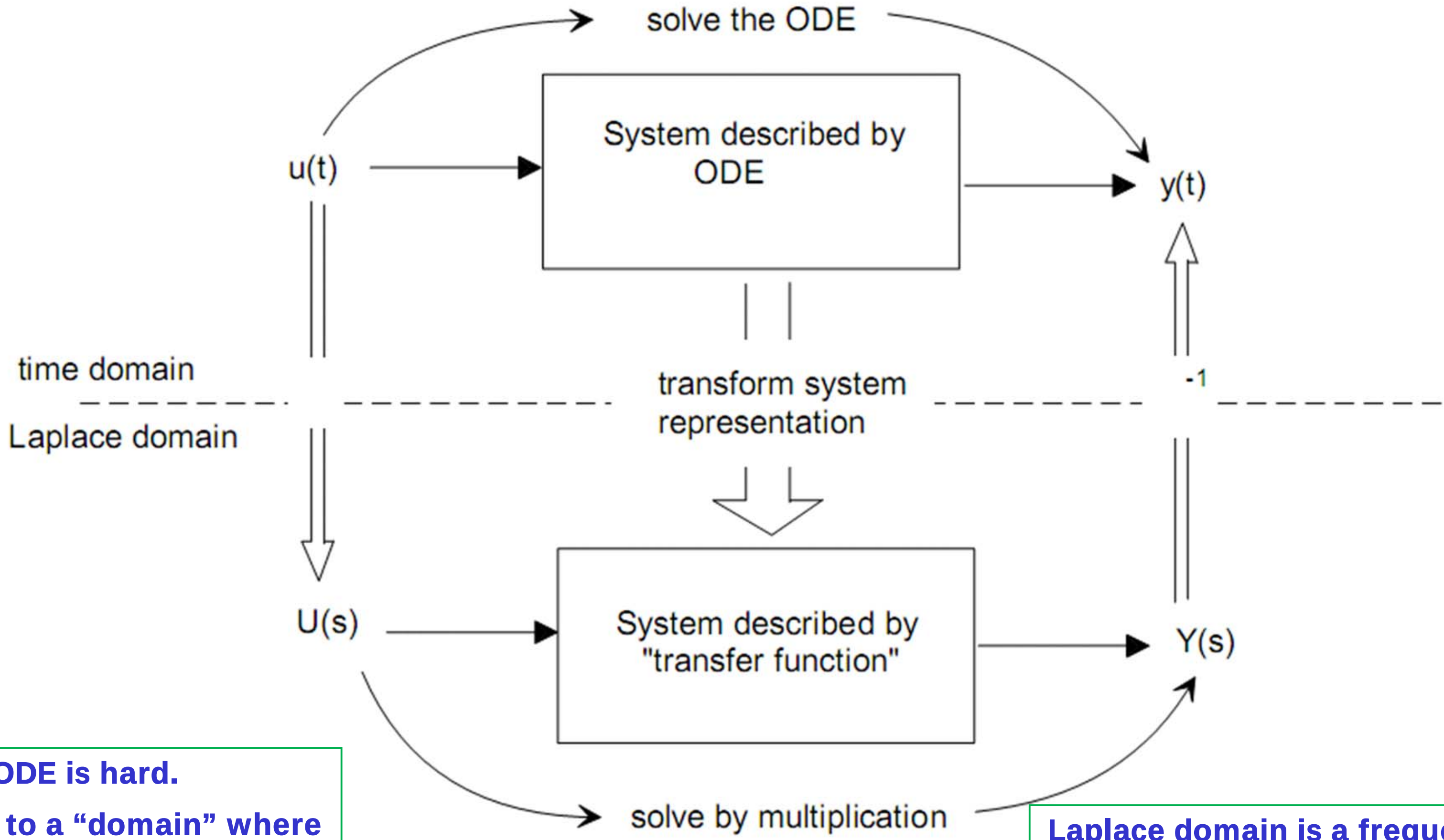
Laplace Transformation II

Why use Laplace Transform?

Algebraic Manipulation of ODE



Solution of ODE is hard.

Tranform in to a “domain” where it’s easier to solve

Solve in the new domain

Perform “inverse” transform.

Laplace domain is a frequency domain.

Integration, Differentiation becomes multiplication, division.

Laplace Transformation

· Definition: $\mathcal{L}\,[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}dt = F(s)$ Integration from 0-

$\mathcal{L}$: f(t) $\Rightarrow$ F(s) , s=σ +jω (complex variable)

f(t) : a time function such that f(t)=0 for t<0

$$\mathcal{L}[1(t)] = \frac{1}{s}$$

$$\mathcal{L}[1(t-t_0)] = \frac{e^{-st}}{s}$$

$$\mathcal{L}[Ae^{-\alpha t}] = \frac{A}{s+\alpha}$$

$$\mathcal{L}[t] = \frac{1}{s^2}$$

$$\mathcal{L}\left[\delta(t)\right] = 1$$

$$\mathcal{L}\left[\delta(t-t_0)\right] = e^{-t_0 s}\cdot 1$$

$$\mathcal{L}[A\sin\omega t] = \frac{A\omega}{s^2+\omega^2}$$

$$\mathcal{L}[A\cos\omega t] = \frac{As}{s^2+\omega^2}$$

Useful Theorems

Theorem 1. Linearity

$$\mathcal{L}[af(t)] = aF(s)$$

Theorem 2. Superposition

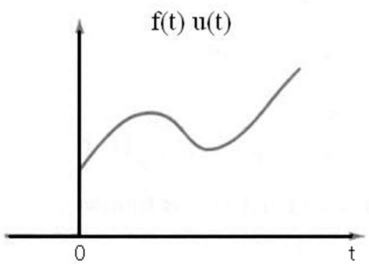
$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

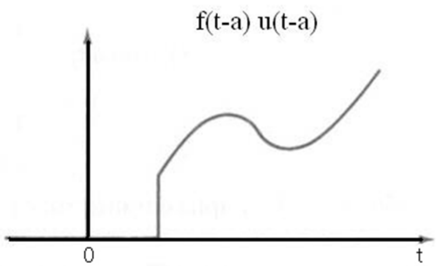
Theorem 3. Translation in time.

$$\mathcal{L}[f(t-a)1(t-a)] = \int_0^\infty f(t-a)1(t-a)e^{-st}dt \qquad (a>0)$$

$$= \int_{-a}^\infty f(\tau)1(\tau)e^{-s(\tau+a)}d\tau \qquad (let.\, t-a=\tau)$$

$$= \int_0^\infty f(\tau)1(\tau)e^{-s\tau}e^{-sa}d\tau = \boxed{e^{-sa}F(s)} \qquad (\because f(\tau)1(\tau)=0 \quad for\ \tau<0)$$





Useful Theorems

Theorem 4. Complex differentiations

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[t \cdot 1] = -\frac{d}{ds}F(s) = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = \mathcal{L}[t \cdot t] = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3} \qquad \mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$

$$proof) \quad let. \;\; F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

$$\frac{d}{ds}F(s) = \int_0^{\infty} \frac{\partial}{\partial s}\left[f(t)e^{-st} \right] dt = -\int_0^{\infty} tf(t)e^{-st}dt = -\mathcal{L}[tf(t)]$$

$$similarly, \quad \mathcal{L}[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}F(s) \qquad \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$$

Theorem 5. Translation in the s-domain

$$\mathcal{L}[e^{at} f(t)] = F(s-a) \qquad \mathcal{L}[e^{at}\cos\omega t] = \frac{(s-a)}{(s-a)^2 + \omega^2}$$

Useful Theorems

Theorem 6. Real Differentiation

$$Df(t) = \frac{d}{dt} f(t) \qquad \mathcal{L}[\frac{d}{dt} f(t)] = s \cdot F(s) - f(0-)$$

$$proof) \quad let. \quad \mathcal{L}[\frac{d}{dt} f(t)] = \int_{0-}^{\infty} \frac{d}{dt} f(t) e^{-st} dt$$

$$= f(t)e^{-st}\Big|_0^{\infty} - \int_{0-}^{\infty} f(t)e^{-st} dt(-s)$$

$$= -f(0-) + s \cdot F(s)$$

$$= \boxed{s \cdot F(s) - f(0-)}$$

$$similarly, \quad \mathcal{L}[D^2 f(t)] = \mathcal{L}[D \cdot Df(t)] = \mathcal{L}[Df'(t)]$$

$$= s\{s \cdot F(s) - f(0-)\} - f'(0-) = s^2 \cdot F(s) - s \cdot f(0-) - f'(0-)$$

$$\mathcal{L}[\frac{d^n}{dt^n} f(t)] = s^n \cdot F(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0-)$$

$$\ldots - f^{(n-1)}(0-)$$

Useful Theorems

Theorem 7. Real Integration

$$\int_0^t f(t)dt = D^{-1}f(t) - D^{-1}f(0)$$

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \int_0^\infty \int_0^t f(\tau)d\tau \cdot e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\int_0^t f(\tau)d\tau\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}f(t)dt$$

$$= \frac{1}{s}\int_0^t f(\tau)d\tau\Big|_{t=0} + \frac{1}{s}\int_0^\infty e^{-st}f(t)dt = \boxed{\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}}$$

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$$

Theorem 8. Complex Integration

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$$

Useful Theorems

Theorem 9. Final value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Theorem 10. Initial value Theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Inverse Laplace Transformation

$\mathcal{L} : f(t) \rightarrow F(s)$

$\mathcal{L}^{-1} : F(s) \rightarrow f(t)$

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} F(s)e^{st}ds$$ (c : real constant)

⁕ **Inverse Laplace Transformation by Partial Fraction Method**

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_0}{s^n + b_{n-1}s^{n-1} + \cdots b_1 s + b_0} \qquad (n \geq m)$$

$s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0$ $\Rightarrow$ real complex conjugate, a , b : real num.

$= (s-c_1)(s-c_2)\cdots(s^2 + d_1 s + d_2)$

$$\Rightarrow F(s) = \frac{\alpha_1}{s-c_1} + \frac{\alpha_2}{s-c_2} + \cdots + \frac{\beta_1 s + \beta_2}{s^2 + d_1 s + d_2} + \cdots$$

Examples of Inverse Laplace Transformation

ex) $F(s)=\dfrac{1}{(s+2)^2(s+3)}=\dfrac{a}{(s+2)}+\dfrac{b}{(s+2)^2}+\dfrac{c}{(s+3)}$

$a(s+2)(s+3)+b(s+3)+c(s+2)^2=1$

$let\ s=-2, than\ b=1,\ \ let\ s=-3, than\ c=1$

$a(s+2)+b+c\cdot\dfrac{s+2}{s+3}=\dfrac{1}{s+3}\quad\xrightarrow{\frac{d}{ds}}\quad a+c\cdot\dfrac{(s+3)-(s+2)}{s+3}=\dfrac{-1}{(s+3)^2}$

$\therefore\ a=-1,\ b=1,\ c=1,\qquad F(s)=\dfrac{-1}{(s+2)}+\dfrac{1}{(s+2)^2}+\dfrac{1}{(s+3)}$

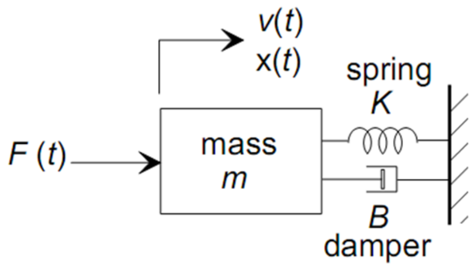
=>   Inverse Laplace Transformation

$f(t)=-e^{-2t}+te^{-2t}+e^{-3t}$ **(partial fraction method)**

ex) $F(s)=\dfrac{10}{s^2+6s+25}=\dfrac{10\times\frac{1}{4}\times 4}{(s+3)^2+4^2}=\dfrac{10}{4}\dfrac{4}{(s+3)^2+4^2}$

$\therefore\ f(t)=\dfrac{10}{4}\sin 4te^{-3t}$

Solution of Differential Equation by Laplace Transformation



$$m\ddot{x} + B\dot{x} + Kx = F.$$

$$y'' + 2y' + 4y = 1 \quad y(0) = 0\ ,\ y'(0) = 2$$

L.T: $$s^2Y(s) - sy(0) - y'(0) + 2\{sY(s) - y(0)\} + 4Y(s) = \frac{1}{s}$$

$$(s^2 + 2s + 4)Y(s) = \frac{1}{s} + 2 = \frac{2s+1}{s}$$

$$Y(s) = \frac{2s+1}{s(s^2+2s+4)} = \frac{1}{4s} - \frac{1}{4}\frac{s+1-1}{(s+1)^2 + \left(\sqrt{3}\right)^2}$$

$$\therefore y(t) = \frac{1}{4} - \frac{1}{4}\cos\sqrt{3}te^{-t} + \frac{1}{4\sqrt{3}}\sin\sqrt{3}te^{-t}$$

Laplace Transform Table

Item no.	$f(t)$	$F(s)$
1.	$\delta(l)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$l^n u(t)$	$\frac{n1}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Nise Ch.2

Laplace Transform Theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{1} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to \infty} sF(s)$	Initial value theorem[2]

End of Laplace Transform