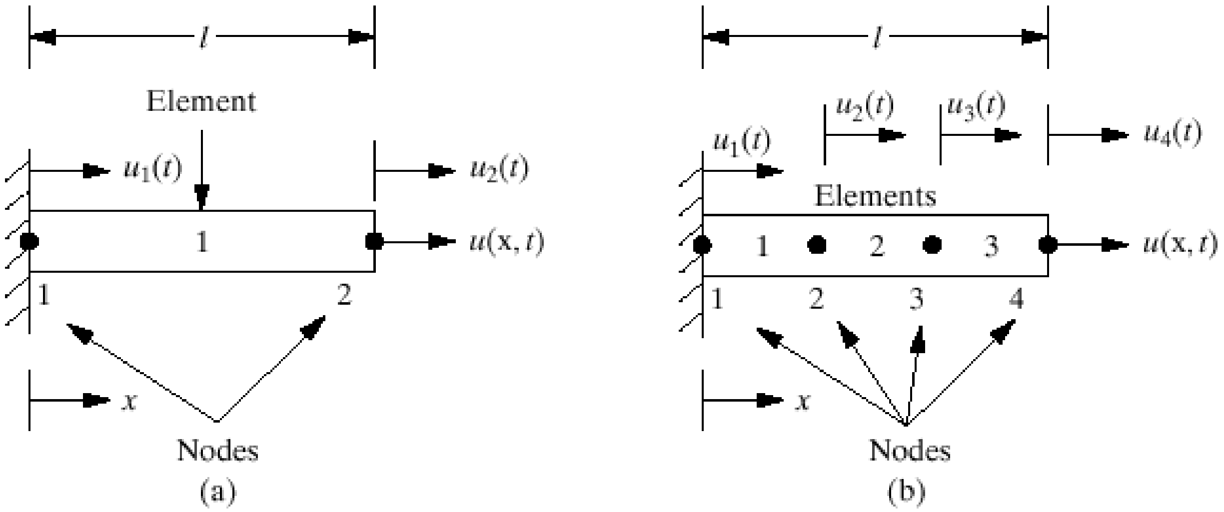
Chapter 8 Dynamic Finite Element Methods (FEM)

- **A method of discretizing continuous systems, such as Rayleigh-Ritz**
- **Also, and more importantly, a method of modeling complex systems**
- **Numerous large commercial codes are available based on these ideas**
- **Here we examine the fundamentals**

Section 8.1 Example: The Bar

- **Recall the longitudinal vibration of a bar from section 6.3**
- **A two node FEM is presented**



Consider the Static Case

$$EA\frac{d^2u(x)}{dx^2} = 0$$

$$\Rightarrow u(x) = c_1 x + c_2$$

Two approximations

1) element/ mesh size

2) static solution

Here c_1 and c_2 are constants of integration

But we treat them as functions of time

$c_1(t)$ and $c_2(t)$

Nodal Displacements

- $u_1(t)$ and $u_2(t)$ are called nodal displacements
- They are functions of time
- Eventually we will solve for these
- The solution is approximated by the spatial solution $u(x)$ and the two temporal functions $u_1(t)$ and $u_2(t)$ matched at the nodes

Solving for the functions of integration

- **Recall:** $u(x) = c_1 x + c_2$
- **From the sketch:** $u(0) = u_1(t) = c_1(0) + c_2$

$$\Rightarrow c_2(t) = u_1(t)$$

$$u(l) = u_2(t) = c_1 l + c_2$$

$$\Rightarrow c_1(t) = \tfrac{1}{l}[u_2(t) - u_1(t)]$$

The approximate solution is

Nodal coordinates

$$u(x,t) = \left(1 - \frac{x}{l}\right) u_1(t) + \frac{x}{l} u_2(t)$$

Shape functions

Here we do not yet know the values of $u_1(t)$ and $u_2(t)$. If we did, we would be done!

Finding an Equation of Motion for $u_i(t)$

Write the strain energy and kinetic energy and use the Lagrangian approach.

- **The strain energy is** $V(t) = \frac{1}{2}\int_0^l EA\left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx$
- **Substitute in the approximate value of *u*(*x*,*t*):**

$$V(t) = \frac{1}{2}\int_0^l \frac{EA}{l^2}\left[-u_1(t) + u_2(t)\right]^2 dx$$

$$= \frac{EA}{2l}\left(u_1^2 - 2u_1u_2 + u_2^2\right)$$

Recognize the Quadratic Form

$$\frac{EA}{2l}\left(u_1^2 - 2u_1u_2 + u_2^2\right) =$$

$$\frac{1}{2}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{2}\mathbf{u}^T K\mathbf{u}$$

Where: $\mathbf{u} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad K = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Calculate the Kinetic Energy

$$T(t) = \frac{1}{2}\int_0^l A\rho(x)\left[\frac{\partial u(x,t)}{\partial t}\right]^2 dx$$

$$u(x,t) = \left(1 - \frac{x}{l}\right)u_1(t) + \frac{x}{l}u_2(t)$$

$$\Rightarrow u_t(x,t) = \left(1 - \frac{x}{l}\right)\dot{u}_1(t) + \frac{x}{l}\dot{u}_2(t)$$

Substitute and Expand to get:

$$T(t) = \frac{1}{2}\frac{\rho A l}{3}\left(\dot{u}_1^2 + \dot{u}_1\dot{u}_2 + \dot{u}_2^2\right)$$

$$= \frac{1}{2}\dot{\mathbf{u}}^T M \dot{\mathbf{u}}$$

Where

$$\dot{\mathbf{u}} = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}, \qquad M = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Apply the boundary condition:

$$u_1(t) = 0$$

$$\Rightarrow V(t) = \frac{1}{2}\frac{EA}{l}u_2^2, \quad T(t) = \frac{1}{2}\frac{\rho Al}{3}\dot{u}_2^2$$

Apply the Lagrangian: $$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{u}_i}\right) - \frac{\partial T}{\partial u_i} + \frac{\partial V}{\partial u_i} = f_i(t)$$

$$\Rightarrow \frac{\rho Al}{3}\ddot{u}_2(t) + \frac{EA}{l}u_2(t) = 0$$

Now we can solve to get:

$$u_2(t) = A\sin(\frac{1}{l}\sqrt{\frac{3E}{\rho}}t + \theta)$$

$$\Rightarrow u(x,t) = \frac{x}{l}A\sin(\frac{1}{l}\sqrt{\frac{3E}{\rho}}t + \theta)$$

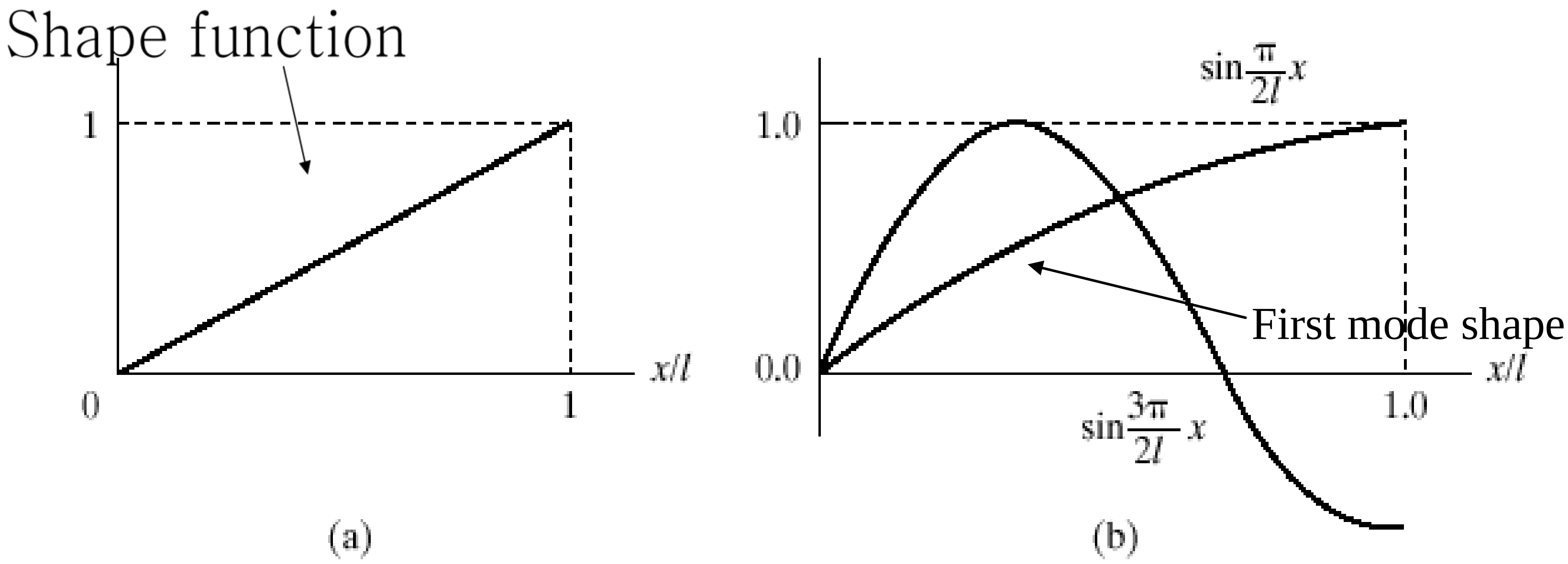
An approximate solution to the bar

using one finite element!

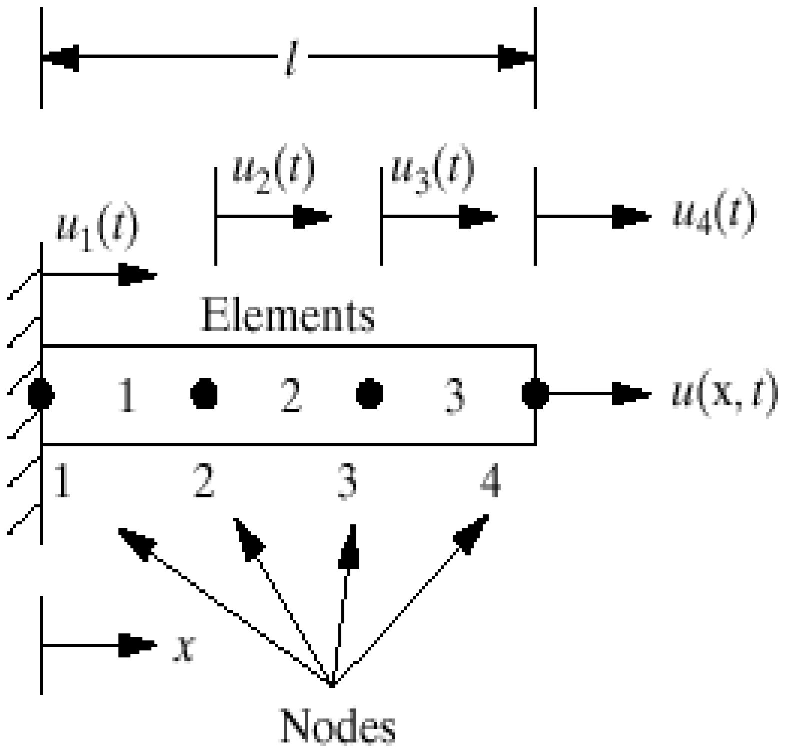
Comparison Between Exact and FEM

- **The frequency for FEM is:** $\omega_{\text{FEM}} = \dfrac{1.732}{l}\sqrt{\dfrac{E}{\rho}}$

- **The exact frequencies are:** $\omega_1 = \dfrac{1.57}{l}\sqrt{\dfrac{E}{\rho}}, \omega_2 = \dfrac{4.712}{l}\sqrt{\dfrac{E}{\rho}}$



8.2 Three Element Bar



- **Next we expand the one element solution to three (or more)**
- **Use the same strain energy computations for each element**
- **Change only the coordinates and length of the element**

Strain Energy for Each Element

Replace $l \rightarrow \dfrac{l}{3}$

$$V_1(t) = \frac{3EA}{2l}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$V_2(t) = \frac{3EA}{2l}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

$$V_3(t) = \frac{3EA}{2l}\begin{bmatrix} u_3 \\ u_4 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

Expand to Compute the Total Strain Energy

$$V_1 = \frac{3EA}{2l}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$V_2 = \frac{3EA}{2l}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$V_3 = \frac{3EA}{2l}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Now add these as they are all in the same coordinate system:

$$V_T = \frac{3EA}{2l}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Apply the Boundary Condition

$$V_T = \frac{3EA}{2l}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$= \frac{3EA}{2l}\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix}^T \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$V_T(t) = \frac{1}{2}\mathbf{u}^T K \mathbf{u}$$

Defines the *Global Stiffness*

Matrix *K*

Now Compute the Kinetic Energy

From Eq. (8.10): $$T(t) = \frac{\rho A l}{12} \mathbf{u}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{u}$$

Let: $l \rightarrow \frac{l}{3}$ **and get:** $$T_i(t) = \frac{\rho A l}{36} \mathbf{u}_i^{\ T} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{u}_1$$

- **This defines the kinetic energy per element.**
- **Next write these for each element and,**
- **Assemble the elements to get the global kinetic energy.**

Elemental Kinetic Energy:

$$T_1 = \frac{\rho A l}{36}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\end{bmatrix}^T\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\end{bmatrix}$$

$$T_2 = \frac{\rho A l}{36}\begin{bmatrix}\dot{u}_2\\ \dot{u}_3\end{bmatrix}^T\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_2\\ \dot{u}_3\end{bmatrix}$$

$$T_3 = \frac{\rho A l}{36}\begin{bmatrix}\dot{u}_3\\ \dot{u}_4\end{bmatrix}^T\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_3\\ \dot{u}_4\end{bmatrix}$$

Expand and Assemble Global KE

$$\left.\begin{aligned}
T_1 &= \frac{\rho Al}{36}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}^T\begin{bmatrix}2&1&0&0\\1&2&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}\\
T_2 &= \frac{\rho Al}{36}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}^T\begin{bmatrix}0&0&0&0\\0&2&1&0\\0&1&2&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}\\
T_3 &= \frac{\rho Al}{36}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}^T\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&2&1\\0&0&1&2\end{bmatrix}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}
\end{aligned}\right\} \Rightarrow T_T = \frac{1}{2}\frac{\rho Al}{18}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}^T\begin{bmatrix}2&1&0&0\\1&4&1&0\\0&1&4&1\\0&0&1&2\end{bmatrix}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}$$

Apply the Clamped BC

$$T_T = \frac{1}{2}\frac{\rho A l}{18}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}^T \begin{bmatrix}2 & 1 & 0 & 0\\ 1 & 4 & 1 & 0\\ 0 & 1 & 4 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}$$

$$T_T(t) = \frac{1}{2}\frac{\rho A l}{18}\begin{bmatrix}\dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}^T \begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\end{bmatrix}$$

$$T_T = \frac{1}{2}\dot{\mathbf{u}}^T M \dot{\mathbf{u}}$$

Use Lagrange Implementation for EOM

$$T = \tfrac{1}{2}\dot{\mathbf{u}}^{\mathrm{T}} M \dot{\mathbf{u}} \quad \text{and} \quad V = \tfrac{1}{2}\mathbf{u}^{\mathrm{T}} K \mathbf{u}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{u}}}\right) - \frac{\partial T}{\partial \mathbf{u}} + \frac{\partial V}{\partial \mathbf{u}} = 0 \text{ where } \mathbf{u} = \text{all gen. coords}$$

$$\frac{\partial T}{\partial \dot{\mathbf{u}}} = M\dot{\mathbf{u}},\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{u}}}\right) = M\ddot{\mathbf{u}},\ \frac{\partial T}{\partial \mathbf{u}} = 0,\ \frac{\partial V}{\partial \mathbf{u}} = K\mathbf{u}$$

$$\therefore \underline{M\ddot{\mathbf{u}} + K\mathbf{u} = 0}$$

Just what we expected!

The Equation of Motion is:

$$\frac{\rho A l}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \frac{3EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \mathbf{0}$$

Comparison with Analytical Solution

Consider a 1 meter Al bar, *E* =7.0x10^{10} N/m^2

And ρ = 2700 kg/m^3.

FEM-3 Elements	Exact (% error)
$\omega_1 = 8092$ rad/s	$\omega_1 = 7998$ rad/s (0.55%)
$\omega_2 = 26,458$ rad/s	$\omega_2 = 23,994$ rad/s (9.64%)
$\omega_3 = 47,997$ rad/s	$\omega_3 = 39,900$ rad/s (19.3%)

Rule of thumb: need at least twice as many elements as you want accurate frequencies

System-Level Assembly

- **How can we assemble the system stiffness matrix directly?**
- **First, define a system DOF vector containing all the degrees of freedom**

$$u = \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \text{list of all DOFs in system}$$

ndof = 3 here

Global Stiffness Matrix

- **In this case, each "element" attaches to two coordinates**
- **Positions in the global matrix are determined by the coordinate vector**

$$K = \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}_{u_2 \quad u_3 \quad u_4} \begin{matrix} u_2 \\ u_3 \\ u_4 \end{matrix} \quad \text{since} \quad \mathbf{u} = \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Element Stiffness Matrix

- **Note that for the i^th^ element,**

$$V_i = \tfrac{1}{2} k_i \left(u_{i+1} - u_i\right)^2 = \tfrac{1}{2} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}^{\mathrm{T}} \underbrace{\begin{bmatrix} k_i & -k_i \\ -k_i & k_i \end{bmatrix}}_{\text{stiffness matrix for i}^{\text{th}}\text{ element}} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} = \tfrac{1}{2} \left(u_i^e\right)^T k_i^e u_i^e$$

$$K = \underbrace{\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{element 1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}_{\text{element 2}} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

Basic Elements of a FE Code (I)

- **Mesh generation (1D)**
  - **Choose length of beam;**
  - **Choose number of nodes or elements (each implies the other) and create uniform-length elements (W.L.O.G.)**
- **Define material and mass properties**
  - **Thickness, width, modulus, density,**
- **Define loads (spatial information)**

MATLAB Suggestions (I)

- **Presize matrices**

  ndof = 3; % number of nodes in this problem
  M=zeros(ndof,ndof); K=zeros(ndof,ndof); etc

- **Define element stiffness matrices**

  k = [k1   k2];% list of element (spring) stiffnesses

  kelem = k(j) * [1   -1   ;   -1   1]; % for jth element

- **Use pointers to locate rows and columns**

  point = [j   j+1]; % for jth element

  K(point,point) = K(point,point) + kelem ;

MATLAB Suggestions (II)

- **Use pointers again to enforce boundary conditions. This point to rows and cols**

```
active = [2 3];% the active DOFs in our problem
```

- **Can create a reduced-size matrix**

```
Mr = M(active,active); % Mr is a 2x2
Kr = K(active,active); % Kr is a 2x2
[V,D] = eig(Kr,Mr);% the direct EV problem
```

- **Can just point to the active DOFs. This is equivalent to above; no new matrices needed**

```
[V,D] = eig(K(active,active), M(active,active));
```

Sample FE Code (I)

```
% Simple spring-mass FE code.    EMA 11/17/99

clear all

num_elem = 2;      % define number of elements (springs)

% write your code so that any num_elem will work!

num_nodes = num_elem + 1;

ndof = num_nodes;          % number of DOFs

nodal_forces = zeros(ndof,1); nodal_forces(ndof) = 1;% put a unit force on the last DOF

nodal_masses = ones(ndof,1); % make all mass values = 1.0

spring_stiffnesses = 100 * ones(ndof,1); % all k's = 100.0

% presize the system-level matrices

M = zeros(ndof,ndof);

K = zeros(ndof,ndof);

F = zeros(ndof,1);
```

Sample FE Code (II)

```
% loop on elements and assemble the system matrices
for j = 1:num_elem
    point = [j j+1];      % point to the global DOFs for this element
    kelem = spring_stiffnesses(j) * [1    -1 ; -1    1];      % element k
    K(point,point) = K(point,point) + kelem;
end    % looping on elements
% Apply the discrete masses and forces to appropriate DOF
for i = 1:ndof
    M(i,i) = M(i,i) + nodal_masses(i);
    F(i) = F(i) + nodal_forces(i);
end
```

Sample FE Code (III)

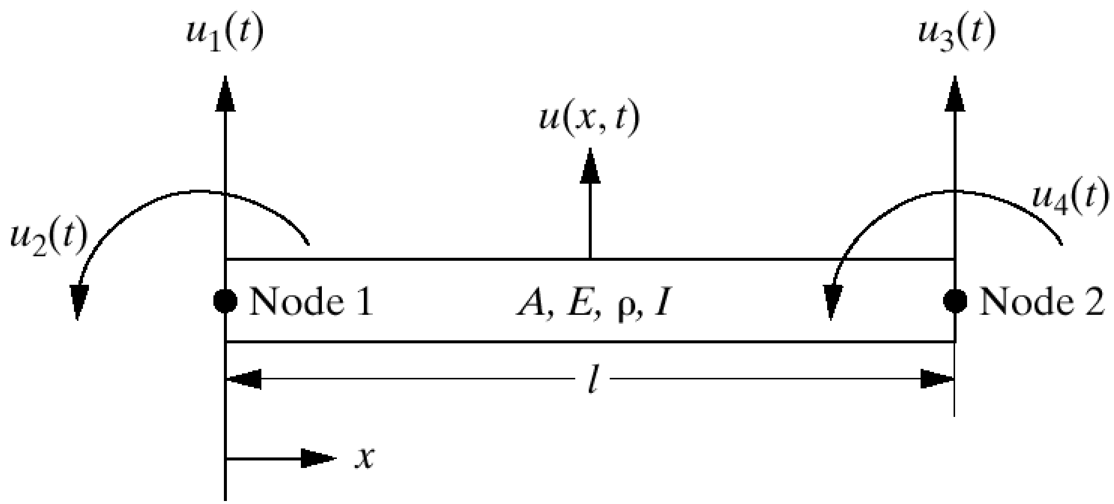
```
% Enforce boundary conditions by defining the active DOFs as
% everything but the clamped (first) DOF
active = 2:ndof;
% What is the static displaced shape?
% ur is "reduced" length, i.e., ndof - number of BC
ur = K(active,active)\F(active);% inv(K)*F in MATLAB
% recreate the full-size solution vector
u_static = zeros(ndof,1);
u_static(active) = ur;      % just that simple!
```

- **Just treat like any other MDOF   M, K, and F matrices**

8.3 Beam Elements

- **We just did the bar element: i.e. motion in the longitudinal direction**
- **Next we will do a beam element to capture motion in the transverse direction**
- **Note from Chapter 6 that there are other elements to consider: plate, shell, etc.**

The Bending Beam Element



- **Note the 4 coordinates to approximate *u*(*x*,*t*)**
- **The transverse static displacement is:**

$$\frac{\partial^2}{\partial x^2}\left[EI \frac{\partial^2 u(x,t)}{\partial x^2} \right] = 0$$

Approximate Bending Solution

Integrating the static deflection expression yields:

$$u(x,t) = c_1(t)x^3 + c_2(t)x^2 + c_3(t)x + c_4(t)$$

This is subject to the following end conditions:

$$u(0,t) = u_1(t) \qquad u_x(0,t) = u_2(t)$$

$$u(l,t) = u_3(t) \qquad u_x(l,t) = u_4(t)$$

Use the End Conditions to find the "Constants" of Integration

$$c_4 = u_1(t) \quad c_3 = u_2(t)$$

$$c_2 = \frac{1}{l^2}[3(u_3 - u_1) - l(2u_2 + u_4)]$$

$$c_1 = \frac{1}{l^2}[2(u_1 - u_3) + l(u_2 + u_4)] \Rightarrow$$

Shape Functions

$$u(x,t) = \left[1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}\right]u_1(t) + l\left[\frac{x}{l} - 2\frac{x^2}{l^2} + \frac{x^3}{l^3}\right]u_2(t)$$

$$+\left[3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right]u_3(t) + l\left[-\frac{x^2}{l^2} + \frac{x^3}{l^3}\right]u_4(t)$$

Shape Functions

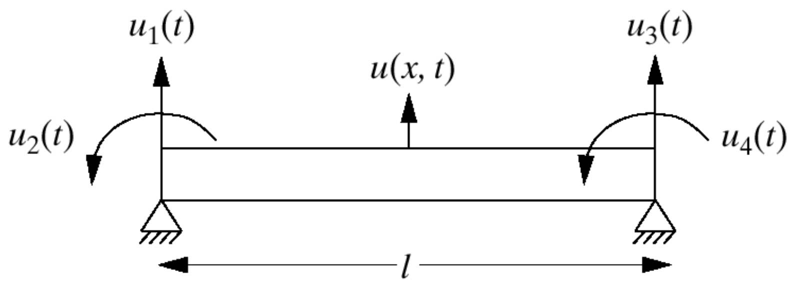
Following the Energy Method as Before Yields the Beam Element Energy, Mass and Stiffness:

$$M = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

$$K = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Example 8.3.1: Simply supported Beam

- **Using a single element**
- **Note from the sketch that $u_1(t)$ and $u_3(t)$ are fixed**
- **Deleting these from the mass and stiffness expressions yields:**



$$M = \frac{\rho A l^3}{420}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$$

$$K = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

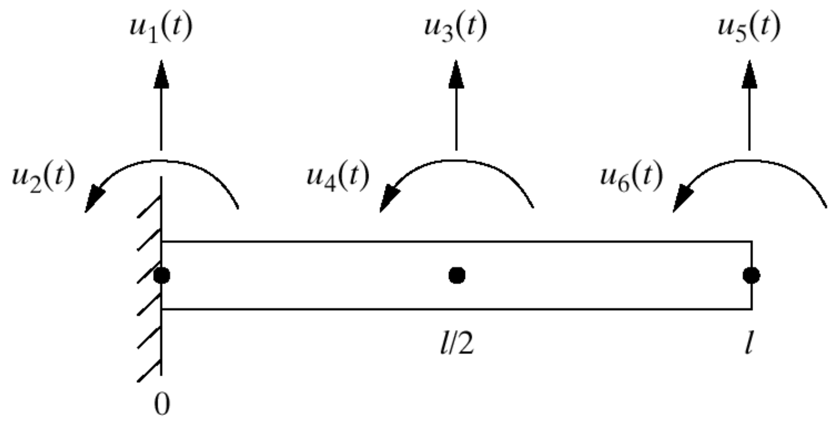
Equation of Motion and Frequencies

$$\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} \ddot{u}_2(t) \\ \ddot{u}_4(t) \end{bmatrix} + \frac{840EI}{\rho A l^4}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u_2(t) \\ u_4(t) \end{bmatrix} = \mathbf{0}$$

$$\omega_1 = 10.95445\sqrt{\frac{EI}{\rho A l^4}} \qquad (9.86965\sqrt{\frac{EI}{\rho A l^4}})$$

$$\omega_2 = 50.199605\sqrt{\frac{EI}{\rho A l^4}} \qquad (39.47860\sqrt{\frac{EI}{\rho A l^4}})$$

Example 8.3.3 Derive the global mass and stiffness matrices for a clamped free beam using 3 nodes and two elements.



- **Obtain global matrices directly from the beam FEM of equations (8.53) and (8.56)**
- **Change *l* for *l*/2, write in global coordinates and apply the clamped constraint**

The First Element Changing l for l/2 yields:

$$M_1 = \frac{\rho A l}{840}\begin{bmatrix} 156 & 11l & 54 & -\frac{13}{2}l \\ 11l & l^2 & \frac{13}{2}l & -\frac{3}{4}l^2 \\ 54 & \frac{13}{2}l & 156 & -11l \\ -\frac{13}{2}l & -\frac{3}{4}l^2 & -11l & l^2 \end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$K_1 = \frac{8EI}{l^3}\begin{bmatrix} 12 & 3l & -12 & 3l \\ 3l & l^2 & -3l & 0.5l^2 \\ -12 & -3l & 12 & -3l \\ 3l & 0.5l^2 & -3l & l^2 \end{bmatrix}$$

The Second Element Changing l for $l/2$ yields:

$$M_2 = \frac{\rho A l}{840}\begin{bmatrix} 156 & 11l & 54 & -\frac{13}{2}l \\ 11l & l^2 & \frac{13}{2}l & -\frac{3}{4}l^2 \\ 54 & \frac{13}{2}l & 156 & -11l \\ -\frac{13}{2}l & -\frac{3}{4}l^2 & -11l & l^2 \end{bmatrix}$$

$$\mathbf{u}_2 = \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

$$K_2 = \frac{8EI}{l^3}\begin{bmatrix} 12 & 3l & -12 & 3l \\ 3l & l^2 & -3l & 0.5l^2 \\ -12 & -3l & 12 & -3l \\ 3l & 0.5l^2 & -3l & l^2 \end{bmatrix}$$

Expand to Global Coordinates and Add to get Global *M* and *K*

In this case the global coordinates are: $\mathbf{u} = \mathbf{u}_2$

$$M = M_1 + M_2 = \frac{\rho A l}{840}\begin{bmatrix} 312 & 0 & 54 & -\frac{13}{2}l \\ 0 & 2l^2 & \frac{13}{2}l & -\frac{3}{4}l^2 \\ 54 & \frac{13}{2}l & 156 & -11l \\ -\frac{13}{2}l & -\frac{3}{4}l^2 & -11l & l^2 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

$$K = K_1 + K_2 = \frac{8EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 3l \\ 0 & 2l^2 & -3l & 0.5l^2 \\ -12 & -3l & 12 & -3l \\ 3l & 0.5l^2 & -3l & l^2 \end{bmatrix}$$

8.5 Trusses

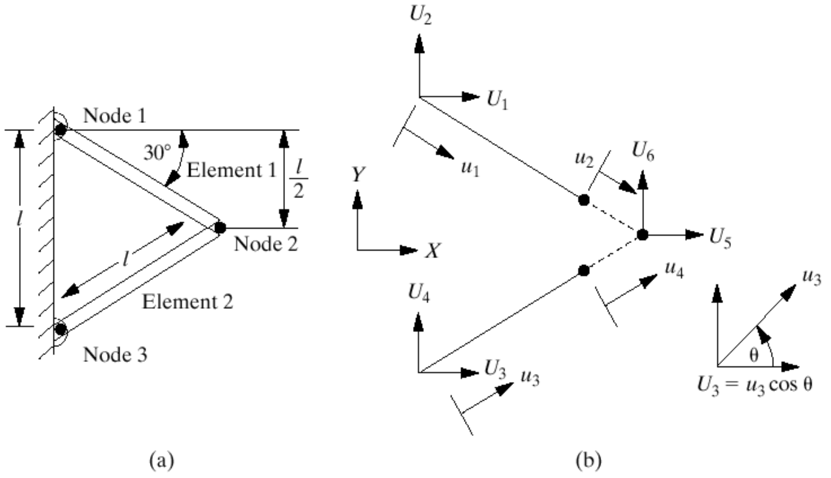
- **As an example of using FEM for structures that do not have analytical solutions consider a simple truss.**
- **The ability of FEM to model very complex structures and hence to provide solutions where analytical solutions do not exist makes the method a key engineering tool.**

A Simple Frame

- Considered to be formed of bar elements
- Local coordinates pointing in different directions
- But system can vibrate in the x-y plane

Local coordinates: $u_i(t)$

Global coordinates: $U_i(t)$



Global to local transformation for element 2:

$$u_3(t) = U_3 \cos\theta + U_4 \sin\theta$$
$$u_4(t) = U_5 \cos\theta + U_6 \sin\theta$$

Developing the Stiffness For Bar 2

$$\underbrace{\begin{bmatrix} u_3 \\ u_4 \end{bmatrix}}_{\mathbf{u}_2} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}}_{\Gamma} \underbrace{\begin{bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix}}_{\mathbf{U}_2}$$

Local/global transformation matrix

Local coordinate vector for element 2

$$\mathbf{u}_2 = \Gamma \mathbf{U}_2$$

Global coordinate vector for element 2

Element Potential Energy:

$$V(t) = \frac{1}{2}\mathbf{u}^T K_e \mathbf{u} = \frac{1}{2}\mathbf{U}^T \underbrace{\Gamma^T K_e \Gamma}_{K_{(2)}} \mathbf{U}$$

Element 2 Potential In Global Coordinates

$$\Gamma^T K_e \Gamma = \frac{EA}{l}\begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}$$

$$K_{(2)} = \frac{EA}{l}\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Element 1 Stiffness

$$K_{(1)} = \frac{EA}{l}\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta & \sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & \sin\theta\cos\theta & \cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin^2\theta & -\sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Corresponding to:

$$\mathbf{U}_1 = \begin{bmatrix} U_1(t) \\ U_2(t) \\ U_5(t) \\ U_6(t) \end{bmatrix}$$

Expand to Complete Element Stiffness in Global Coordinates

$$K'_{(1)} = \frac{EA}{l}\begin{bmatrix} \cos^2\theta & -\sin\theta\cos\theta & 0 & 0 & -\cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta & 0 & 0 & \sin\theta\cos\theta & -\sin^2\theta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\cos^2\theta & \sin\theta\cos\theta & 0 & 0 & \cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin^2\theta & 0 & 0 & -\sin\theta\cos\theta & -\sin^2\theta \end{bmatrix}$$

$$\mathbf{U}^T = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \end{bmatrix}$$

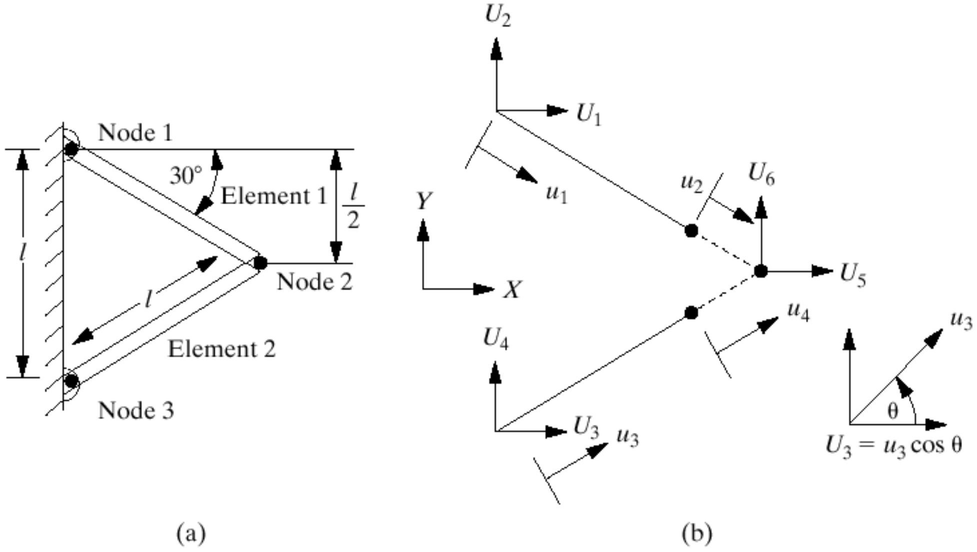
Compute a similar expression for element 2.

For θ =30° add Expanded local to get Global Stiffness

$$K\mathbf{U} =$$

$$\frac{EA}{l}\begin{bmatrix} 0.75 & -0.4330 & 0 & 0 & -0.75 & 0.4330 \\ -0.4330 & 0.25 & 0 & 0 & 0.4330 & -0.25 \\ 0 & 0 & 0.75 & 0.4330 & -0.75 & -0.4330 \\ 0 & 0 & 0.4330 & 0.25 & -0.4330 & -0.75 \\ -0.75 & 0.4330 & -0.75 & -0.4330 & 1.5 & 0 \\ 0.4330 & -0.25 & -0.4330 & -0.25 & 0 & 0.5 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix}$$

Apply the BC at Nodes 1 & 3



$$\Rightarrow U_1 = U_2 = U_3 = U_4 = 0$$

$$\Rightarrow K = \frac{EA}{l}\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} U_5 \\ U_6 \end{bmatrix}$$

Compute the Global Mass Matrix

$$M = \rho A l \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Compute the natural frequencies:

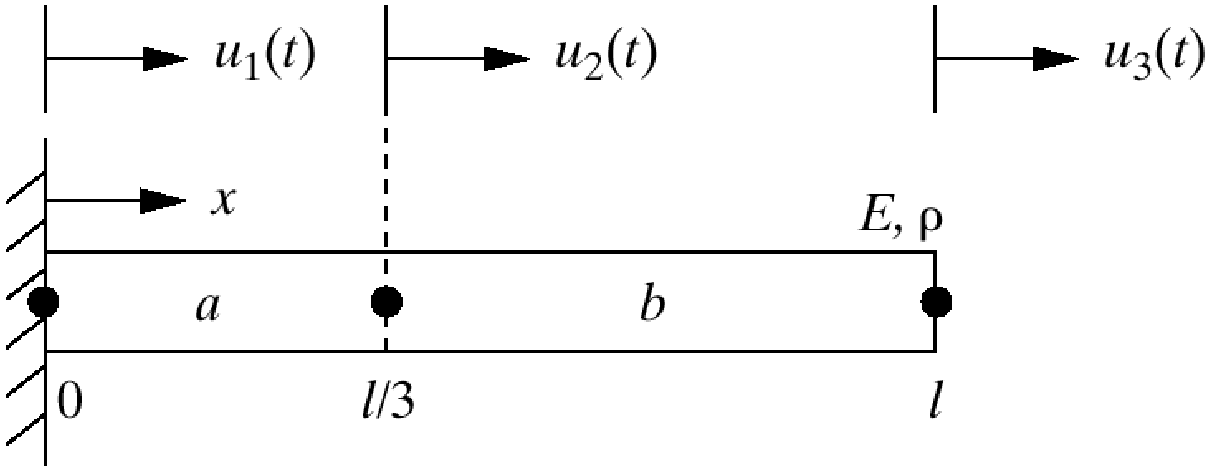
$$\omega_1 = \frac{1}{l}\sqrt{\frac{0.5E}{\rho}}, \omega_2 = \frac{1}{l}\sqrt{\frac{1.5E}{\rho}}$$

Several Other Features

- **Variable Grid Sizes**
- **Inconsistent Mass Matrices**
- **Model Reduction**
- **Damping**

Variable Grid Size: Example 8.2.2

Can use a variable grid size to increase accuracy without increasing the size of the model.



Comparison

FIX THIS SLIDE

	3 elements	2 non uniform Elements	Exact	
$\omega_1 =$	8092,	8367,	7998	
$\omega_2 =$	26458,	26460,	23994	rad/s

Variable grid size can accommodate non uniformities and works to increase accuracy and or reduce size

8.4 Lumped Mass Matrix

- **An alternative way to compute the mass matrix that avoids**
  - **having to compute integrals**
  - **off diagonal elements**
- **The energy approach we have used is call the “consistent mass matrix”**
- **The simplified version we are about to compute is also called the “inconsistent mass matrix**

Bar Element

For two elements, lump half the mass at each element, for 3 elements, a third

$$M_{2\text{ elements}} = \frac{\rho A l}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M_{3\text{ elements}} = \frac{\rho A l}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Beam Element

Divide mass among TRANSVERSE coordinates

$$M = \frac{\rho A l}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A singular matrix!

8.6 Model Reduction

- **To increase accuracy, the order of the system often makes computation (simulation or modal computations) prohibitive.**
- **Thus many techniques have been developed to eliminate coordinates in meaningful ways.**

Removing "insignificant" terms via partitioning:

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_1 \\ \ddot{\mathbf{u}}_2 \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$$

Coordinates deemed not significant

Re-Arranged so that those with low mass or thought to be insignificant are placed last in the model

Energy Expressions:

$$V = \frac{1}{2}\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}^T \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

$$T = \frac{1}{2}\begin{bmatrix} \dot{\mathbf{u}}_1 \\ \dot{\mathbf{u}}_2 \end{bmatrix}^T \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}_1 \\ \dot{\mathbf{u}}_2 \end{bmatrix}$$

Next assume f_2 is zero, which implies that the variation of V with u_2 is Zero (actually an assumption)

Constraint that *V* does not change with respect to $\mathbf{u}_2$ yields:

$$\frac{\partial V}{\partial \mathbf{u}_2}\left(\mathbf{u}_1^T K_{11}\mathbf{u}_1 + \mathbf{u}_1^T K_{12}\mathbf{u}_2 + \mathbf{u}_2^T K_{21}\mathbf{u}_1 + \mathbf{u}_2^T K_{22}\mathbf{u}_2\right) = 0$$

$$\Rightarrow \mathbf{u}_2 = -K_{22}^{-1}K_{21}\mathbf{u}_1$$

Allows us to remove $\mathbf{u}_2$ in terms of $\mathbf{u}_1$ and suggests the Transformation: $Q = \begin{bmatrix} I \\ -K_{22}^{-1}K_{21} \end{bmatrix}$

Computing the Reduced Model

Letting u = $Q\mathbf{u}_1$ in the equation of motion and multiplying by Q^T yields:

$$Q^T M Q \ddot{\mathbf{u}}_1 + Q^T K Q \mathbf{u}_1 = Q^T \mathbf{f} \quad \text{of dimension of } \mathbf{u}_1$$

where:

$$Q^T M Q = M_{11} - K_{21}^T K_{22}^{-1} M_{21} - M_{12} K_{22}^{-1} K_{21} + K_{21}^T K_{22}^{-1} M_{22} K_{22}^{-1} K_{21}$$

$$Q^T K Q = K_{11} - K_{12} K_{22}^{-1} K_{21}$$

Note that Q does not preserve eigenvalues but this lower order system does contain some of the original dynamics.

Example 8.6.1

$$M = \frac{1}{420}\begin{bmatrix} 312 & 54 & 0 & -13 \\ 54 & 156 & 13 & -22 \\ 0 & 13 & 8 & -3 \\ -13 & -22 & -3 & 4 \end{bmatrix}, \quad K = \begin{bmatrix} 24 & -6 & 0 & 6 \\ -6 & 12 & -6 & -6 \\ 0 & -6 & 10 & 4 \\ 6 & -6 & 4 & 4 \end{bmatrix}$$

$$M_{11} = \frac{1}{420}\begin{bmatrix} 312 & 54 \\ 54 & 156 \end{bmatrix}, M_{12} = \frac{1}{420}\begin{bmatrix} 0 & -13 \\ 13 & -22 \end{bmatrix} = M_{21}^T, M_{22} = \frac{1}{420}\begin{bmatrix} 8 & -3 \\ -3 & 4 \end{bmatrix}$$

$$K_{11} = \begin{bmatrix} 24 & -6 \\ -6 & 12 \end{bmatrix}, K_{22} = \begin{bmatrix} 10 & 4 \\ 4 & 4 \end{bmatrix}, K_{12} = \begin{bmatrix} 0 & 6 \\ -6 & -6 \end{bmatrix} = K_{21}^T$$

The Reduced Model is then:

$$Q^T M Q = \begin{bmatrix} 1.021 & 0.198 \\ 0.198 & 0.236 \end{bmatrix}, \qquad Q^T K Q = \begin{bmatrix} 9 & 3 \\ 3 & 3 \end{bmatrix}$$

$$\lambda_1^{rom} = 6.981, \qquad \lambda_2^{rom} = 12.916$$

$$\lambda_1 = 6.965, \qquad \lambda_2 = 12.196,$$

$$\lambda_3 = 230.934, \qquad \lambda_4 = 3.833 \times 10^3$$