10-4

Linear Systems Analysis in the Time Domain IV

- Transient Response -

Matrix Exponential

$$\mathrm{x}(t) = e^{At}\,\mathrm{x}(0) \qquad \cdots(\alpha)\,, \qquad\qquad \dot{\mathrm{x}} = \mathrm{A}\,\mathrm{x} \qquad \cdots(\beta)$$

Matrix exponential e^{At}

$$e^{At} = \mathrm{I} + \mathrm{A}\,t + \frac{1}{2!}\mathrm{A}^2 t^2 + \frac{1}{3!}\mathrm{A}^3 t^3 + \cdots \qquad (*1)$$

$$\left(\because\ e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots \right)$$

$$\frac{d}{dt}(e^{At}) = \mathrm{A} + \mathrm{A}^2 t + \frac{1}{2}\mathrm{A}^3 t^2 + \cdots = \mathrm{A}\left\{ \mathrm{I} + \mathrm{A}\,t + \frac{1}{2}\mathrm{A}^2 t^2 + \cdots \right\}$$

$$= \mathrm{A}\,e^{At}$$

$$\dot{\mathrm{x}}(t) = \frac{d}{dt}\left(e^{At}\right)\mathrm{x}(0) = \mathrm{A}\,e^{At}\,\mathrm{x}(0) = \mathrm{A}\,\mathrm{x}(t)$$

$\Rightarrow (\alpha)$ is the solution of the matrix differential equation (β)

Matrix Exponential

- $e^{At} = \exp(\mathrm{A}\,t) = \sum_{k=0}^{\infty} \frac{\mathrm{A}^k t^k}{k!}$

- $\frac{d\left(e^{At}\right)}{dt} = \mathrm{A}\,e^{At}$

- $e^{(A+B)t} = e^{At}e^{Bt} \qquad if \quad \mathrm{AB} = \mathrm{BA}$

  $e^{(A+B)t} \neq e^{At}e^{Bt} \qquad if \quad \mathrm{AB} \neq \mathrm{BA}$

$$\Rightarrow e^{(A+B)t} = \mathrm{I} + (\mathrm{A}+\mathrm{B})t + \frac{(\mathrm{A}+\mathrm{B})^2}{2!}t^2 + \frac{(\mathrm{A}+\mathrm{B})^3}{3!}t^3 + \cdots$$

$$e^{At}e^{Bt} = \left(\mathrm{I} + \mathrm{A}\,t + \frac{\mathrm{A}^2 t^2}{2!} + \frac{\mathrm{A}^3 t^3}{3!} + \cdots\right)\left(\mathrm{I} + \mathrm{B}\,t + \frac{\mathrm{B}^2 t^2}{2!} + \frac{\mathrm{B}^3 t^3}{3!} + \cdots\right)$$

$$= \mathrm{I} + (\mathrm{A}+\mathrm{B})t + \frac{\mathrm{A}^2 t^2}{2!} + \mathrm{AB}\,t^2 + \frac{\mathrm{B}^2 t^2}{2!} + \frac{\mathrm{B}^3 t^3}{3!} + \frac{\mathrm{A}^2 \mathrm{B}\,t^3}{2!} + \frac{\mathrm{AB}^2 t^3}{2!} + \frac{\mathrm{B}^3 t^3}{3!} \cdots$$

How to Evaluate e^{At}

Hence,

$$e^{(A+B)t} - e^{At}e^{Bt} = \frac{BA - AB}{2!}t^2 + \frac{BA^2 + ABA + B^2A + BAB - 2A^2B - 2AB^2}{3!}t^3 + \cdots$$

The difference between $e^{(A+B)t}$ and $e^{At}e^{Bt}$ vanishes,
if A and B commute.

How to Evaluate e^{At}

$(*1) \Rightarrow$ a) Diagonalized Form $\quad A = \begin{bmatrix} \lambda_1 & & \underline{0} \\ & \lambda_2 & \\ \underline{0} & & \lambda_3 \end{bmatrix}$

$$e^{At} = I + At + \frac{1}{2}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}^2 t^2 + \frac{1}{3!}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}^3 t^3 + \cdots$$

$$= \begin{bmatrix} 1 + \lambda_1 t + \frac{1}{2}\lambda_1{}^2 t^2 + \frac{1}{3!}\lambda_1{}^3 t^3 + \cdots & \\ & \ddots \\ & \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_3 t} \end{bmatrix}$$

How to Evaluate e^{At}

b) Jordan Form

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \qquad \rightarrow \qquad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \qquad \rightarrow \qquad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2 e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

How to Evaluate e^{At}

c) General A : diagonalize !!

$\dot{\mathrm{x}} = \mathrm{A\,x} + \mathrm{B}u$ i) $Q(\lambda) = \det[\lambda I - A] = 0$: Characteristic equation.

$$\mathrm{A} : n \times n$$

$$n \;\; sol \;\; = \lambda_i \qquad i = 1, \cdots, n$$

ii) λ_i : eigenvalue of A

n *distinct eigenvalues.*

iii) $(\lambda_i \mathrm{I} - \mathrm{A}) p_i = 0$ $\qquad p_i : eigen\ vector$

$$\mathrm{A} p_i = \lambda_i p_i$$

$$\mathrm{A}\begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \qquad \mathrm{A\,P} = \mathrm{P}\,\Lambda$$

$$\Rightarrow \;\; \mathrm{P}^{-1}\mathrm{AP} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \Lambda$$

Diagonalizable if $\lambda_i : n - distint\ eigenvalues$

How to Evaluate e^{At}

$$\mathrm{A}p_i = \lambda_i p_i$$

$if \quad p_i^T p_i = 1, \quad p_i$ *is a normalized eigenvector.*

Orthogonality of eigenvector

$$if\ \mathrm{A} = \mathrm{A}^T$$

$$\begin{cases} \mathrm{A}\, p_i = \lambda_i p_i \\ \mathrm{A}\, p_j = \lambda_j p_j \end{cases} \rightarrow \begin{cases} p_j^T\, \mathrm{A}\, p_i = \lambda_i p_j^T p_i = 0 \\ p_i^T\, \mathrm{A}\, p_j = \lambda_j p_i^T p_j = 0 \end{cases}$$

$$\mathrm{A}\left[p_1 \quad \cdots \quad p_n\right] = \left[p_1 \quad \cdots \quad p_n\right]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\mathrm{P}^{\mathrm{T}}\mathrm{A}\mathrm{P} = \Lambda$$

How to Evaluate e^{At}

$$let \quad P\hat{x} = x, \qquad \dot{x} = P\dot{\hat{x}}$$

$$\dot{x} = A\,x + B\,u$$

$$P\,\dot{\hat{x}} = AP\hat{x} + B\,u$$

$$\dot{\hat{x}} = P^{-1}\,AP\hat{x} + P^{-1}\,B\,u = \Lambda\,\hat{x} + P^{-1}\,B\,u$$

$$\hat{x}(t) = e^{\Lambda t}\,\hat{x}(0) + \int_0^t e^{\Lambda(t-\tau)}\,P^{-1}\,B\,u(\tau)d\tau$$

$$P\hat{x}(t) = P\,e^{\Lambda t}\,P^{-1}\,x(0) + \int_0^t P\,e^{\Lambda(t-\tau)}\,P^{-1}\,B\,u(\tau)d\tau$$

$$\therefore\ x(t) = P\,e^{\Lambda t}\,P^{-1}\,x(0) + \int_0^t P\,e^{\Lambda(t-\tau)}\,P^{-1}\,B\,u(\tau)d\tau$$

How to Evaluate e^{At}

d) General A (-> Jordan form)

repeated λ_i : multiple eigenvalue

$$\det[\lambda_i \mathrm{I} - \mathrm{A}] = (\lambda - \lambda_1)(\lambda - \lambda_2)^2 \qquad \mathrm{A}: 3\times 3$$

$$(\lambda_1 \mathrm{I} - \mathrm{A})p_1 = 0 \quad \Rightarrow \quad p_1$$

$$(\lambda_2 \mathrm{I} - \mathrm{A})p_2 = 0 \quad \Rightarrow \quad p_2$$

$$if\ \ rank(\lambda_2 \mathrm{I} - \mathrm{A}) = 2 \quad then \quad p_3\,?$$

Find p *such that*

$$\mathrm{AP} = \mathrm{P}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & 1 \\ & & \lambda_2 \end{bmatrix} \quad \Rightarrow \quad \mathrm{A}\, p_1 = \lambda_1 p_1,$$

$$\mathrm{A}\, p_2 = \lambda_2 p_2$$

$$\mathrm{A}\, p_3 = p_2 + \lambda_2 p_3, \qquad (\mathrm{A} - \lambda_2 \mathrm{I})p_3 = p_2$$

Multiple Eigenvalue - Diagonal Form

ex1)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad Q(\lambda) = \det\left|\lambda \mathrm{I} - A\right| = \begin{vmatrix} \lambda - 1 & 0 & 1 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix}$$

$$= (\lambda - 1)^2 (\lambda - 2)$$

$$eigenvalue \ : \ \lambda_1 = 1, \ \lambda_2 = 1, \ \lambda_3 = 2$$

$$eigenvector : \ Ap_i = \lambda_i p_i \qquad (\lambda_i \mathrm{I} - A) p_i = 0$$

$$\lambda_{1,2} = 1, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} p_{1,2} = 0 \ \Rightarrow \ choose \ \ p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 2, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} p_3 = 0 \ \Rightarrow \ p_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Multiple Eigenvalue - Diagonal Form

$$\mathrm{P} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathrm{P}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \mathrm{P}^{-1}\mathrm{AP} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \Lambda$$

$$\dot{\mathrm{x}} = \mathrm{A\,x}$$

$$\dot{\hat{\mathrm{x}}} = \mathrm{P}^{-1}\,\mathrm{AP}\hat{\mathrm{x}} = \Lambda\,\hat{\mathrm{x}}$$

$$\hat{\mathrm{x}}(t) = e^{\Lambda t}\,\hat{\mathrm{x}}(0), \qquad \mathrm{P}\,\hat{\mathrm{x}} = \mathrm{x} \qquad \Rightarrow \qquad \mathrm{P}^{-1}\mathrm{x}(t) = e^{\Lambda t}\,\mathrm{P}^{-1}\,\mathrm{x}(0)$$

$$\therefore\ \mathrm{x}(t) = \mathrm{P}\,e^{\Lambda t}\,\mathrm{P}^{-1}\,\mathrm{x}(0) = e^{\mathrm{A}t}\,\mathrm{x}(0)$$

$$e^{\mathrm{A}t} = \mathrm{P}\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}\mathrm{P}^{-1} = \begin{bmatrix} e^t & 0 & e^t \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

Multiple Eigenvalue - Jordan Form

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \qquad Q(\lambda) = \det\left|\lambda\,\mathrm{I} - \mathrm{A}\right| = \begin{vmatrix} \lambda - 1 & -1 & -2 \\ 0 & \lambda - 1 & -3 \\ 0 & 0 & \lambda - 2 \end{vmatrix}$$

$$= \left(\lambda - 1\right)^2 \left(\lambda - 2\right)$$

$$eigenvalue \;:\; \lambda_{1,2} = 1, \quad \lambda_3 = 2$$

$$eigenvector : \; \mathrm{A}\, p_i = \lambda_i\, p_i \qquad \left(\lambda_i\,\mathrm{I} - \mathrm{A}\right) p_i = 0$$

$$\lambda_{1,2} = 1, \qquad \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix} p_{1,2} = 0 \quad \Rightarrow \quad p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We can find only 1 eigenvector for $\lambda = 1$

$$\lambda_3 = 2, \qquad \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} p_3 = 0 \quad \Rightarrow \quad p_3 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

Multiple Eigenvalue - Jordan Form

If not able to find 3 independent vectors, find p_2 which transforms A as Jordan form.

$$\mathrm{AP} = \mathrm{PJ} = \mathrm{P}\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\mathrm{A}\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \mathrm{A}\,p_1 & \mathrm{A}\,p_2 & \mathrm{A}\,p_3 \end{bmatrix} = \begin{bmatrix} p_1\lambda_1 & p_1 + \lambda_1 p_2 & \lambda_3 p_3 \end{bmatrix}$$

So that, $\mathrm{A}\,p_1 = \lambda_1 p_1 \qquad \left[\lambda_1 \mathrm{I} - \mathrm{A}\right] p_1 = 0 \quad \Rightarrow \quad p_1$

$\mathrm{A}\,p_1 = p_1 + \lambda_1 p_2 \qquad \left[\mathrm{A} - \lambda_1 \mathrm{I}\right] p_2 = p_1 \quad \Rightarrow \quad p_2$

$\mathrm{A}\,p_3 = \lambda_3 p_3 \qquad \left[\mathrm{A} - \lambda_3 \mathrm{I}\right] p_3 = 0 \quad \Rightarrow \quad p_3$

Multiple Eigenvalue - Jordan Form

$$\left[\mathrm{A}-\lambda_1 \mathrm{I}\right] p_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} p_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad let, \quad p_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$b + 2c = 1, \quad 3c = 0, \quad c = 0 \qquad \rightarrow \qquad b = 1, \qquad choose \quad p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathrm{P} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathrm{P}^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad \mathrm{P}^{-1}\mathrm{AP} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Jordan form

Multiple Eigenvalue - Jordan Form

$$\textit{if}\quad \dot{\mathrm{x}} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \mathrm{x} = \mathrm{A}\mathrm{x}\,, \qquad \mathrm{x}(t) = e^{\mathrm{A}t}\,\mathrm{x}(0)\ \ ?$$

$$\textit{let}\quad \mathrm{P}\hat{\mathrm{x}} = \mathrm{x}, \qquad \dot{\mathrm{x}} = \mathrm{P}\dot{\hat{\mathrm{x}}}$$

$$\dot{\hat{\mathrm{x}}}(t) = \mathrm{P}^{-1}\,\mathrm{A}\mathrm{P}\hat{\mathrm{x}} = \mathrm{J}\,\hat{\mathrm{x}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \hat{\mathrm{x}}$$

$$e^{\mathrm{J}t} = \begin{bmatrix} e^{t} & te^{t} & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}, \qquad \hat{\mathrm{x}}(t) = e^{\mathrm{J}t}\,\hat{\mathrm{x}}(0)$$

$$\mathrm{x}(t) = \mathrm{P}\,e^{\mathrm{J}t}\,\mathrm{P}^{-1}\mathrm{x}(0) = e^{\mathrm{A}t}\,\mathrm{x}(0)$$

Multiple Eigenvalue - Jordan Form

$$\therefore e^{\mathrm{A}t} = \mathrm{P}\, e^{\mathrm{J}t}\, \mathrm{P}^{-1} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & te^{t} & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & te^{t} & 5e^{2t} \\ 0 & e^{t} & 3e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & te^{t} & \left(-5e^{t} - 3te^{t} + 5e^{2t}\right) \\ 0 & e^{t} & \left(-3e^{t} + 3e^{2t}\right) \\ 0 & 0 & e^{2t} \end{bmatrix}$$

Summary

1. Diagonal Form

$$A = \begin{bmatrix} \lambda_1 & & \underline{0} \\ & \lambda_2 & \\ \underline{0} & & \lambda_3 \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_3 t} \end{bmatrix}$$

2. Jordan Form

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} \end{bmatrix},$$

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2 e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

Summary

3. General A

$$\dot{\mathrm{x}} = \mathrm{A}\,\mathrm{x} + \mathrm{B}u \qquad \mathrm{A}\,p_i = \lambda_i p_i, \qquad \mathrm{P}^{-1}\mathrm{A}\mathrm{P} = \Lambda$$

$$let \quad \mathrm{x} = \mathrm{P}\hat{\mathrm{x}}$$

$$\dot{\hat{\mathrm{x}}} = \mathrm{P}^{-1}\,\mathrm{A}\mathrm{P}\hat{\mathrm{x}} + \mathrm{P}^{-1}\,\mathrm{B}u = \Lambda\,\hat{\mathrm{x}} + \mathrm{P}^{-1}\,\mathrm{B}u$$

$$\hat{\mathrm{x}}(t) = e^{\Lambda t}\,\hat{\mathrm{x}}(0) + \int e^{\Lambda(t-\tau)}\,\mathrm{P}^{-1}\,\mathrm{B}u(\tau)d\tau$$

$$\mathrm{P}\hat{\mathrm{x}}(t) = \mathrm{x}(t) = \mathrm{P}\,e^{\Lambda t}\,\mathrm{P}^{-1}\,\mathrm{x}(0) + \int \mathrm{P}\,e^{\Lambda(t-\tau)}\,\mathrm{P}^{-1}\,\mathrm{B}u(\tau)d\tau$$

$$\mathrm{P}\,e^{\Lambda t}\,\mathrm{P}^{-1} = e^{\mathrm{A}t}$$

Laplace Transformation Method

$$\dot{\mathrm{x}} = \mathrm{A}\,\mathrm{x} + \mathrm{B}u\,, \qquad x(t) = e^{\mathrm{A}t}x_0 + \int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)d\tau$$

Laplace Transformation

$$s\,\mathrm{X}(s) - x_0 = \mathrm{A}\,\mathrm{X}(s) + \mathrm{B}u(s)\,, \qquad \mathrm{X}(s) = (s\,\mathrm{I} - \mathrm{A})^{-1}x_0 + (s\,\mathrm{I} - \mathrm{A})^{-1}\mathrm{B}u(s)$$

$$\therefore e^{\mathrm{A}t} = \mathcal{L}^{-1}\left[(s\,\mathrm{I} - \mathrm{A})^{-1}\right]$$

$$(s\,\mathrm{I} - \mathrm{A})^{-1} = \frac{1}{s}\left(\mathrm{I} - \frac{1}{s}\mathrm{A}\right)^{-1}$$

$$= \frac{1}{s}\left(\mathrm{I} + \frac{1}{s}\mathrm{A} + \frac{1}{s^2}\mathrm{A}^2 + \frac{1}{s^3}\mathrm{A}^3 \cdots\right)$$

$$= \frac{1}{s}\mathrm{I} + \frac{1}{s^2}\mathrm{A} + \frac{1}{s^3}\mathrm{A}^2 + \frac{1}{s^4}\mathrm{A}^3 \cdots$$

Laplace Transformation Method

Laplace transformation table

$$\mathcal{L}[t] = \frac{1}{s^2}, \quad \mathcal{L}\left[\frac{1}{(n-1)!}t^{n-1}\right] = \frac{1}{s^n}, \quad \mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\Rightarrow \mathcal{L}^{-1}\left[(s\mathrm{I}-\mathrm{A})^{-1}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\mathrm{I} + \frac{1}{s^2}\mathrm{A} + \frac{1}{s^3}\mathrm{A}^2 + \frac{1}{s^4}\mathrm{A}^3 \cdots\right]$$

$$= \mathrm{I} + \mathrm{A}t + \frac{1}{2!}\mathrm{A}^2 t^2 + \frac{1}{3!}\mathrm{A}^3 t^3 \cdots$$

$$\therefore e^{\mathrm{A}t} = \mathrm{I} + \mathrm{A}t + \frac{1}{2!}\mathrm{A}^2 t^2 + \frac{1}{3!}\mathrm{A}^3 t^3 + \cdots$$

Solution of Linear (Time Invariant) State Equation

Method 1. Diagonalization

Method 2. Laplace Transformation

Method 3. Sylvester's Interpolation Formula

End of Lecture 10-4