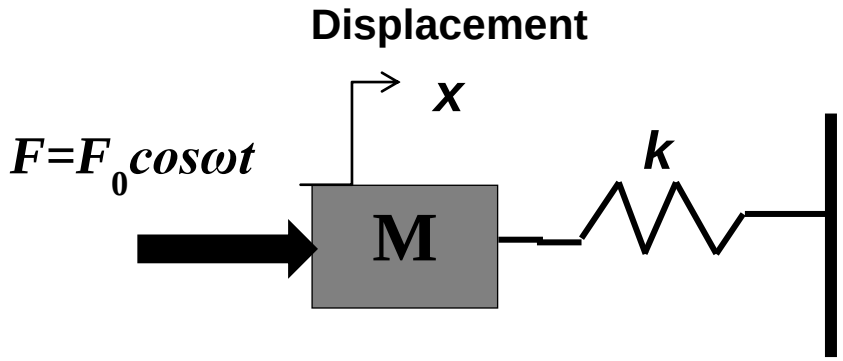
Chapter 2 Response to Harmonic Excitation

Introduces the important concept of resonance

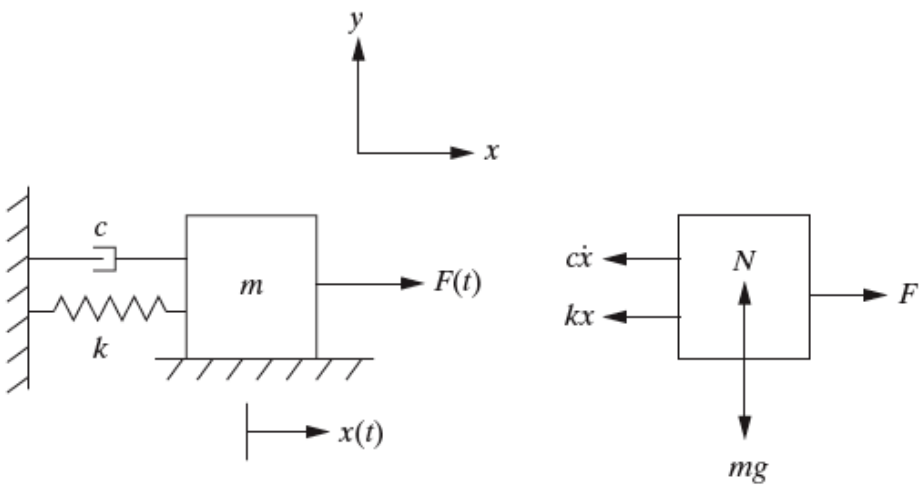
2.1 Harmonic Excitation of Undamped Systems

- **Consider the usual spring mass damper system with applied force $F(t)=F_0\cos\omega t$**
- **ω is the driving frequency**
- **F_0 is the magnitude of the applied force**
- **We take c = 0 to start with**



Equations of motion

- **Solution is the sum of *homogenous* and *particular* solution**
- **The *particular* solution assumes form of forcing function (physically the input wins):**



$$m\ddot{x}(t) = -kx(t) + F_0 \cos(\omega t)$$

$$\ddot{x}(t) + \omega_n^2 x(t) = f_0 \cos(\omega t)$$

where $f_0 = F_0 / m, \quad \omega_n = \sqrt{k/m}$

$$x_p(t) = X \cos(\omega t)$$

Substitute *particular* solution into the equation of motion:

$$x_p(t) = X\cos(\omega t)$$

$$\overbrace{-\omega^2 X\cos\omega t}^{\ddot{x}_p} + \overbrace{\omega_n^2 X\cos\omega t}^{\omega_n^2 x_p} = f_0\cos\omega t$$

solving yields: $X = \dfrac{f_0}{\omega_n^2 - \omega^2}$

Thus the particular solution has the form:

$$x_p(t) = \frac{f_0}{\omega_n^2 - \omega^2}\cos(\omega t)$$

Add particular and homogeneous solutions to get general solution:

$$x(t) =$$

$$\underbrace{A_1 \sin \omega_n t + A_2 \cos \omega_n t}_{\text{homogeneous}} + \overbrace{\frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t}^{particular} \qquad \textbf{(2.8)}$$

A_1 and A_2 are constants of integration.

Apply the initial conditions to evaluate the constants

$$x(0) = A_1 \sin 0 + A_2 \cos 0 + \frac{f_0}{\omega_n^2 - \omega^2} \cos 0 = A_2 + \frac{f_0}{\omega_n^2 - \omega^2} = x_0$$

$$\Rightarrow \underline{A_2 = x_0 - \frac{f_0}{\omega_n^2 - \omega^2}}$$

$$\dot{x}(0) = \omega_n (A_1 \cos 0 - A_2 \sin 0) - \frac{f_0}{\omega_n^2 - \omega^2} \sin 0 = \omega_n A_1 = v_0$$

$$\Rightarrow \underline{A_1 = \frac{v_0}{\omega_n}} \qquad \Rightarrow \qquad \textbf{(2.11)}$$

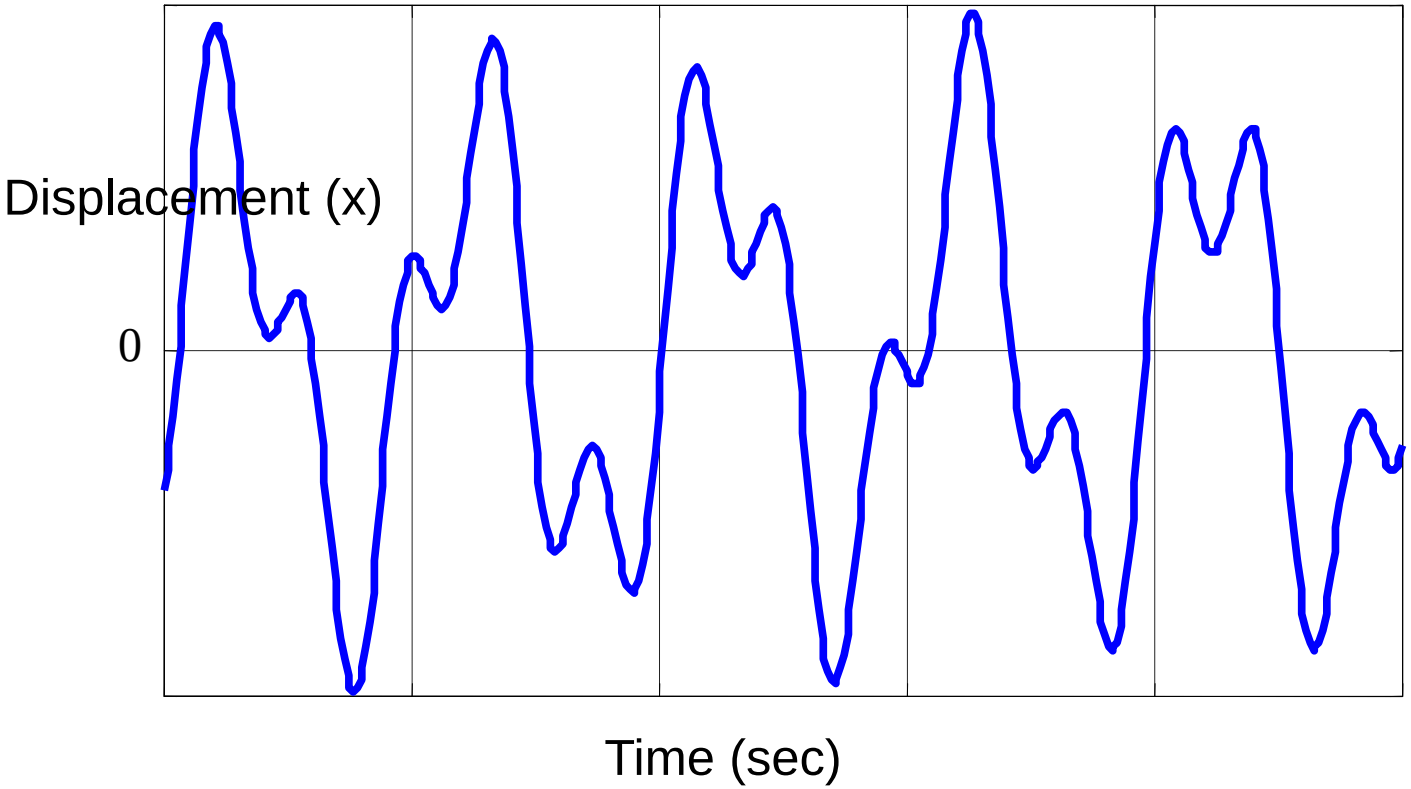
$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \right) \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

Comparison of free and forced response

- **Sum of two harmonic terms of different frequency**
- **Free response has amplitude and phase effected by forcing function**
- **Our solution is <u>not</u> defined for $\omega_n = \omega$ because it produces division by 0.**
- **If forcing frequency is close to natural frequency the amplitude of *particular* solution is very large**

Response for m=100 kg, k=1000 N/m, F=100 N, $\omega = \omega_n + 5$

v_0=0.1m/s and x_0= -0.02 m.



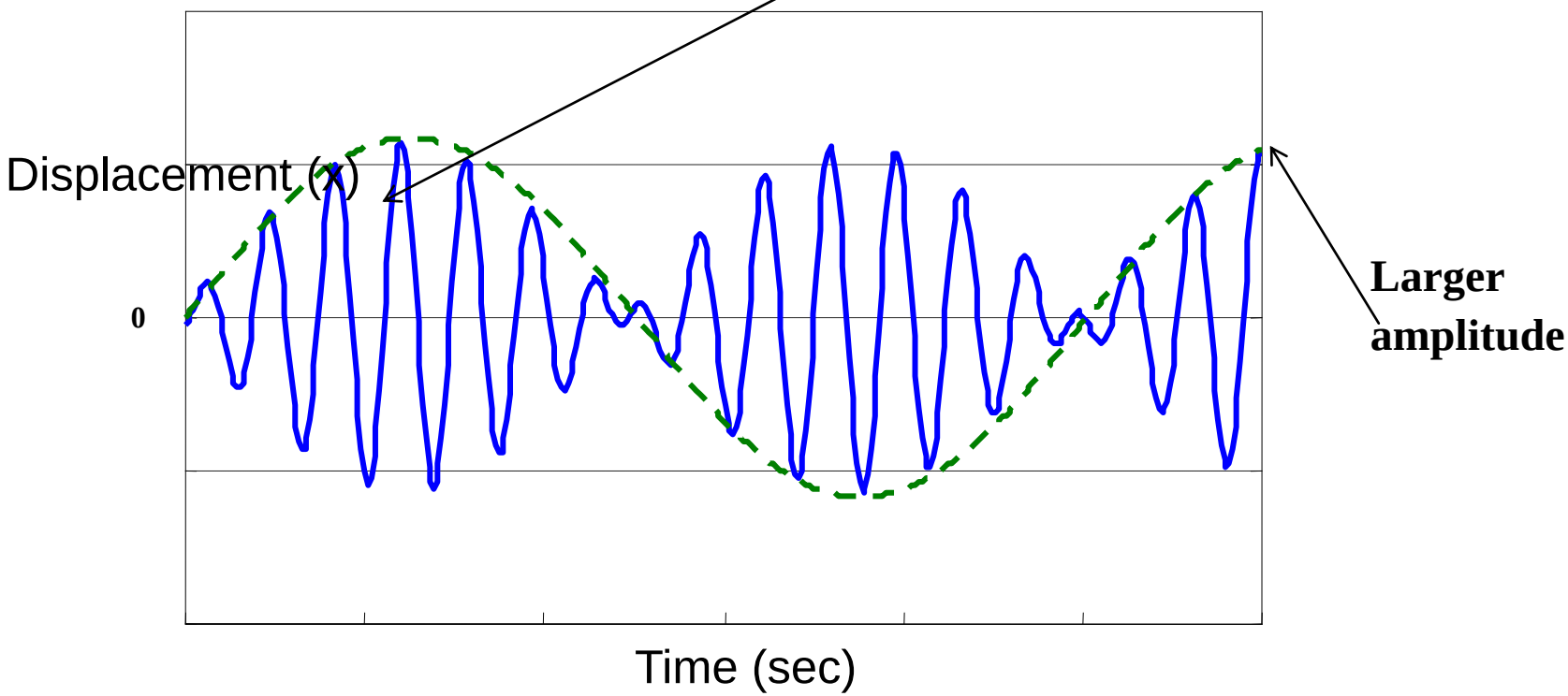
Note the obvious presence of two harmonic signals

What happens when ω is near ω_n?

$$x(t) = \frac{2 f_0}{\omega_n^2 - \omega^2} \sin\left(\frac{\omega_n - \omega}{2} t\right) \sin\left(\frac{\omega_n + \omega}{2} t\right) \quad (2.13)$$

When the drive frequency and natural frequency are close a beating phenomena occurs

$$\frac{2 f_0}{\omega_n^2 - \omega^2} \sin\left(\frac{\omega_n - \omega}{2} t\right)$$



What happens when ω is ω_n?

$$x_p(t) = tX\sin(\omega t)$$

substitute into eq. and solve for X

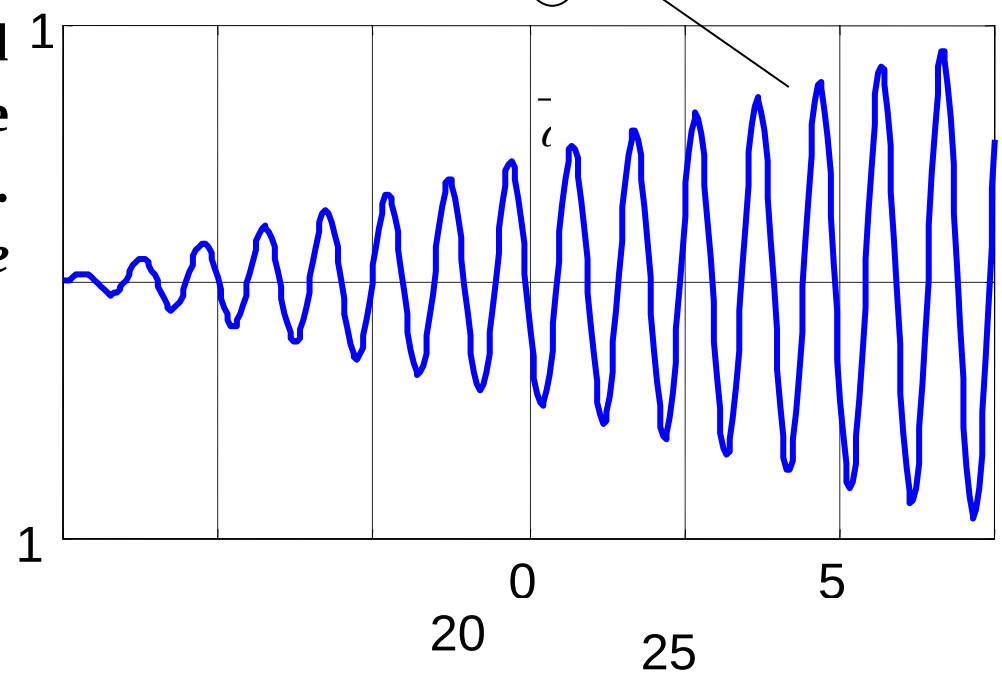
$$X = \frac{f_0}{2\omega}$$

$$x(t) = A_1 \sin wt + A_2 \cos wt + \overbrace{\frac{f_0}{2w} t \sin(wt)}^{\text{grows with out bound}}$$

When the drive frequency and natural frequency are the same the amplitude of the vibration grows without bounds. This is known as a *resonance* condition.

The most important

concept in Chapter 2!



Example 2.1.1: Compute and plot the response for m=10 kg, k=1000 N/m, x_0=0, v_0=0.2 m/s, F=23 N, $\omega = 2\omega_n$.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000 \text{ N/m}}{10 \text{ kg}}} = 10 \text{ rad/s}, \ \omega = 2\omega_n = 20 \text{ rad/s}$$

$$f_0 = \frac{F}{m} = \frac{23 \text{ N}}{10 \text{ kg}} = 2.3 \text{ N/kg}, \quad \frac{v_0}{\omega_n} = \frac{0.2 \text{ m/s}}{10 \text{ rad/s}} = 0.02 \text{ m}$$

$$\frac{f_0}{\omega_n^2 - \omega^2} = \frac{2.3 \text{ N/kg}}{(10^2 - 20^2) \text{ rad}^2/\text{s}^2} = -7.9667 \times 10^{-3} \text{ m}$$

Equation (2.11) then yields:

$$\underline{x(t) = 0.02 \sin 10t + 7.9667 \times 10^{-3} (\cos 10t - \cos 20t)}$$

Example 2.1.2 **Given zero initial conditions a harmonic input of 10 Hz with 20 N magnitude and k= 2000 N/m, and measured response amplitude of 0.1m, compute the mass of the system.**

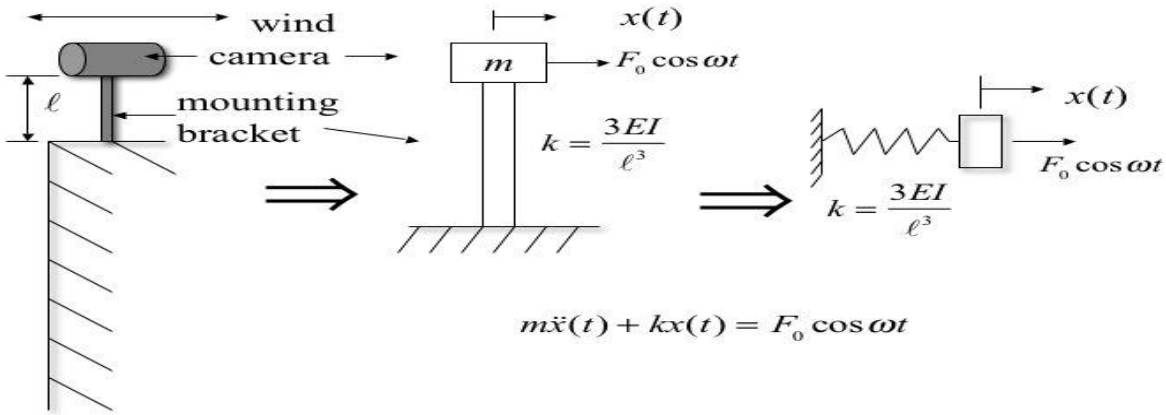
$$x(t) = \frac{f_0}{\omega_n^2 - \omega^2}\left(\cos 20\pi t - \cos \omega_n t\right) \text{ for zero initial conditions}$$

$$\text{trig identity} \Rightarrow x(t) = \underbrace{\frac{2 f_0}{\omega_n^2 - \omega^2}}_{0.1\text{ m}} \sin\left(\frac{\omega_n - \omega}{2} t\right) \sin\left(\frac{\omega_n + \omega}{2} t\right)$$

$$\Rightarrow \frac{2 f_0}{\omega_n^2 - \omega^2} = 0.1 \Rightarrow \frac{2(20/m)}{\left(2000/m\right) - (20\pi)^2} = 0.1$$

$$\Rightarrow \qquad \underline{m = 0.405 \text{ kg}}$$

Example 2.1.3 Design a rectangular mount for a security camera.



0.02 x 0.02 m

in cross section

Compute l > 0.5 m so that the mount keeps the camera from vibrating more then 0.01 m of maximum amplitude under a wind load of 15 N at 10 Hz. The mass of the camera is 3 kg.

Solution: Modeling the mount and camera as a beam with a tip mass, and the wind as harmonic, the equation of motion becomes: $m\ddot{x} + \frac{3EI}{\ell^3} x(t) = F_0 \cos \omega t$

From strength of materials: $I = \frac{bh^3}{12}$

Thus the frequency expression is: $\omega_n^2 = \frac{3Ebh^3}{12m\ell^3} = \frac{Ebh^3}{14m\ell^3}$

Here we are interested computing l that will make the amplitude less then 0.01m:

$$\left| \frac{2f_0}{\omega_n^2 - \omega^2} \right| < 0.01 \Rightarrow \begin{cases} (a) & -0.01 < \frac{2f_0}{\omega_n^2 - \omega^2}, \quad \text{for } \omega_n^2 - \omega^2 < 0 \\ (b) & \frac{2f_0}{\omega_n^2 - \omega^2} < 0.01, \quad \text{for } \omega_n^2 - \omega^2 > 0 \end{cases}$$

Case (a) (assume aluminum for the material):

$$-0.01 < \frac{2f_0}{\omega_n^2 - \omega^2} \Rightarrow 2f_0 < 0.01\omega^2 - 0.01\omega_n^2 \Rightarrow 0.01\omega^2 - 2f_0 > 0.01\frac{Ebh^3}{4m\ell^3}$$

$$\Rightarrow \ell^3 > 0.01\frac{Ebh^3}{4m(0.01\omega^2 - 2f_0)} = 0.02 \Rightarrow \ell > 0.272 \text{ m}$$

Case (b):

$$\frac{2f_0}{\omega_n^2 - \omega^2} < 0.01 \Rightarrow 2f_0 < 0.01\omega_n^2 - 0.01\omega^2 \Rightarrow 2f_0 + 0.01\omega^2 < 0.01\frac{Ebh^3}{4m\ell^3}$$

$$\Rightarrow \ell^3 < 0.01\frac{Ebh^3}{4m(2f_0 + 0.01\omega^2)} = 0.012 \Rightarrow \ell < 0.229 \text{ m}$$

Remembering the constraint that the length must be at least 0.2 m, (a) and (b) yield

$$0.2 < \ell < 0.229, \quad \text{or choose} \quad \ell = 0.22 \text{ m}$$

To check, note that

$$\omega_n^2 - \omega^2 = \frac{3Ebh^3}{12m\ell^3} - (20\pi)^2 = 1609 > 0$$

$$m = \rho \ell b h$$
$$= (2.7 \times 10^3)(0.22)(0.01)(0.01)$$
$$= 0.149 \text{ kg}$$

A harmonic force may also be represented by sine or a complex exponential. How does this change the solution?

$$m\ddot{x}(t) + kx(t) = F_0 \sin \omega t \quad \text{or} \quad \ddot{x}(t) + \omega_n^2 x(t) = f_0 \sin \omega t \qquad (2.18)$$

The particular solution then becomes a sine:

$$x_p(t) = X \sin \omega t \qquad (2.19)$$

Substitution of (2.19) into (2.18) yields: $x_p(t) = \dfrac{f_0}{\omega_n^2 - \omega^2} \sin \omega t$

Solving for the homogenous solution and evaluating the constants yields

$$x(t) = x_0 \cos \omega_n t + \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \frac{f_0}{\omega_n^2 - \omega^2} \right) \sin \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \sin \omega t \qquad (2.25)$$

Section 2.2 Harmonic Excitation of Damped Systems

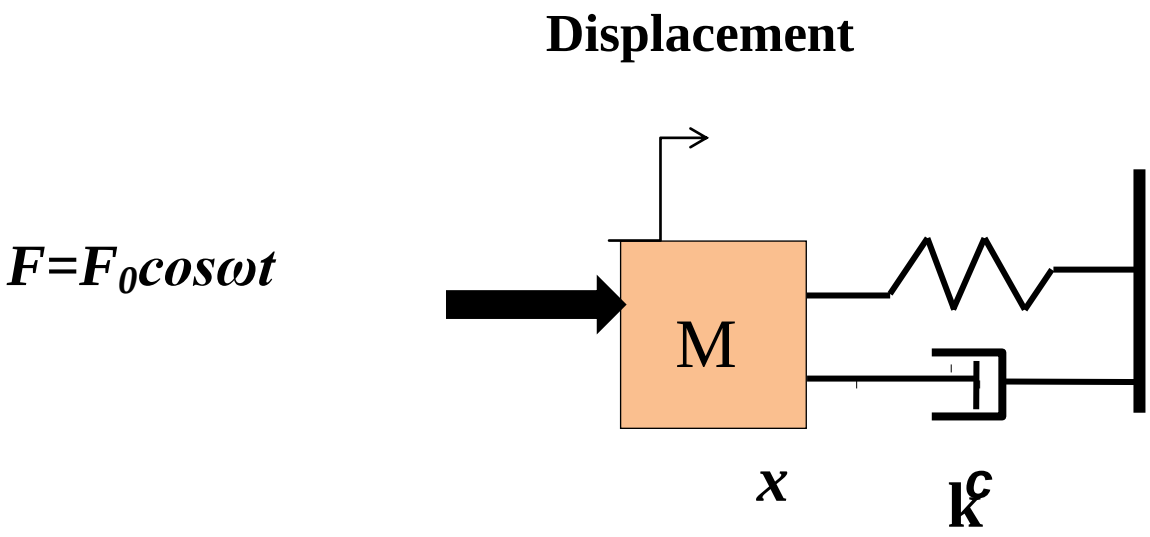
Extending resonance and response calculation to damped systems

2.2 Harmonic excitation of damped systems

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos \omega t \qquad \textbf{(2.26)}$$

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = f_0 \cos \omega t \qquad \textbf{(2.27)}$$

$$x_p(t) = \underbrace{X \cos(\omega t - \theta)}_{\text{now includes a phase shift}}$$



Let x_p have the form:

$$x_p(t) = A_s \cos \omega t + B_s \sin \omega t$$

$$X = \sqrt{A_s^{\,2} + B_s^{\,2}}, \theta = \tan^{-1}\left(\frac{B_s}{A_s}\right)$$

$$\dot{x}_p = -\omega A_s \sin \omega t + \omega B_s \cos \omega t$$

$$\ddot{x}_p = -\omega^2 A_s \cos \omega t - \omega^2 B_s \sin \omega t$$

Note that we are using the rectangular form, but we could use one of the other forms of the solution.

Substitute into the equations of motion

$$(-\omega^2 A_s + 2\zeta\omega_n\omega B_s + \omega_n^2 A_s - f_0)\cos\omega t$$

$$+\left(-\omega^2 B_s + 2\zeta\omega_n\omega A_s + \omega_n^2 B_s\right)\sin\omega t = 0$$

for all time. Specifically for $t = 0, 2\pi/\omega \Rightarrow$

$$(\omega_n^2 - \omega^2)A_s + (2\zeta\omega_n\omega)B_s = f_0$$

$$(-2\zeta\omega_n\omega)A_s + (\omega_n^2 - \omega^2)B_s = 0$$

Write as a matrix equation:

$$\begin{bmatrix} (\omega_n^2 - \omega^2) & 2\zeta\omega_n\omega \\ -2\zeta\omega_n\omega & (\omega_n^2 - \omega^2) \end{bmatrix} \begin{bmatrix} A_s \\ B_s \end{bmatrix} = \begin{bmatrix} f_0 \\ 0 \end{bmatrix}$$

Solving for A_s and B_s:

$$A_s = \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$B_s = \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

Substitute the values of A_s and B_s into x_p:

$$x_p(t) = \underbrace{\frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}}_{X} \cos(\omega t - \underbrace{\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)}_{\theta})$$

Add homogeneous and particular to get total solution:

$$x(t) = \underbrace{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}_{\text{homogeneous or transient solution}} + \underbrace{X\cos(\omega t - \theta)}_{\substack{\text{particular or} \\ \text{steady state solution}}}$$

Note: that A and Φ will not have the same values as in Ch 1, for the free response. Also as t gets large, transient dies out.

Things to notice about damped forced response

- **If $\zeta = 0$, undamped equations result**
- **Steady state solution prevails for large *t***
- **Often we ignore the transient term (how large is ζ, how long is *t*?)**
- **Coefficients of transient terms (constants of integration) are effected by the initial conditions AND the forcing function**
- **For underdamped systems at resonance the, amplitude is finite.**

Example 2.2.3: ω_n = 10 rad/s, ω = 5 rad/s, ζ = 0.01, F_0= 1000 N, m = 100 kg, and the initial conditions x_0 = 0.05 m and v_0 = 0. Compare amplitude A and phase Φ for forced and unforced case:

Using the equations on slide 23:

$$X = 0.133, \theta = -0.013$$

$$x(t) = Ae^{-0.1t}\sin(9.99t + \phi) + 0.133\cos(5t - 0.013)$$

$$x(0) = 0.05$$

$$= A\sin\phi + 0.133\cos(-0.013)$$

$$v(0) = 0 = -0.01A\sin\phi$$

$$+ 9.999A\cos\phi + 0.665\sin 0.013$$

Differentiating yields:

$$v(t) = -\ 0.01Ae^{-0.1t}\sin(9.999t + \phi)$$

$$+ 9.999Ae^{-0.1t}\cos(9.999t + \phi)$$

$$- 0.665\sin(5t - 0.013)$$

applying the intial conditions:

$$A = -\ 0.083\ (0.05),\ \ \phi = 1.55\ (1.561)$$

Proceeding with ignoring the transient

- **Always check to make sure the transient is not significant**
- **For example, transients are very important in earthquakes**
- **However, in many machine applications transients may be ignored**

Proceeding with ignoring the transient

Magnitude:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \quad \textbf{(2.39)}$$

Frequency ratio: $r = \dfrac{\omega}{\omega_n}$

Non dimensional Form:

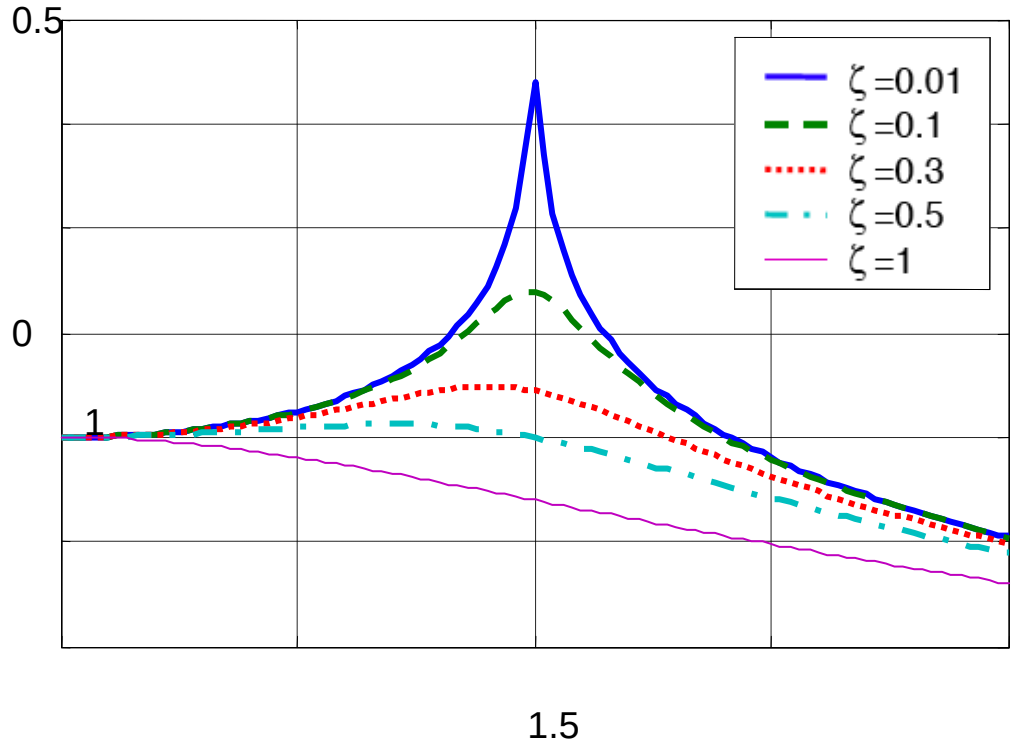
$$\frac{Xk}{F_0} = \frac{X\omega_n^2}{f_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \textbf{(2.40)}$$

Phase: $\theta = \tan^{-1}\left({2\zeta r}/{1-r^2} \right)$

Magnitude plot

- **Resonance is close to *r* = 1**
- **For ζ = 0, *r* =1 defines resonance**
- **As ζ grows resonance moves *r* <1, and *X* decreases**
- **The exact value of *r*, can be found from differentiating the magnitude**

$$X = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$



Phase plot

- **Resonance occurs at Φ = π/2**
- **The phase changes more rapidly when the damping is small**
- **From low to high values of *r* the phase always changes by 180^0 or π radians**

$$\theta = \tan^{-1}\left({2\zeta r}/{1-r^2} \right)$$

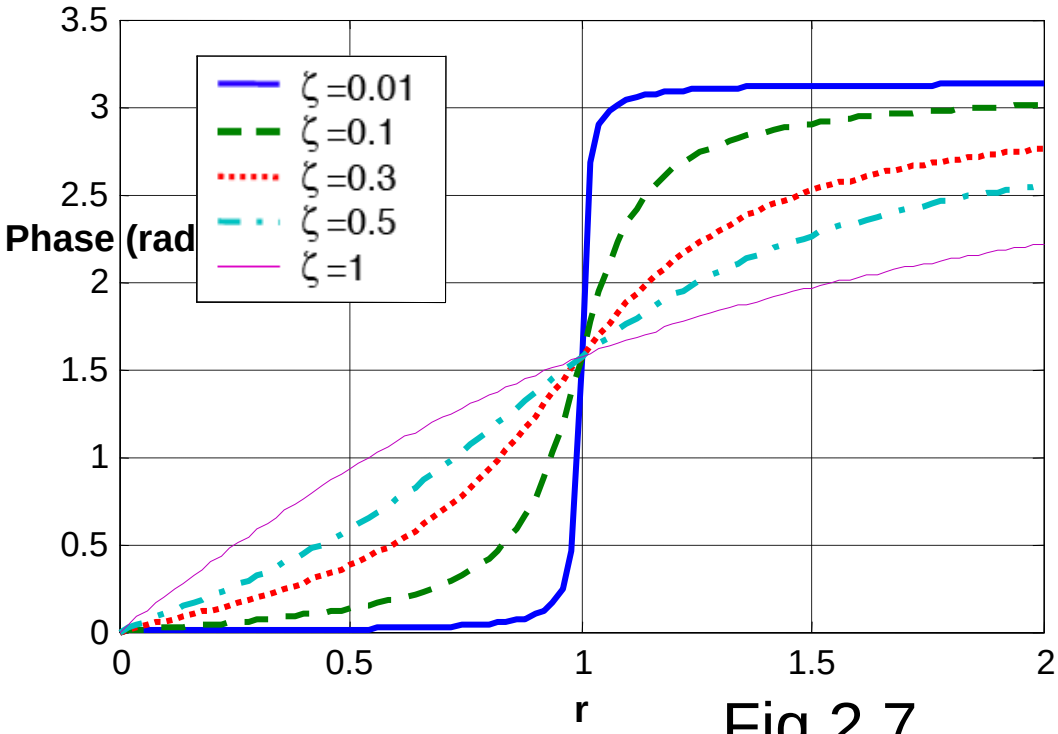


Fig 2.7

Example 2.2.3 **Compute max peak by differentiating:**

$$\frac{d}{dr}\left(\frac{Xk}{F_0}\right)=\frac{d}{dr}\left(\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\right)=0\Rightarrow$$

$$r_{\text{peak}}=\sqrt{1-2\zeta^2}<1\Rightarrow\zeta<1/\sqrt{2} \qquad \textbf{(2.41)}$$

$$\left(\frac{Xk}{F_0}\right)_{\max}=\frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \textbf{(2.42)}$$

Effect of Damping on Peak Value

- **The top plot shows how the peak value becomes very large when the damping level is small**
- **The lower plot shows how the frequency at which the peak value occurs reduces with increased damping**
- **Note that the peak value is only defined for values $\zeta<0.707$**

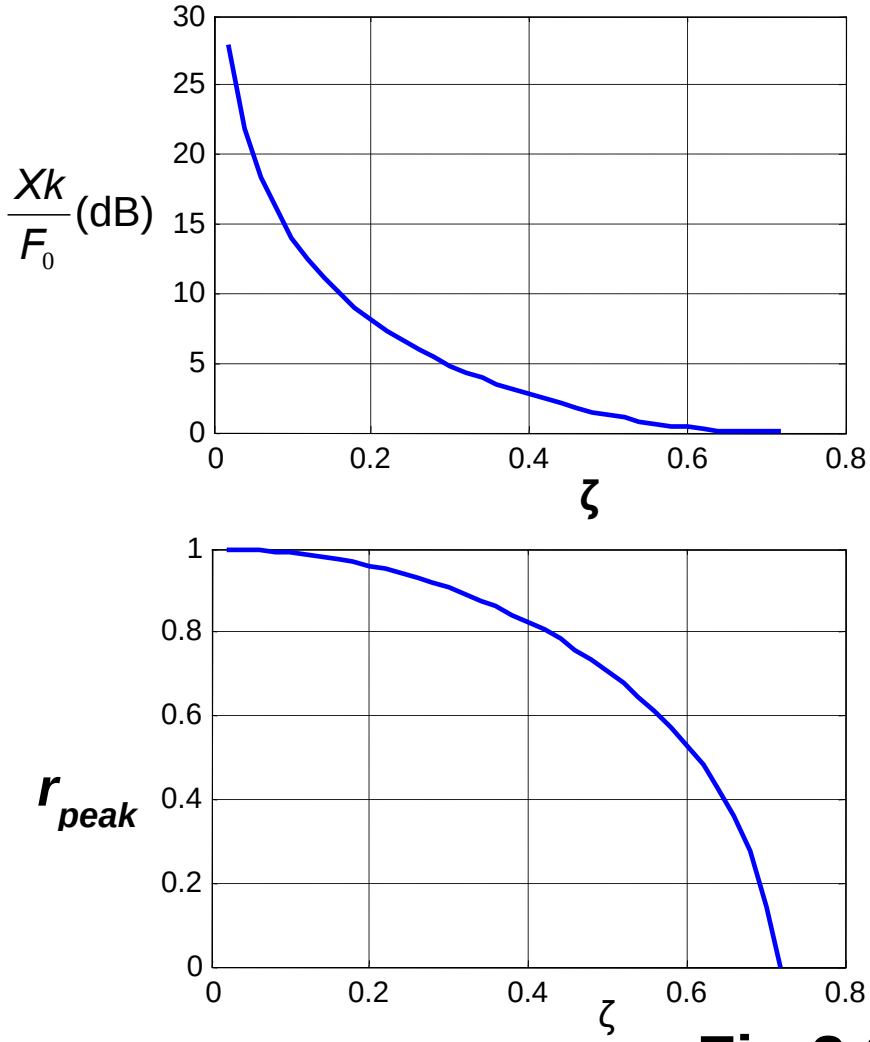


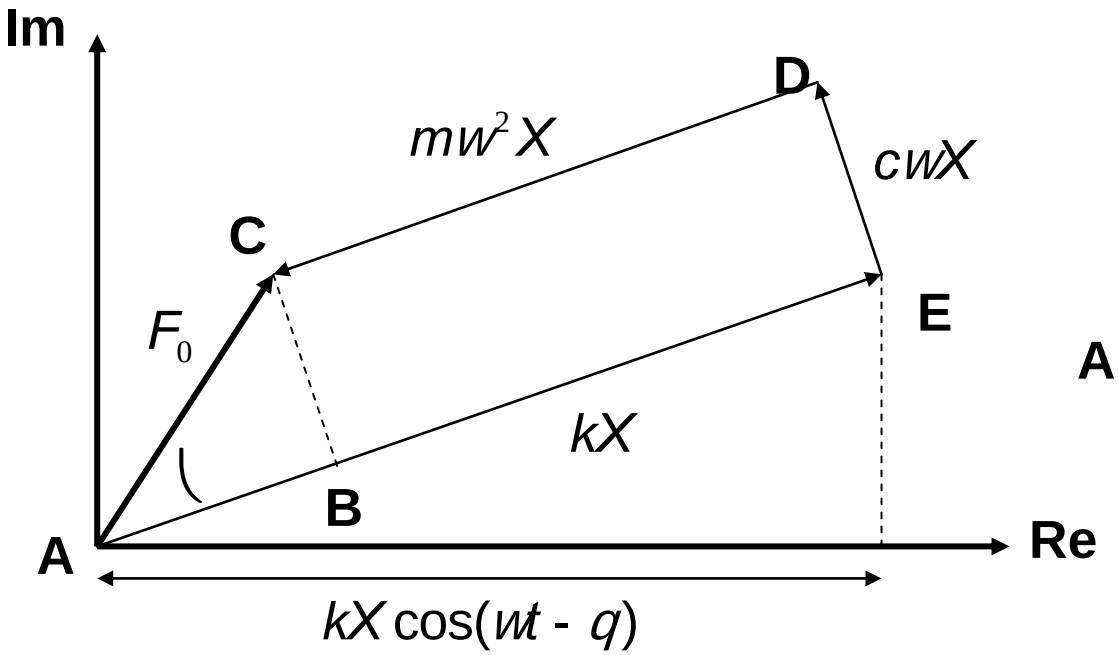
Fig 2.9

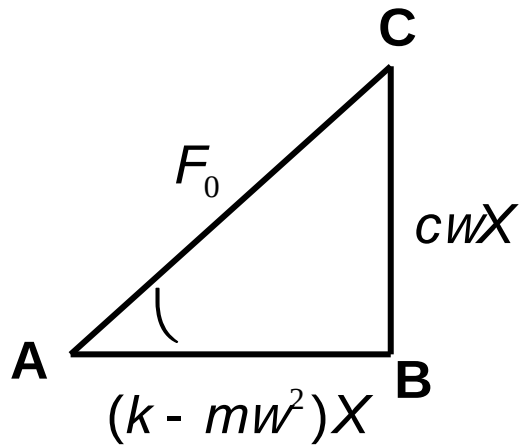
Section 2.3 Alternative Representations

- **A variety methods for solving differential equations**
- **So far, we used the method of undetermined coefficients**
- **Now we look at 3 alternatives:**
  a geometric approach
  a frequency response approach
  a transform approach
- **These also give us some insight and additional useful tools.**

2.3.1 Geometric Approach

- **Position, velocity and acceleration phase shifted each by $\pi/2$**
- **Therefore write each as a vector**
- **Compute X in terms of F_0 via vector addition**





Using vector addition on the diagram:

$$F_0^2 = (k - m\omega^2)^2 X^2 + (c\omega)^2 X^2$$

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

At resonance:

$$\theta = \frac{\pi}{2}, \quad X = \frac{F_0}{c\omega}$$

2.3.2 Complex response method

$$Ae^{j\omega t} = \underbrace{A\cos\omega t}_{\text{real part}} + \underbrace{(A\sin\omega t)}_{\text{imaginary part}}\, j \qquad \textbf{(2.47)}$$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \underbrace{F_0 e^{j\omega t}}_{\text{harmonic input}} \qquad \textbf{(2.48)}$$

Real part of this complex solution corresponds to the physical solution

Choose complex exponential as a solution

$$x_p(t) = Xe^{j\omega t} \tag{2.49}$$

$$(-\omega^2 m + cj\omega + k)Xe^{j\omega t} = F_0 e^{j\omega t} \tag{2.50}$$

$$X = \frac{F_0}{(k - m\omega^2) + (c\omega)j} = H(j\omega)F_0 \tag{2.51}$$

$$H(j\omega) = \frac{1}{\underbrace{(k - m\omega^2) + (c\omega)j}_{\text{the frequency response function}}} \tag{2.52}$$

Note: These are all complex functions

Using complex arithmetic:

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} e^{-j\theta} \qquad \textbf{(2.53)}$$

$$\theta = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \qquad \textbf{(2.54)}$$

$$x_p(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} e^{j(\omega t - \theta)} \qquad \textbf{(2.55)}$$

Has real part = to previous solution

Comments:

- **Label *x*-axis $\mathrm{Re}(e^{j\omega t})$ and *y*-axis $\mathrm{Im}(e^{j\omega t})$ results in the graphical approach**
- **It is the real part of this complex solution that is physical**
- **The approach is useful in more complicated problems**

Example 2.3.1: Use the frequency response approach to compute the particular solution of an undamped system

The equation of motion is written as

$$m\ddot{x}(t) + kx(t) = F_0 e^{j\omega t} \Rightarrow \ddot{x}(t) + \omega_n^2 x(t) = f_0 e^{j\omega t}$$

$$\text{Let } x_p(t) = Xe^{j\omega t}$$

$$\Rightarrow \left(-\omega^2 + \omega_n^2\right) Xe^{j\omega t} = f_0 e^{j\omega t}$$

$$\Rightarrow X = \frac{f_0}{\left(\omega_n^2 - \omega^2\right)}$$

2.3.3 Transfer Function Method

The Laplace Transform

- **Changes ODE into algebraic equation**
- **Solve algebraic equation then compute the inverse transform**
- **Rule and table based in many cases**
- **Is used extensively in control analysis to examine the response**
- **Related to the frequency response function**

The Laplace Transform approach:

- **See appendix B and section 3.4 for details**
- **Transforms the time variable into an algebraic, complex variable**
- **Transforms differential equations into an algebraic equation**
- **Related to the frequency response method**

$$X(s) = \mathcal{L}(x(t)) = \int_0^\infty x(t)e^{-st}dt$$

Take the transform of the equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \Rightarrow$$

$$(ms^2 + cs + k)X(s) = \frac{F_0 s}{s^2 + \omega^2}$$

Now solve algebraic equation in *s* for *X*(*s*)

$$X(s) = \frac{F_0 s}{(ms^2 + cs + k)(s^2 + \omega^2)}$$

To get the time response this must be "inverse transformed"

Transfer Function Method

With zero initial conditions:

$$(ms^2 + cs + k)X(s) = F(s) \Rightarrow$$

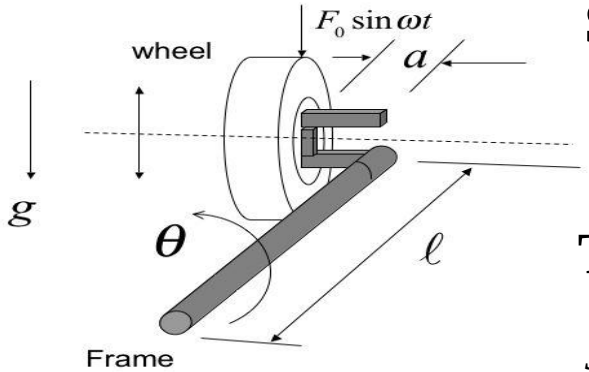
The transfer function

$$\frac{X(s)}{F(s)} = H(s) = \frac{1}{ms^2 + cs + k} \qquad \textbf{(2.59)}$$

$$H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \qquad \textbf{(2.60)}$$

$$= \text{frequency response function}$$

Example 2.3.2 Compute forced response of the suspension system shown using the Laplace transform



Summing moments about the shaft:

$$J\ddot{\theta}(t) + k\theta(t) = aF_0 \sin \omega t$$

Taking the Laplace transform

$$Js^2 X(s) + kX(s) = aF_0 \frac{\omega}{s^2 + \omega^2}$$

$$\Rightarrow X(s) = a\omega F_0 \frac{1}{\left(s^2 + \omega^2\right)\left(Js^2 + k\right)}$$

Taking the inverse Laplace transform:

$$\theta(t) = a\omega F_0 L^{-1}\left(\frac{1}{\left(s^2 + \omega^2\right)\left(Js^2 + k\right)}\right)$$

$$\Rightarrow \theta(t) = \frac{a\omega F}{J}\frac{1}{\omega^2 - \omega_n^2}\left(\frac{1}{\omega}\sin \omega t - \frac{1}{\omega_n}\sin \omega_n t\right), \quad \omega_n = \sqrt{\frac{k}{J}}$$

Notes on Phase for Homogeneous and Particular Solutions

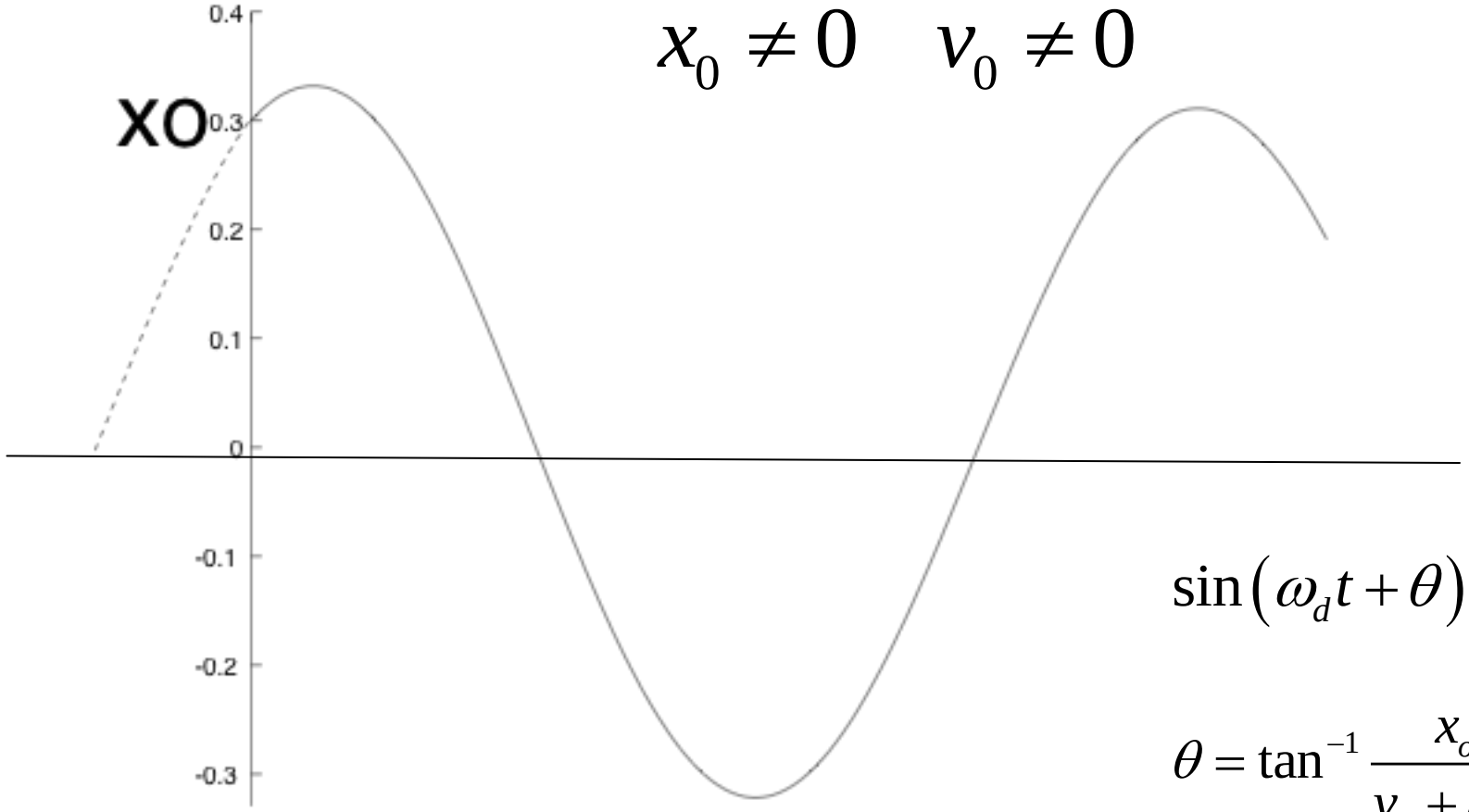
- **Equation (2.37) gives the full solution for a harmonically driven underdamped SDOF oscillator to be**

$$x(t) = \underbrace{Ae^{-\zeta\omega_n t}\sin\left(\omega_d t + \theta\right)}_{\text{homogeneous solution}} + \underbrace{X\cos\left(\omega t - \phi\right)}_{\text{particular solution}}$$

How do we interpret these phase angles?
Why is one added and the other subtracted?

Non-Zero initial conditions

$$x_0 \neq 0 \quad v_0 \neq 0$$

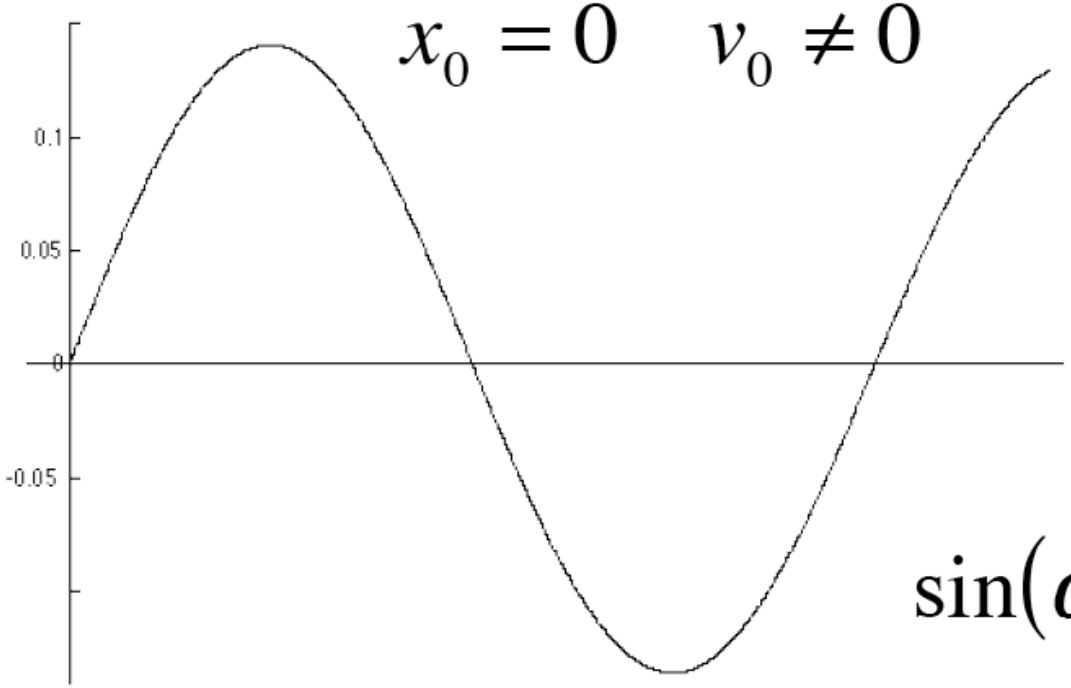


$$\sin\left(\omega_d t + \theta\right)$$

$$\theta = \tan^{-1} \frac{x_o \omega_d}{v_o + \zeta \omega_n x_o}$$

Zero initial displacement

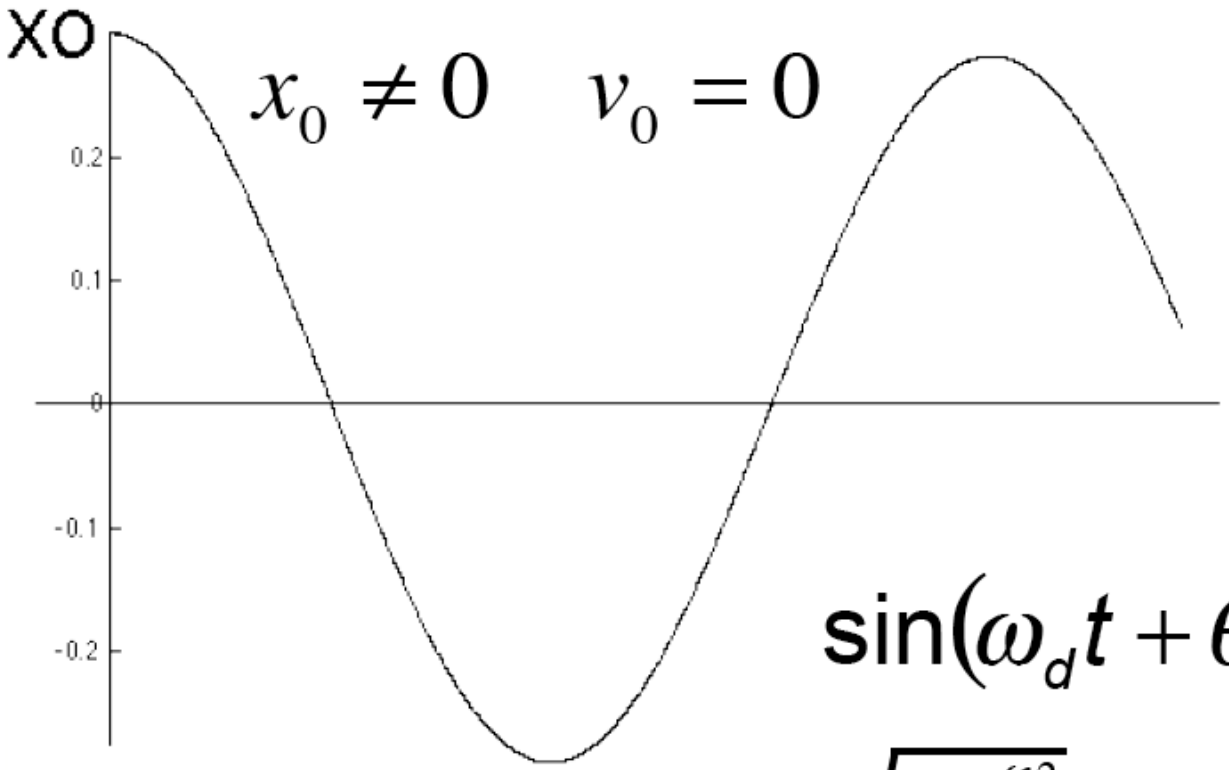
$$x_0 = 0 \quad v_0 \neq 0$$



$$\sin(\omega_d t + \theta)$$

$$\theta = \tan^{-1}\frac{x_o\omega_d}{v_o + \zeta\omega_n x_o} = \tan^{-1}\frac{0}{v_o} = 0$$

Zero initial velocity



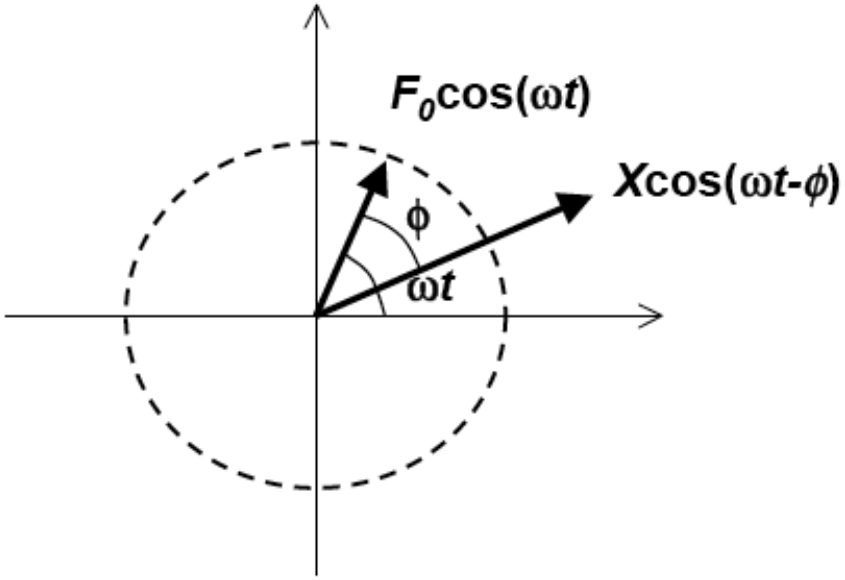
$$x_0 \neq 0 \qquad v_0 = 0$$

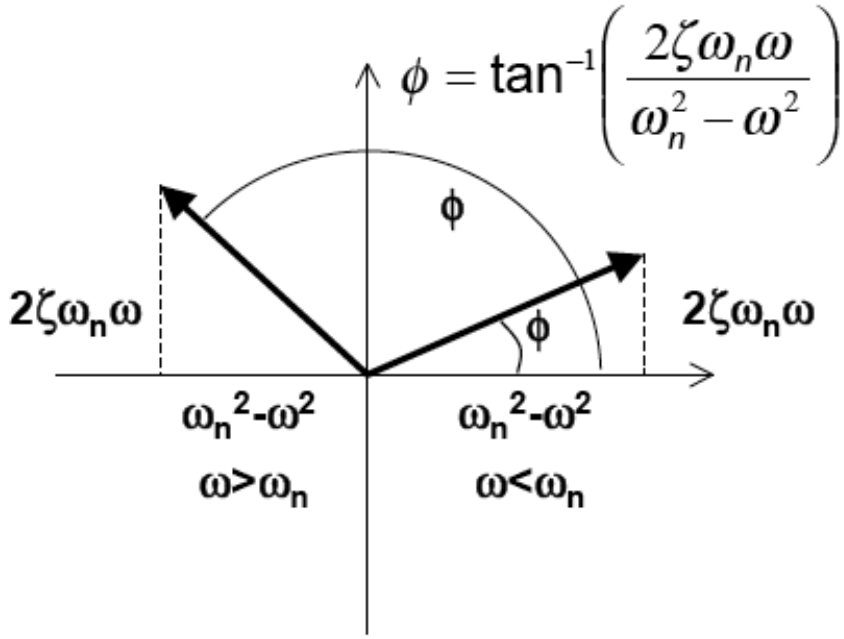
$$\sin(\omega_d t + \theta)$$

$$\theta = \tan^{-1} \frac{x_o \omega_d}{v_o + \zeta \omega_n x_o} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Phase on Particular Solution

$$x_p(t) = X\cos(\omega t - \phi)$$





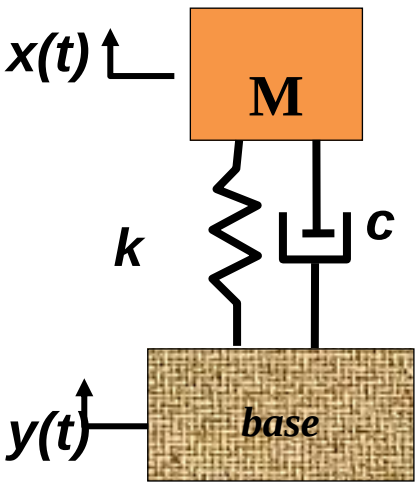
- **Simple "atan" gives $-\pi/2 < \Phi < \pi/2$**
- **Four-quadrant "atan2" gives $0 < \Phi < \pi$**

2.4 Base Excitation

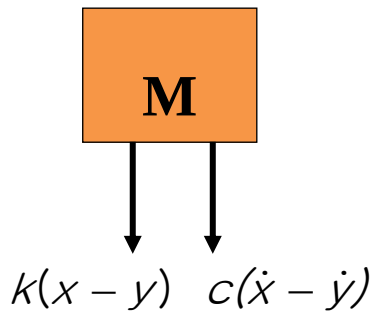
- **Important class of vibration analysis**
  - **Preventing excitations from passing from a vibrating base through its mount into a structure**
- **Vibration isolation**
  - **Vibrations in your car**
  - **Satellite operation**
  - **Disk drives, etc.**

FBD of SDOF Base Excitation

System Sketch



System FBD



$$\sum F = -k(x-y) - c(\dot{x}-\dot{y}) = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \qquad (2.61)$$

SDOF Base Excitation (cont)

Assume: $y(t) = Y\sin(\omega t)$ and plug into Equation(2.61)

$$m\ddot{x}+c\dot{x} + kx = \underbrace{c\omega Y\cos(\omega t) + kY\sin(\omega t)}_{\text{harmonic forcing functions}} \qquad (2.63)$$

For a car, $\omega = \dfrac{2\pi}{\tau} = \dfrac{2\pi V}{\lambda}$

The steady-state solution is just the superposition of the two individual particular solutions (system is linear).

$$\ddot{x}+2\zeta\omega_n\dot{x} + \omega_n^2 x = \overbrace{\underbrace{2\zeta\omega_n\omega Y\cos(\omega t)}}^{f_{0c}} + \overbrace{\underbrace{\omega_n^2 Y\sin(\omega t)}}^{f_{0s}} \qquad (2.64)$$

Particular Solution (sine term)

With a sine for the forcing function,

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x = f_{0s}\sin\omega t$$

$$x_{ps} = \underline{A_s\cos\omega t + B_s\sin\omega t} = X_s\sin(\omega t-\phi_s)$$

where

$$A_s = \frac{-2\zeta\omega_n\omega f_{0s}}{(\omega_n^2-\omega^2)^2+\left(2\zeta\omega_n\omega\right)^2}$$

$$B_s = \frac{(\omega_n^2-\omega^2) f_{0s}}{(\omega_n^2-\omega^2)^2+\left(2\zeta\omega_n\omega\right)^2}$$

Use rectangular form to make it easier to add the cos term

Particular Solution (cos term)

With a cosine for the forcing function, we showed

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x=f_{0c}\cos\omega t$$

$$x_{pc}=\underline{A_c\cos\omega t+B_c\sin\omega t}=X_c\cos(\omega t-\phi_c)$$

where

$$A_c=\frac{(\omega_n^2-\omega^2)f_{0c}}{(\omega_n^2-\omega^2)^2+\left(2\zeta\omega_n\omega\right)^2}$$

$$B_c=\frac{2\zeta\omega_n\omega f_{0c}}{(\omega_n^2-\omega^2)^2+\left(2\zeta\omega_n\omega\right)^2}$$

Magnitude X/Y

Now add the sin and cos terms to get the magnitude of the full particular solution

$$X = \sqrt{\frac{f_{0c}^2 + f_{0s}^2}{(\omega_n^2 - \omega^2)^2 + \left(2\zeta\omega_n\omega\right)^2}} = \omega_n Y \sqrt{\frac{(2\zeta\omega)^2 + \omega_n^2}{(\omega_n^2 - \omega^2)^2 + \left(2\zeta\omega_n\omega\right)^2}}$$

where $f_{0c} = 2\zeta\omega_n\omega Y$ and $f_{0s} = \omega_n^2 Y$

if we define $r = \omega/\omega_n$ this becomes $X = Y\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2 + \left(2\zeta r\right)^2}}$ (2.70)

$$\frac{X}{Y} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2 + \left(2\zeta r\right)^2}} \qquad (2.71)$$

The relative magnitude plot of X/Y versus frequency ratio: Called the *Displacement Transmissibility*

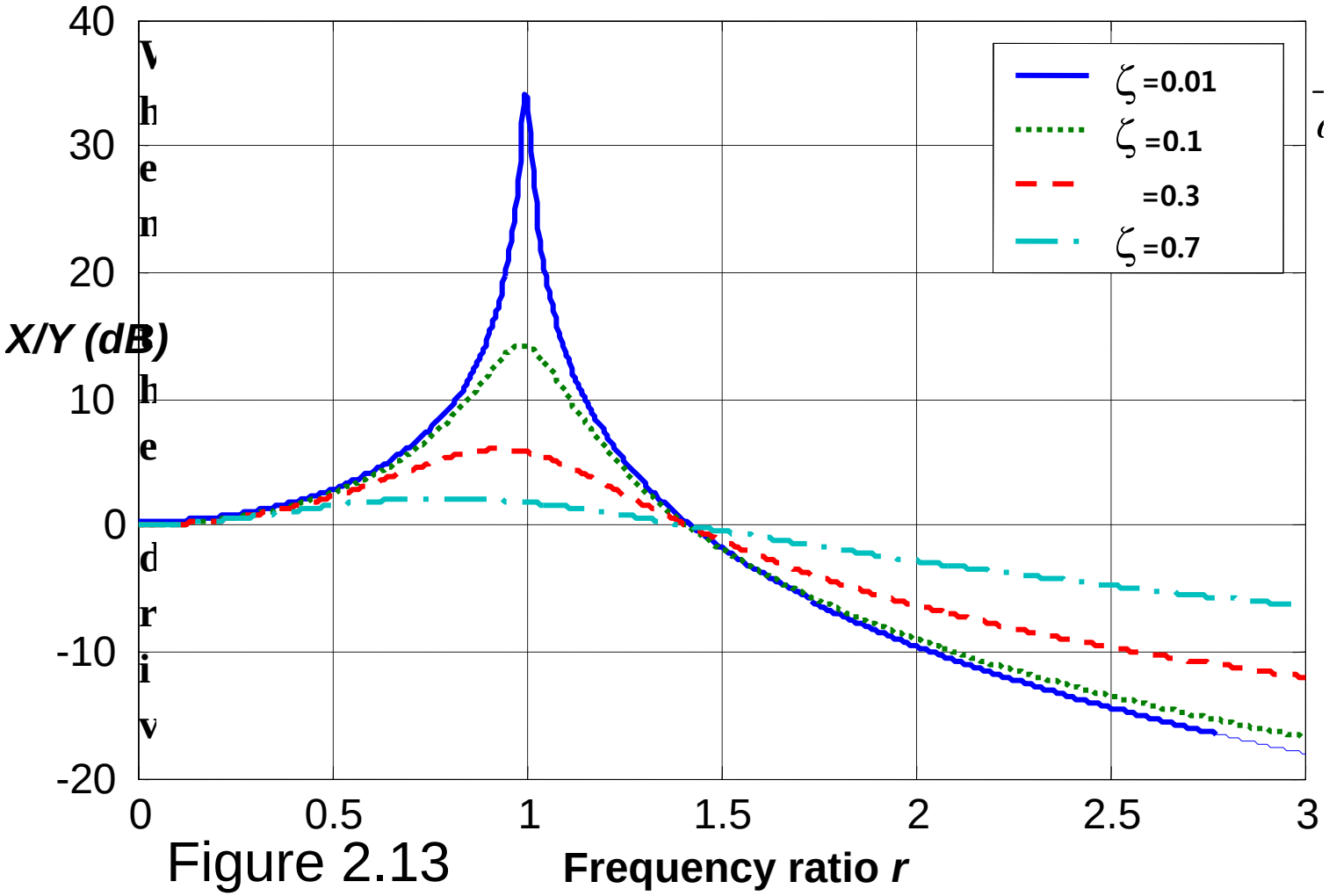


Figure 2.13

From the plot of relative Displacement Transmissibility observe that:

- **X/Y is called Displacement Transmissibility Ratio**
- **Potentially severe amplification at resonance**
- **Attenuation for *r* > sqrt(2) Isolation Zone**
- **If *r* < sqrt(2) transmissibility decreases with damping ratio Amplification Zone**
- **If *r* >> 1 then transmissibility increases with damping ratio $X_p \sim 2Y\zeta/r$**

Next examine the *Force Transmitted to the mass* as a function of the frequency ratio

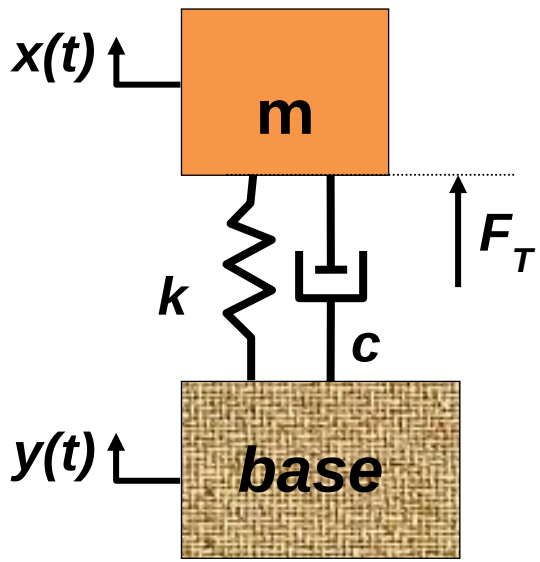
$$F_T = -k(x-y) - c(\dot{x}-\dot{y}) = m\ddot{x}$$

From FBD

At steady state, $x(t) = X\cos(\omega t - \phi)$,

so $\ddot{x} = -\omega^2 X \cos(\omega t - \phi)$

$$\left|F_T\right| = m\omega^2 X = k r^2 X$$



Plot of Force Transmissibility (in dB) versus frequency ratio

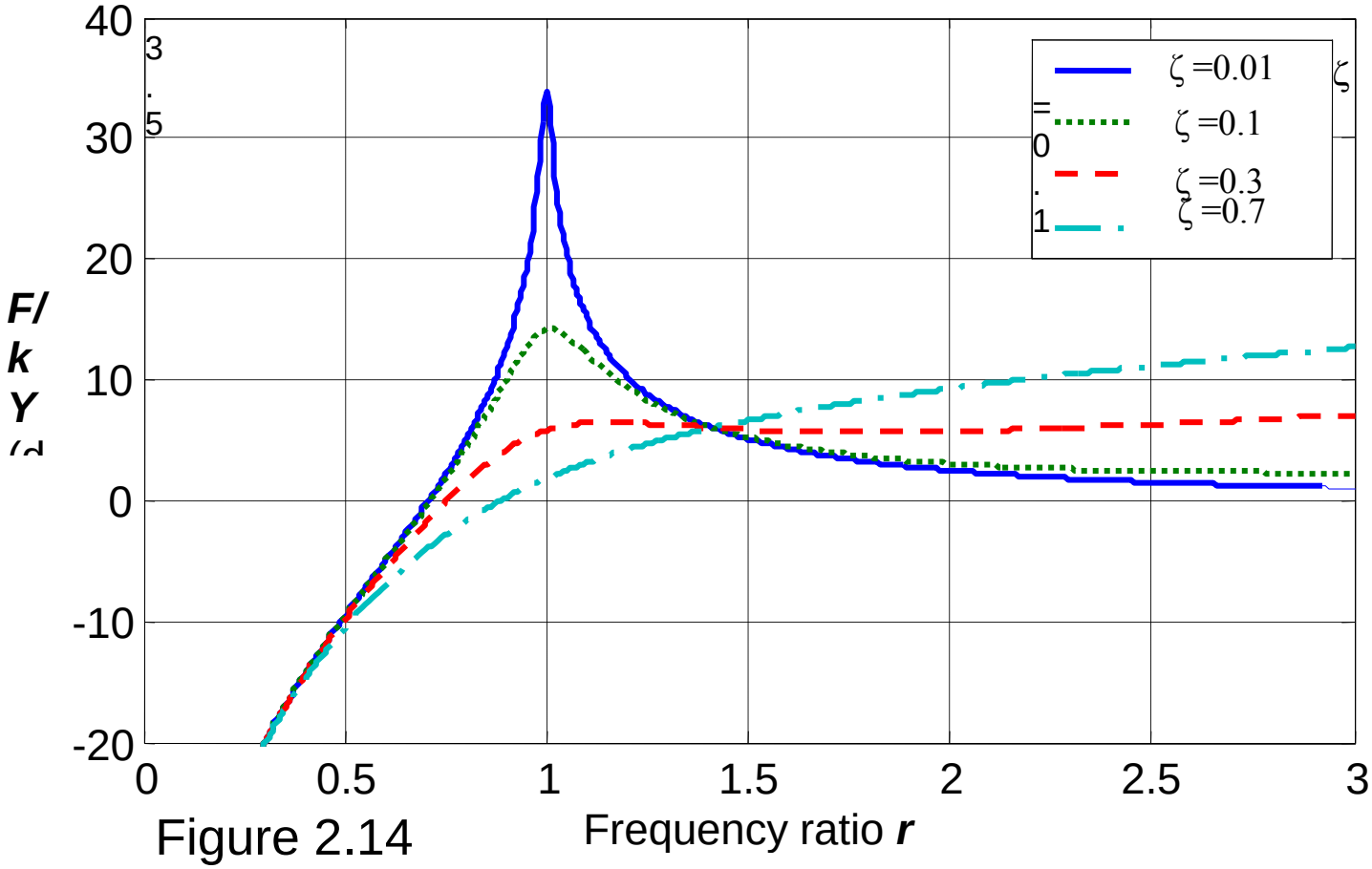
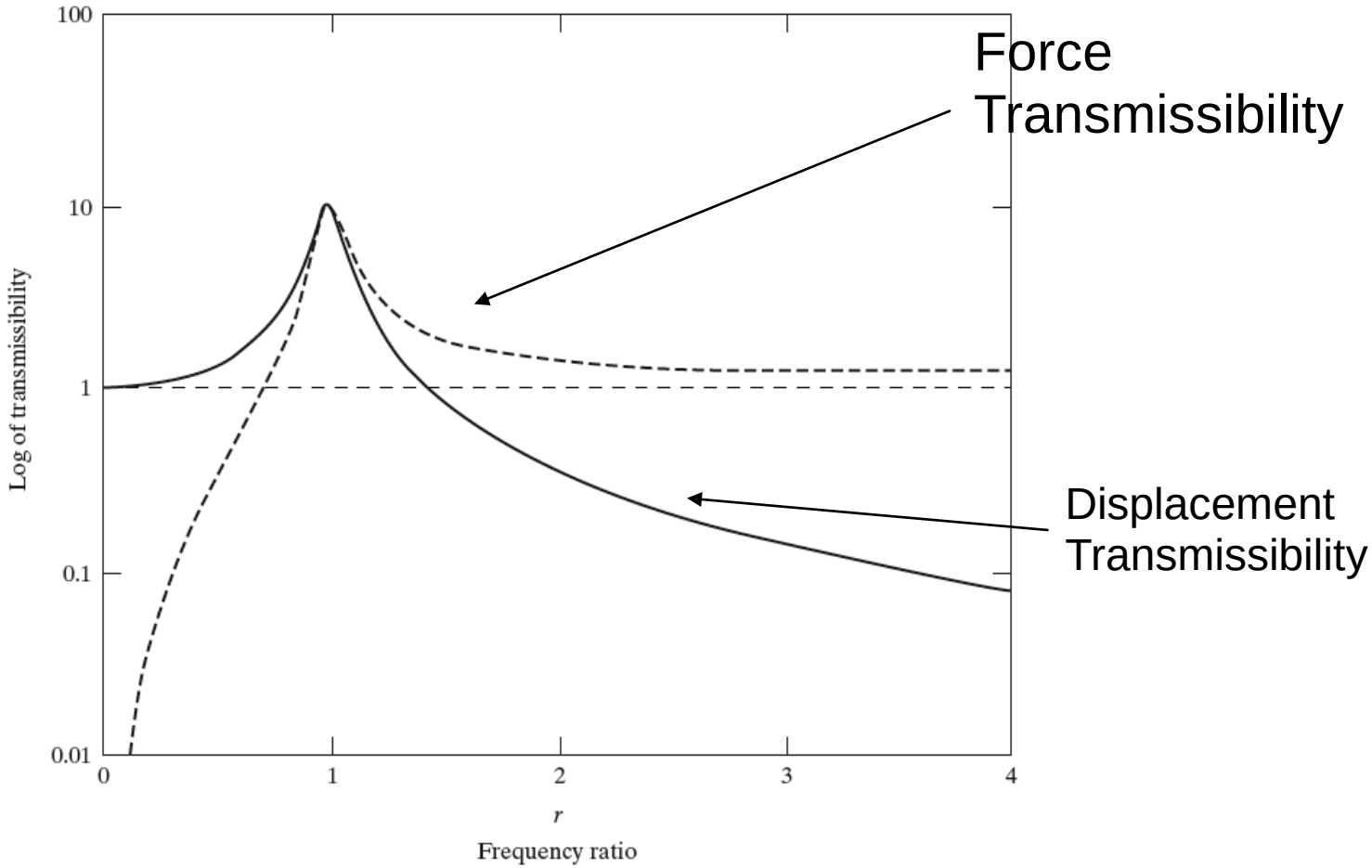


Figure 2.14

Figure 2.16 Comparison between force and displacement transmissibility



Example 2.4.2: Effect of speed on the amplitude of car vibration

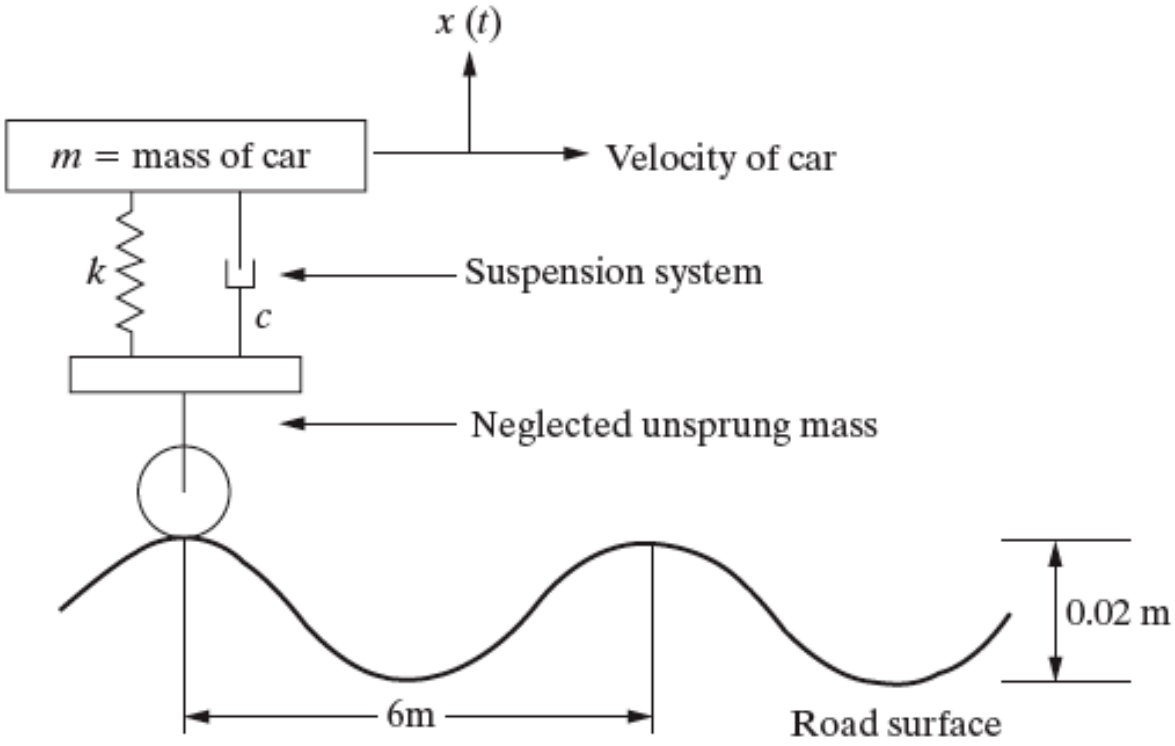


Figure 2.17

Model the road as a sinusoidal input to base motion of the car model

Approximation of road surface:

$$y(t) = (0.01\text{ m})\sin\omega_b t$$

$$\omega_b = v(\text{km/hr})\left(\frac{1}{0.006\text{ km}}\right)\left(\frac{\text{hour}}{3600\text{ s}}\right)\left(\frac{2\pi\text{ rad}}{\text{cycle}}\right) = 0.2909v\text{ rad/s}$$

$$\omega_b(20\text{km/hr}) = 5.818\text{ rad/s}$$

From the data give, determine the frequency and damping ratio of the car suspension:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\times10^4\text{ N/m}}{1007\text{ kg}}} = 6.303\text{ rad/s} \quad (\approx 1\text{ Hz})$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{2000\text{ Ns/m}}{2\sqrt{\left(4\times10^4\text{ N/m}\right)\left(1007\text{ kg}\right)}} = 0.158$$

From the input frequency, input amplitude, natural frequency and damping ratio use equation (2.70) to compute the amplitude of the response:

$$r = \frac{\omega_b}{\omega} = \frac{5.818}{6.303}$$

$$X = Y\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

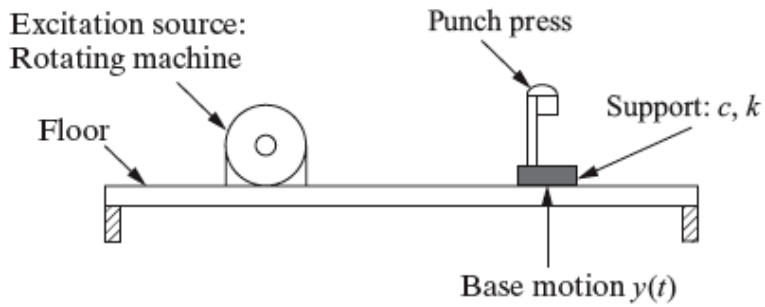
$$= (0.01\text{ m})\sqrt{\frac{1+\left[2(0.158)(0.923)\right]^2}{\left(1-(0.923)^2\right)^2+\left(2(0.158)(0.923)\right)^2}} = 0.0319\text{ m}$$

What happens as the car goes faster? See Table 2.1.

Example 2.4.3: Compute the force transmitted to a machine through base motion at resonance

From (2.77) at r =1:

$$\frac{F_T}{kY} = \left[\frac{1+(2\zeta)^2}{(2\zeta)^2}\right]^{1/2} \Rightarrow F_T = \frac{kY}{2\zeta}\sqrt{1+4\zeta^2}$$



From given m, c, and k:

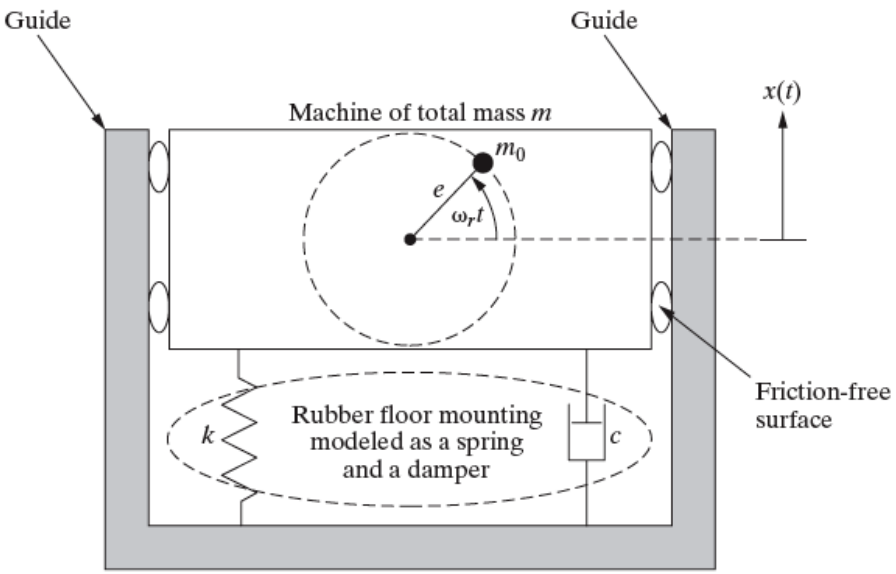
$$\zeta = \frac{c}{2\sqrt{km}} = \frac{900}{2\sqrt{40,000 \bullet 3000}} \cong 0.04$$

From measured excitation Y = 0.001 m:

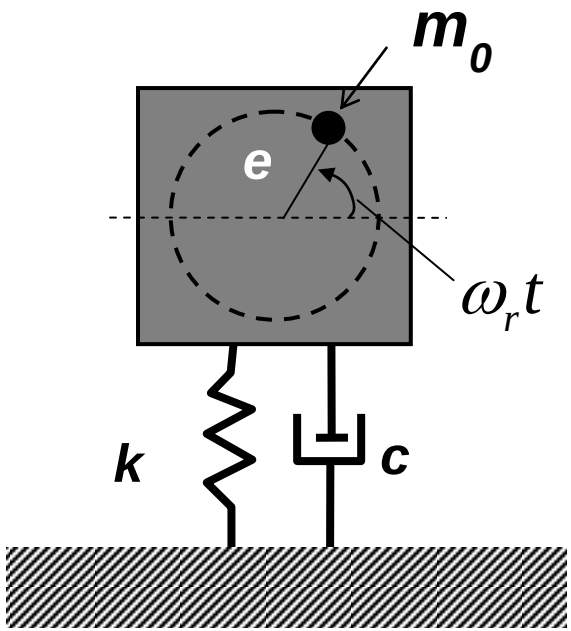
$$F_T = \frac{kY}{2\zeta}\sqrt{1+4\zeta^2} = \frac{(40,000 \text{ N/m})(0.001 \text{ m})}{2(0.04)}\sqrt{1+4(0.04)^2} = \underline{501.6 \text{ N}}$$

2.5 Rotating Unbalance

- **Gyros**
- **Cryo-coolers**
- **Tires**
- **Washing machines**



Machine of total mass *m* i.e. m_0 included in *m*

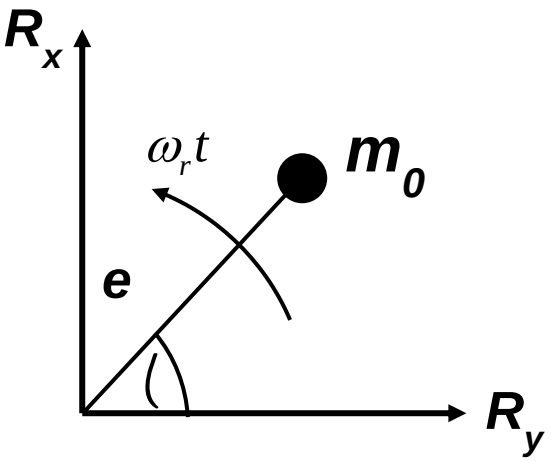


e = eccentricity

m_o = mass unbalance

$\omega_r t$ =rotation frequency

Rotating Unbalance (cont)



What force is imparted on the structure? Note it rotates with x component:

$$x_r = e\sin\omega_r t$$

$$\Rightarrow a_x = \ddot{x}_r = -e\omega_r^2 \sin\omega_r t$$

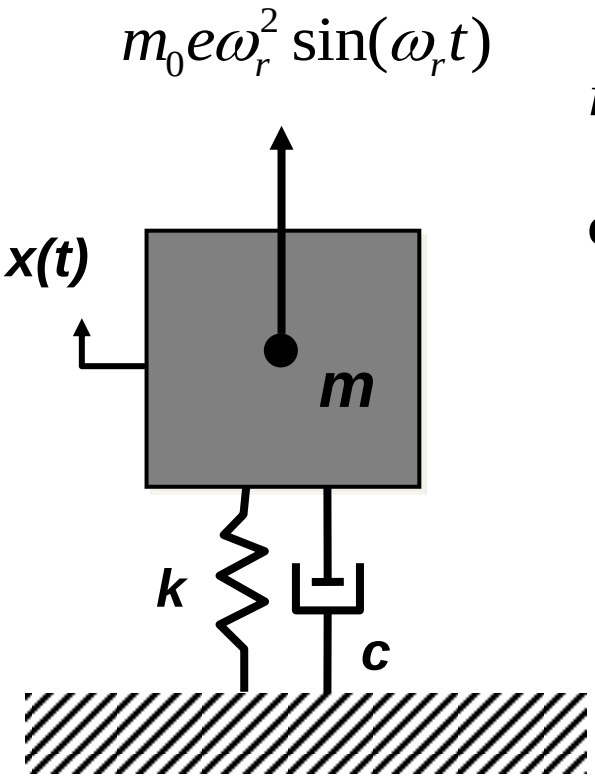
From sophomore dynamics,

$$R_x = m_0 a_x = -m_o e\omega_r^2 \sin\theta = -m_o e\omega_r^2 \sin\omega_r t$$

$$R_y = m_0 a_y = -m_o e\omega_r^2 \cos\theta = -m_o e\omega_r^2 \cos\omega_r t$$

Rotating Unbalance (cont)

The problem is now just like any other SDOF system with a harmonic excitation



$$m\ddot{x} + c\dot{x} + kx = m_o e\omega_r^2 \sin \omega_r t \qquad (2.82)$$

$$\text{or} \quad \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{m_o}{m} e\omega_r^2 \sin \omega_r t$$

Note the influences on the forcing function **(we are assuming that the mass *m* is held in place in the *y* direction as indicated in Figure 2.19)**

Rotating Unbalance (cont)

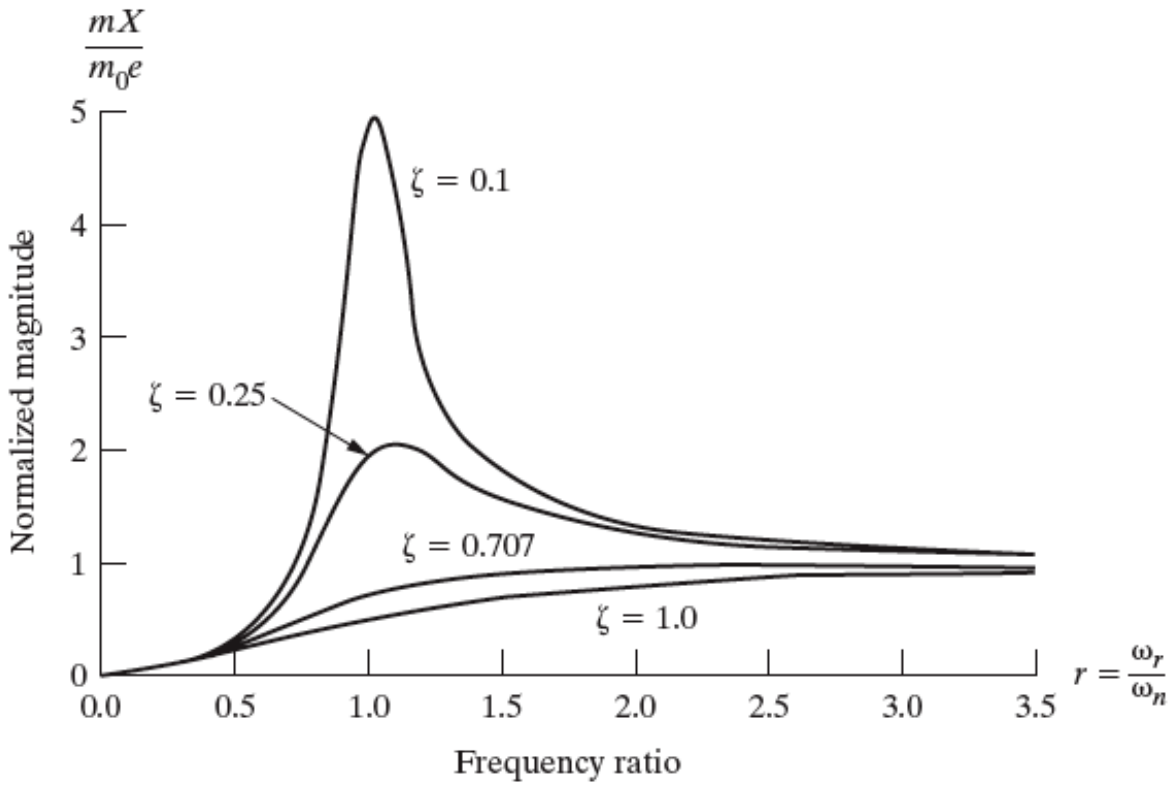
- **Just another SDOF oscillator with a harmonic forcing function**
- **Expressed in terms of frequency ratio *r***

$$x_p(t) = X\sin(\omega_r t - \phi) \qquad (2.83)$$

$$X = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad (2.84)$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \qquad (2.85)$$

Figure 2.21: Displacement magnitude vs frequency caused by rotating unbalance



Example 2.5.1: Given the deflection at resonance (0.1m), $\zeta = 0.05$ and a 10% out of balance, compute *e* and the amount of added mass needed to reduce the maximum amplitude to 0.01 m.

At resonance $r = 1$ and

$$\frac{mX}{m_0 e} = \frac{1}{2\zeta} = \frac{1}{2(0.05)} \Rightarrow 10\frac{0.1\text{ m}}{e} = \frac{1}{2\zeta} = 10 \Rightarrow \underline{e = 0.1\text{ m}}$$

Now to compute the added mass, again at resonance;

$$\frac{m}{m_0}\left(\frac{X}{0.1\text{ m}}\right) = 10$$

Use this to find Δm so that X is 0.01:

$$\frac{m+\Delta m}{m_0}\left(\frac{0.01\text{ m}}{0.1\text{ m}}\right) = 10 \Rightarrow \frac{m+\Delta m}{(0.1)m} = 100 \Rightarrow \underline{\Delta m = 9m}$$

Example 2.5.2 Helicopter rotor unbalance

Given

$k = 1\times10^5$ N/m

$m_{tail} = 60$ kg

$m_{rot} = 20$ kg

$m_0 = 0.5$ kg

$\zeta = 0.01$

Fig 2.22

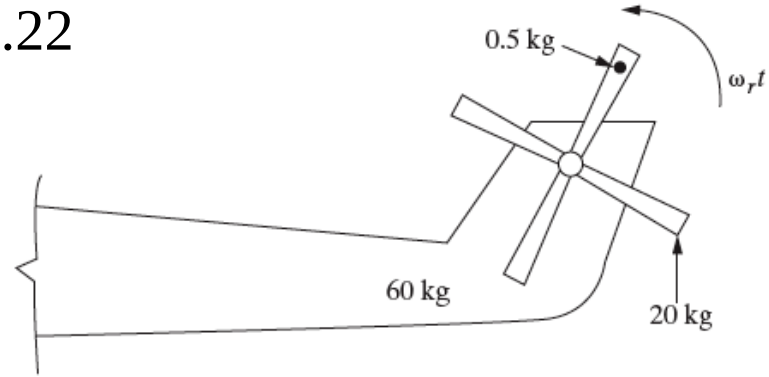
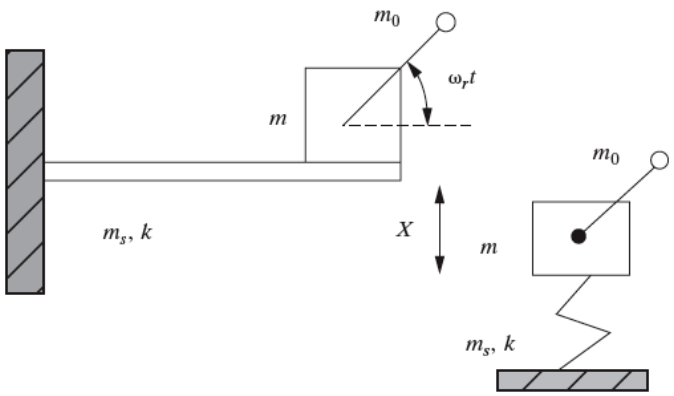


Fig 2.23



Compute the deflection at 1500 rpm and find the rotor speed at which the deflection is maximum

Example 2.5.2 Solution

The rotating mass is 20 + 0.5 or 20.5. The stiffness is provided by the Tail section and the corresponding mass is that determined in the example of a heavy beam. So the system natural frequency is

$$\omega_n = \sqrt{\frac{k}{m + \frac{33}{140} m_{tail}}} = \sqrt{\frac{10^5 \text{ N/m}}{20.5 + \frac{33}{140} 60\text{kg}}} = 53.72\text{rad/s}$$

The frequency of rotation is

$$\omega_r = 1500 \text{ rpm} = 1500 \frac{\text{rev}}{\text{min}} \frac{\text{min}}{60 \text{ s}} \frac{2\pi \text{ rad}}{\text{rev}} = 157 \text{ rad/s}$$

$$\Rightarrow \; r = \frac{157 \text{ rad/s}}{53.96 \text{ rad/s}} = 2.92$$

Now compute the deflection at r = 2.91 and ζ =0.01 using eq (2.84)

$$X = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$= \frac{(0.5\text{ kg})(0.15\text{ m})}{34.64\text{ kg}} \frac{(2.92)^2}{\sqrt{\left(1-(2.92)^2\right)^2 - \left(2(0.01)(2.92)\right)^2}} = \underline{0.002\text{ m}}$$

At around r = 1, the max deflection occurs:

$$r = 1 \Rightarrow \omega_r = 53.72\text{ rad/s} = 53.72 \frac{\text{rad}}{\text{s}} \frac{\text{rev}}{2\pi\text{ rad}} \frac{60\text{ s}}{\text{min}} = \underline{515.1\text{ rpm}}$$

At r = 1:

$$X = \frac{m_0 e}{m_{\text{eq}}} \frac{1}{2\zeta} = \frac{(0.5\text{ kg})(0.15\text{ m})}{34.34\text{ kg}} \frac{1}{2(0.01)} = 0.108\text{ m} \text{ or } \underline{10.8\text{ cm}}$$

2.6 Measurement Devices

- **A basic transducer used in vibration measurement is the *accelerometer.***
- **This device can be modeled using the base equations developed in the previous section**

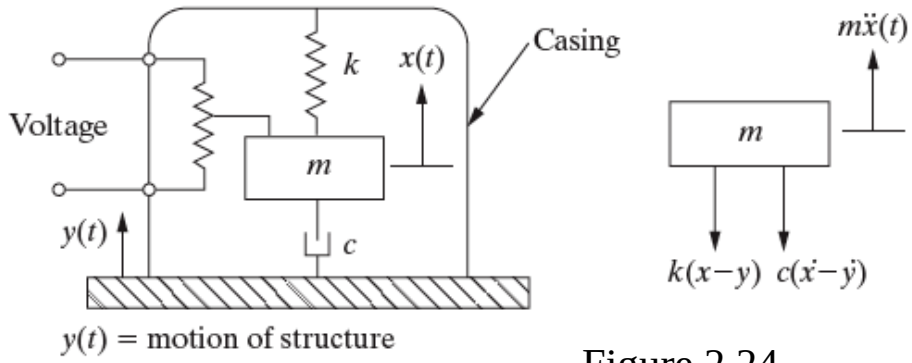


Figure 2.24

$$\sum F = -k(x-y) - c(\dot{x}-\dot{y}) = m\ddot{x}$$

$$\Rightarrow m\ddot{x} = -c(\dot{x}-\dot{y}) - k(x-y)$$

(2.86) and (2.61)

Here, *y*(*t*) is the measured response of the structure

Base motion applied to measurement devices

Let $z(t) = x(t) - y(t)$ (2.87):

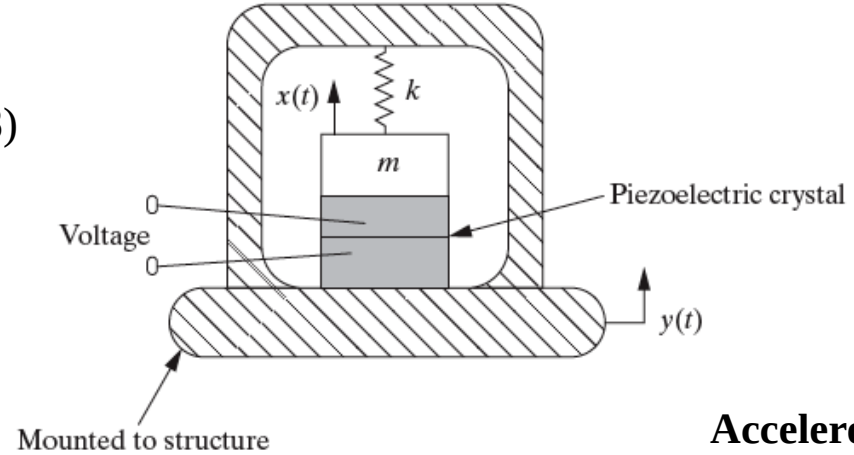
$\Rightarrow$

$$m\ddot{z} + c\dot{z}(t) + kz(t) = m\omega_b^2 Y \cos\omega_b t \quad (2.88)$$

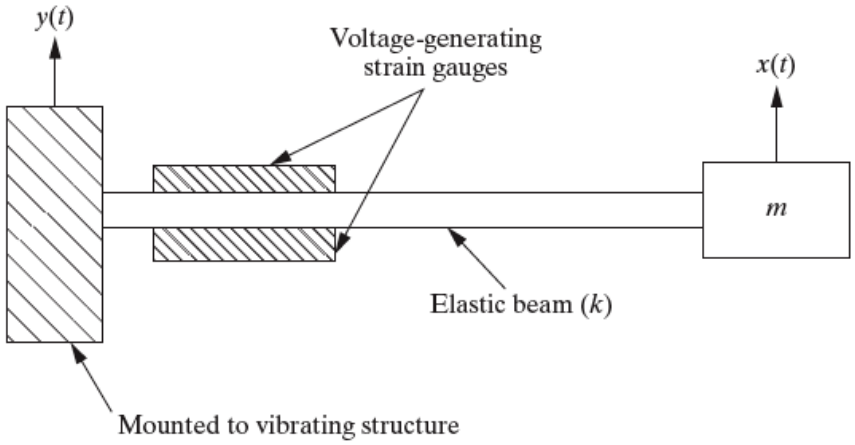
$$\Rightarrow \frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad (2.90)$$

and

$$\theta = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \quad (2.91)$$



Accelerometer



These equations should be familiar from base motion.
Here they describe measurement!

Magnitude and sensitivity plots for accelerometers.

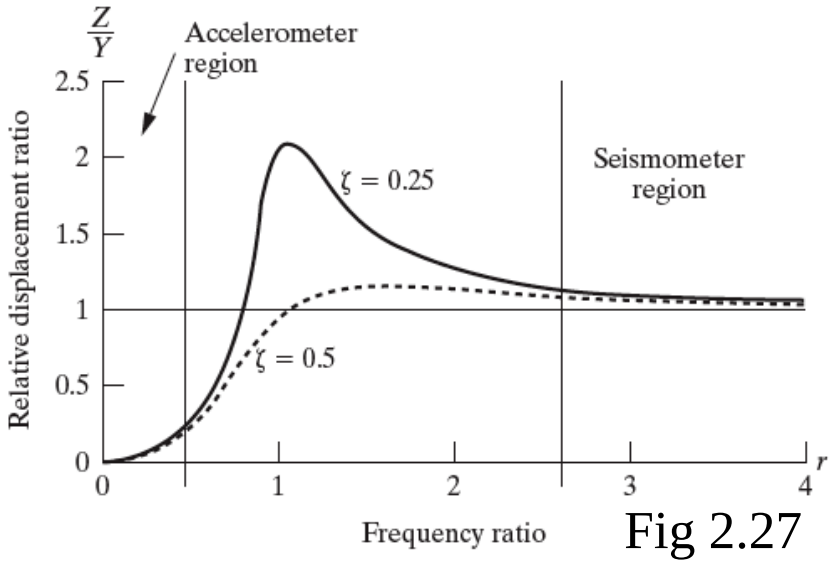


Fig 2.27

Effect of damping on proportionality constant

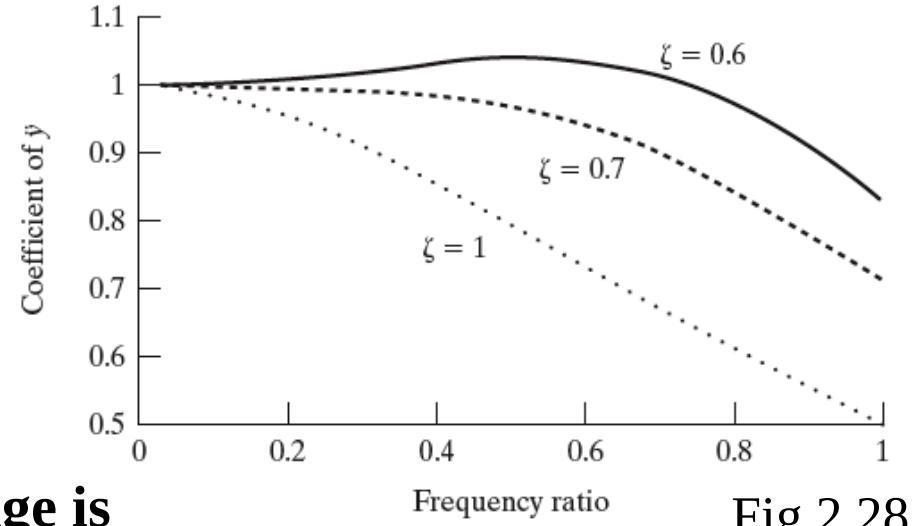


Fig 2.28

Magnitude plot showing Regions of measurement

In the accel region, output voltage is nearly proportional to displacement

2.7 Other forms of damping

TABLE 2.2 DAMPING MODELS

Name	Damping Force	c_{eq}	Source
Linear viscous damping	$c\dot{x}$	c	Slow fluid
Air damping	$a\,\mathrm{sgn}(\dot{x})\dot{x}^2$	$\dfrac{8a\omega X}{3\pi}$	Fast fluid
Coulomb damping	$\beta\,\mathrm{sgn}\,\dot{x}$	$\dfrac{4\beta}{\pi\omega X}$	Sliding friction
Displacement-squared damping	$d\,\mathrm{sgn}(\dot{x})x^2$	$\dfrac{4dX}{3\pi\omega}$	Material damping
Solid, or structural, damping	$b\,\mathrm{sgn}(\dot{x})\lvert x\rvert$	$\dfrac{2b}{\pi\omega}$	Internal damping

These various other forms of damping are all nonlinear. They can be compared to linear damping by the method of "equivalent viscous damping" discussed next. A numerical treatment of the exact response is given in section 2.9.

The method of equivalent viscous damping: consists of comparing the energy dissipated during one cycle of forced response

Assume a stead state resulting from a harmonic input and compute the energy dissipated per one cycle

$$x_{ss} = X \sin \omega t$$

The energy per cycle for a viscously damped system is

$$\Delta E = \oint F_d dx = \int_0^{2\pi/\omega} c\dot{x} \frac{dx}{dt} dt = \int_0^{2\pi/\omega} c\dot{x}^2 dt \qquad (2.99)$$

$$x_{ss} = X \sin \omega t \Rightarrow \dot{x} = \omega X \cos \omega t \Rightarrow \qquad (2.101)$$

$$\Delta E = c \int_0^{2\pi/\omega} (\omega X \cos \omega t)^2 dt = \underline{\pi c \omega X^2}$$

Next compute the energy dissipated per cycle for Coulomb damping:

$$\Delta E = \mu mg \int_0^{2\pi/\omega} \operatorname{sgn}(\dot{x})\dot{x}dt = \mu mg$$

$$= \mu mgX\left(\int_0^{\pi/2} \cos u du - \int_{\pi/2}^{3\pi/2} \cos u du + \int_{3\pi/2}^{2\pi} \cos u du\right) = \underline{4\mu mgX}$$

Here we let *u* = ω*t* and *du* =ω*dt* and split up the integral according to the sign changes in velocity.

Next compare this energy to that of a viscous system:

$$\pi c_{eq}\omega X^2 = 4\mu mgX \Rightarrow \underline{c_{eq} = \frac{4\mu mg}{\pi\omega X}} \qquad (2.105)$$

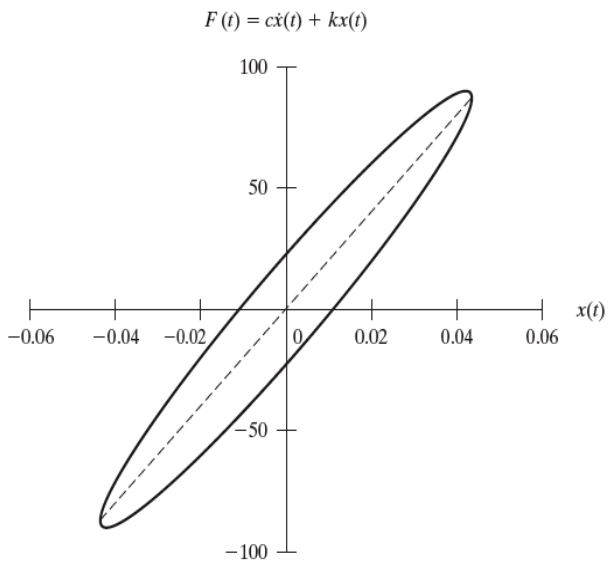
This yields a linear viscous system dissipating the same amount of energy per cycle.

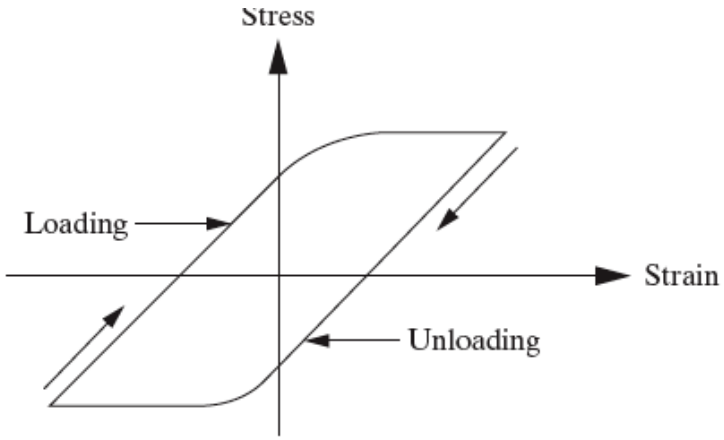
Using the equivalent viscous damping calculations, each of the systems in Table 2.2 can be approximated by a linear viscous system

In particular, c_{eq} can be used to derive amplitude expressions. However, as indicated in Section 2.8 and 2.9 the response can be simulated numerically to provide more accurate magnitude and response information.

Hysteresis: an important concept characterizing damping

- **A plot of displacement versus spring/damping force for viscous damping yields a loop**
- **At the bottom is a stress strain plot for a system with material damping of the hysteretic type**
- **The enclosed area is equal to the energy lost per cycle**





The measured area yields the energy dissipated. For some materials, called hysteretic this is

$$\Delta E = \pi k \beta X^2 \qquad (2.120)$$

Here the constant β, a measured quantity is called the hysteretic damping constant, *k* is the stiffness and *X* is the amplitude.

Comparing this to the viscous energy yields:

$$c_{eq} = \frac{k\beta}{\omega}$$

Hysteresis gives rise to the concept of complex stiffness

Substitution of the equivalent damping coefficient and using the complex exponential to describe a harmonic input yields:

$$m\ddot{x} + \frac{k\beta}{\omega}\dot{x} + kx = F_0 e^{j\omega t}$$

Assuming $x(t) = Xe^{j\omega t}$ and $\dot{x}(t) = Xj\omega e^{j\omega t}$

yields

$$m\ddot{x}(t) + \underbrace{k(1 + j\beta)}_{\text{complex stiffness}} x(t) = F_0 e^{j\omega t}$$

2.8 Numerical Simulation and Design

- **Four things we can do computationally to help solve, understand and design vibration problems subject to harmonic excitation**
- **Symbolic manipulation**
- **Plotting of the time response**
- **Solution and plotting of the time response**
- **Plotting magnitude and phase**

Symbolic Manipulation

Let

$$A = \begin{bmatrix} \omega_n^2 - \omega^2 & 2\zeta\omega_n\omega \\ -2\zeta\omega_n\omega & \omega_n^2 - \omega^2 \end{bmatrix} \quad \textbf{and} \quad x = \begin{bmatrix} f_0 \\ 0 \end{bmatrix}$$

What is

$$A_n = A^{-1}x$$

This can be solved using Matlab, Mathcad or Mathematica

Symbolic Manipulation

Solve equations (2.34) using Mathcad symbolics :

Enter this $\longrightarrow$

$$\begin{bmatrix} \omega n^2 - \omega^2 & 2\cdot\zeta\cdot\omega n\cdot\omega \\ -\left(2\cdot\zeta\cdot\omega n\cdot\omega\right) & \omega n^2 - \omega^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} f0 \\ 0 \end{bmatrix}$$

Choose evaluate under symbolics to get this

$$\begin{bmatrix} \dfrac{\left(\omega n^2 - \omega^2\right)}{\left(\omega n^4 - 2\cdot\omega n^2\cdot\omega^2 + \omega^4 + 4\cdot\zeta^2\cdot\omega n^2\cdot\omega^2\right)}\cdot f0 \\ 2\cdot\zeta\cdot\omega n\cdot\dfrac{\omega}{\left(\omega n^4 - 2\cdot\omega n^2\cdot\omega^2 + \omega^4 + 4\cdot\zeta^2\cdot\omega n^2\cdot\omega^2\right)}\cdot f0 \end{bmatrix}$$

In MATLAB Command Window

```
>> syms z wn w f0
>> A=[wn^2-w^2 2*z*wn*w;-2*z*wn*w wn^2-w^2];
>> x=[f0 ;0];
>> An=inv(A)*x
   An =
   [ (wn^2-w^2)/(wn^4-2*wn^2*w^2+w^4+4*z^2*wn^2*w^2)*f0]
   [  2*z*wn*w/(wn^4-2*wn^2*w^2+w^4+4*z^2*wn^2*w^2)*f0]
>> pretty(An)

            [            2   2             ]
            [         (wn  - w ) f0         ]
            [ ------------------------------- ]
            [   4      2  2    4     2   2  2 ]
            [ wn  - 2 wn  w  + w  + 4 z  wn  w  ]
            [                                ]
            [            z wn w f0          ]
            [2 ---------------------------------]
            [    4      2  2    4     2   2  2]
            [  wn  - 2 wn  w  + w  + 4 z  wn  w ]
```

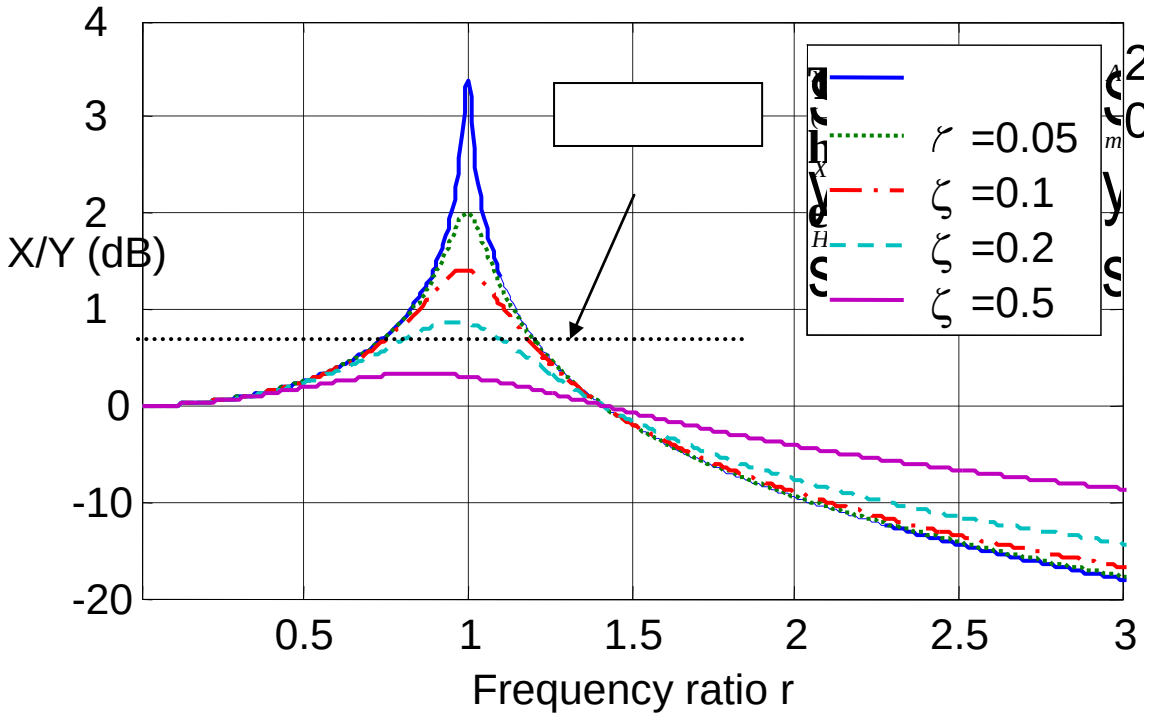
Magnitude plots: Base Excitation

```
%m-file to plot base excitation to mass vibration
r=linspace(0,3,500);
ze=[0.01;0.05;0.1;0.20;0.50];
X=sqrt( ((2*ze*r).^2+1) ./ ( (ones(size(ze))*(1-r.*r).^2) + (2*ze*r).^2) );
figure(1)
plot(r,20*log10(X))
```

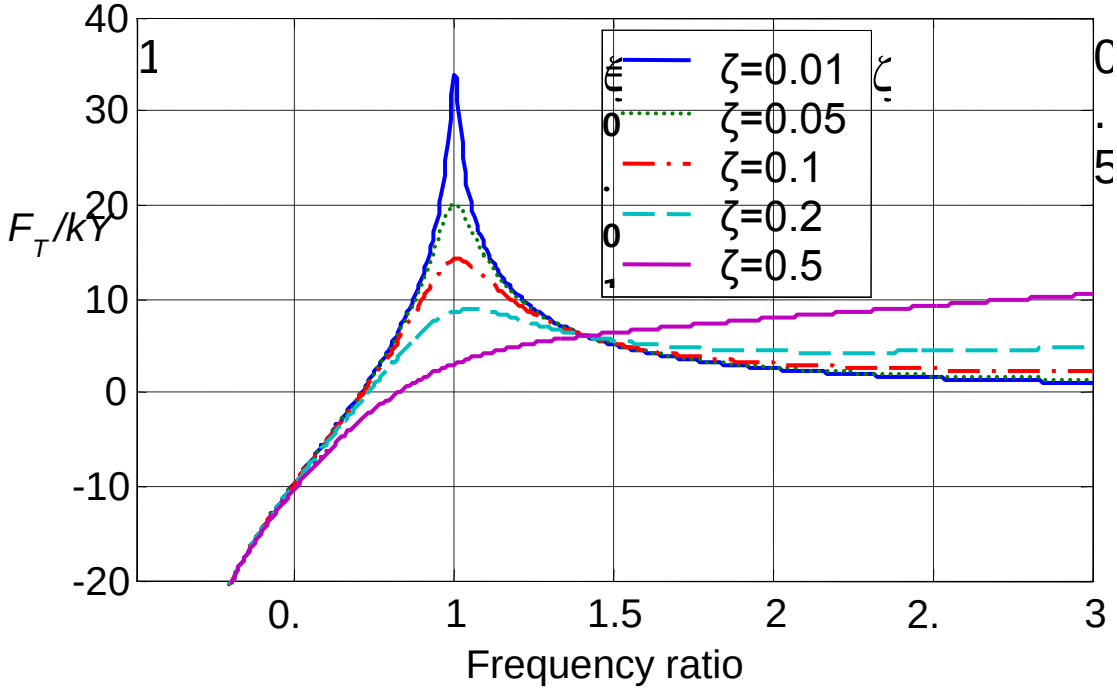
The values of z can then be chosen directly off of the plot.

For Example:

If the T.R. needs to be less than 2 (or 6dB) and *r* is close to 1 then z must be more than 0.2 (probably about 0.3).



Force Magnitude plots: Base Excitation



Numerical Simulation

We can put the forced case:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos \omega t$$

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = f_0 \cos \omega t$$

Into a state space form

$$\dot{x}_1 = x_2$$

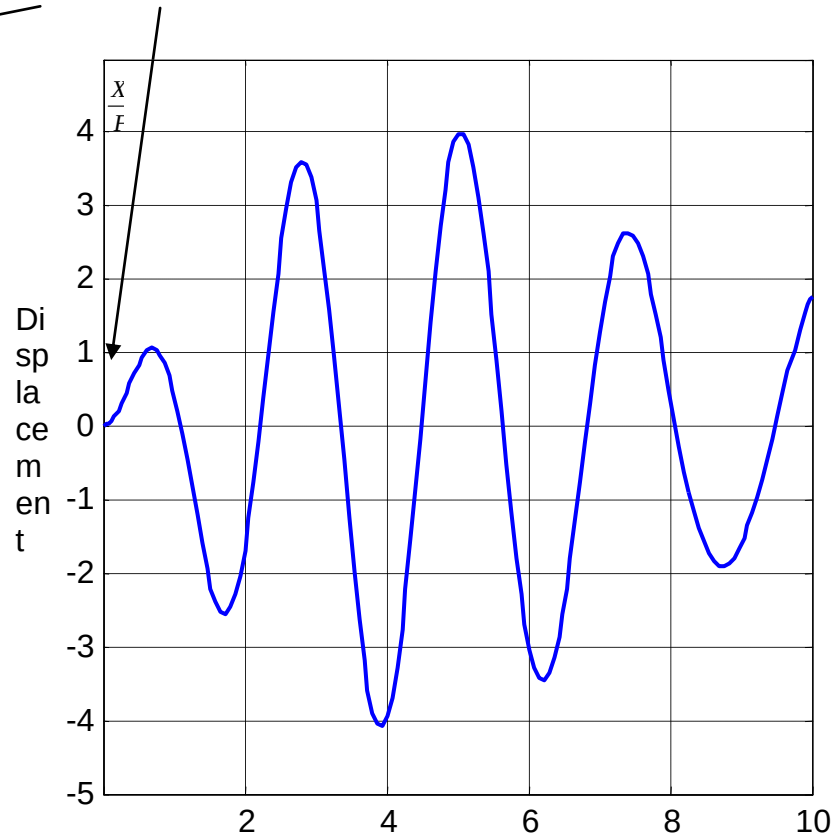
$$\dot{x}_2 = -2\zeta\omega_n x_2 - \omega_n^2 x_1 + f_0 \cos \omega t$$

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ f_0 \cos \omega t \end{bmatrix}$$

Numerical Integration

$$\text{Euler:}\quad \mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + A\mathbf{x}(t_i)\Delta t + \mathbf{f}(t_i)\Delta t$$

```
>>TSPAN=[0 10];
>>Y0=[0;0];
>>[t,y] =ode45('num_for',TSPAN,Y0);
>>plot(t,y(:,1))
```



Including forcing

```
function Xdot=num_for(t,X)
m=100;k=1000;c=25;
ze=c/(2*sqrt(k*m));
wn=sqrt(k/m);
w=2.5;F=1000;f=F/m;
f=[0 ;f*cos(w*t)];
A=[0 1;-wn*wn -2*ze*wn];
Xdot=A*X+f;
```

Example 2.8.2: Design damping for an electronics model

- **100 kg mass, subject to 150cos(5t) N**
- **Stiffness k=500 N/m, c = 10kg/s**
- **Usually x_0=0.01 m, v_0 = 0.5 m/s**
- **Find a new c such that the max transient value is 0.2 m.**

Response of the board is;

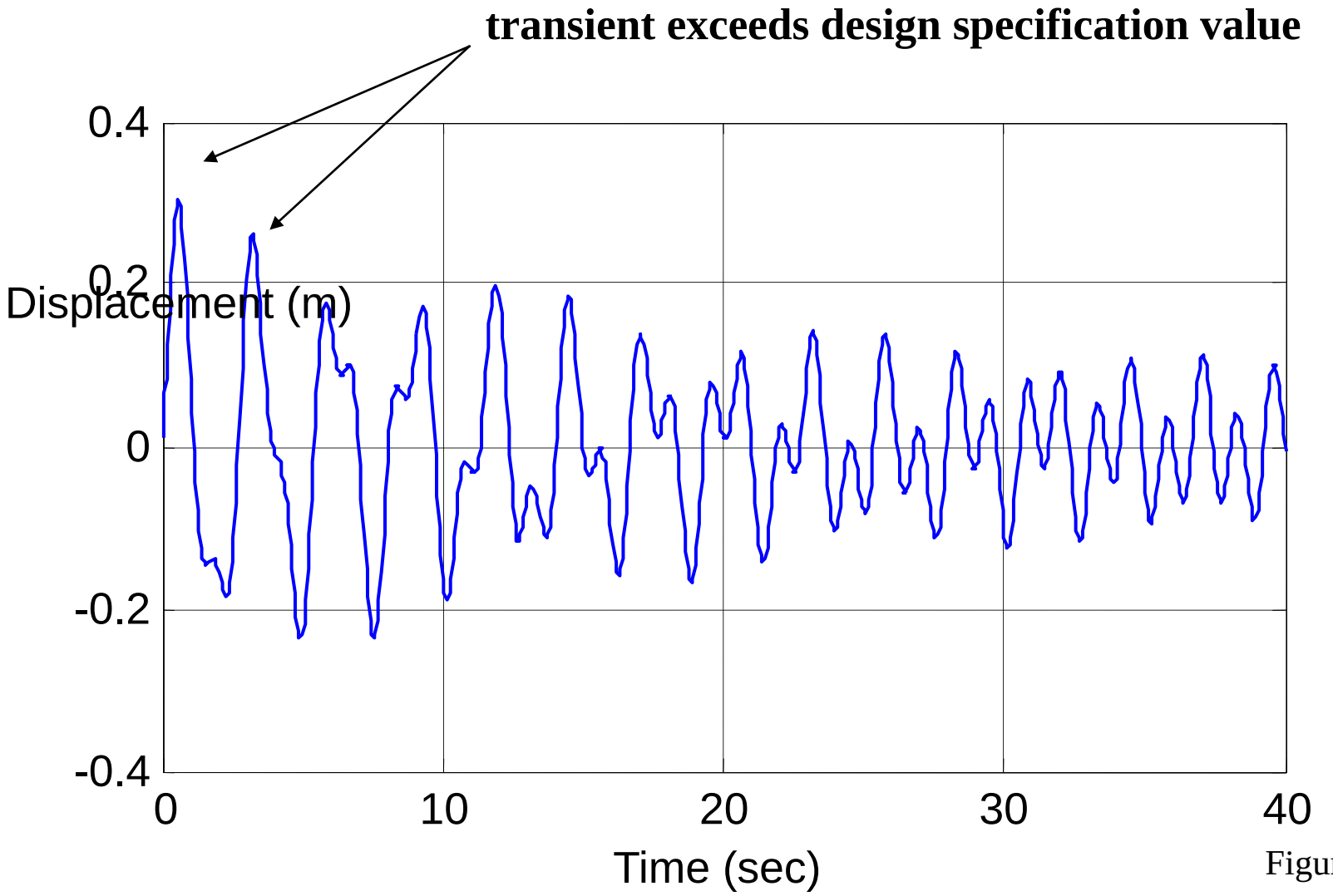


Figure 2.33

To run this use the following file:

Create function to model forcing

```
function Xdot=num_for(t,X)
m=100;k=500;c=10;
ze=c/(2*sqrt(k*m));
wn=sqrt(k/m);
w=5;F=150;f=F/m;
f=[0 ;f*cos(w*t)];
A=[0 1;-wn*wn -2*ze*wn];
Xdot=A*X+f;
```

Matlab command window

```
>>TSPAN=[0 40];
>> Y0=[0.01;0.5];
>>[t,y] = ode45('num_for',TSPAN,Y0);
>> plot(t,y(:,1))
>> xlabel('Time (sec)')
>> ylabel('Displacement (m)')
>> grid
```

Rerun this code, increasing *c* each time until a response that satisfies the design limits results.

Solution: code it, plot it and change *c* until the desired response bound is obtained.

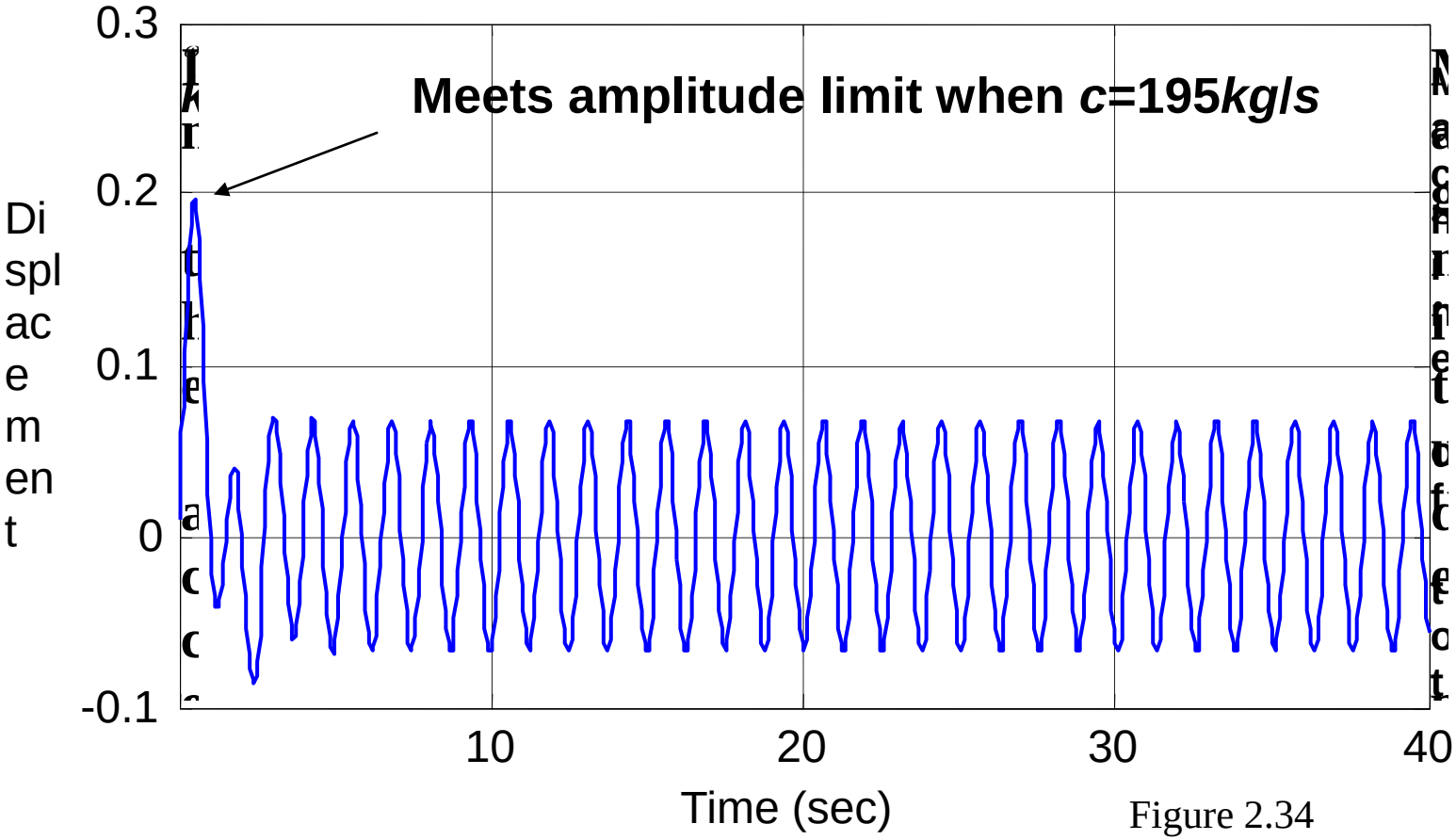


Figure 2.34

2.9 Nonlinear Response Properties

- **More than one equilibrium**
- **Steady state depends on initial conditions**
- **Period depends on I.C. and amplitude**
- **Sub and super harmonic resonance**
- **No superposition**
- **Harmonic input resulting in nonperiodic motion**
- **Jumps appear in response amplitude**

Computing the forced response of a non-linear system

A non-linear system has a equation of motion given by:

$$\ddot{x}(t) + f(x, \dot{x}) = f_0 \cos \omega t$$

Put this expression into state-space form:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -f(x_1, x_2) + f_0 \cos \omega t \end{cases}$$

In vector form:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}) + \mathbf{f}(t)$$

Numerical form

Vector of nonlinear dynamics

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x_2(t) \\ -f(x_1, x_2) \end{bmatrix},$$

Input force vector

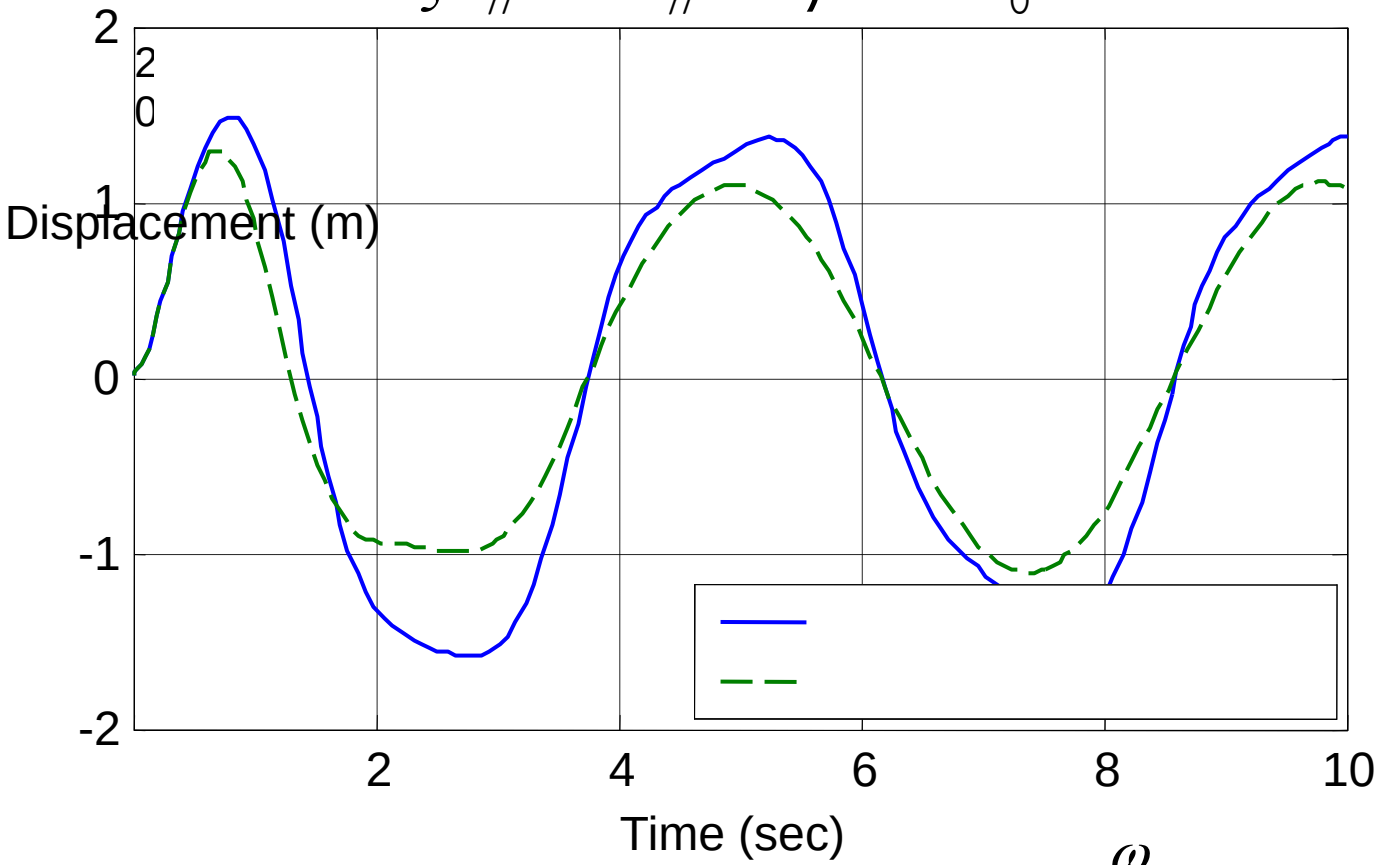
$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ f_0 \cos \omega t \end{bmatrix}$$

Euler equation is

$$\mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + \mathbf{F}(\mathbf{x}(t_i))\Delta t + \mathbf{f}(t_i)\Delta t$$

Cubic nonlinear spring (2.9.1)

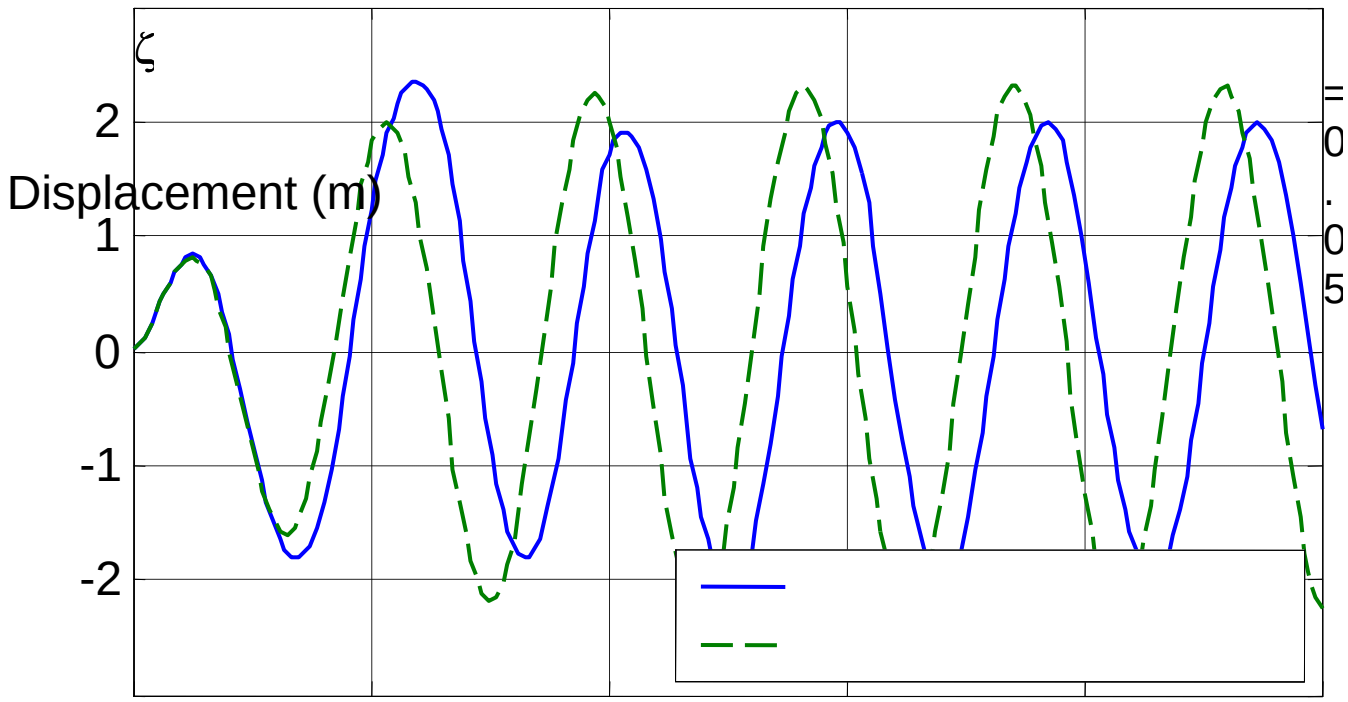
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x - \beta x^3 = f_0 \cos\omega t$$



$$\omega = \frac{\omega_n}{2.964}$$

Cubic nonlinear spring near resonance

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x - \beta x^3 = f_0 \cos\omega t$$



$$\omega = \frac{\omega_n}{1.09}$$