Lecture 11-2

Frequency Domain Analysis II

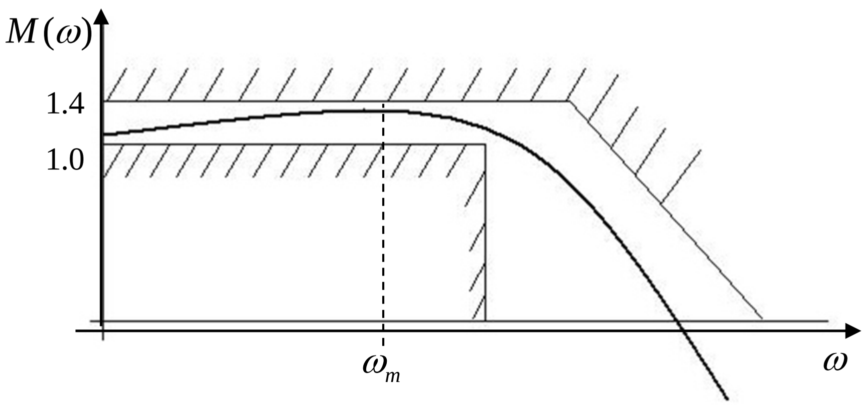
Control System Design in Frequency Response

1. large $\omega_m \rightarrow$ $\boxed{\text{fast time response}}$

2. $M_m,\ M_p$: $\boxed{\text{function of damping ratio } \zeta}$
   large $M_m \rightarrow$ large M_p

3. good damping characteristics $1 < M_m < 1.4$

4. minimum effect of any undesirable noise



Experimental Determination of Transfer Function

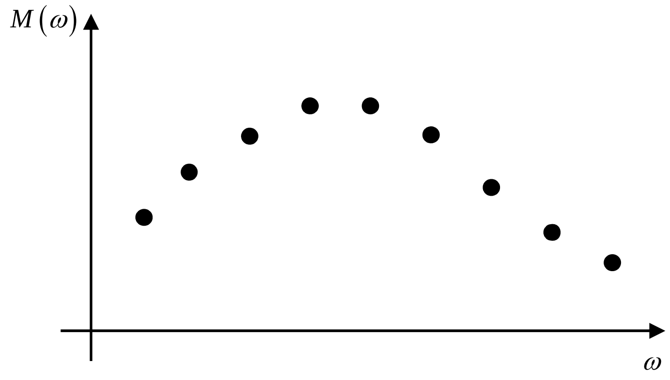
$$r(t) \longrightarrow \boxed{\begin{array}{c}\textit{Unknown}\\ \textit{System}\end{array}} \longrightarrow y(t)$$

$$r(t) = \sin \omega_i t \quad ; \quad y(t) = M(\omega_i)\sin\left(\omega_i t - \phi\right)$$

some frequency ω_i $\rightarrow$ measure $M\left(\omega_i\right)$
steady state

$$G(s) = G\left(j\omega\right)$$

$$M(\omega) = \left|G\left(j\omega\right)\right|$$



Transfer Functions and Frequency Magnitude Plot

- Bode Plot (Logarithmic Plots)

i) Log Magnitude $L_m G(j\omega) = \boxed{20\log\left|G(j\omega)\right| dB}$

ii) dB : decibel, Logarithm of the magnitude

iii) 1 decade : $1Hz \sim 10Hz$ frequency width

$2.5Hz \sim 25Hz$

- Drawing the Bode Plots

The reason of using logarithm – mathematical operation $\times,\ \div \rightarrow +,\ -$

– typical three types : simple straight line

asymtotic

approximations

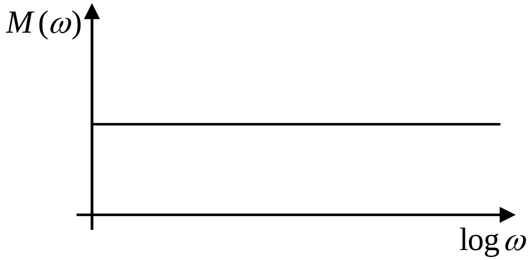
Transfer Functions and Frequency Magnitude Plot

ex) $$G(s)=\frac{K(1+sT_1)(1+sT_2)}{s^2(1+sT_3)(1+as+bs^2)}$$

$$L_mG(j\omega)=L_mK+L_m(1+j\omega T_1)+L_m(1+j\omega T_2)$$
$$-2L_m(j\omega)-L_m(1+j\omega T_3)-L_m\left\{1+aj\omega+b(j\omega)^2\right\}$$

$i)\ L_m\cdot K=\boxed{20\log K\ dB}$

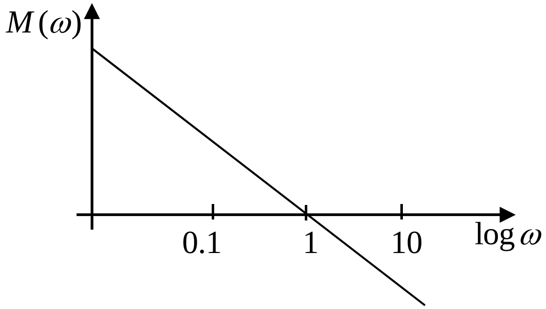
straight line



$$ii)\ L_m\left(\frac{1}{j\omega}\right)=\boxed{20\log\left|\frac{1}{j\omega}\right|=-20\log\omega}$$

straight line

With negative slope -20dB/decade



Transfer Functions and Frequency Magnitude Plot

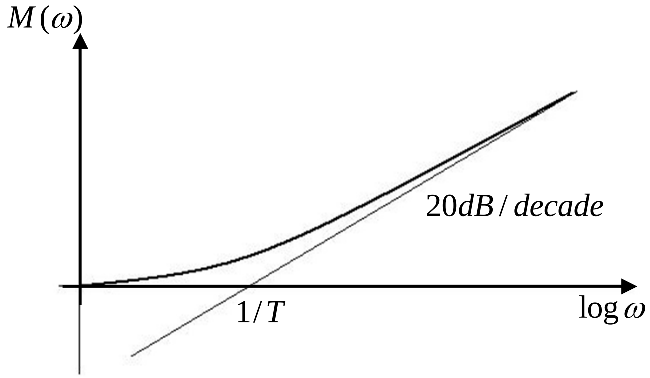
$$iii)\quad (1+j\omega T)$$

$$L_m(1+j\omega T) = \boxed{20\log|1+j\omega T|} = \boxed{20\log\sqrt{1+\omega^2T^2}}$$

$$\omega T \ll 1 \quad \rightarrow \quad L_m(1+j\omega T) \cong 20\log 1 = 0dB$$

$$\omega T \gg 1 \quad \rightarrow \quad L_m(1+j\omega T) \cong 20\log \omega T$$

$$= 20\log\omega + 20\log T$$



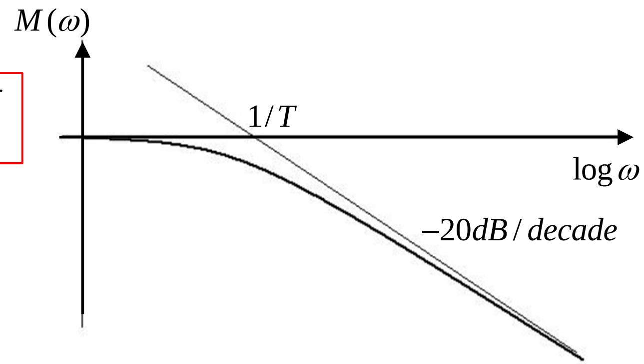
$$iv)\quad \frac{1}{1+j\omega T}$$

$$L_m(1+j\omega T)^{-1} = \boxed{-20\log|1+j\omega T| = -20\log\sqrt{1+\omega^2T^2}}$$

$$\omega T \ll 1 \quad \rightarrow \quad L_m(1+j\omega T)^{-1} \cong -20\log 1 = 0dB$$

$$\omega T \gg 1 \quad \rightarrow \quad L_m(1+j\omega T)^{-1} \cong -20\log \omega T$$

$$= -20\log\omega - 20\log T$$



Transfer Functions and Frequency Magnitude Plot

$$v)\quad \frac{1}{\left(1+\frac{2\zeta}{\omega_n}s+\frac{1}{\omega_n^{2}}s^{2}\right)} \quad\Rightarrow\quad \frac{1}{\left[1+\frac{2\zeta}{\omega_n}j\omega+\frac{1}{\omega_n^{2}}(j\omega)^{2}\right]}$$

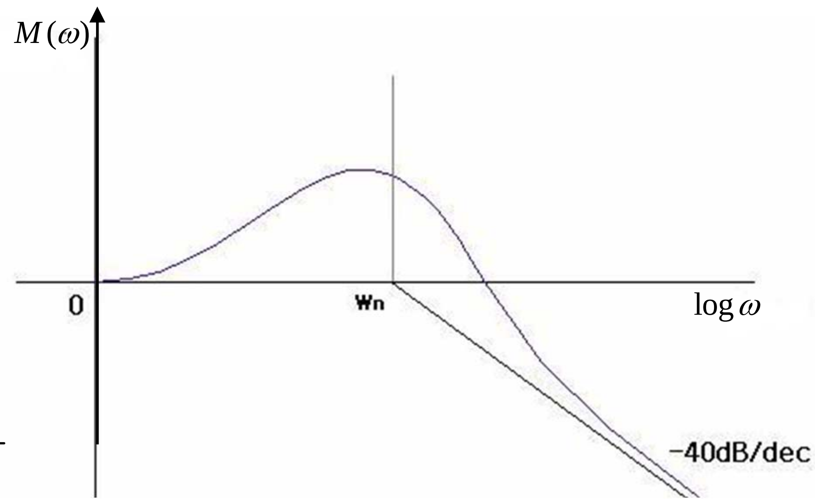
$$L_m\frac{1}{\left[1+\frac{2\zeta}{\omega_n}j\omega+\frac{1}{\omega_n^{2}}(j\omega)^{2}\right]}=\boxed{20\log\frac{1}{\left[1+\frac{2\zeta}{\omega_n}j\omega+\frac{1}{\omega_n^{2}}(j\omega)^{2}\right]}=-20\log\left[\left(1-\frac{\omega^{2}}{\omega_n^{2}}\right)+\left(\frac{2\zeta\omega}{\omega_n}\right)^{2}\right]^{1/2}}$$

$$\frac{\omega}{\omega_n}<<1$$

$$\rightarrow\quad L_m\left[G(j\omega)\right]\cong-20\log 1=0dB$$

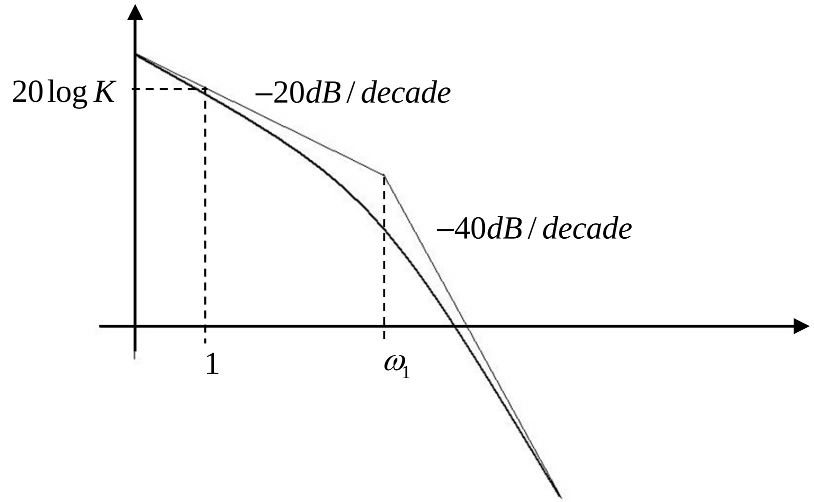
$$\frac{\omega}{\omega_n}>>1$$

$$\rightarrow\quad L_m\left[G(j\omega)\right]\cong-20\log\frac{\omega^{2}}{\omega_n^{2}}=-40\log\frac{\omega}{\omega_n}$$

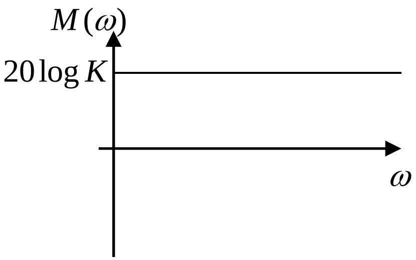


Transfer Functions and Frequency Magnitude Plot

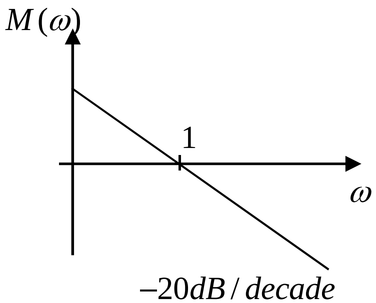
ex)



$=$



$+$



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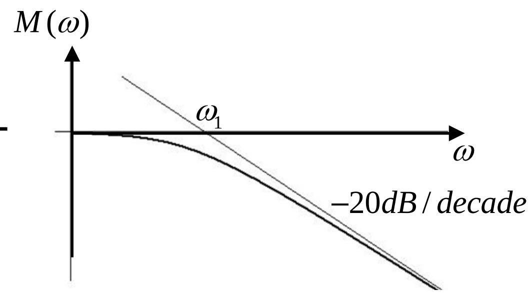


$$= \boxed{\frac{K}{(j\omega)\left(1+\dfrac{j\omega}{\omega_1}\right)} \quad \Rightarrow \quad \frac{K}{s\left(1+\dfrac{1}{\omega_1}s\right)}}$$

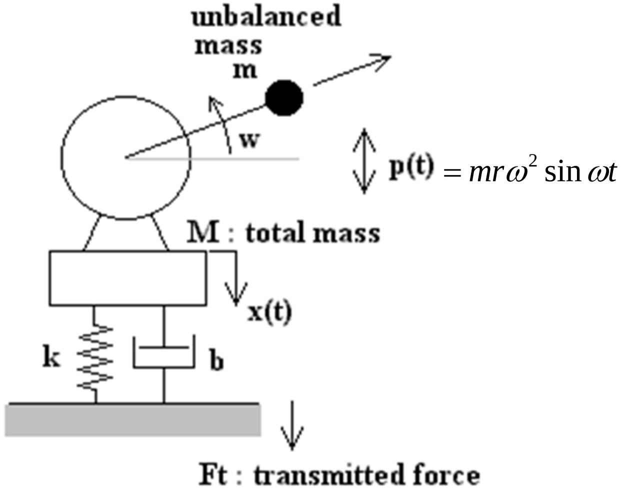
Vibration Isolation in Rotating Systems

Vibration due to rotating unbalance

$$(1) \quad M\ddot{x} + b\dot{x} + kx = p(t)$$

$$= mr\omega^2 \sin \omega t$$

$$\frac{X(s)}{P(s)} = \frac{1}{Ms^2 + bs + k} = G(s)$$



Vibration Isolation in Rotating Systems

(2) Frequency response

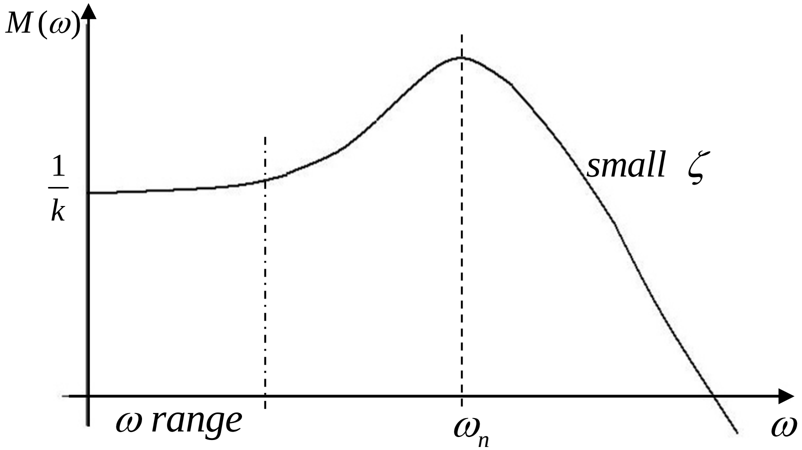
$$\frac{x(j\omega)}{p(j\omega)} = \frac{1}{-M\omega^2 + bj\omega + k} = G(j\omega)$$

$$x(t) = X\sin(\omega t + \phi) = \left|G(j\omega)\right| \sin(\omega t + \phi) \cdot mr\omega^2$$

$$\phi = -\tan^{-1}\frac{b\omega}{k - M\omega^2} = -\tan^{-1}\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^{\ 2}}$$

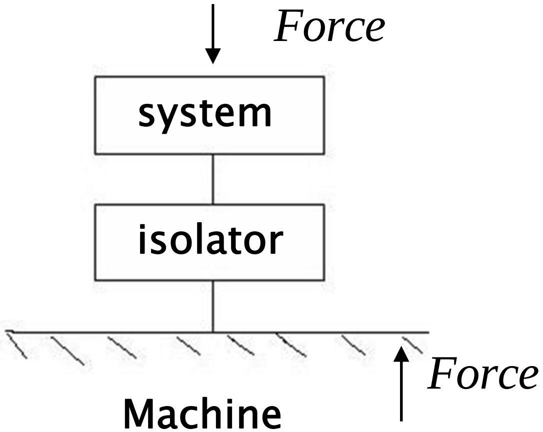
$$\left|G(j\omega)\right| = \frac{1}{\sqrt{\left(k - M\omega^2\right)^2 + b^2\omega^2}}$$

$$= \frac{1/k}{\sqrt{\left(1-\omega^2/\omega_n^{\ 2}\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}}$$

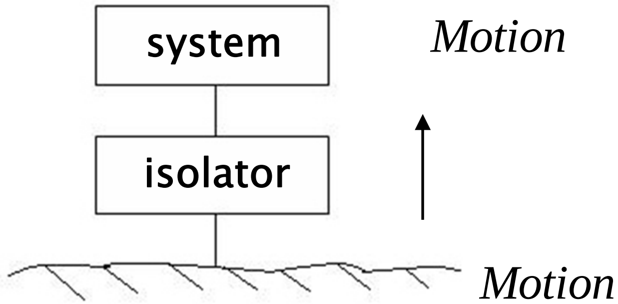


Vibration Isolation in Rotating Systems

Vibration isolators



Machine



Suspensions

Reduce the magnitude of force transmitted from a machine to its foundation

Reduce the magnitude of motion transmitted from a vibratory foundation to a system

Isolator i) - load supporting elements, spring.

- energy-dissipating elements, damper.

ii) - Synthetic rubber : both load supporting and energy dissipating

Transmissibility

Transmissibility :

A measure of the reduction of transmitted force or motion afforded by an isolator.

$$TR(machine) = \frac{the\ \ force\ amplitude\ transmitted\ to\ the\ \ foundation}{the\ amplitude\ of\ \ the\ exciting\ \ force}$$

$$TR(suspensions) = \frac{the\ vibration\ amplitude\ of\ \ the\ system}{the\ vibration\ amplitude\ of\ \ the\ \ foundation}$$

Transmissibility for Force Excitation

$$p(t) = mr\omega^2 \sin \omega t = F_0 \sin \omega t$$

$$f(t) = b\dot{x} + kx = F_t \sin\left(\omega t + \phi\right)$$

$$X(s) = \frac{1}{Ms^2 + bs + k} P(s)$$

$$F_t(s) = (bs + k) X(s) = \frac{bs + k}{Ms^2 + bs + k} \cdot P(s)$$

Frequency response

$$\frac{F_t(j\omega)}{p(j\omega)} = \frac{bj\omega + k}{-M\omega^2 + bj\omega + k} = \frac{\left(b/M\right) j\omega + \left(k/M\right)}{-\omega^2 + \left(b/M\right) j\omega + k/M}$$

Transmissibility for Force Excitation

$$k/M=\omega_n^{\,2}\,,\qquad b/M=2\zeta\omega_n \qquad \Rightarrow \frac{F_t(j\omega)}{p(j\omega)}=\frac{1+j\left(2\zeta\omega/\omega_n\right)}{1-\omega^2/\omega_n^{\,2}+j\left(2\zeta\omega/\omega_n\right)}$$

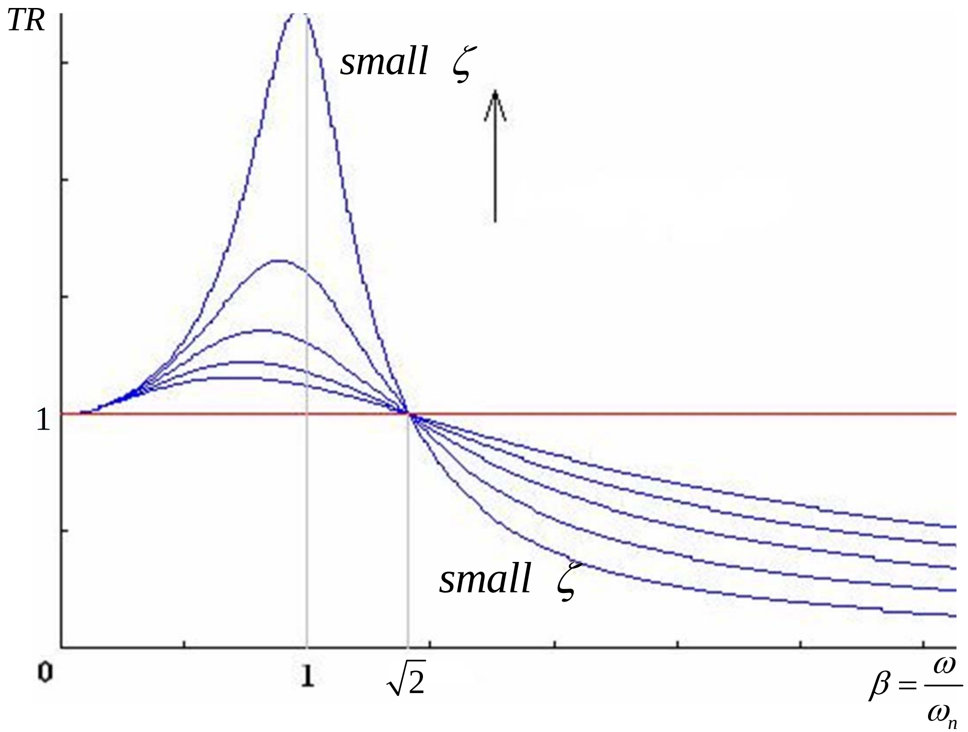
$$TR=\frac{F_t}{F_0}=\left|\frac{F_t(j\omega)}{p(j\omega)}\right|$$

$$=\frac{\sqrt{1+\left(2\zeta\omega/\omega_n\right)^2}}{\sqrt{\left(1-\omega^2/\omega_n^{\,2}\right)^2+\left(2\zeta\omega/\omega_n\right)^2}}$$

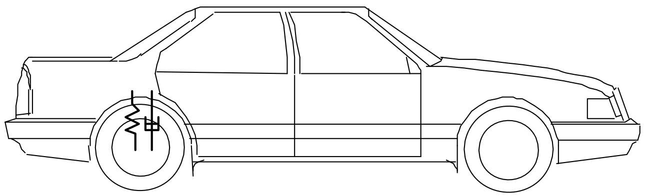
$$=\frac{\sqrt{1+\left(2\zeta\beta\right)^2}}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}$$

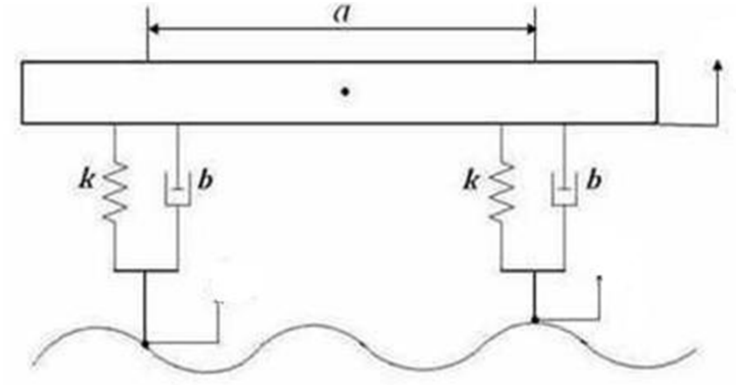
Note, $\beta=\sqrt{2}\,,\qquad TR=1$

for any ζ

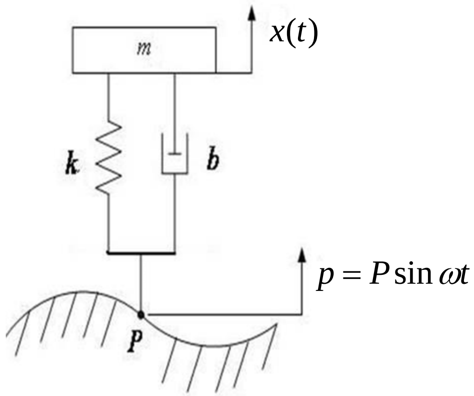


Automobile Suspension Systems





Simplified model



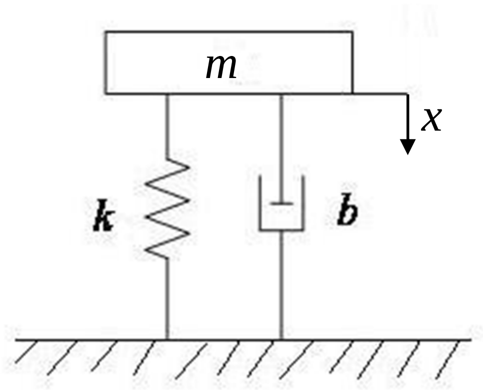
$$m\ddot{x} + b(\dot{x} - p) + k(x - p) = 0$$

$$\frac{X(s)}{P(s)} = \frac{bs + k}{ms^2 + bs + k}, \qquad \frac{x(j\omega)}{p(j\omega)} = \frac{bj\omega + k}{-m\omega^2 + bj\omega + k}$$

$$TR = \left| \frac{x(j\omega)}{p(j\omega)} \right| = \frac{\sqrt{b^2\omega^2 + k^2}}{\sqrt{\left(k - m\omega^2\right)^2 + b^2\omega^2}}$$

$$= \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{\left(1 - \beta^2\right)^2 + (2\zeta\beta)^2}}$$

Dynamic Vibration Absorbers



$$P = mr\omega^2$$

$$TR = \frac{\left(mr\omega^2\right)\sqrt{b^2\omega^2 + k^2}}{\sqrt{\left(k - m\omega^2\right)^2 + b^2\omega^2}}$$

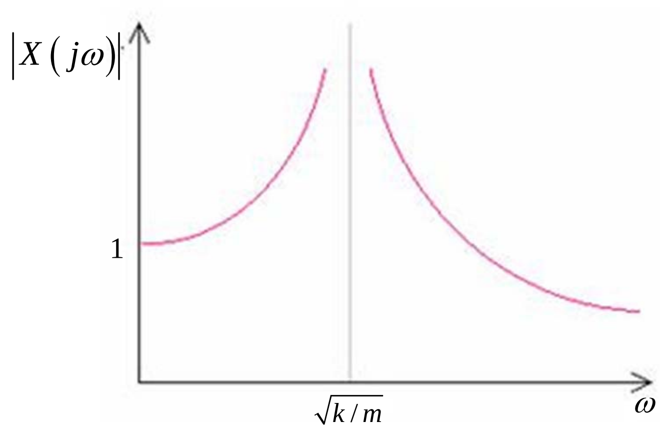
If b is small and $\omega = \omega_n$

Then resonance – excessive vibration

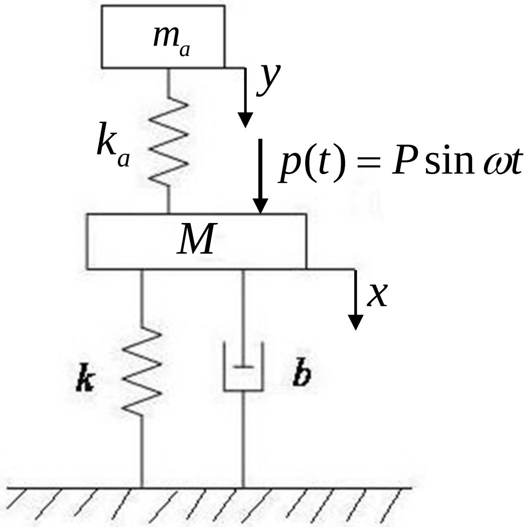
– extremely large force transmitted

ω is very close to ω_n , critical speed,

=> A dynamic vibration absorber



Dynamic Vibration Absorbers



$$M\ddot{x} = -kx - b\dot{x} - k_a\left(x - y\right) + p(t)$$

$$m_a\ddot{y} = -k_a\left(y - x\right)$$

Laplace Transform

$$\left(Ms^2 + bs + k + k_a\right)X(s) - k_aY(s) = P(s)$$

$$\left(m_as^2 + k_a\right)Y(s) - k_aX(s) = 0$$

$$\Rightarrow \quad \frac{X(s)}{P(s)} = \frac{m_as^2 + k_a}{\left(Ms^2 + bs + k + k_a\right)\left(m_as^2 + k_a\right) - k_a^{\,2}}$$

Frequency response

$$\frac{X(j\omega)}{P(j\omega)} = \frac{-m_a\omega^2 + k_a}{\left(-M\omega^2 + k + k_a\right)\left(-m_a\omega^2 + k_a\right) - k_a^{\,2}} \qquad \textit{for small } b$$

The transmitted force

$$f(t) = kx + b\dot{x} \cong kx$$

Dynamic Vibration Absorbers

$$|X(j\omega)| = \left| \frac{k_a - m_a\omega^2}{\left(-M\omega^2 + k + k_a\right)\left(k_a + m_a\omega^2\right) - k_a{}^2} \right| |P(j\omega)|$$

$$= \left| \frac{mr\omega^2\left(k_a - m_a\omega^2\right)}{\left(-M\omega^2 + k + k_a\right)\left(k_a + m_a\omega^2\right) - k_a{}^2} \right|$$

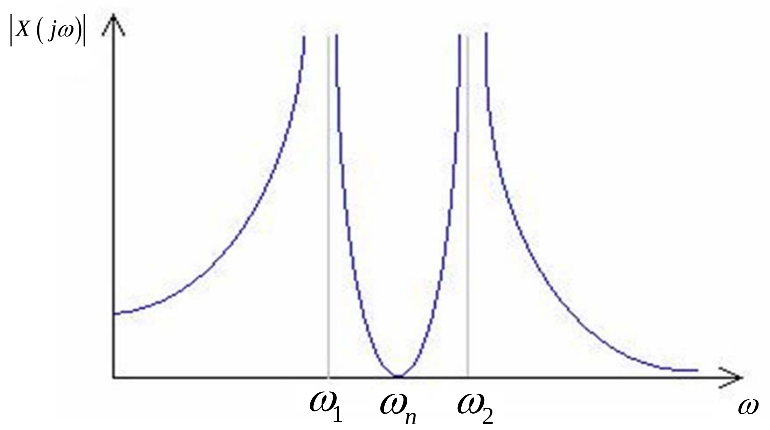
If we choose $k_a,\ m_a$ so that,

$$k_a - m_a\omega^2 = 0$$

then, $|X(j\omega)| = 0,\ |f(t)| = 0$ at this frequency.

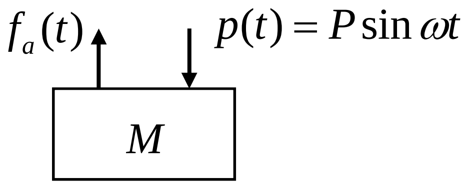
If $\omega_{exc} = \omega_n$

we use dynamic vibration absorbers.



Dynamic Vibration Absorbers

Physically,



$$f_a(t) = -k_a\left(x - y\right) = k_a y = p(t), \qquad \left(at \;\; \omega_{exc} = \omega_n, \quad x \cong 0\right)$$

Spring force $k_a y$ cancels $p(t)$.

$$\frac{Y(j\omega)}{P(j\omega)} = \frac{k_a}{\left(-M\omega^2 + k + k_a\right)\left(-m_a\omega^2 + k_a\right) - k_a{}^2}$$

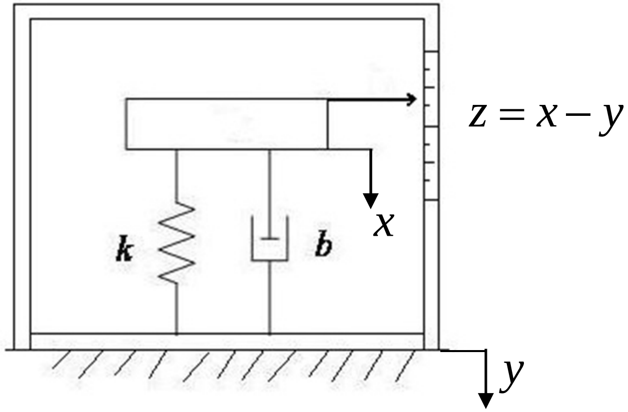
If we choose m_a, k_a so that, $k_a - m_a\omega^2 = 0$

then $\dfrac{Y(j\omega)}{P(j\omega)} = -\dfrac{1}{k_a}$ magnitude ratio $\left|\dfrac{1}{k_a}\right|$

$$p(t) = \sin\omega t$$

$$y(t) = \frac{1}{k_a}P\sin\left(\omega t - 180^\circ\right) = -\frac{P}{k_a}\sin\omega t \qquad \left(phase \;\; -180^\circ\right)$$

Seismograph



A device used to measure ground displacement during earthquakes

Equation of motion : $m\ddot{x} = k\left(y - x\right) + b\left(\dot{y} - \dot{x}\right)$

measurement : $z = x - y$

$$\Rightarrow m\left(\ddot{y} + \ddot{z}\right) + b\dot{z} + kz = 0$$

$$m\ddot{z} + b\dot{z} + kz = -m\ddot{y}$$

$$\Rightarrow Transfer\ Function\ :\ \frac{Z(s)}{Y(s)} = \frac{-ms^2}{ms^2 + bs + k} = \frac{-s^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

$$\left(\frac{k}{m} = \omega_n{}^2,\quad \frac{b}{m} = 2\zeta\omega_n\right)$$

Seismograph

$$\frac{Z(j\omega)}{Y(j\omega)} = \frac{\omega^2}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n{}^2} = \frac{\beta^2}{1-\beta^2 + 2\zeta\beta j} \qquad \left(\beta = \omega/\omega_n\right)$$

We hope, $z(t) \cong y(t)$

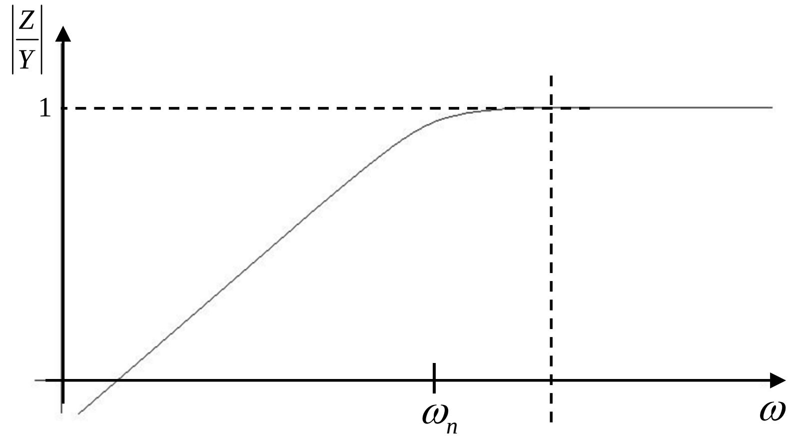
If, $\beta >> 1,\quad \dfrac{Z(j\omega)}{Y(j\omega)} = -1$

$z(t) \cong y(t) \qquad for \quad \omega >> \omega_n$

Choose ω_n as small as possible.

(large mass, soft spring)

$$\left(\omega_n = \sqrt{\frac{k}{m}}\right)$$



Accelerometer

The system configuration is basically the same as the seismograph, but choice of undamped natural frequency is different.

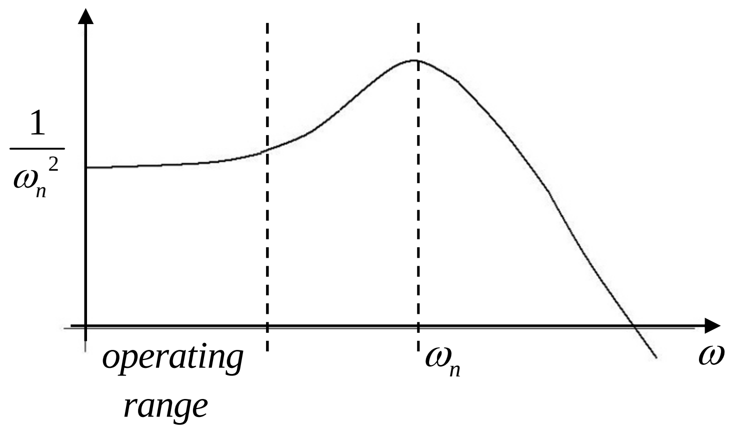
Equation of motion : the same as the seismograph, $m\ddot{z} + b\dot{z} + kz = -m\ddot{y}$

$$\Rightarrow \textit{Transfer Function} : \frac{Z(s)}{s^2 Y(s)} = \frac{-m}{ms^2 + bs + k} = \frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

Frequency response

$\omega_n >> \omega$

$$\frac{Z(s)}{s^2 Y(s)} \cong -\frac{1}{\omega_n}$$



End of Lecture 11-2