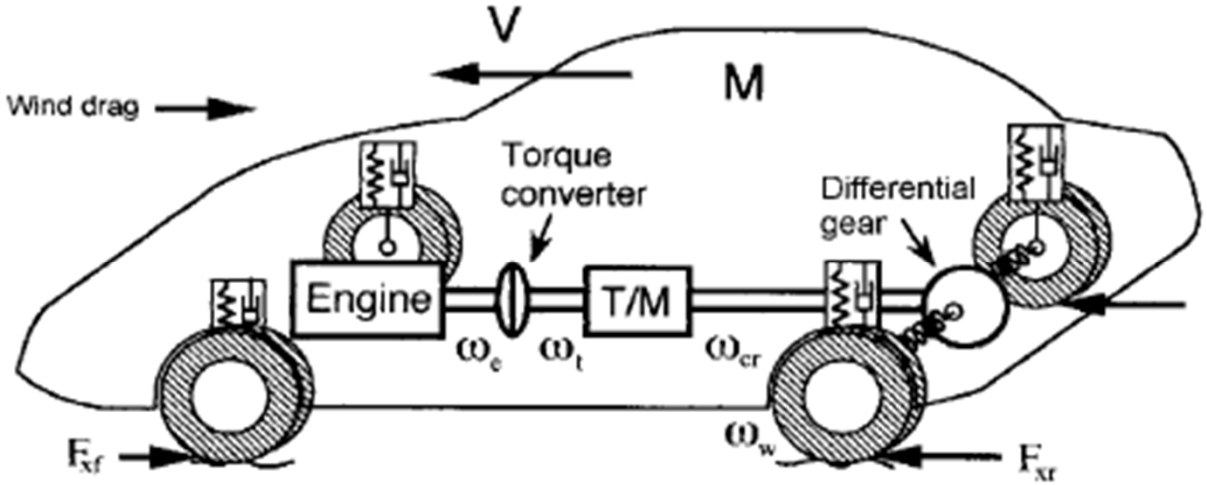
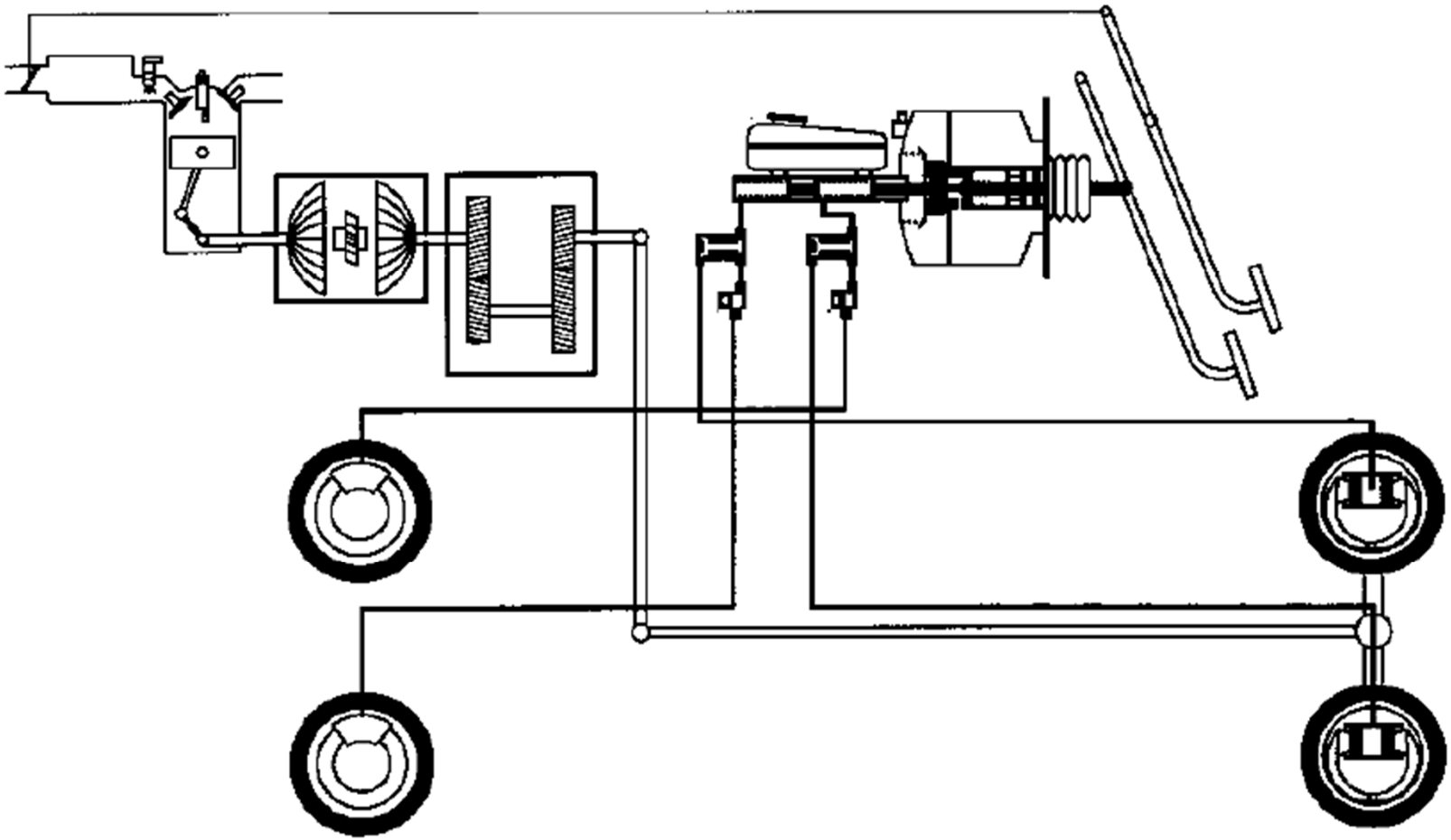
Mechanical Systems I

Vehicle Suspension

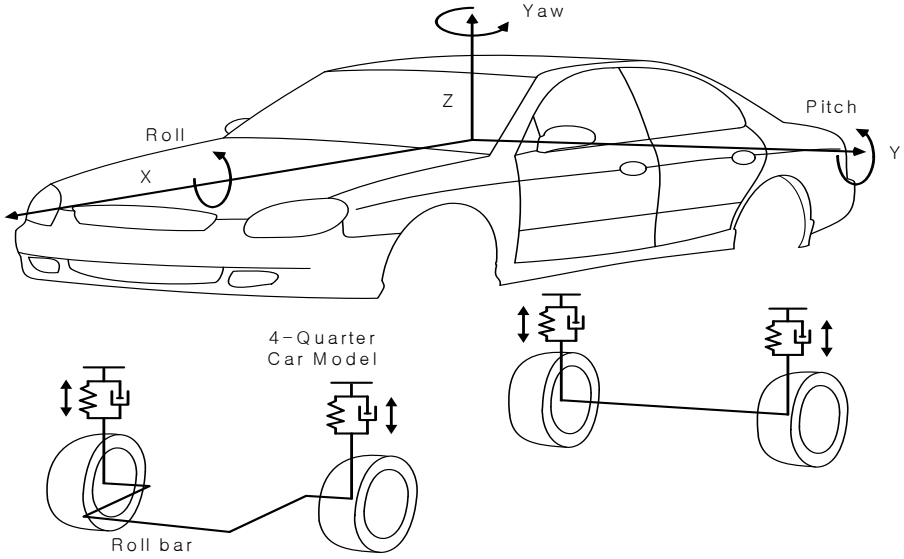
Vehicle Suspension

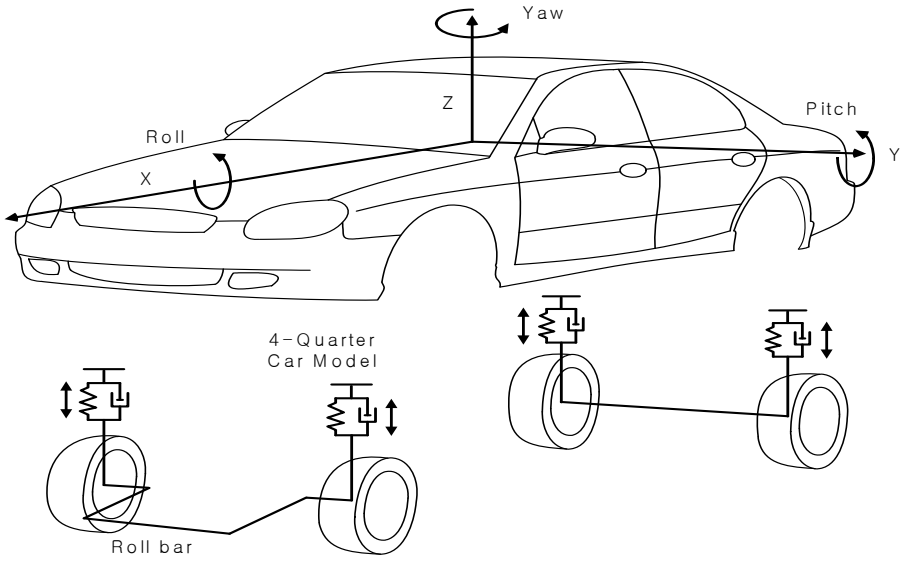


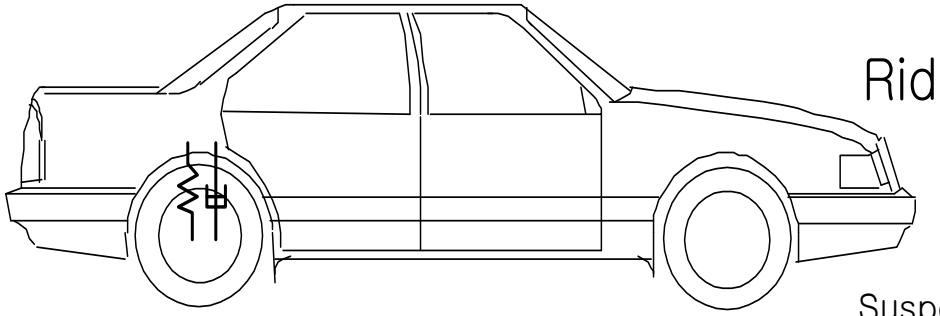
- **Engine Model**
- **Torque Converter**
- **Transmission**
- **Axle Shaft**
- **Differential Gear**



Vehicle Suspension







Ride Quality

Suspension

Newton's Laws

1)First law : conservation of momentum

no external force

➔ no momentum change

linear momentum : mv

angular momentum : $J\omega$

2) Second law : $\sum F = ma = m\frac{dv}{dt}$

$$\sum T = J\alpha = J\frac{d\omega}{dt}$$

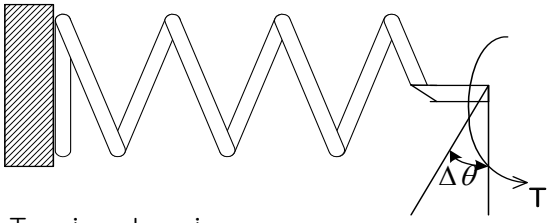
Three Basic Elements in Modeling Mechanical Systems

i) Inertial elements (kinetic energy):

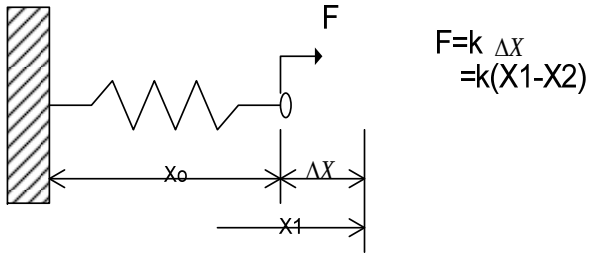
masses: M moments of inertial : J

ii) Spring elements (Potential energy)

$$T = k_\theta \cdot \Delta\theta = k(\theta_1 - \theta_0)$$

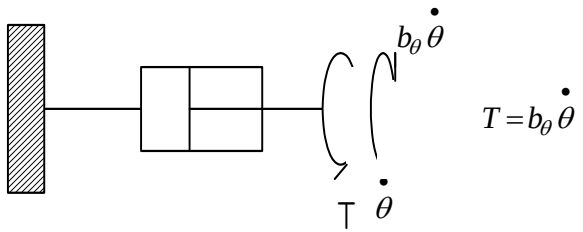


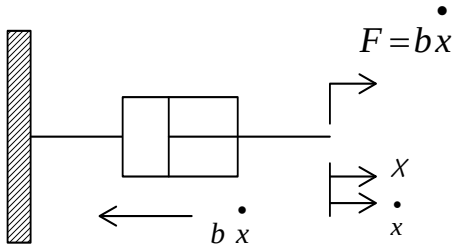
Torsional spring



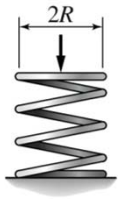
k : stiffness

iii) Damper elements (Energy dissipation)





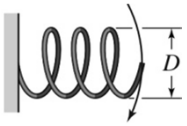
Spring Elements



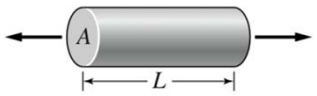
$$k = \frac{Gd^4}{64nR^3}$$

$d = wire\ diameter$

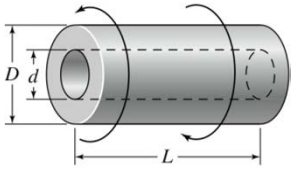
$n = number\ of\ coils$



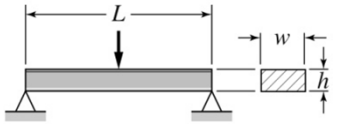
$$k_T = \frac{Ed^4}{64nD}$$



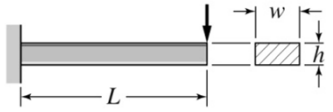
$$k = \frac{EA}{L}$$



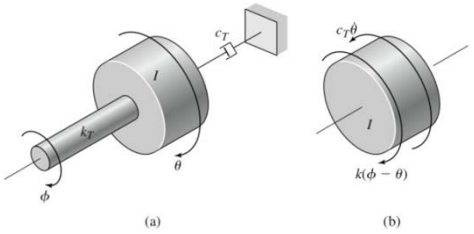
$$k_T = \frac{\pi G(D^4 - d^4)}{32L}$$



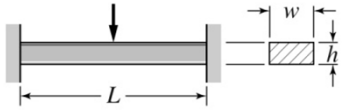
$$k = \frac{4Ewh^3}{L^3}$$



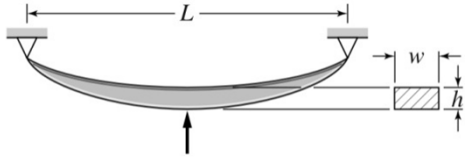
$$k = \frac{Ewh^3}{4L^3}$$



$$k_T = \frac{\pi GD^4}{32L}$$



$$k = \frac{16Ewh^3}{L^3}$$

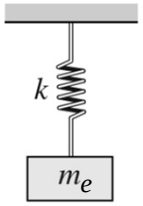


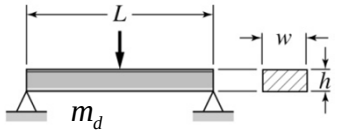
$$k = \frac{2Ewh^3}{L^3}$$

E: Young's modulus

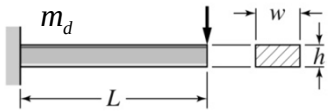
G: Shear modulus

Equivalent masses of common elements

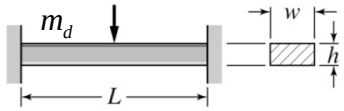




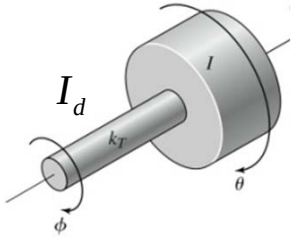
$$k = \frac{4Ewh^3}{L^3} \qquad m_e = 0.38m_d$$



$$k = \frac{Ewh^3}{4L^3} \qquad m_e = 0.23m_d$$



$$k = \frac{16Ewh^3}{L^3} \qquad m_e = 0.5m_d$$



$$I_e = I + I_d / 3$$

$$k_T = \frac{\pi GD^4}{32L}$$

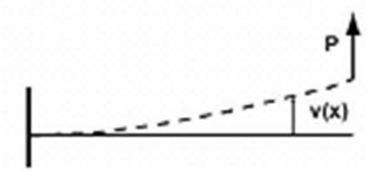
E: Young's modulus

G: Shear modulus

Tip-Loaded Cantilever: Lateral Deflections

curvature / moment relations:

$$\begin{aligned} v''(x) &= \frac{1}{EI} M(x) \\ &= \frac{1}{EI} (P(L-x)) \Rightarrow \\ v'(x) &= \frac{P}{EI} \left(Lx - x^2/2 + C_1 \right) \Rightarrow \\ v(x) &= \frac{P}{EI} \left(Lx^2/2 - x^3/6 + C_1 x + C_2 \right) \end{aligned}$$



geometric boundary conditions

$$\begin{aligned} \phi(0) = v'(0) &= 0 \Rightarrow C_1 = 0 \\ v(0) &= 0 \Rightarrow C_2 = 0 \\ v(x) &= \frac{Px^2}{6EI} (3L - x) \end{aligned}$$

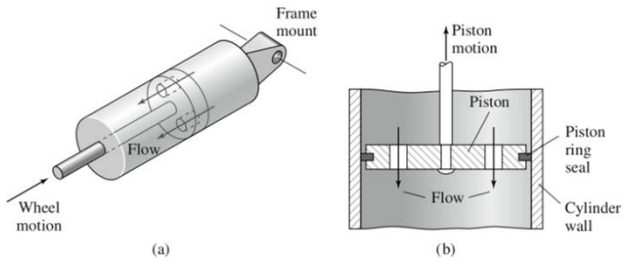
tip deflection and rotation:

$$\begin{aligned} \Delta \equiv v(L) &= \frac{PL^3}{3EI} \\ \Phi \equiv v'(L) &= \frac{PL^2}{2EI} \end{aligned}$$

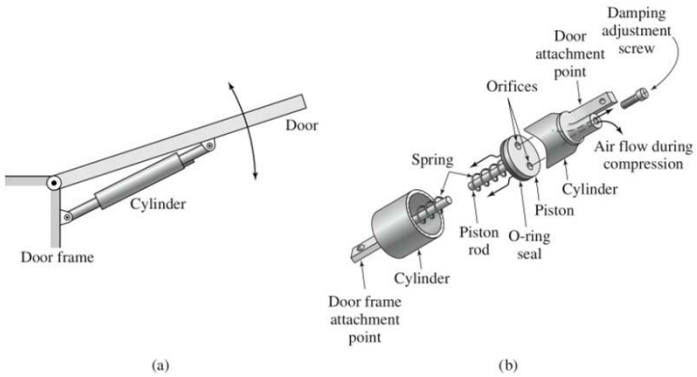
stiffness and modulus:

$$\begin{aligned} k \equiv \frac{P}{\Delta} &= \frac{3EI}{L^3} \\ E &= \frac{kL^3}{3I} = \frac{PL^3}{3I\Delta} \end{aligned}$$

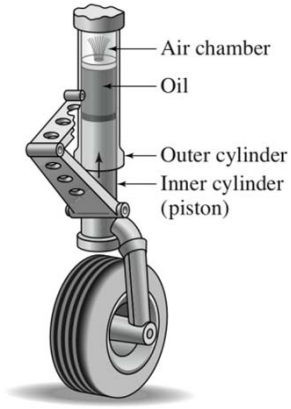
Damping Elements



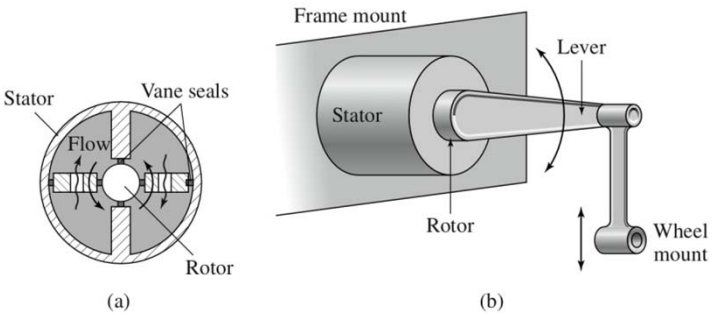
Piston damper



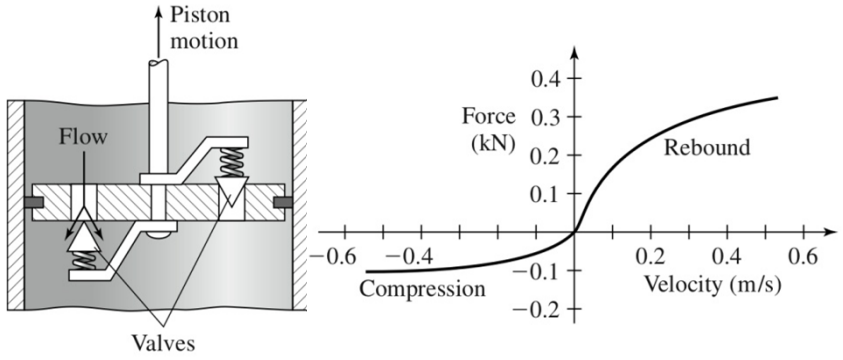
Pneumatic door closer



Oleo strut



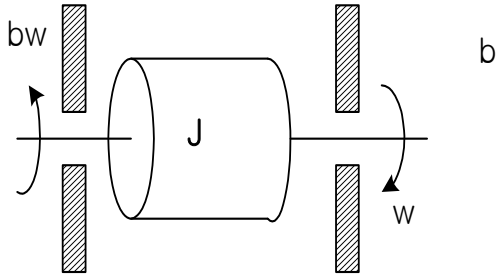
Rotary damper



Damper with spring loaded valves

Examples of Modeling Mechanical Systems

ex1) $t = 0, \quad \omega(0) = \omega_0$



equation of motion :

$$J\frac{d\omega}{dt} = -b\omega$$

$$J\frac{d\omega}{dt} + b\omega = 0 \rightarrow \frac{d\omega}{dt} + \frac{b}{J}\omega = 0$$

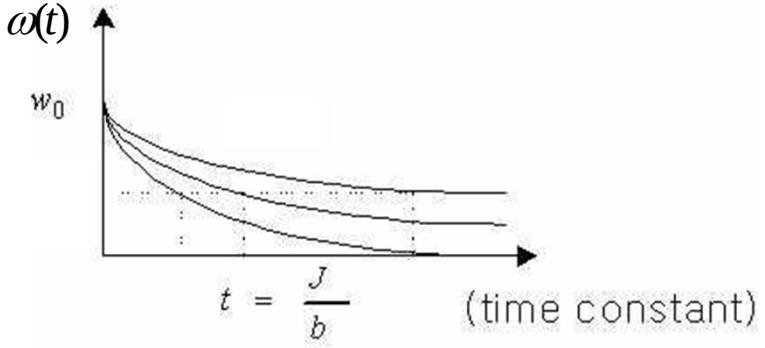
$$let,\ \omega(t) = ce^{\lambda t} \rightarrow \lambda e^{\lambda t} + \frac{b}{J}e^{\lambda t} = 0$$

$$\lambda = -\frac{b}{J}$$

$$\omega_0 e^{-1} = 0.368\omega_0$$

$$\omega(t) = ce^{-\frac{b}{J}t},\ t = 0,\ \omega(0) = \omega_0 = C$$

$$\therefore \omega(t) = \omega_o e^{-\frac{b}{J}t}$$

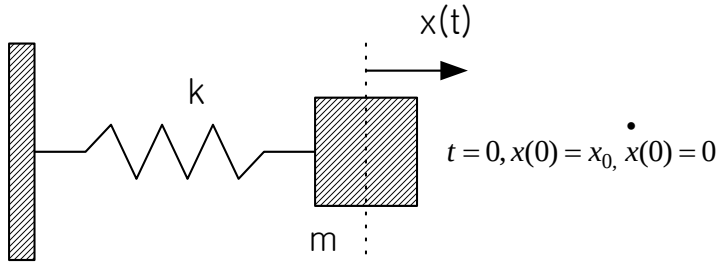


T=Time constant: time to reach 63.2%

Examples of Modeling Mechanical Systems

Spring-Mass

ex2)



$$m\frac{d^2x}{dt^2}=\sum F=-kx$$

$$m\ddot{x}+kx=0 \quad \rightarrow \quad \ddot{x}+\frac{k}{m}x=0$$

$$x(t)=A\cos\sqrt{\frac{k}{m}}t+B\sin\sqrt{\frac{k}{m}}t$$

$$x(0)=A=x_o$$

$$\dot{x}(0)=-A\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t+B\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}t \quad =B\sqrt{\frac{k}{m}}=0$$

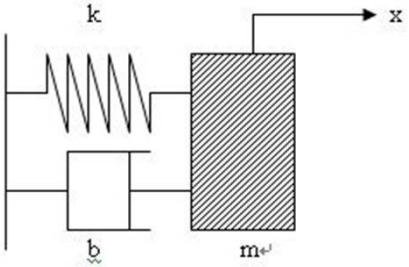
$$x(t)=x_0\cos\sqrt{\frac{k}{m}}t=x_0\cos\omega_n t$$

- **Period** $T=\frac{2\pi}{\sqrt{k/m}}[\text{sec}]$
- **Frequency** $f=\frac{1}{T}=\frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad [Hz]$
- **Natural frequency**

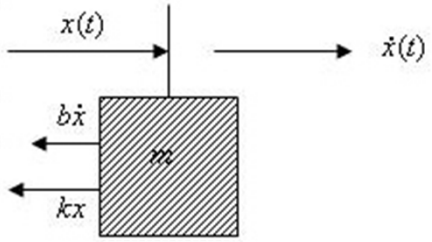
$$\omega_n=2\pi f=\sqrt{\frac{k}{m}} \quad [rad/\text{sec}]$$

Examples of Modeling Mechanical Systems

Spring-mass-damper



$$x(0) = x_0$$
$$\dot{x}(0) = \dot{x}_0$$



$$m\frac{d^2x}{dt^2} = -kx - b\dot{x}$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$let,\quad \frac{k}{m} = \omega_n^2, \qquad \frac{b}{m} = 2\zeta\omega_n, \qquad \frac{b}{2\sqrt{mk}} = \zeta$$

Damping ratio=

Exponential decay

Natural frequency

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \qquad x(t) = c\cdot e^{\lambda t} \qquad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 x = 0$$

$$\therefore \lambda = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\omega_n$$

Examples of Modeling Mechanical Systems

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \qquad x(0) = x_0 \quad \dot{x}(0) = \dot{x}_0 = 0$$

$$let, \quad \frac{k}{m} = \omega_n^2, \qquad \frac{b}{m} = 2\zeta\omega_n, \qquad \frac{b}{2\sqrt{mk}} = \zeta$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Laplace Transform

$$(s^2X(s) - s\cdot x_0 - \dot{x}_0) + 2\zeta\omega_n(sX(s) - x_0) + \omega_n^2 X(s) = 0$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)X(s) - s\cdot x_0 - 2\zeta\omega_n x_0 = 0$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)X(s) = s\cdot x_0 + 2\zeta\omega_n x_0$$

$$X(s) = \frac{s\cdot x_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Examples of Modeling Mechanical Systems

$$X(s) = \frac{s \cdot x_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s \cdot x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$s^2 + 2\zeta\omega_n s + \omega_n^2$: charateristic polynomial

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$: characteistic equation

$$s = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$: charateristic roots

1) $\zeta < 1$ underdamped

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$X(s) = \frac{(s + \zeta\omega_n) \cdot x_0}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} + \frac{\zeta\omega_n x_0}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\zeta}{\sqrt{1-\zeta^2}} x_0 \sin\sqrt{-\zeta^2}\,\omega_n t \right)$$

Examples of Modeling Mechanical Systems

2) $\zeta > 1$ overdamped

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$X(s) = \frac{(s + 2\zeta\omega_n)\cdot x_0}{\left(s + \zeta\omega_n - \omega_n^2\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n + \omega_n^2\sqrt{\zeta^2 - 1}\right)}$$

$$x(t) = k_1 e^{\left(-\zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n\right)t} + k_2 e^{\left(-\zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n\right)t}$$

3) $\zeta = 1$ critically damped

$$s = -\zeta\omega_n$$

$$X(s) = \frac{(s + 2\zeta\omega_n)\cdot x_0}{\left(s + \zeta\omega_n\right)^2}$$

$$x(t) = k_1 e^{(-\zeta\omega_n)t} + k_2 t e^{(-\zeta\omega_n)t}$$

Examples of Modeling Mechanical Systems

1) $\zeta < 1$ **Underdamped**

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\zeta}{\sqrt{1-\zeta^2}} x_0 \sin\sqrt{-\zeta^2}\,\omega_n t \right)$$

$$\begin{cases} \omega_n : & \textbf{natural frequency} \\ \zeta : & \textbf{damping ratio} \end{cases} \qquad \omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

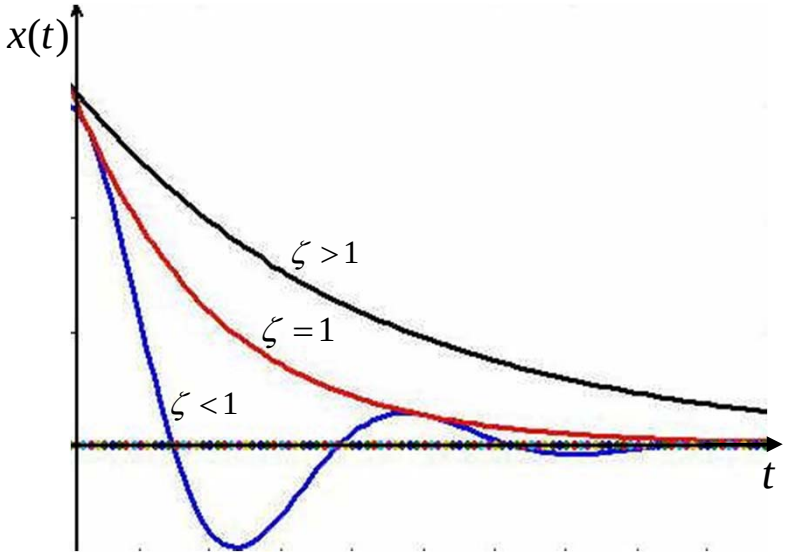
2) $\zeta > 1$ **Overdamped**

$$x(t) = k_1 e^{\left(-\zeta\omega_n + \sqrt{\zeta^2-1}\omega_n\right)t} + k_2 e^{\left(-\zeta\omega_n - \sqrt{\zeta^2-1}\omega_n\right)t}$$

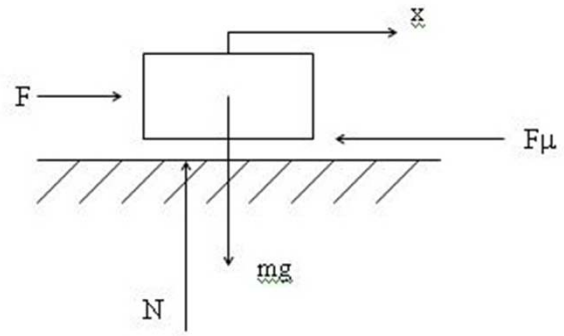
$$\lambda = -\zeta\omega_n \pm \sqrt{\zeta^2-1}\omega_n$$

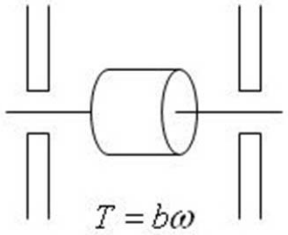
3) $\zeta = 1$ $\qquad \lambda = -\zeta\omega_n$

$$x(t) = k_1 e^{-\zeta\omega_n t} + k_2 t e^{-\zeta\omega_n t}$$



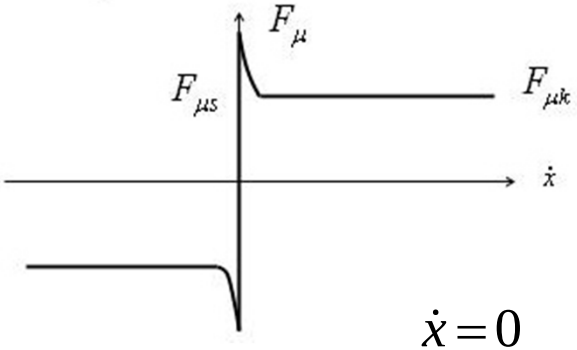
Dry Friction (no lubricant)





- $F\mu s$ = Static Friction Force

  $F\mu k$ = Kinetic Friction Force



- $F\mu s = \mu s \cdot N$ μs : Static Friction Coefficient

  $F\mu k = \mu k \cdot N$ μk : Kinetic Friction Coefficient

$\dot{x} = 0$

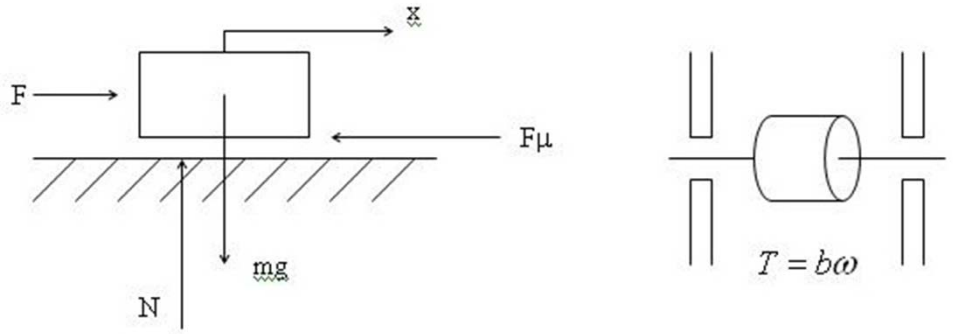
$$F_\mu = \begin{cases} F & \text{if } F \le F_{\mu s} \\ F_{\mu s}\,\mathrm{sgn}(F) & \text{if } F > F_{\mu s} \end{cases}$$

$\dot{x} \neq 0$

$$F_\mu = F_{\mu k}\,\mathrm{sgn}(\dot{x})$$

ε

Friction (with lubricant)



$$F_\mu = \begin{cases} b\dot{x} + G \cdot N \,\mathrm{sgn}(\dot{x}) & \forall \dot{x} > \varepsilon \\ F & \text{if } |\dot{x}| < \varepsilon \text{ and } |\mathrm{F}| \le (F_s + G \cdot N) \\ (F_s + G \cdot N)\,\mathrm{sgn}(F) & \textit{otherwise} \end{cases} \qquad \text{i.e., if } |\dot{x}| < \varepsilon \text{ and } |\mathrm{F}| > (F_s + G \cdot N)$$

b: viscous friction coefficient

G: load-dependent factor

N: normal force

Fs: the maximum static friction

ε : a small bound for zero velocity detection

End of lecture 3-1