

**PROBABILISTIC ENGINEERING  
ANALYSIS AND DESIGN  
(446.779)**

**INSTRUCTOR: PROF. YOUN, BYENG DONG  
OFFICE: BUILDING 301, ROOM 1514**

**COURSE SYLLABUS**

## Syllabus

|                            |   |  |                 |                         |   |         |       |
|----------------------------|---|--|-----------------|-------------------------|---|---------|-------|
| Course ID                  | 446.779   | Lecture No.  | 001             | Course Name             | Probabilistic Engineering Analysis and Design | Credits | 3     |
| Instructor                 | Professor Youn Byeng Dong   |  |                 | Website: shrm.snu.ac.kr |   |         |       |
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| TA                         | Keun Su Kim (E-mail: keunshu@gmail.com, Tel: 880-1664, Office: 301-215)   |  |                 |                         |   |         |       |
| 1. Objective               | The course covers three important issues encountered in practical engineering fields: uncertainty characterization, probabilistic engineering (or reliability) analysis, design under uncertainty, and system management under uncertainty (or health monitoring). Probabilistic engineering analysis includes advanced topics in statistics, uncertainty characterization, and test/simulation-based probabilistic engineering (or reliability) analysis. Design under uncertainty includes probability sensitivity analysis, surrogate modeling, and advanced methodologies for design under uncertainty. Some health diagnostics and prognostics techniques are briefly introduced for the purpose of system management under uncertainty. |  |                 |                         |   |         |       |
| 2. Textbook and References | <ul style="list-style-type: none"> <li>● Youn, B.D. and Hu, C., <i>Engineering Analysis and Design under Uncertainty</i>, SNU Print, 2012.</li> <li>● Haldar, A., and Mahadevan, S., <i>Probability, Reliability, and Statistical Methods in Engineering Design</i>, John Wiley &amp; Sons Inc., 2000.</li> <li>● Arora, J.S. <i>Introduction to Optimum Design</i>, Second Edition, Elsevier, 2004.</li> <li>● Myers and Montgomery, <i>Response Surface Methodology</i>, Wiley, 1995.</li> <li>● G. Vachtsevanos G, et al., <i>Intelligent Fault Diagnosis and Prognosis for Engineering Systems</i>, 1st edition. Hoboken, New Jersey, USA: John Wiley &amp; Sons, 2006.</li> </ul>  |  |                 |                         |   |         |       |
| 3. Evaluation method       | Homework  | Exam I (10.19)   | Exam II (11.25) | Project I               | Project II                                    |         | Total |
|                            | 20%   | 20%  | 20%             | 20%                     | 30%   |         | 110%  |
| 4. Lecture Plan            | Wk  | Contents   |                 |                         |   |         |       |
|                            | 1   | Course introduction; Concepts of uncertainty, reliability and risk; Basic probability theory |                 |                         |   |         |       |
|                            | 2   | Graphical methods for exploratory data analysis  |                 |                         |   |         |       |
|                            | 3   | Uncertainty characterization   |                 |                         |   |         |       |
|                            | 4   | Definition of reliability; Reliability analysis (time-independent);                          |                 |                         |   |         |       |
|                            | 5   | Numerical methods for probabilistic engineering analysis (MCS, FORM methods)                 |                 |                         |   |         |       |
|                            | 6   | Case studies of reliability analysis (cellular phone, LCD, and others)                       |                 |                         |   |         |       |
|                            | 7   | Reliability modeling (time-dependent);   |                 |                         |   |         |       |
|                            | 8   | Exam I & Project I review  |                 |                         |   |         |       |
|                            | 9   | Accelerated life testing; Accelerated life testing;  |                 |                         |   |         |       |
|                            | 10  | Bayesian analysis;   |                 |                         |   |         |       |
|                            | 11  | Design optimization review;  |                 |                         |   |         |       |
|                            | 12  | Design optimization review;  |                 |                         |   |         |       |
|                            | 13  | Design under uncertainty (methodology)   |                 |                         |   |         |       |
|                            | 14  | Design under uncertainty (formulation; numerical methods); Exam II                           |                 |                         |   |         |       |
| 15                         | Prognostics and Health Management (PHM): reasoning function; Course review  |  |                 |                         |   |         |       |
| 5. Consideration           | <ul style="list-style-type: none"> <li>- Prerequisites : Engineering Statistics, Design Optimization</li> <li>- 2 open-book exams and one individual project</li> </ul>   |  |                 |                         |   |         |       |
| 6. Rules                   | All students are presumed upon enrollment to have an understanding of the Honor System. Academic dishonesty by a student will be treated in accordance with the SNU procedures. A score of "0" can be assigned for the corresponding test/assignment and/or a course grade of 'F' can be assigned.  |  |                 |                         |   |         |       |

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**Chapter 7. Design under Uncertainty**

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## CHAPTER 1. INTRODUCTION

### 1.1 Instances of Catastrophic Failures



Chernobyl disaster, April 26 1986  
 Adverse events due to **Faulty operation**  
 Consequence: **4K deaths & 600K contaminated**, hundreds of billion dollar



I-35 Bridge Failure, August 2007  
 Adverse events due to **Faulty design and maintenance**  
 Consequence: 13 deaths, 145 injured, **\$2 Billion annual loss**



Wind turbine failure, Feb. 22 2008  
 Adverse events due to **Faulty design (controller)**  
 Consequence: **Collapse of whole wind turbine**



Power transformer failure, July 6, 2002  
 Adverse events due to **faulty bushing design**  
 Consequence: **\$5 million property & business loss**



UPS Flight 1307 fire, Feb. 7, 2006,  
 Adverse events possibly due to **faulty Lithium-ion battery design**  
 Consequence: **3 injured, loss of whole airplane**

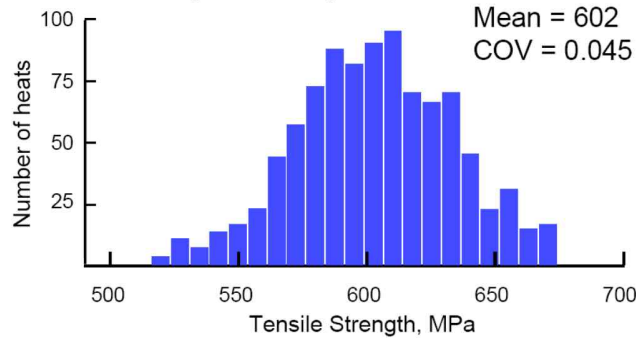
### Research Questions:

- Q1. Is it possible to predict engineering performances on a reliable basis, i.e., failure rates and reliability?
- Q2. Is it possible to design engineered systems reliable and robust under various uncertainties?
- Q3. What technologies make it possible to a reliable and robust design under uncertainty?

**1.2 Uncertainty (inferable) or variability (measurable)**




**1.2.1 Sources of uncertainty or variability**

- **Physical uncertainty or variability:** Actual variability (or uncertainty) of physical quantities, such as loads, material properties, and dimensions. This variability can be described in terms of probability distributions or stochastic processes.

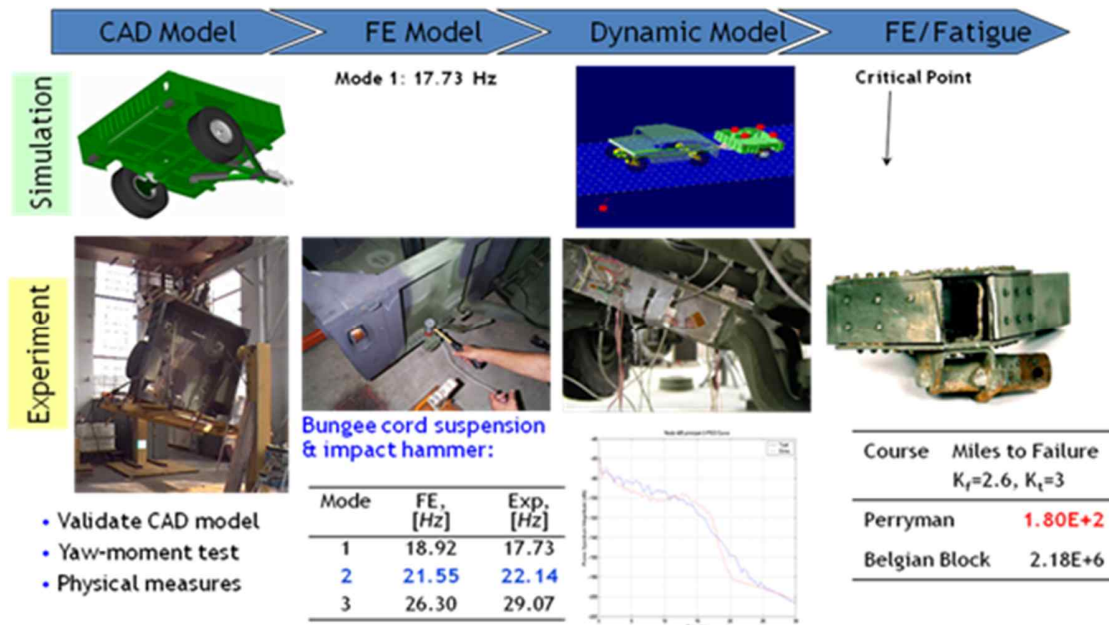


**Figure 1.1:** Histogram of Tensile Strength, MPa

- **Statistical uncertainty or variability:** This uncertainty arises solely as a result of the lack of information. Data must be collected to build a probabilistic model of the physical variability of physical quantities. The probabilistic model is composed of a probability distribution and its parameters. When the amount of data is small, the distribution and parameters are considered to be uncertain.
- **Model uncertainty or variability:** This source of uncertainty occurs as a result of simplifying assumptions, unknown boundary conditions, and their interactions which are not included in the model. To minimize this uncertainty, Verification and Validation (V&V) is quite important.

| Uncertainty Sources  | Meaning   |
|--|---|
| <b>Physical Uncertainty</b>     | <ul style="list-style-type: none"> <li>- <b>Inherent variation</b> in physical quantity</li> <li>- Description by probability distribution</li> <li>- Ex) material property, manufacturing tolerance, loading condition, boundary condition, ...</li> </ul> |
| <b>Statistical Uncertainty</b>  | <ul style="list-style-type: none"> <li>- <b>Imprecise statistical estimation</b> (probability distribution type, parameters, ...)</li> <li>- Only depending on the sample size and location</li> <li>- Ex) lack of data, improper sampling</li> </ul>       |
| <b>Modeling Uncertainty</b>     | <ul style="list-style-type: none"> <li>- Uncertainty from <b>invalid modeling</b></li> <li>- Ex) improper approximation, inaccurate boundary condition, ...</li> </ul>  |

## Uncertainty is ubiquitous!



**Figure 1.2:** Model Uncertainty in Fatigue Analysis

### Homework 1: Sources of uncertainty in a vibration problem

Let us consider an undamped system with a lumped mass and spring. The motion behavior of the system can be ideally modeled using a second-order ordinary differential equation as

$$my''(t) + ky(t) = 0; \quad y(0) = -1.5, \quad y'(0) = 0$$

where  $m$  and  $k$  are the mass and spring coefficient of the system, respectively. According to the manufacturer of the system, the mass and spring coefficient are believed to be 10 kg and 1000 N/m, respectively. At time  $t = 1$  second, ten experimental tests show a set of  $y$  data as (1.1202, 1.2474, 1.3472, 1.1767, 1.3113, 1.2890, 1.3171, 1.1244, 1.1421, 1.2539). Answer the following questions:

- (1) Please explain why experimentally measured  $y$  values are scattered.
- (2) Identify all possible sources of uncertainties involved in this problem.
- (3) Also, provide possible reasons for what causes the difference between experimental and analytical  $y$  values.

### 1.2.2 Types of uncertainty

- **Aleatory uncertainty:**

It is an uncertainty with the sufficient size of data, referred to as objective uncertainty or irreducible uncertainty. It means the degree of uncertainty is irreducible with more relevant data.

**Table 1.1:** Aleatory Uncertainty (or Objective Uncertainty)

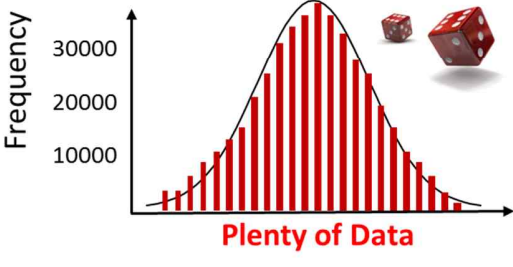
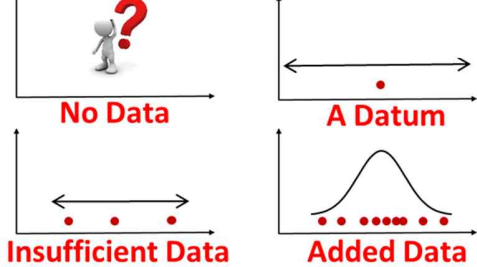
```
>> thickness_a = normrnd(1,0.05,100000,5); mean(thickness_a), std(thickness_a)
ans =
    1.0003    1.0001    1.0000    1.0001    1.0001
ans =
    0.0501    0.0499    0.0501    0.0501    0.0500
```

- **Epistemic uncertainty:**

It is mainly an uncertainty due to the lack of data, referred to as subjective uncertainty or reducible uncertainty. It means the degree of uncertainty is reducible with more relevant data.

**Table 1.2:** Epistemic Uncertainty (or Subjective Uncertainty)

```
>> thickness_e = normrnd(1,0.05,10,5); mean(thickness_e), std(thickness_e)
ans =
    0.9995    1.0150    1.0217    1.0024    0.9763
ans =
    0.0514    0.0410    0.0375    0.0394    0.0546
```

| Aleatory Uncertainty  | Epistemic Uncertainty  |
|---|--|
| <ul style="list-style-type: none"> <li>- <b>Inherent randomness</b> in physical properties</li> <li>- Latin alea: The rolling of dice</li> <li>- <b>Irreducible</b> with the acquisition of additional data</li> <li>- Ex) material properties, product geometry, loading condition, boundary condition, ...</li> </ul>  | <ul style="list-style-type: none"> <li>- Due to the <b>lack of knowledge</b></li> <li>- <b>The smaller sample size, the wider confidence interval</b> in statistical parameter estimation</li> <li>- <b>Reducible</b> with additional information</li> <li>- Ex) manufacturing tolerance, material property, expert opinion in case of knowledge absence, ..</li> </ul>  |

| Epistemic Uncertainty   |  |
|---|--|
| Recognized Uncertainty  | Unrecognized Uncertainty   |
| <ul style="list-style-type: none"> <li>- “Know what I don’t know” type</li> <li>- Recognized uncertainty <b>without accurate information</b></li> <li>- Come from a <b>conscious decision making</b></li> <li>- Ex) surrogate model error, boundary condition modeling, plastic/fatigue material property, ...</li> </ul> | <ul style="list-style-type: none"> <li>- “Don’t know what I don’t know” type</li> <li>- Existent in physical quantity but <b>unrecognized</b> uncertainty</li> <li>- Come from being incognizant of knowledge incompleteness or necessity of knowledge</li> <li>- Ex) mistakes, blunders, errors, ...</li> </ul> |

1.2.3 Probability Theory

- *Frequentist probability theory – a conventional probability theory*
- *Bayesian probability theory*

**1.3 Reliability, Risk, Availability, Maintainability, Durability, etc.**

• **Reliability:**

The ability of a **system** or component to perform its required functions under stated conditions for a specified period of time. Commonly used techniques include: Accelerated (life) testing, Weibull analysis, Simulation-based analysis under



uncertainty, Product qualification tests, Failure mode and effects analysis (FMEA), Fault tree analysis, Root cause analysis, etc.

- **Risk:**

A measure of the **loss** due to **potential failures** of social, built, and natural systems (or components). Related research topics include: qualitative risk analysis, quantitative risk analysis, risk management, etc.

- **Availability:**

The degree to which a **system, subsystem**, or equipment is in a specified operable and committable state at the start of a mission, often described as a ratio of expected value of the uptime of a system to the aggregate of the expected values of up and down time.

- **Maintainability:**

The degree of the ease with which a product can be **maintained** in order to: (i) correct defects, (ii) meet new **requirements**, (iii) make future maintenance easier, or (iv) cope with a changed environment.

- **Durability:**

The ability to perform over a specified period of time while withstanding potential hazards (wear, corrosion, fatigue, etc.).

## 1.4 Introduction to Risk Assessment

### 1.4.1 Definition

A measure of the **loss** due to **potential failures** of social, built, and natural systems (or components).

**The loss is the consequence of potential failure (or the adverse consequence) given in the form of human fatality, property loss, monetary loss, etc.** For a given failure event, the loss is believed to be known or predictable in most situations.

**The potential failure of social, built, and natural systems (or components) is expressed as the probability or frequency of the failure (or hazard) per unit time or space.**

- Risk by natural systems: Hurricanes, floods, wild fires, etc.
- Risk by social systems: company bankruptcy, subprime mortgage, etc.
- Risk by built systems: bridges, space shuttle, nuclear plant, etc.



(a) City Floods due to Hurricane Katarina

(b) Subprime mortgage

(c) Interstate 35W

**Figure 1.3:** Potential failures and their risk

## Risk is everywhere!

From an engineering point of view, risk is the sum of the products of the consequences and their probabilities of engineered system (or component) failures as

$$R = \sum_{i=1}^n f_i c_i \quad (1)$$

**Example:** For instance, according to the U.S. Transportation, 12 million vehicle accidents are reported in 2007; 1 in 320 accidents resulted in death. Assuming average loss of US \$500,000 per death and US \$25,000 of property loss per accident involving fatality, calculate both fatality and economic risk values.

### 1.4.2 Approaches for risk prediction

- **Qualitative risk assessment**

It is easier to perform a qualitative risk analysis because it does not require gathering data. **This approach uses linguistic scales**, such as low, medium, and high. A risk assessment matrix is formed which characterizes risk in form of the frequency (or likelihood) of the loss versus potential magnitudes (amount) of the loss in qualitative scales. Because this type of analysis does not need to rely on actual data and probabilistic treatment of such data, the analysis is far simpler and easier to use and understand, but is extremely subjective. The matrix is then used to make policy and risk management decisions in a conceptual design stage.

**Table 1.3:** Qualitative Risk Assessment Matrix

|               |                     | Catastrophic | Critical | Marginal | Negligible |
|---------------|---------------------|--------------|----------|----------|------------|
| Very frequent | $> 0.5$             | H            | H        | H        | I          |
| Frequent      | $0.5 - 10^{-1}$     | H            | H        | I        | L          |
| Probable      | $10^{-1} - 10^{-2}$ | H            | H        | L        | L          |
| Occasional    | $10^{-2} - 10^{-4}$ | H            | H        | L        | L          |
| Improbable    | $10^{-4} - 10^{-6}$ | H            | I        | L        | T          |
| Rarely        | $< 10^{-6}$         | I            | I        | T        | T          |

H (high risk); I (intermediate risk); L (low risk); T (trivial risk)

- **Quantitative risk assessment (Or Probabilistic Risk Analysis)**

The quantitative risk assessment attempts to estimate the risk in form of the frequency (or likelihood) of the loss versus potential magnitudes (amount) of the loss in quantitative scales. Risk-relevant data must be involved in estimating the frequency of the loss and potential amount of the loss. Risk-relevant data include field data, test data, and other evidences. So, this approach is clearly the preferred approach when risk-relevant data exist to estimate the probability of failure and its consequence.

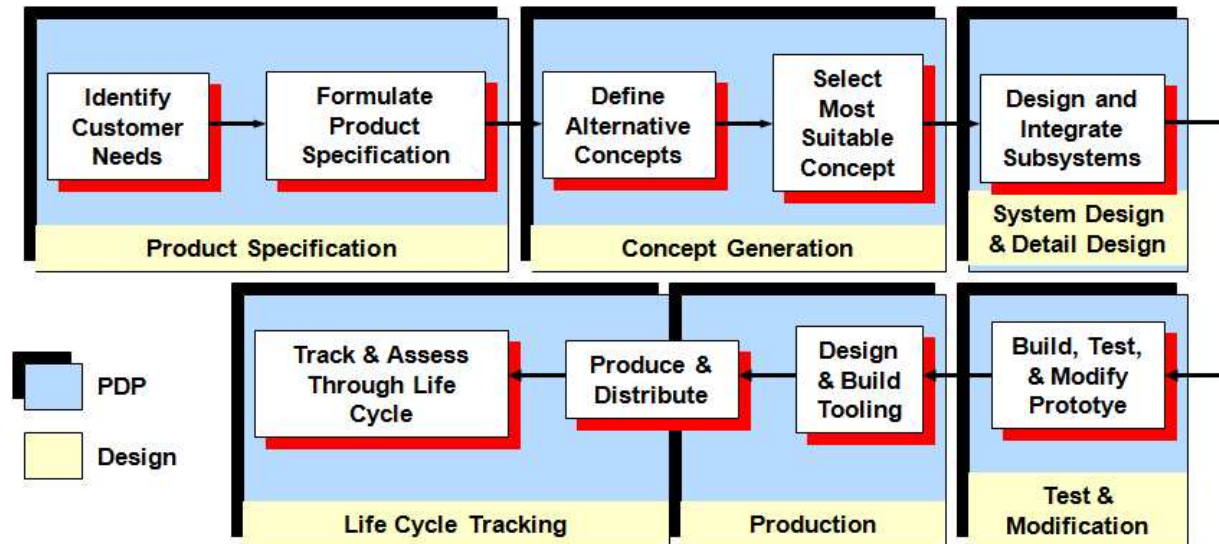
$$\begin{aligned}
 \text{Risk Value} = & \text{Probability of powertrain system failure} \times \text{Consequence (1)} \\
 & + \text{Probability of steering system failure} \times \text{Consequence (2)} \\
 & + \text{Probability of central controller failure} \times \text{Consequence (3)} \\
 & + \text{Probability of air-conditioning failure} \times \text{Consequence (4)} \\
 & + \text{Probability of airbag sensor failure} \times \text{Consequence (5)} \\
 & \dots \\
 & + \text{Probability of tire failure} \times \text{Consequence (n)}
 \end{aligned}$$

## 1.5 Introduction to Engineering Design

Four important factors for success of a product:

- **Performance, price, and quality** from the customer's perspective
- **Time** needed to bring product to the market (Time-to-market)
- **Cost** of product development
- **Cost** to manufacture the product

### 1.5.1 Product Development Process (PDP)



**Figure 1.4** Product Development Process (PDP)

Duration of each program phase depends on product complexity, technology maturity, and specific requirements.

### 1.5.2 Product Design

- Concept generation
  - Define subsystems
  - Find existing concepts (patent survey; Triz)
  - Generate new concepts (Pugh chart)
  - Eliminate poor ideas (Pugh chart)
  - Prepare design proposals
- System Design
  - Need for subsystems – modular concept (product family design)
  - Define the characteristics of subsystems
  - Allocate system design
  - Provide interfacing
- Detail Design
  - Define the characteristics of components
  - Providing interfacing
  - Design for manufacturing
  - Design for assembly
  - Design for maintenance
  - Design optimization

**Homework 2: Reliability engineering problem**

Read the following article carefully.

[http://webcache.googleusercontent.com/search?q=cache:http://www.squaretrade.com/htm/pdf/cell\\_phone\\_comparison\\_study\\_nov\\_10.pdf](http://webcache.googleusercontent.com/search?q=cache:http://www.squaretrade.com/htm/pdf/cell_phone_comparison_study_nov_10.pdf)

Generally, smartphone manufacturers conduct product qualification tests with five samples before shipping the phones to customers. The phone manufacturer begins to ship the phones when the five phones pass the test. The qualification test involves structural, functional, environmental, and accelerated life tests. Discuss why the qualification tests cannot guarantee defect-free products and how this difficulty can be taken care of. Assume that the phone manufacturer cannot increase sample size because of its limited resources.

**Homework 3: Product Development Process**

Review the videos for your understanding of product development process.

1. Customer needs for product specification  
[http://www.youtube.com/watch?feature=player\\_detailpage&v=FA9Yzunsrlc](http://www.youtube.com/watch?feature=player_detailpage&v=FA9Yzunsrlc)
2. Concept generations  
- <https://www.youtube.com/watch?v=7KVbRWF1vtY>
3. Detail design (w/ suspension arm)  
<http://www.youtube.com/watch?v=OHVXvv5GAcI>  
<http://www.youtube.com/watch?NR=1&feature=endscreen&v=qrhbOeJaaxg>
4. Discussion  
Study “embodiment design” and “modular design” and report each with 250 words at minimum in English.

## 1.6 Probabilistic Description of Engineering Performances

### 1.6.1 Description of engineering performance:

Probabilistic performance  
=  $Y(\mathbf{X})$ ;  $\mathbf{X}$  is a vector of uncertainty that affects system performance

Probability ( $Y(\mathbf{X}) \leq Y_U$ ) = Reliability  
= Probability of safety (=success)  
=  $1 -$  Probability of failure

Probability ( $Y_L \leq Y(\mathbf{X}) \leq Y_U$ ) = Robustness

### 1.6.2 Challenges in probabilistic engineering analysis and design:

One of the primary challenges is “how to collect system performance data under uncertainty?”

Engineering analysts and designers want to precisely predict the probability of system performance while designing engineered systems (or components), prior to manufacturing them. However, it is extremely difficult to predict system performance before producing and testing the systems (or components). To predict the probability<sup>1</sup> of a system (or component) performance, numerous testing (or warranty) data must be collected after releasing the system to the market.

**Table 1.4:** Challenge in predicting the probability of system performance

“The designer must predict system performance prior to making design decisions, and data cannot be obtained from physical tests of the system prior to construction of the system.” by George A. Hazelrigg, an NSF program manager.

“It appears the new emission technology may have affected engine performance and quality, as customer satisfaction with both factors dropped significantly in 2005 ... This is a common pattern whenever new technologies are introduced in an industry, and the assumption is that the scores for these engines will improve over time,” by Etchelle, senior manager of commercial vehicle research at J.D. Power and Associates.

**Homework 4:** *Essay to describe the challenges in system design*

Above is the discussion about the challenges in system design. In this work, assume you are an engine designer in the company to make sure the engine performance, lifetime, and reliability before shipping the engines to your

<sup>1</sup> Probability is the likelihood or chance that something is the case or will happen.

customers. Given a four-month period for the engine design and performance (including life/reliability) testing, what challenges are expected for engine design? Write your essay using at least 250 words in English.

Other challenges include:

- Uncertain factors in manufacturing and operational processes.
- Fidelity in techniques to predict system performance and reliability (or probability of safety).
- Effective design and management of engineered systems under uncertainty.
- Precise estimation of the consequence of potential failures.

## Data = Money, Time, Know-how

### 1.6.3 System performance data under uncertainty (Example: Automotive Industries):

- *Reliability & Owner Satisfaction Data from ConsumerReports.org*

<http://www.consumerreports.org/cro/cars/used-cars/reliability/best-worst-in-car-reliability-1005/overview/index.htm>

- Overall, Asian models still dominate in reliability, accounting for 34 of the 39 models in the **Most reliable new car** list. Thirty-one are Japanese and three are South Korean.
- Despite Toyota's problems, the automaker still ranks third overall in reliability, behind only Honda and Subaru, with 17 models in the best list. Honda has seven with a smaller model lineup.
- Only four domestic models made the Most reliable list: the Ford Fusion, Mercury Milan, Pontiac Vibe, and the two-wheel-drive Ford F-150 with the V6 engine. U.S. makes, however, account for almost half the models-- 20 of 44--on the **Least reliable** list. There are 13 from GM, 6 from Chrysler, and 1 from Ford.
- European makes account for 17 models on the Least reliable list. This includes six each from Mercedes-Benz and Volkswagen/Audi.

- *Initial Quality Study (IQS) (with 90 days of ownership) from J.D. Power and Associates*

<http://www.jdpower.com/press-releases/pressrelease.aspx?id=2007088>

- *National Highway Traffic Safety Administration (NHTSA):*

<http://www.nhtsa.gov/>

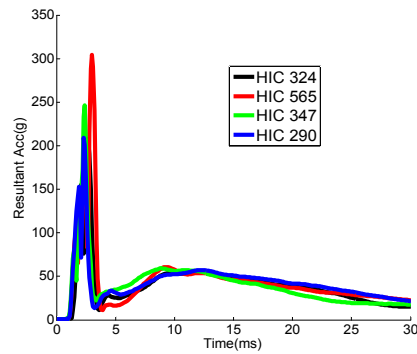
- *Top Safety Picks from Insurance Institute for Highway Safety (IIHS):*

<http://www.iihs.org/ratings/default.aspx>

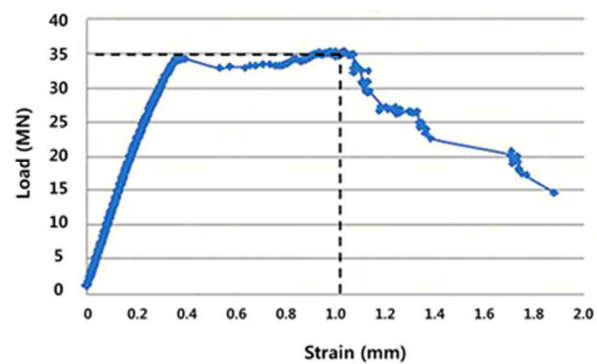
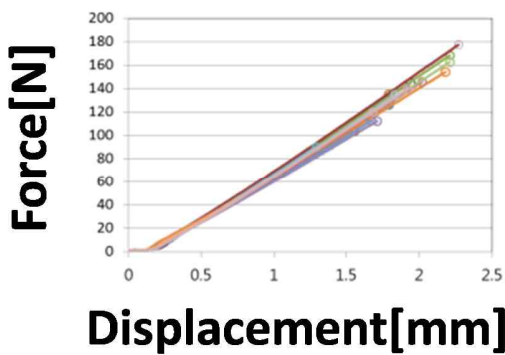
- *Warranty data*

- Testing and measurement data**

Use material test, reliability test, life test, measured tolerance data, etc. They can be found in Standard references disclosed by DOD, NIST, and other professional societies (ASME, IEEE, etc.) and companies.

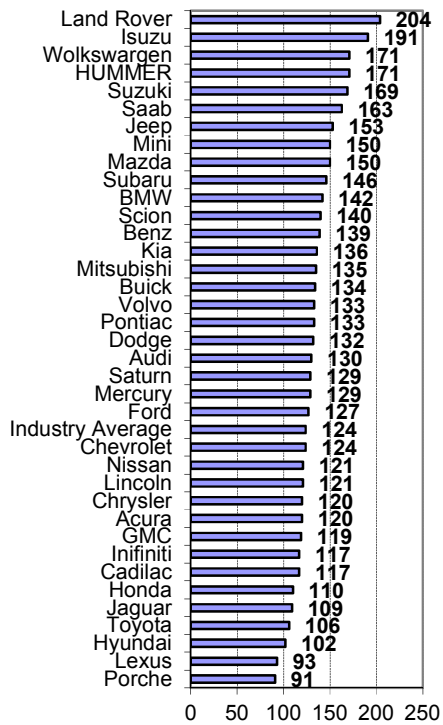


**Tensile Test**



- Customer data from survey, clinic testing, and user-generated contents**





**Figure 1.5:** IQS from J.D. Power and Associates

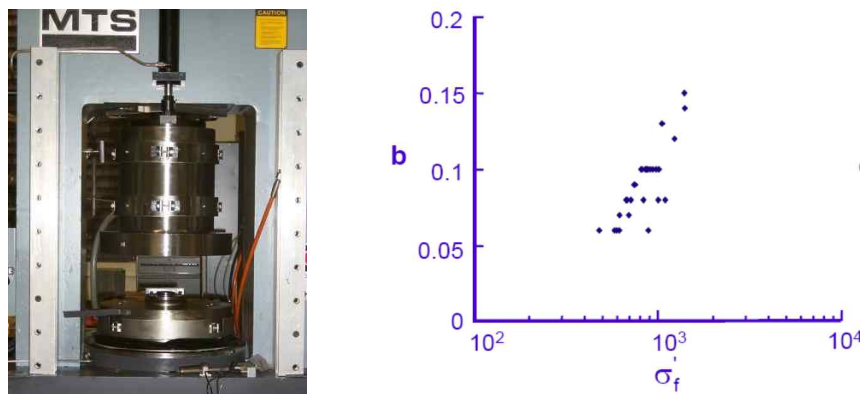
#### 1.6.4 Key elements for probabilistic description of engineering performances (Probability of Failure or inversely Reliability)

- *Variability in manufacturing tolerances, use conditions, etc.*
- *Uncertainty (or variability) and bias in modeling and test results*
- *Complexity in engineering mechanics (test-based, model-based, sensor-based, or hybrid-based)*
- *Uncertainty propagation (or probabilistic engineering analysis)*

## CHAPTER 2 BASIC PROBABILITY THEORY

### 2.1 Sample Space

For any physical quantity, a set of data can be obtained through physical tests or surveys under a homogeneous condition. The set of all possible outcomes of such tests is called the **sample space (or random space)  $\Omega$**  and each individual outcome is a sample point. For the example of fatigue tests, the sample data can be obtained about the physical quantities, as shown in Figure 2.1. **The sample space can be described by a probability distribution (or mass) function or histogram.**



$$\text{Strain-life equation: } \frac{\Delta \varepsilon}{2} = \frac{\sigma'_f}{E} (2N_f)^b + \varepsilon'_f (2N_f)^c : \text{Low-cycle fatigue}$$

**Figure 2.1:** Fatigue Tests and Sample Data Set (Courtesy of Prof. Darrell F. Socie, UIUC, Probabilistic Fatigue, 2005)

### 2.2 Axioms and Theories of Probability

Axiom 1. For any event  $E$

$$0 \leq P(E) \leq 1 \quad (2)$$

where  $P(E)$  is the probability of the event  $E$ .

Axiom 2. Let the sample space be  $\Omega$ . Then,

$$P(\Omega) = 1 \quad (3)$$

Axiom 3. If  $E_1, E_2, \dots, E_n$  are mutually exclusive events then

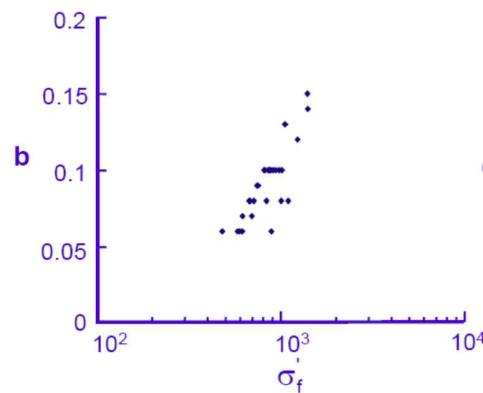
$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i) \quad (4)$$

Some Useful Theorems

$$\begin{aligned}
 P(\bar{E}) &= 1 - P(E) \\
 P(\emptyset) &= 0 \\
 P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\
 P(E_1 | E_2) &= \frac{P(E_1 \cap E_2)}{P(E_2)}: \text{ Conditional Probability} \\
 P(E_1 \cap E_2) &= P(E_1)P(E_2) \text{ if } E_1 \text{ and } E_2 \text{ are independent.}
 \end{aligned}
 \tag{5}$$

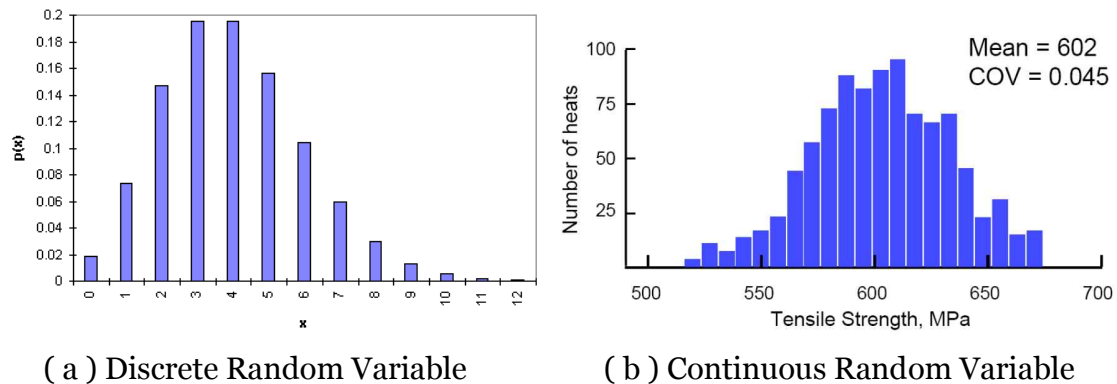
### 2.3 Random Variables

A **random variable** is a function which maps events in the sample space  $\Omega$  into the real value  $R$  where the outcomes of the event can be real or integer, continuous or discrete, success or fail, etc. The random variable is often denoted as  $X: E \rightarrow R$ .



The outcome of an event need not be a number, for example, the outcome of coin toss can be either “head” or “tail”. However, we often want to represent outcomes as numbers.

- **Discrete random variable** (Fig. 2.2a):  
The outcome of an experiment is discrete. For example, specimen tensile tests with 10 kN are conducted one hundred times. Each tensile test employs 20 specimens. Let say, the random variable  $X$  is the number of failed specimens in each tensile test. Then,  $X$  is a discrete random variable.
- **Continuous random variable** (Fig. 2.2b):  
The outcome of an experiment is continuous. For example, an LED light bulb is tested until it burns out. The random variable  $X$  is its lifetime in hours.  $X$  can take any positive real value, so  $X$  is a continuous random variable. Similar examples include the tensile strength of specimen tensile tests.



( a ) Discrete Random Variable

( b ) Continuous Random Variable

**Figure 2.2:** Random Variable

## 2.4 Univariate Distributions

- To understand probability distributions relevant to engineering applications
- To investigate statistical properties of probability distributions
- To make use of Matlab statistical toolbox

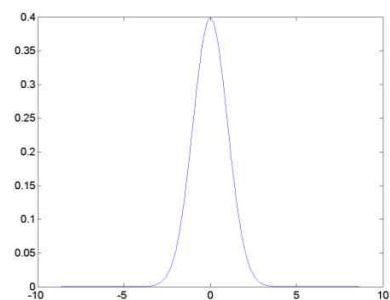
Let  $X$  be a random variable in an engineering application. The probability density function (PDF) and cumulative distribution function (CDF) of  $X$  are denoted by  $f_X$  and  $F_X$ , respectively. Their relationship is  $f_X(x) = \frac{\partial}{\partial x} F_X(x)$ .

### Normal Distribution (or Gaussian Distribution)

$$y(x; \mu, \sigma) = f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

```
>> x=[-10:0.1:10];
>> y=normpdf(x,0,1);
>> plot(x,y)
```

- ✓ Symmetric distribution, skewness=0, kurtosis=3
- ✓ Central limit theorem states that any distribution with finite mean and standard deviation tends to follow normal distribution
- ✓ Special case of chi-squared distribution and gamma distribution
- ✓ Dimension of fabricated part
- ✓ Uncontrolled random quantities (i.e., White Gaussian noise)

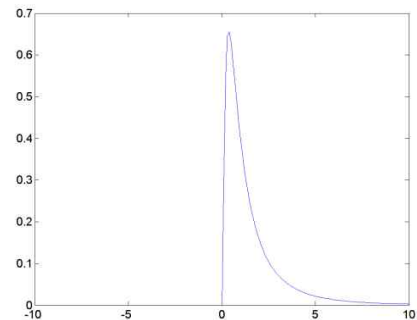


### Lognormal Distribution

$$y(x; \mu, \sigma) = f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

```
>> x=[-10:0.1:10];
>> y=lognpdf(x,0,1);
>> plot(x,y)
```

- ✓ Limited to a finite value at the lower limit
- ✓ Positively skewed
- ✓ Strengths of materials, fracture toughness

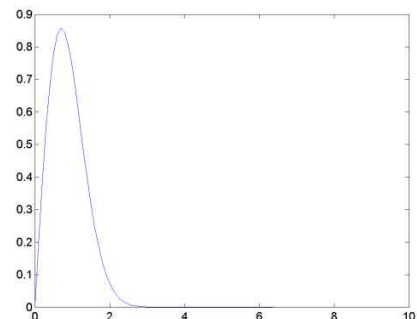


### Weibull Distribution

$$y(x; \nu, k, a) = f_X(x) = \frac{k-a}{\nu-a} \left( \frac{x-a}{\nu-a} \right)^{k-1} e^{-\left( \frac{x-a}{\nu-a} \right)^k}, \quad \text{2 parameter Weibull if } a = 0$$

```
>> x=[0:0.1:10];
>> y=wblpdf(x,1,2);
>> plot(x,y)
```

- ✓  $k$  is a shape parameter;  $\nu$  is a scale parameter;  $a$  is a location parameter
- ✓ Originally proposed for fatigue life
- ✓ Used in analysis of systems with weakest link
- ✓ Wear, fatigue, and fracture

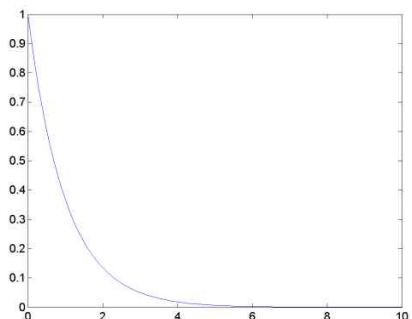


### Exponential Distribution

$$y(x; \mu) = f_X(x) = \frac{1}{\mu} e^{-\frac{x-a}{\mu}}$$

```
>> x=[0:0.1:10];
>> y=expdf(x,1);
>> plot(x,y)
```

- ✓  $a$  is a location parameter;  $\mu$  is a scale parameter
- ✓ Used to model data for time between failures with a constant failure rate
- ✓ Called as “memoryless random distribution”
- ✓ Continuous version of Poisson distribution to describe the number of occurrences per unit time

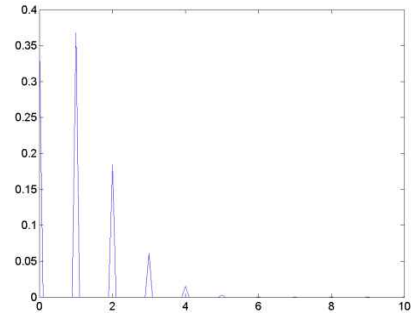


### Poisson Distribution (Discrete)

$$y(x; \lambda) = f_X(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

```
>> x=[0:0.1:10];
>> y=poisspdf(x,1);
>> plot(x,y)
```

- ✓ An event occurrence in a given interval
- ✓ The occurrences are independent; called as “memoryless random distribution”
- ✓ Used to model data for the number of failed specimens (or product defects) in a given batch with a constant failure rate

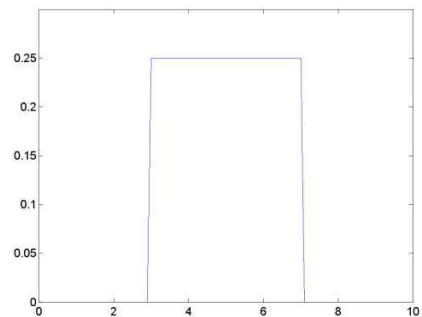


### Uniform Distribution

$$y(x; a, b) = f_X(x) = \frac{1}{b-a}$$

```
>> x=[0:0.1:10];
>> y=unifpdf(x,3,7);
>> plot(x,y)
```

- ✓ Symmetric, skewness=0
- ✓ Equal occurrence
- ✓ Random number generator

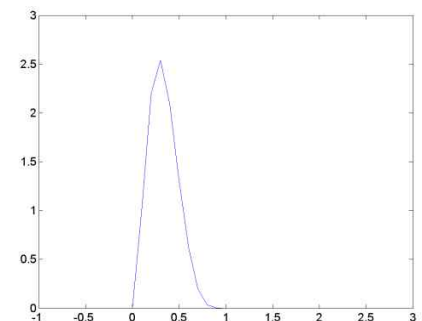


### Beta Distribution

$$y(x; a, b) = f_X(x) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}, \quad B(a, b): \text{Beta function}$$

```
>> x=[-10:0.1:10];
>> y=betapdf(x,3,6);
>> plot(x,y)
```

- ✓ Bounded distributions
- ✓ Related to Gamma distribution
- ✓ Manufacturing tolerance
- ✓ Reliability data in a Bayesian model



Other Distributions in Engineering

Rayleigh distribution, Gamma distribution, Extreme Type I, II distributions, etc. Refer to <http://mathworld.wolfram.com/topics/ProbabilityandStatistics.html> and <http://www.itl.nist.gov/div898/handbook/eda/section3/eda366.htm>.

**Homework 5.1: Statistical uncertainty**

Device a way to quantify the amount of statistical uncertainty. Use  $n=10, 100, 1000$  for  $X \sim N(0,1^2)$ .

**Homework 5.2: Statistical modeling of material strengths**

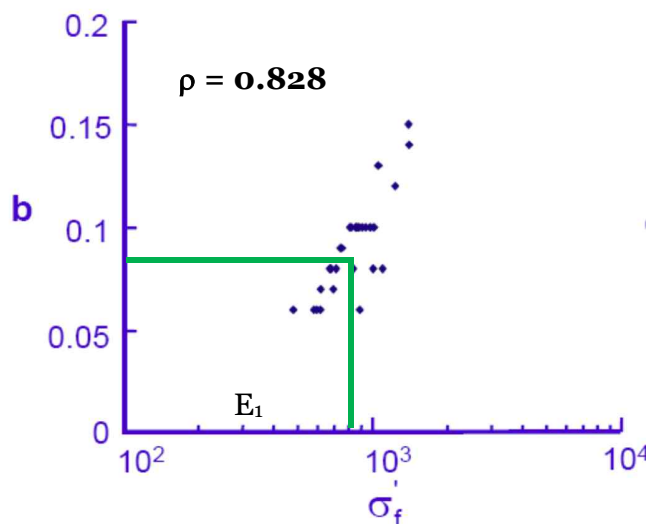
Download the excel file named 'tensile\_test.xlsx' at the ETL. You can find the yield strength and tensile strength data from uniaxial tensile tests. Among the probability distribution functions listed above, you are asked to determine two best candidates to model the yield strength and tensile strength. DO NOT use any advanced techniques but rely on the basic analysis of the distribution types described above. Write your essay with at least 150 words.

**2.5 Random Vectors (Material properties, etc.) – Statistical correlation (related to random vectors)**

Suppose  $X_1$  and  $X_2$  are jointly distributed and joint event is defined as  $X_1 \leq x_1$  and  $X_2 \leq x_2$ . The corresponding bi-variate distribution of a random vector is defined as

$$\text{Joint CDF: } F_{X_1, X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2) \quad (6)$$

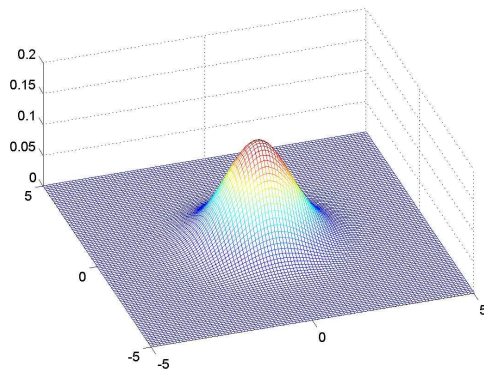
$$\text{Joint PDF: } f_{X_1, X_2}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X_1, X_2}(x_1, x_2)$$



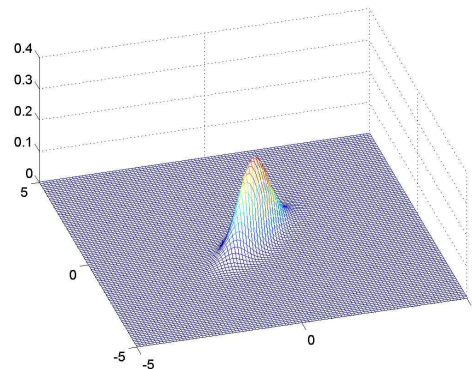
Assume that two random variables are normally distributed. To define the joint PDF of a multivariate distribution, five parameters are required, namely, the mean values of  $X_1$  and  $X_2$ ,  $\mu_{X_1}$  and  $\mu_{X_2}$ , their standard deviations  $\sigma_{X_1}$  and  $\sigma_{X_2}$ , and the correlation coefficient  $\rho_{X_1, X_2}$ . The PDF of the bivariate normal distribution can be expressed as

$$f_{X_1, X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho_{X_1, X_2}^2}} \exp \left\{ -\frac{1}{2(1-\rho_{X_1, X_2}^2)} \left[ \left( \frac{x_1 - \mu_{X_1}}{\sigma_{X_1}} \right)^2 - 2\rho_{X_1, X_2} \frac{(x_1 - \mu_{X_1})(x_2 - \mu_{X_2})}{\sigma_{X_1}\sigma_{X_2}} + \left( \frac{x_2 - \mu_{X_2}}{\sigma_{X_2}} \right)^2 \right] \right\} \quad (7)$$

If  $X_1$  and  $X_2$  are correlated, namely,  $\rho_{X_1, X_2} \neq 0$ ,  $f_{X_1, X_2}(x_1, x_2)$  is not symmetry.



```
>> [x1,x2]=meshgrid(-5:0.1:5);
>> f=1/(2*pi)*exp(-(x1.^2+(x2).^2)/2);
>> mesh(x1,x2,f)
```



```
>> [x1,x2]=meshgrid(-5:0.1:5);
>> f=1/(2*pi*sqrt(1-0.8^2))*exp(-(x1.^2-1.6*x1.*x2+x2.^2)/(2*(1-0.8^2)^2));
>> mesh(x1,x2,f)
```

Bivariate distribution of random vector can be generalized for  $n$ -dimensional random vector,  $\mathbf{X}: \Omega \rightarrow R^n$ . Joint CDF and PDF for  $n$ -dimensional random vector are written as

$$\text{Joint CDF: } F_{\mathbf{X}}(\mathbf{x}) = P\left(\bigcap_{i=1}^n \{X_i \leq x_i\}\right) \quad (8)$$

$$\text{Joint PDF: } f_{\mathbf{X}}(\mathbf{x}) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{\mathbf{X}}(\mathbf{x})$$

A multi-variate normal random vector is distributed as

$$f_{\mathbf{X}}(\mathbf{x}) = (2\pi)^{-\frac{n}{2}} |\Sigma_{\mathbf{X}}|^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}})^T \Sigma_{\mathbf{X}}^{-1} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}}) \right] \quad (9)$$

where  $\boldsymbol{\mu}_{\mathbf{X}}$  and  $\Sigma_{\mathbf{X}}$  are mean and covariance matrix of  $\mathbf{X}$ .



## 2.6 Conditional Probability – Statistical dependence (related to joint events)

The probability of the event  $E_1$  occurrence conditional upon the event  $E_2$  occurrence is defined as:

$$P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \quad (10)$$

Let us recall the example of fatigue tests. The sample data can be obtained about the physical quantities in the damage model below.

$$\frac{\Delta \varepsilon}{2} = \frac{\sigma'_f}{E} (2N_f)^b + \varepsilon'_f (2N_f)^c$$

**Exercise:** Let us consider a 20 data set for the fatigue strength coefficient ( $\sigma'_f$ ) and exponent ( $b$ ) used in the strain-life formula shown above. Two events are defined as

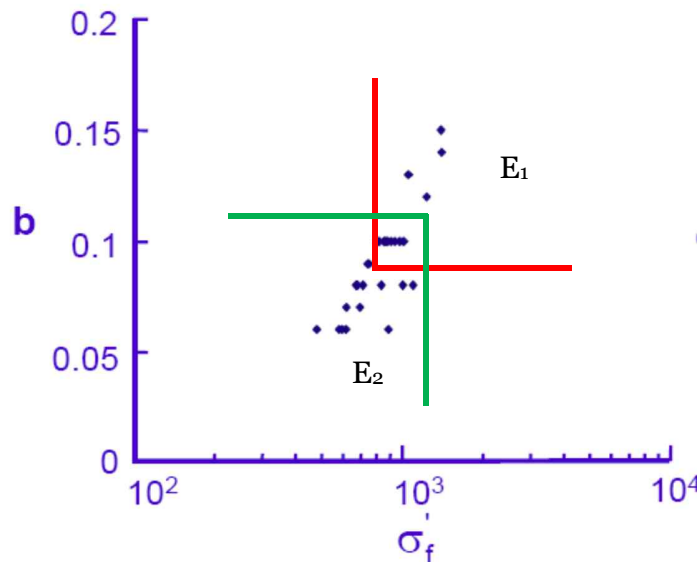
$$E_1 = \{(X_1, X_2) | X_1 > 8 \times 10^2 \text{ and } X_2 > 0.09\}$$

$$E_2 = \{(X_1, X_2) | X_1 < 1.02 \times 10^3 \text{ and } X_2 < 0.11\}$$

$$P(E_1) = 8/20 = 2/5, \quad P(E_2) = 16/20 = 4/5, \quad P(E_1 \cap E_2) = 4/20 = 1/5$$

$$P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{1/5}{4/5} = \frac{1}{4}$$

$$P(E_2 | E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{1/5}{2/5} = \frac{1}{2}$$



**Bayesian statistics (or inference) is based on the conditional probability.** It will be recalled in the Bayesian probability theory.

## 2.7 Statistical Moments – Quantification of randomness

- To understand the statistical moments of a random variable.
- To apply statistical moments to an uncertain response.
- To prepare uncertainty propagation analysis through a system in Sections 4 & 5.

Let  $\mathbf{X} = \{X_1, \dots, X_n\}^T$  be an  $n$ -dimensional random vector and  $g(\mathbf{X})$  be a function of  $\mathbf{X}$ . In general, the  $N^{\text{th}}$  statistical moment of  $g(\mathbf{X})$  is defined as

$$E[g(\mathbf{X})]^N \equiv \int_{\Omega} g^N(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (11)$$

where  $f_{\mathbf{X}}(\mathbf{x})$  is the joint PDF of  $\mathbf{X}$  and  $\Omega$  is a sample (or random) space.

### 2.7.1 Statistical Moments of a Random Vector

First, one special case is considered to find out statistical moments of an input random variable, that is,  $g(\mathbf{X}) = X_i, i = 1, \dots, n$ .

#### Mean of a Random Vector

Let  $g(\mathbf{X}) = X_1$  and set  $N=1$ . The first moment of random variable  $X_1$  is defined as

$$\begin{aligned} E[X_1]^1 &\equiv \int_{-\infty}^{\infty} x_1 f_{X_1}(x_1) dx_1 \\ &= \mu_{X_1} \end{aligned} \quad (12)$$

Similarly,

$$\begin{aligned} E[X_2]^1 &\equiv \int_{-\infty}^{\infty} x_2 f_{X_2}(x_2) dx_2 = \mu_{X_2} \\ &\vdots \\ E[X_n]^1 &\equiv \int_{-\infty}^{\infty} x_n f_{X_n}(x_n) dx_n = \mu_{X_n} \end{aligned}$$

$$\boldsymbol{\mu}_{\mathbf{X}} = \{\mu_{X_1} \quad \dots \quad \mu_{X_n}\}^T$$

### 2.7.2 Covariance of a Random Vector

Let  $g(\mathbf{X}) = (X_i - \mu_i)(X_j - \mu_j)$ . The statistical moment is defined as

$$\begin{aligned} E[(X_i - \mu_i)(X_j - \mu_j)] &\equiv \int_{-\infty}^{\infty} (x_i - \mu_i)(x_j - \mu_j) f_{X_i X_j}(x_i, x_j) dx_i dx_j \\ &= [\Sigma_{ij}] = \boldsymbol{\Sigma}_{\mathbf{X}} \end{aligned} \quad (13)$$

where  $f_{X_i X_j}(x_i, x_j)$  and  $\Sigma_{ij}$  are the joint PDF and the covariance matrix of  $X_i$  and  $X_j$ , respectively.

When  $i = j$ , the diagonal terms in the covariance matrix are obtained as

$$\begin{aligned}
 E[X_1 - \mu_1]^2 &\equiv \int_{-\infty}^{\infty} (x_1 - \mu_1)^2 f_{X_1}(x_1) dx_1 \\
 &= \sigma_{X_1}^2 = \Sigma_{11} \\
 &\vdots \\
 E[X_n - \mu_n]^2 &\equiv \int_{-\infty}^{\infty} (x_n - \mu_n)^2 f_{X_n}(x_n) dx_n \\
 &= \sigma_{X_n}^2 = \Sigma_{nn}
 \end{aligned} \tag{14}$$

If  $i \neq j$ , the off-diagonal terms in the covariance matrix are obtained as

$$\begin{aligned}
 E[(X_1 - \mu_1)(X_2 - \mu_2)] &\equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \mu_1)(x_2 - \mu_2) f_{X_1, X_2}(x_1, x_2) dx_1 dx_2 \\
 &= \Sigma_{12} \\
 &\vdots \\
 E[(X_n - \mu_n)(X_{n-1} - \mu_{n-1})] &\equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_n - \mu_n)(x_{n-1} - \mu_{n-1}) f_{X_n, X_{n-1}}(x_n, x_{n-1}) dx_n dx_{n-1} \\
 &= \Sigma_{nn-1}
 \end{aligned} \tag{15}$$

The covariance matrix is written as

$$\Sigma_{\mathbf{X}} = \begin{bmatrix} \Sigma_{11} & \cdots & \Sigma_{1n} \\ \vdots & \ddots & \vdots \\ \Sigma_{n1} & \cdots & \Sigma_{nn} \end{bmatrix}$$

### 2.7.2 Higher moments

Skewness and Kurtosis (3<sup>rd</sup> and 4<sup>th</sup> order moments)

$$\text{skewness} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] \quad \text{or} \quad \text{skewness} = \frac{\sum_{i=1}^N (x_i - \bar{X})^3}{(N-1)s^3} \tag{16}$$

$$\text{kurtosis} = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] \quad \text{or} \quad \text{kurtosis} = \frac{\sum_{i=1}^N (x_i - \bar{X})^4}{(N-1)s^4} \tag{17}$$

### 2.7.3 Properties of Covariance Matrix, $\Sigma_{\mathbf{X}}$

- $\Sigma_{\mathbf{X}}$  is symmetric, i.e.,  $\Sigma_{\mathbf{X}} = \Sigma_{\mathbf{X}}^T$
- Variance of  $X_i$  is the  $i^{\text{th}}$  diagonal element of  $\Sigma_{\mathbf{X}}$ , i.e.,  $\sigma_{X_i}^2 = \Sigma_{ii}$
- $\Sigma_{\mathbf{X}}$  is a positive semi-definite matrix, i.e.,  $\mathbf{A}^T \Sigma_{\mathbf{X}} \mathbf{A} \geq 0, \quad \forall \mathbf{A} \in R^n$

### 2.7.4 Correlation Coefficient, $\rho_{ij}$

The correlation coefficient  $\rho_{ij}$  is defined as

$$\rho_{ij} \equiv \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}}\sqrt{\Sigma_{jj}}} = \frac{\Sigma_{ij}}{\sigma_i\sigma_j} \quad (18)$$

The correlation coefficient  $\rho_{ij}$  is a degree of correlation between two random variables. Note that  $\Sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$  represents the off-diagonal elements of covariance matrix,  $\Sigma_X$ .

- If  $X_i$  and  $X_j$  are independent (i.e.,  $f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$ ), then  $X_i$  and  $X_j$  are uncorrelated (i.e.  $\rho_{ij} = 0$ ), but vice versa is not true.
- $-1 \leq \rho_{ij} \leq +1$
- If  $X_j = aX_i + b$ ,  $\rho_{ij} = \pm 1 = \text{sgn}(a)$ .

### 2.7.5 Coefficient of Variation, $\text{COV}(X) \equiv \sigma_X / \mu_X$

#### **Homework 6: Statistical moments and joint PDF**

Use the same excel file named 'tensile\_test.xlsx' at the ETL. Calculate the sampled means and standard deviations of yield strength and tensile strength. With the calculated means, standard deviations, and correlation coefficient, you can plot a joint pdf of yield strength and tensile strength. ASSUME the yield strength and tensile strength follow normal distribution.

#### **Homework 7: Read Chapter 2 of the Textbook**

Read Chapter 2 to reinforce your knowledge about the fundamentals of the engineering statistics.

## CHAPTER 3. UNCERTAINTY CHARACTERIZATION

This chapter discusses statistical analysis based on available sample data that characterizes uncertain data in a statistical form. Specifically, it introduces statistical procedures to determine an appropriate probability distribution for a random variable based on a limited set of sample data. There are two approaches in the statistical data analysis techniques: (a) conventional statistical methods (graphical methods and statistical hypothesis tests) and (b) Bayesian methods.

### 3.1 Conventional (or Frequentist) Statistical Methods

The conventional statistical methods impose models (both deterministic and probabilistic) on the data. Deterministic models include, for example, regression models and analysis of variance (ANOVA) models. The most common probabilistic models include the graphical methods and quantitative methods.

#### 3.1.1 Graphical Methods

- Histogram (Fig. 3.1)  
The purpose of a histogram is to graphically summarize the distribution of a univariate data set. This histogram graphically shows the following:

1. center (i.e., the location) of the data;
2. spread (i.e., the variation) of the data;
3. skewness of the data;
4. presence of outliers; and
5. presence of multiple modes in the data.

These features provide strong indications of the proper distributional model for the data. The probability plot or a goodness-of-fit test can be used to verify the distributional model.

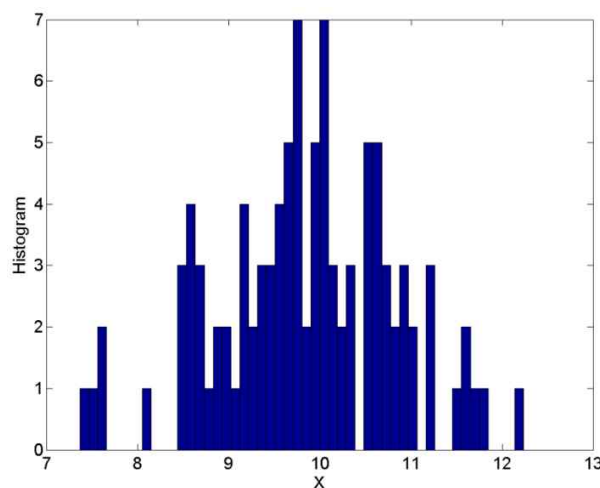
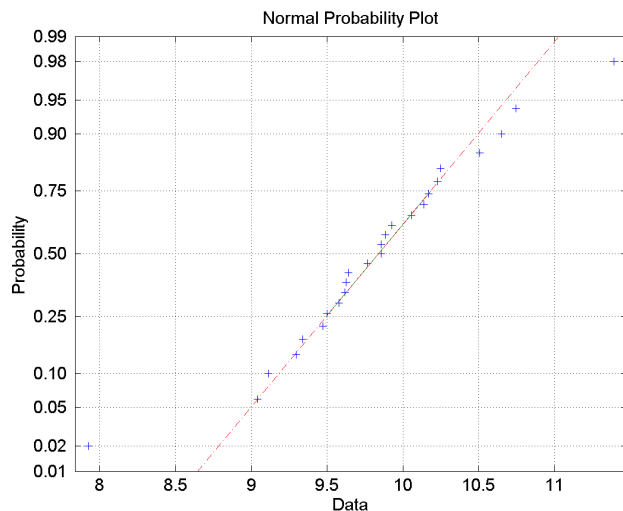


Figure 3.1: Histogram

- Normal probability plot

The normal probability plot is a graphical technique for assessing whether or not a data set can be approximated as a normal distribution. The data are plotted against a theoretical normal distribution in such a way that the points should form an approximate straight line. **Departures from this straight line indicate departures from normality.** The normal probability plot is a special case of the probability plot.

```
>> x = normrnd(10,1,25,1);
>> normplot(x);
```



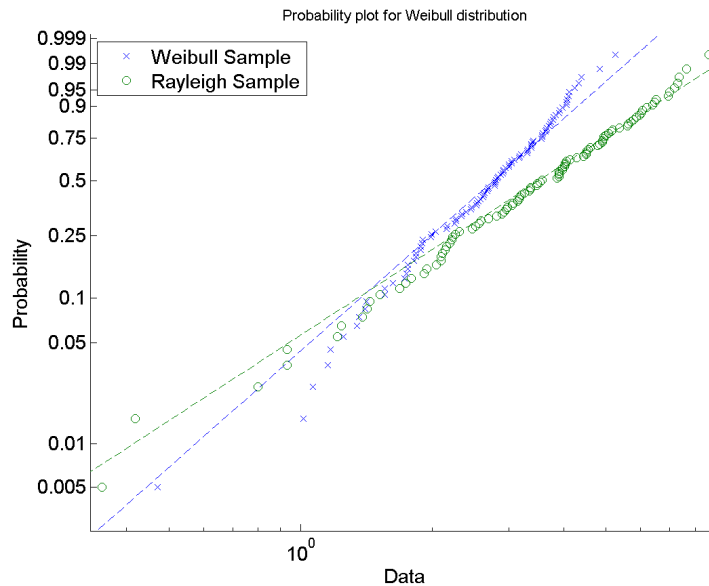
- Probability plot

The uniform distribution has a linear relationship between ordered physical data and probability. So any probability distribution can be used for approximating a given data set if a probability distribution is related to the uniform distribution. The relationship can be defined as

$$U(i) = G(P(x(i)))$$

where  $P(i)$  is the probability of the event  $E = \{X \mid x(i) \in \Omega\}$  and  $U(i)$  follows a uniform distribution.

```
>> x1 = wblrnd(3,2,100,1);
>> x2 = raylrnd(3,100,1);
>> probplot('weibull',[x1 x2])
>> legend('Weibull Sample','Rayleigh Sample','Location','NW')
```



Rayleigh distribution is a special case of weibull distribution when a shape parameter is 2. Therefore both distributions follow the straight lines very closely.

### **Homework 8: Graphical methods**

Use the data set for elastic modulus and yield strength in the excel file named 'tensile\_test.xlsx'. Build histograms and plot each data set on the normal probability plot to determine if they follow a normal distribution. Discuss your observation.

#### 3.1.2 Quantitative Methods

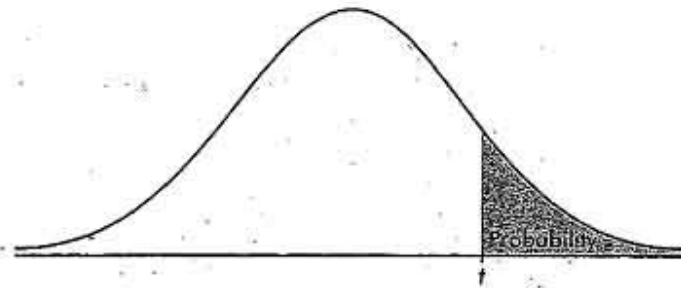
- Statistical Moments:  
**First-order moment (e.g., mean, location)**

$$\text{mean}(\bar{X}) = \frac{\sum_{i=1}^N x_i}{N} \quad (19)$$

- a. Confidence limits (or interval) for the mean (T-test)

$$\bar{X} \pm t_{(\alpha/2, N-1)} s / \sqrt{N} \quad (20)$$

where  $\bar{X}$  and  $s$  are the sampled mean and standard deviation,  $N$  is the sample size,  $\alpha$  is the desired significance level (or  $1-\alpha =$  confidence level), and  $t_{(\alpha/2, N-1)}$  is the critical value of the  $t$ -distribution with  $N-1$ .



**TABLE B: *t*-DISTRIBUTION CRITICAL VALUES**

| df   | Tail probability <i>p</i> |       |       |       |       |       |       |       |       |       |       |       |
|------|---------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|      | .25                       | .20   | .15   | .10   | .05   | .025  | .02   | .01   | .005  | .0025 | .001  | .0005 |
| 1    | 1.000                     | 1.376 | 1.963 | 3.078 | 6.314 | 12.71 | 15.89 | 31.82 | 63.66 | 127.3 | 318.3 | 636.6 |
| 2    | .816                      | 1.061 | 1.386 | 1.886 | 2.920 | 4.303 | 4.849 | 6.965 | 9.925 | 14.09 | 22.33 | 31.60 |
| 3    | .765                      | .978  | 1.250 | 1.638 | 2.353 | 3.182 | 3.482 | 4.541 | 5.841 | 7.453 | 10.21 | 12.92 |
| 4    | .741                      | .941  | 1.190 | 1.533 | 2.132 | 2.776 | 2.999 | 3.747 | 4.604 | 5.598 | 7.173 | 8.610 |
| 5    | .727                      | .920  | 1.156 | 1.476 | 2.015 | 2.571 | 2.757 | 3.365 | 4.032 | 4.773 | 5.893 | 6.869 |
| 6    | .718                      | .906  | 1.134 | 1.440 | 1.943 | 2.447 | 2.612 | 3.143 | 3.707 | 4.317 | 5.208 | 5.959 |
| 7    | .711                      | .896  | 1.119 | 1.415 | 1.895 | 2.365 | 2.517 | 2.998 | 3.499 | 4.029 | 4.785 | 5.408 |
| 8    | .706                      | .889  | 1.108 | 1.397 | 1.860 | 2.306 | 2.449 | 2.896 | 3.355 | 3.833 | 4.501 | 5.041 |
| 9    | .703                      | .883  | 1.100 | 1.383 | 1.833 | 2.262 | 2.398 | 2.821 | 3.250 | 3.690 | 4.297 | 4.781 |
| 10   | .700                      | .879  | 1.093 | 1.372 | 1.812 | 2.228 | 2.359 | 2.764 | 3.169 | 3.581 | 4.144 | 4.587 |
| 11   | .697                      | .876  | 1.088 | 1.363 | 1.796 | 2.201 | 2.328 | 2.718 | 3.106 | 3.497 | 4.025 | 4.437 |
| 12   | .695                      | .873  | 1.083 | 1.356 | 1.782 | 2.179 | 2.303 | 2.681 | 3.055 | 3.428 | 3.930 | 4.318 |
| 13   | .694                      | .870  | 1.079 | 1.350 | 1.771 | 2.160 | 2.282 | 2.650 | 3.012 | 3.372 | 3.852 | 4.221 |
| 14   | .692                      | .868  | 1.076 | 1.345 | 1.761 | 2.145 | 2.264 | 2.624 | 2.977 | 3.326 | 3.787 | 4.140 |
| 15   | .691                      | .866  | 1.074 | 1.341 | 1.753 | 2.131 | 2.249 | 2.602 | 2.947 | 3.286 | 3.733 | 4.073 |
| 16   | .690                      | .865  | 1.071 | 1.337 | 1.746 | 2.120 | 2.235 | 2.583 | 2.921 | 3.252 | 3.686 | 4.015 |
| 17   | .689                      | .863  | 1.069 | 1.333 | 1.740 | 2.110 | 2.224 | 2.567 | 2.898 | 3.222 | 3.646 | 3.965 |
| 18   | .688                      | .862  | 1.067 | 1.330 | 1.734 | 2.101 | 2.214 | 2.552 | 2.878 | 3.197 | 3.611 | 3.922 |
| 19   | .688                      | .861  | 1.066 | 1.328 | 1.729 | 2.093 | 2.205 | 2.539 | 2.861 | 3.174 | 3.579 | 3.883 |
| 20   | .687                      | .860  | 1.064 | 1.325 | 1.725 | 2.086 | 2.197 | 2.528 | 2.845 | 3.153 | 3.552 | 3.850 |
| 21   | .686                      | .859  | 1.063 | 1.323 | 1.721 | 2.080 | 2.189 | 2.518 | 2.831 | 3.135 | 3.527 | 3.819 |
| 22   | .686                      | .858  | 1.061 | 1.321 | 1.717 | 2.074 | 2.183 | 2.508 | 2.819 | 3.119 | 3.505 | 3.792 |
| 23   | .685                      | .858  | 1.060 | 1.319 | 1.714 | 2.069 | 2.177 | 2.500 | 2.807 | 3.104 | 3.485 | 3.768 |
| 24   | .685                      | .857  | 1.059 | 1.318 | 1.711 | 2.064 | 2.172 | 2.492 | 2.797 | 3.091 | 3.467 | 3.745 |
| 25   | .684                      | .856  | 1.058 | 1.316 | 1.708 | 2.060 | 2.167 | 2.485 | 2.787 | 3.078 | 3.450 | 3.725 |
| 26   | .684                      | .856  | 1.058 | 1.315 | 1.706 | 2.056 | 2.162 | 2.479 | 2.779 | 3.067 | 3.435 | 3.707 |
| 27   | .684                      | .855  | 1.057 | 1.314 | 1.703 | 2.052 | 2.158 | 2.473 | 2.771 | 3.057 | 3.421 | 3.690 |
| 28   | .683                      | .855  | 1.056 | 1.313 | 1.701 | 2.048 | 2.154 | 2.467 | 2.763 | 3.047 | 3.408 | 3.674 |
| 29   | .683                      | .854  | 1.055 | 1.311 | 1.699 | 2.045 | 2.150 | 2.462 | 2.756 | 3.038 | 3.396 | 3.659 |
| 30   | .683                      | .854  | 1.055 | 1.310 | 1.697 | 2.042 | 2.147 | 2.457 | 2.750 | 3.030 | 3.385 | 3.646 |
| 40   | .681                      | .851  | 1.050 | 1.303 | 1.684 | 2.021 | 2.123 | 2.423 | 2.704 | 2.971 | 3.307 | 3.551 |
| 50   | .679                      | .849  | 1.047 | 1.299 | 1.676 | 2.009 | 2.109 | 2.403 | 2.678 | 2.937 | 3.261 | 3.496 |
| 60   | .679                      | .848  | 1.045 | 1.296 | 1.671 | 2.000 | 2.099 | 2.390 | 2.660 | 2.915 | 3.232 | 3.460 |
| 80   | .678                      | .846  | 1.043 | 1.292 | 1.664 | 1.990 | 2.088 | 2.374 | 2.639 | 2.887 | 3.195 | 3.416 |
| 100  | .677                      | .845  | 1.042 | 1.290 | 1.660 | 1.984 | 2.081 | 2.364 | 2.626 | 2.871 | 3.174 | 3.390 |
| 1000 | .675                      | .842  | 1.037 | 1.282 | 1.646 | 1.962 | 2.056 | 2.330 | 2.581 | 2.813 | 3.098 | 3.300 |
| ∞    | .674                      | .841  | 1.036 | 1.282 | 1.645 | 1.960 | 2.054 | 2.326 | 2.576 | 2.807 | 3.091 | 3.291 |
|      | 50%                       | 60%   | 70%   | 80%   | 90%   | 95%   | 96%   | 98%   | 99%   | 99.5% | 99.8% | 99.9% |
|      | Confidence level <i>C</i> |       |       |       |       |       |       |       |       |       |       |       |



From the formula, it is clear that the width of the interval is controlled by two factors:

- ✓ As  $N$  increases, the interval gets narrower from the  $\sqrt{N}$  term and  $t_{(\alpha/2, N-1)}$ . That is, one way to obtain more precise estimates for the mean is to increase the sample size.
- ✓ The larger the sample standard deviation, the larger the confidence interval. This simply means that noisy data, i.e., data with a large standard deviation, are going to generate wider intervals than data with a smaller standard deviation.

To test whether the population mean has a specific value,  $\mu_0$ , against the two-sided alternative that it does not have a value  $\mu_0$ , the confidence interval is converted to hypothesis-test form. The test is a one-sample  $t$ -test, and it is defined as:

|                     |   |
|---------------------|---|
| H <sub>0</sub> :    | $\bar{X} = \mu_0$   |
| H <sub>1</sub> :    | $\bar{X} \neq \mu_0$  |
| Tested statistics:  | $T = (\bar{X} - \mu_0) / (s / \sqrt{N})$                                    |
| Significance level: | $\alpha$ (=0.05 is most commonly used.)                                     |
| Critical region:    | Reject the null hypothesis that the mean is a specified value, $\mu_0$ , if |
|                     | $T < -t_{(\alpha/2, N-1)}$ or $T > t_{(\alpha/2, N-1)}$                     |

*Let's say the null hypothesis is rejected. The p-value indicates the probability that the rejection of the null hypothesis is wrong.*

```
>> x1 = normrnd(0.1, 1, 1, 100);
>> [h, p, ci] = ttest(x1, 0)
h =
    0
p =
    0.8323
ci =
   -0.1650    0.2045
```

The test fails to reject the null hypothesis at the default  $\alpha$ . The 95% confidence interval on the mean contains 0.

```
>> x2 = normrnd(0.1,1,1,1000);
>> [h,p,ci] = ttest(x2,0)
h =
    1
p =
    0.0160
ci =
    0.0142    0.1379
```

The test rejects the null hypothesis at the default  $\alpha$ . The p-value has fallen below  $\alpha = 0.05$  and 95% confidence interval on the mean does not contain 0.

b. 1-factor ANOVA (Analysis of Variance)

<http://www.itl.nist.gov/div898/handbook/eda/section3/eda354.htm>

### Second-order moment (e.g., variation)

$$\text{variation}(s^2) = \frac{\sum_{i=1}^N (x_i - \bar{X})^2}{(N-1)} \quad (21)$$

a. Bartlett's test: <http://www.itl.nist.gov/div898/handbook/eda/section3/eda357.htm>

b. Chi-Square test: <http://www.itl.nist.gov/div898/handbook/eda/section3/eda358.htm>

c. F-test: <http://www.itl.nist.gov/div898/handbook/eda/section3/eda359.htm>

d. Levene test: <http://www.itl.nist.gov/div898/handbook/eda/section3/eda35a.htm>

The formula for computing the covariance of the variables X and Y is

$$\text{COV} = \frac{\sum_{i=1}^N (x_i - \bar{X})(y_i - \bar{Y})}{N-1} \quad (22)$$

- Maximum Likelihood Estimation (MLE):

The principle behind the MLE method is that for a random variable  $X$ , if  $x_1, x_2, \dots, x_n$  are the  $N$  observations or sample values, then the estimated value of the parameter is the value most likely to produce these observed values. Consider the density function of  $X$  to be  $f_X(x, \theta)$ , where  $\theta$  is the unknown parameter(s). In random sampling, the  $x_i$ 's are assumed to be independent. If the likelihood of observing  $x_i$ 's is proportional to their corresponding density functions, the likelihood function can be defined as

$$\begin{aligned} L(x_1, x_2, \dots, x_n | \theta) &= \prod_{i=1}^N f_X(x_i | \theta) \\ &= f_X(x_1 | \theta) f_X(x_2 | \theta) \cdots f_X(x_n | \theta) \end{aligned} \quad (23)$$

The MLE can be formulated as

To determine  $\theta$ , maximize  $L(x_1, x_2, \dots, x_n)$

**Homework 9: Quantitative methods**

Use the data set for elastic modulus and yield strength in the excel file named 'tensile\_test.xlsx'. Test whether or not the population mean has a specific value,  $\mu_0=200$  GPa, for a quality control. Let's assume the elastic modulus follow a normal distribution. Determine the optimal mean and standard deviation using the maximum likelihood method.

- Distributional Measures:

### **Chi-squared Goodness-of-Fit (GOF) Tests:**

The chi-square test is used to test if sampled data come from a population with a specific distribution. An attractive feature of the chi-square GOF test is that it can be applied to both continuous and discrete distributions. The chi-square GOF test is applied to binned data (i.e., data put into classes). So the values of the chi-square test statistic are dependent on how the data is binned. Another disadvantage of the chi-square test is that it requires a sufficient sample size in order for the chi-square approximation to be valid.

|                     |  |
|---------------------|--|
| H <sub>0</sub> :    | The data follow a specified distribution.  |
| H <sub>1</sub> :    | The data do not follow the specified distribution.   |
| Significance level: | $\alpha$ (=0.05 is most commonly used.)  |
| Test statistics:    | For the chi-square goodness-of-fit computation, the data are divided into $k$ bins and the test statistics is defined as |

$$\chi^2 = \sum_{i=1}^k (O_i - E_i)^2 / E_i \quad (24)$$

where  $O_i$  is the observed frequency for bin  $i$  and  $E_i$  is the expected frequency for bin  $i$ . The expected frequency is calculated by

$$E_i = N \cdot (F(X_u) - F(X_l))_i = N \cdot \bar{f}_i \quad (25)$$

where  $F$  is the cumulative distribution function (CDF) for the distribution being tested,  $X_u$  is the upper limit for bin  $i$ ,  $X_l$  is the lower limit for a bin  $i$ , and  $N$  is the sample size.

|                  |   |
|------------------|---|
| Critical region: | The hypothesis that the data are from a population with the specified distribution is rejected if |
|------------------|---|

$$\chi^2 > \chi^2_{(\alpha, k-c)}$$

where  $\chi^2_{(\alpha, k-c)}$  is the chi-square percent point function with  $k-c$  degrees of freedom and a significant level of  $\alpha$ .  $k$  is the number of non-empty cells and  $c$  = the number of estimated parameters (including location and scale parameters and shape parameters) for the distribution.

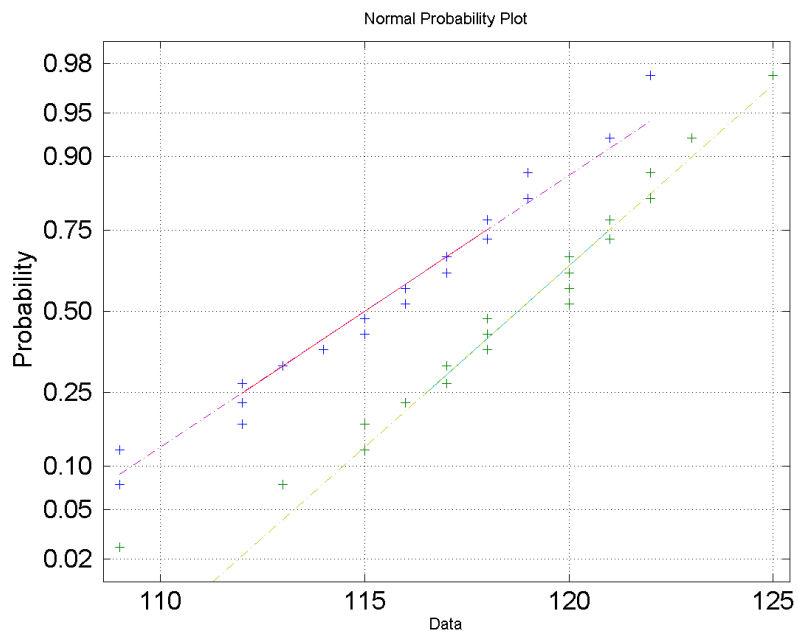
### **Anderson-Darling (A-D) Goodness-of-Fit Test:**

<http://www.itl.nist.gov/div898/handbook/eda/section3/eda35e.htm>

### **Kolmogorov-Smirnov (K-S) Goodness-of-Fit Test:**

<http://www.itl.nist.gov/div898/handbook/eda/section3/eda35g.htm>

```
>> load gas
>> prices = [price1 price2];
>> normplot(prices)
```



```
>> sample_means = mean(prices)
sample_means =
    115.1500  118.5000
>> [h,pvalue,ci] = ttest(price2/100,1.1515)
h =
    1
pvalue =
    4.9517e-004
ci =
    1.1675  1.2025
```

```
>> price2=normrnd(118,3.8,100,1);
>> [h,p] = chi2gof(price2,'cdf',{@normcdf,mean(price2),std(price2)})
>> [h,p] = chi2gof(price2,'cdf',{@normcdf,119,3.5})
```

```
>> x = randn(100,1);
>> [h,p,st] = chi2gof(x,'cdf',@normcdf)
```

h =

0

p =

0.370

st =

chi2stat: 7.5909

df: 7

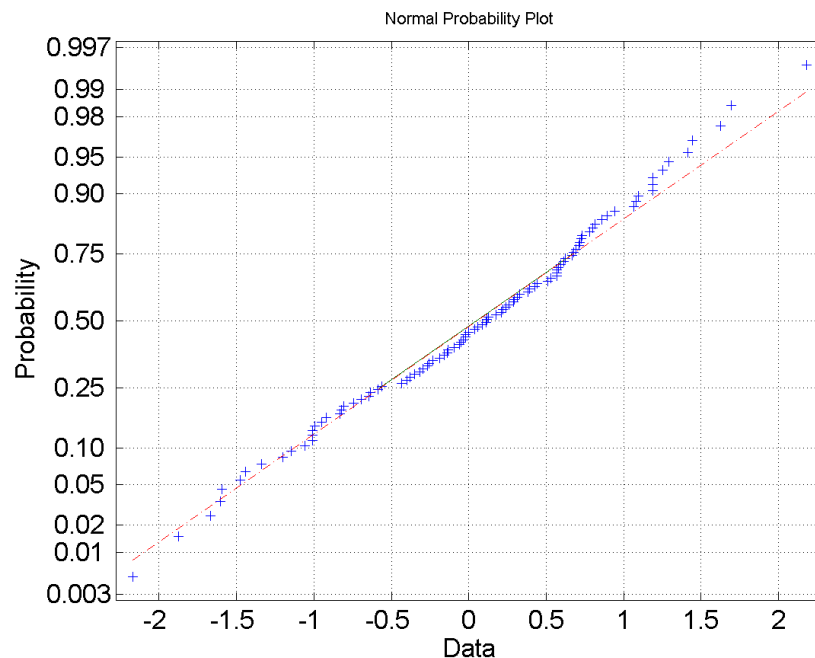
edges: [-2.1707 -1.2999 -0.8645 -0.4291 0.0063 0.4416 0.8770 1.3124 2.1832]

O: [8 9 10 19 18 21 10 5]

E: [9.6817 9.6835 14.0262 16.8581 16.8130 13.9138 9.5546 9.4690]

| Degrees of Freedom | Probability |      |      |      |      |       |       |       |       |       |       |
|--------------------|-------------|------|------|------|------|-------|-------|-------|-------|-------|-------|
|                    | 0.95        | 0.90 | 0.80 | 0.70 | 0.50 | 0.30  | 0.20  | 0.10  | 0.05  | 0.01  | 0.001 |
| 1                  | 0.004       | 0.02 | 0.06 | 0.15 | 0.46 | 1.07  | 1.64  | 2.71  | 3.84  | 6.64  | 10.83 |
| 2                  | 0.10        | 0.21 | 0.45 | 0.71 | 1.39 | 2.41  | 3.22  | 4.60  | 5.99  | 9.21  | 13.82 |
| 3                  | 0.35        | 0.58 | 1.01 | 1.42 | 2.37 | 3.66  | 4.64  | 6.25  | 7.82  | 11.34 | 16.27 |
| 4                  | 0.71        | 1.06 | 1.65 | 2.20 | 3.36 | 4.88  | 5.99  | 7.78  | 9.49  | 13.28 | 18.47 |
| 5                  | 1.14        | 1.61 | 2.34 | 3.00 | 4.35 | 6.06  | 7.29  | 9.24  | 11.07 | 15.09 | 20.52 |
| 6                  | 1.63        | 2.20 | 3.07 | 3.83 | 5.35 | 7.23  | 8.56  | 10.64 | 12.59 | 16.81 | 22.46 |
| 7                  | 2.17        | 2.83 | 3.82 | 4.67 | 6.35 | 8.38  | 9.80  | 12.02 | 14.07 | 18.48 | 24.32 |
| 8                  | 2.73        | 3.49 | 4.59 | 5.53 | 7.34 | 9.52  | 11.03 | 13.36 | 15.51 | 20.09 | 26.12 |
| 9                  | 3.32        | 4.17 | 5.38 | 6.39 | 8.34 | 10.66 | 12.24 | 14.68 | 16.92 | 21.67 | 27.88 |
| 10                 | 3.94        | 4.86 | 6.18 | 7.27 | 9.34 | 11.78 | 13.44 | 15.99 | 18.31 | 23.21 | 29.59 |

```
>> normplot(x)
```



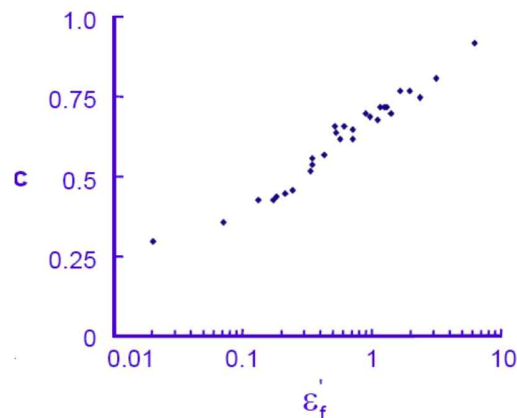
**Homework 10: Probability Distribution & Statistical Moments**

Let us recall the example of fatigue tests. The sample data can be obtained about the physical quantities in the damage model below.

$$\frac{\Delta \varepsilon}{2} = \frac{\sigma'_f}{E} (2N_f)^b + \varepsilon'_f (2N_f)^c$$

Let us consider a 30 data set (Table 3.1) for the fatigue ductility coefficient ( $\varepsilon'_f$ ) and exponent ( $c$ ) used in the strain-life formula shown above. Answer the following questions and **provide a matlab code**:

- (1) Construct the covariance matrix and find out the coefficient of correlation using the data set given in Table 3.1.
- (2) Use normal, weibull, and lognormal distributions. Determine the most suitable parameters of three distributions for the fatigue ductility coefficient ( $\varepsilon'_f$ ) and exponent ( $c$ ) using the MLE method.
- (3) Find out the most suitable distributions for the data set ( $\varepsilon'_f, c$ ) using a GOF test.
- (4) Verify the results with the graphical methods (histogram and probability plots).



**Figure 3.2:** Statistical Correlation

**Table 3.1:** Data for the fatigue ductility coefficient and exponent

| $(\varepsilon'_f, c)$ |       | $(\varepsilon'_f, c)$ |       | $(\varepsilon'_f, c)$ |       | $(\varepsilon'_f, c)$ |       | $(\varepsilon'_f, c)$ |       |
|-----------------------|-------|-----------------------|-------|-----------------------|-------|-----------------------|-------|-----------------------|-------|
| 0.022                 | 0.289 | 0.253                 | 0.466 | 0.539                 | 0.630 | 0.989                 | 0.694 | 1.611                 | 0.702 |
| 0.071                 | 0.370 | 0.342                 | 0.531 | 0.590                 | 0.621 | 1.201                 | 0.690 | 1.845                 | 0.760 |
| 0.146                 | 0.450 | 0.353                 | 0.553 | 0.622                 | 0.653 | 1.304                 | 0.715 | 1.995                 | 0.759 |
| 0.185                 | 0.448 | 0.354                 | 0.580 | 0.727                 | 0.635 | 1.388                 | 0.717 | 2.342                 | 0.748 |
| 0.196                 | 0.452 | 0.431                 | 0.587 | 0.729                 | 0.645 | 1.392                 | 0.716 | 3.288                 | 0.821 |
| 0.215                 | 0.460 | 0.519                 | 0.655 | 0.906                 | 0.703 | 1.426                 | 0.703 | 6.241                 | 0.894 |

### 3.2 Bayesian

We have discussed methods of statistical inference which view the probability as relative frequency and exclusively rely on the sample data to estimate the underlying probability distribution of the population. In addition to these frequentist statistical methods, the Bayesian approach utilizes some prior information in conjunction with the sample information. The Bayesian inference is capable of continuously updating the prior information with evolving sample data to obtain the posterior information.

#### 3.2.1 Bayes' Theorem

Bayes' theorem (also known as Bayes' rule or Bayes' law) is developed based on conditional probabilities. If  $A$  and  $B$  denote two events,  $P(A|B)$  denotes the conditional probability of  $A$  occurring, given that  $B$  occurs. An important application of Bayes' theorem is that it gives a rule how to update or revise a prior belief to a posterior belief. Bayes' theorem relates the conditional and marginal probabilities of stochastic events  $A$  and  $B$ :

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \quad (26)$$

Each term in Bayes' theorem has a conventional name:

- $P(A)$  is the prior probability or marginal probability of  $A$ . The prior probability can be treated as the subjective probability which expresses our belief prior to the occurrence of  $A$ . It is "prior" in the sense that it does not take into account any information about  $B$ .
- $P(B)$  is the prior or marginal probability of  $B$ , and acts as a normalizing constant.
- $P(A|B)$  is the conditional probability of  $A$ , given  $B$ . It is also called the posterior probability of  $A$ , given  $B$  because it depends upon the specified value of  $B$ .
- $P(B|A)$  is the conditional probability of  $B$  given prior information of  $A$ .

An important application of Bayes' theorem is that it gives a rule how to update or revise a *prior* belief to a *posterior* belief. Let us take a look at an interesting example to get a better understanding.

#### Example 3.1

There are three doors and behind two of the doors are goats and behind the third door is a new car with each door equally likely to provide the car. Thus the probability of selecting the car for each door at the beginning of the game is simply  $1/3$ . After you have picked a door, say  $A$ , before showing you what is behind that door, Monty opens another door, say  $B$ , revealing a goat. At this point, Monty gives you the opportunity to switch doors from  $A$  to  $C$  if you want to. What should you do? (Given that Monty is trying to let you get a goat.)

#### Solution

The question is whether the probability is 0.5 to get the car since only two doors left, or mathematically,  $P(A|B_{\text{Monty}}) = P(C|B_{\text{Monty}}) = 0.5$ . Basically we need to determine the



probabilities of two event  $E_1 = \{A|B_{\text{Monty}}\}$ ,  $E_2 = \{C|B_{\text{Monty}}\}$ . We elaborate the computation in the following steps:

1. The prior probabilities read  $P(A) = P(B) = P(C) = 1/3$ .
2. We also have some useful conditional probabilities  $P(B_{\text{Monty}}|A) = 1/2$ ,  $P(B_{\text{Monty}}|B) = 0$ , and  $P(B_{\text{Monty}}|C) = 1$ .
3. We can compute the probabilities of joint events as  $P(B_{\text{Monty}}, A) = 1/2 \times 1/3 = 1/6$ ,  $P(B_{\text{Monty}}, B) = 0$ , and  $P(B_{\text{Monty}}, C) = 1 \times 1/3 = 1/3$ .
4. Finally, with the denominator computed as  $P(B_{\text{Monty}}) = 1/6 + 0 + 1/3 = 1/2$ , we then get  $P(A|B_{\text{Monty}}) = 1/3$ ,  $P(C|B_{\text{Monty}}) = 2/3$ . Thus, it is better to switch to C.

### 3.2.2 Bayesian Inference

Let  $X$  and  $\theta$  be random variables with a joint probability density function  $f(x, \theta)$ ,  $\theta \in \Omega$ . When the amount of data for  $X$  is small or  $X$  is rapidly evolving, its statistical parameter  $\theta$  (e.g.,  $\mu, \sigma$ ) is considered to be random. From the Bayesian point of view,  $\theta$  is interpreted as a realization of a random variable  $\Theta$  with a probability density  $f_{\Theta}(\theta)$ . Based on the Bayes' theorem, the posterior distribution of  $\theta$  given a new observation  $X$  can be expressed as

$$f_{\Theta|X}(\theta|x) = \frac{f_{X,\Theta}(x,\theta)}{f_X(x)} = \frac{f_{X|\Theta}(x|\theta) \cdot f_{\Theta}(\theta)}{f_X(x)} \quad (27)$$

It can be seen that the Bayesian inference employs both the prior distribution of  $\theta$ ,  $f_{\Theta}(\theta)$ , and the conditional probability distribution of the sample (evidence or likelihood),  $f_{X|\Theta}(x|\theta)$ , to find a posterior distribution of  $\theta$ ,  $f_{\Theta|X}(\theta|x)$ . Let us consider a normal inference model as one example to illustrate the Bayesian inference process.

**Example 3.2:** Suppose that we have a set of random samples  $\mathbf{x} = \{x_1, x_2, \dots, x_M\}$  from a normal PDF  $f_X(x; \mu, \sigma)$  of a random variable  $X$ , where  $\mu$  is unknown and  $\sigma$  is known. Assume that the prior distribution of  $\mu$ ,  $f_M(\mu)$ , is a normal distribution with its mean,  $\mu$ , and variance,  $\tau^2$ . Determine the posterior distribution of  $\mu$ ,  $f_{M|X}(\mu|\mathbf{x})$ .

#### Solution

Firstly, we compute the conditional probability of obtaining  $\mathbf{x}$  given  $\mu$  as

$$\begin{aligned} f_{X|M}(\mathbf{x}|\mu) &= \prod_{i=1}^M \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right] \\ &= (2\pi\sigma^2)^{-M/2} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^M (x_i - \mu)^2\right] \end{aligned} \quad (28)$$

Next, we compute the joint probability of  $\mathbf{x}$  and  $\mu$  as

$$\begin{aligned}
f_{X,M}(\mathbf{x}, \mu) &= f_{X|M}(\mathbf{x} | \mu) f_M(\mu) \\
&= (2\pi\sigma^2)^{-M/2} (2\pi\tau^2)^{-1/2} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^M (x_i - \mu)^2 - \frac{1}{2\tau^2} (\mu - u)^2\right] \\
&= K_1(x_1, \dots, x_M, \sigma, u, \tau) \exp\left[-\left(\frac{M}{2\sigma^2} + \frac{1}{2\tau^2}\right) \mu^2 + \left(\frac{M\bar{x}}{\sigma^2} + \frac{u}{\tau^2}\right) \mu\right]
\end{aligned}$$

We then set up a square with  $\mu$  in the exponent as

$$\begin{aligned}
f_{X,M}(\mathbf{x}, \mu) &= K_2(x_1, \dots, x_M, \sigma, u, \tau) \exp\left[-\frac{1}{2}\left(\frac{M}{\sigma^2} + \frac{1}{\tau^2}\right) \left(\mu - \frac{\frac{M\bar{x}}{\sigma^2} + \frac{u}{\tau^2}}{\frac{M}{\sigma^2} + \frac{1}{\tau^2}}\right)^2\right] \\
&= K_2(x_1, \dots, x_M, \sigma, u, \tau) \exp\left[-\frac{1}{2}\left(\frac{M}{\sigma^2} + \frac{1}{\tau^2}\right) \left(\mu - \frac{M\tau^2\bar{x} + \sigma^2 u}{M\tau^2 + \sigma^2}\right)^2\right]
\end{aligned}$$

Since the denominator  $f_X(x_1, x_2, \dots, x_M)$  does not depend on  $\mu$ , we then derive the posterior distribution of  $\mu$  as

$$f_{M|X}(\mu | \mathbf{x}) = K_3(x_1, \dots, x_M, \sigma, u, \tau) \exp\left[-\frac{1}{2}\left(\frac{M}{\sigma^2} + \frac{1}{\tau^2}\right) \left(\mu - \frac{M\tau^2\bar{x} + \sigma^2 u}{M\tau^2 + \sigma^2}\right)^2\right]$$

Clearly, this is a normal distribution with the mean and variance as

$$\hat{\mu} = \frac{M\tau^2\bar{x} + \sigma^2 u}{M\tau^2 + \sigma^2}, \quad \hat{\tau} = \left(\frac{M}{\sigma^2} + \frac{1}{\tau^2}\right)^{-1} = \frac{\sigma^2\tau^2}{M\tau^2 + \sigma^2} \quad (29)$$

Therefore, the Bayes estimate of  $\mu$  is essentially a weighted-sum of the sample mean and the prior mean. In contrast, the maximum likelihood estimator is only the sample mean. As the number of samples  $M$  approaches the infinity, the Bayes estimate becomes equal to the maximum likelihood estimator since the sample data tend to have a predominant influence over the prior information. However, for the case of a small sample size, the prior information often plays an important role, especially when the prior variance  $\tau^2$  is small (or we have very specific prior information).

### 3.2.3 Conjugate Bayes Models

As can be seen in the Example 3.2, the Bayes inference and the maximum likelihood estimation essentially provide the same estimator if we have a very large sample size. In engineering practice, however, we often have very limited sample data possibly due to the high expense to obtain the data. In such cases, the maximum likelihood estimation may not give an accurate or even reasonable estimator. In contrast, the Bayesian inference would give much better estimator if we assume a reasonable prior assumption. By “reasonable”, we mean that the prior assumption is at least consistent

with the underlying distribution of the population. If there is no such consistency, the Bayesian inference may give an erroneous estimator due to the misleading prior information.

Another important observation we can make from Example 3.2 is that the posterior distribution shares a similar form (i.e., normal distribution) with the prior. In this case, we say that the prior is *conjugate* to the likelihood. If we have a conjugate prior, the posterior distribution can be obtained in an explicit form. Looking back to Example 3.2, we note that the normal or Gaussian family is conjugate to itself (or self-conjugate): if the likelihood function is normal, choosing a normal prior will ensure that the posterior distribution is also normal. Other conjugate Bayes inference models include the binomial inference, exponential inference, and Poisson inference. Among these inferences, the binomial inference is the most widely used. Consider a Bernoulli sequence of  $n$  experimental trials with  $x$  occurrences of an outcome whose probability of occurrence  $p_0$  is unknown. We assume a beta prior  $B(a,b)$  for the unknown binomial probability  $p_0$ , expressed as

$$f_{p_0}(p_0) = \frac{\Gamma(a,b)}{\Gamma(a)\Gamma(b)} p_0^{a-1} (1-p_0)^{b-1}$$

The likelihood function can be expressed according to a binomial distribution as

$$L(x; n, p_0) = C(n, x) p_0^x (1-p_0)^{n-x} \propto p_0^x (1-p_0)^{n-x}$$

We can easily obtain the posterior distribution of  $p_0$  as a beta distribution, expressed as

$$f_{p_0|x}(p_0 | x) = \frac{\Gamma(x+a, n+b-x)}{\Gamma(x+a)\Gamma(n+b-x)} p_0^{x+a-1} (1-p_0)^{n+b-x-1}$$

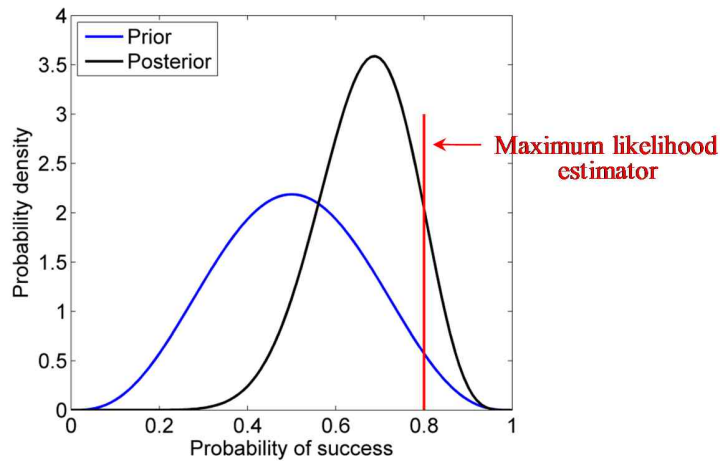
The posterior distribution has the same form (beta distribution) as the prior distribution, leading to the conjugacy condition. Let us take a look at the use of this inference with a simple reliability analysis problem.

### Example 3.3

Suppose that we intend to quantify the reliability of a product by conducting a sequence of 10 repeated tests. The product passes 8 of these tests and fails at the other two. We assume a beta prior  $B(4, 4)$  for the probability of success (or reliability)  $p_0$  in each test. Compute the posterior distribution of  $p_0$  with the reliability test data.

### Solution

Clearly, the parameters in this example take the following values:  $a = 4$ ,  $b = 4$ ,  $x = 8$ ,  $n = 10$ . Then the posterior distribution can be obtained as  $B(x+a, n+b-x)$ , or  $B(12, 6)$ . The prior and posterior distributions of  $p_0$  are plotted in Figure 3.3, where we can see the posterior distribution combines the prior information and the testing information (evidence) and achieves a compromise between the prior distribution and the maximum likelihood estimator.

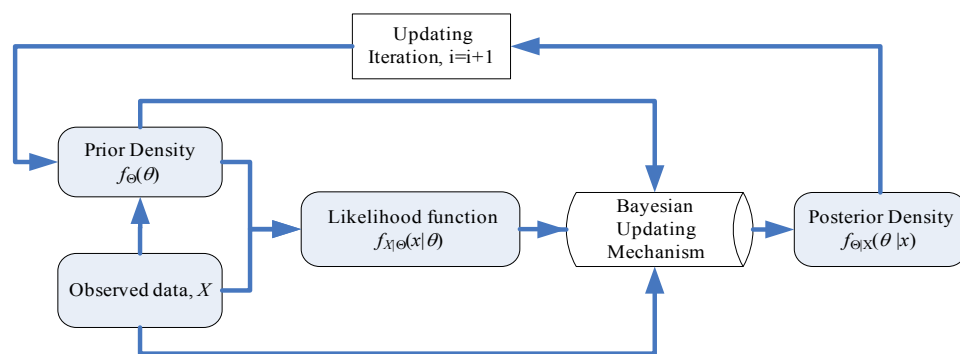


**Figure 3.3** Prior and posterior distributions

### Homework 11. Matlab coding for Bayesian statistics

Build your own Matlab coding for accomplishing the Example 3.3 (results and figure) above.

In many engineering problems, the conjugacy condition does not hold and explicit solutions cannot be readily obtained with simple mathematical manipulations. In such cases, we can build the posterior distributions by random sampling. A commonly used simulation method for drawing samples from the posterior distribution is referred to as *Markov chain Monte Carlo (MCMC)* in which the two most common techniques, the Metropolis–Hastings algorithm and Gibbs sampling, are used. Others include *particle filtering*, *(extended) Karman filtering*, etc. An in-depth theoretical discussion of these techniques is beyond the scope of this book. Readers are recommended to refer to some Bayesian statistics books for detailed information.



**Figure 3.4:** Process of Bayesian Updating

The Bayesian approach is used for updating information about the parameter  $\theta$ . First, a prior distribution of  $\Theta$  must be assigned before any future observation of  $X$  is taken. Then, the prior distribution of  $\Theta$  is updated to the posterior distribution as the new data for  $X$  is employed. The posterior distribution is set to a new prior distribution and this process can be repeated with an evolution of data sets. This updating process can be briefly illustrated in Fig. 3.4.

Markov model is widely used in various fields such as word recognition, voice recognition and gesture recognition in which sequence of the data is very meaningful. Markov chain which consists of Markov model defines probability of posterior event given the prior events. For example, 1<sup>st</sup> Markov chain considers just last event and 2<sup>nd</sup> Markov chain take last two events into consideration to calculate probability of the current event, expressed as

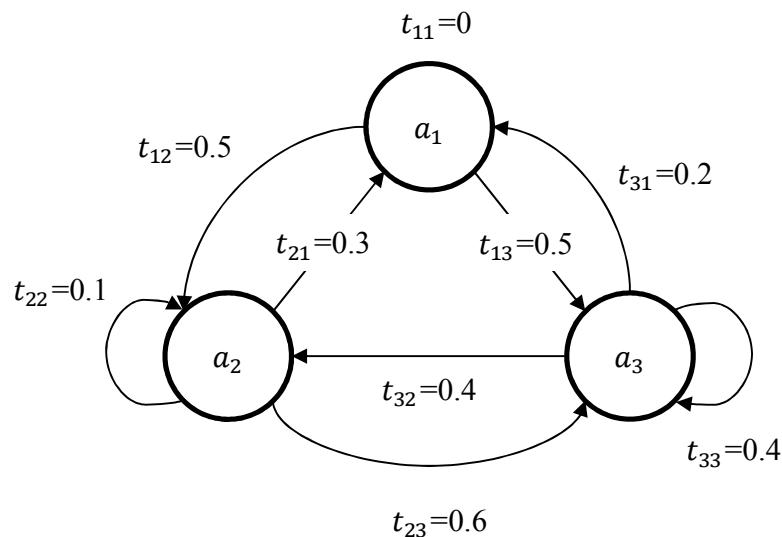
1<sup>st</sup> Markov chain

$$: P(X_t = x | X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}, \dots, X_1 = x_1) = P(X_t = x | X_{t-1} = x_{t-1})$$

2<sup>nd</sup> Markov chain

$$: P(X_t = x | X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}, \dots, X_1 = x_1) = P(X_t = x | X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2})$$

A state diagram for a simple example of the 1<sup>st</sup> Markov chain is shown in the **Figure 3.5**.



**Figure 3.5:** State diagram of a simple Markov chain

' $a$ ' represents the observations which can be obtained from the model, and ' $t_{ij}$ ' is probability that  $a_j$  occurs when  $a_i$  is given. For example, probability that the posterior event  $X_t$  becomes  $a_2$  can be defined based on prior events as follows

$$\begin{aligned}
 P(X_t = a_2) &= P(X_t = a_2 | X_{t-1} = a_1) \times P(X_t = a_2 | X_{t-1} = a_2) \times P(X_t = a_2 | X_{t-1} = a_3) \\
 &= t_{12} \times t_{22} \times t_{32} = 0.5 \times 0.1 \times 0.4 = 0.02
 \end{aligned}$$

For more convenient interpretation of the model, transition matrix can be defined as

$$T = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.3 & 0.1 & 0.6 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

It can be noticed that sum of the probability of all posterior events given one prior event is 1.

### Example 3.4 (Gambler's ruin)

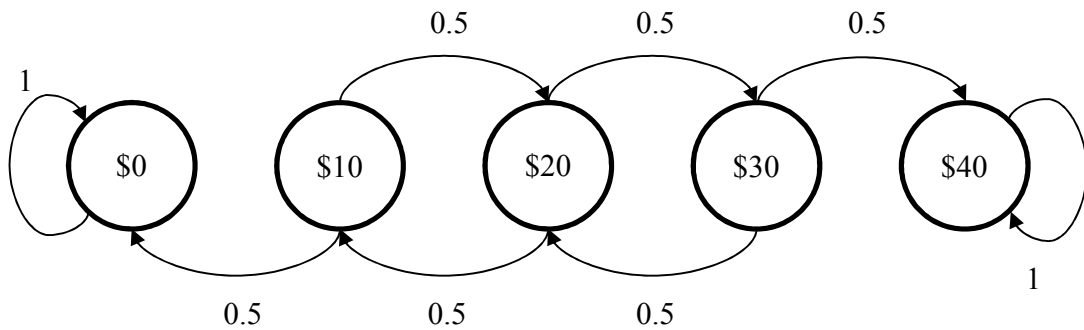
Suppose that a gambler having \$20 is going to gamble at roulette in a Casino. The gambler bets \$10 on odd number, and makes \$10 when it occurs. If even number occurs, he loses the money betting the roulette.

He has to leave the Casino if he loses his entire money or make \$20 to have \$40 in his pocket. What is likelihood that the gambler lose his entire money from ten times of the roulette game given that probability of winning at each game is 50%?



### Solution

First, we have to develop Markov chain to solve the example. 1<sup>st</sup> Markov chain is used in this example. Graphical model can be illustrated as



**Figure 3.6** Markov chain for the Example 3.4

And the corresponding transition matrix is

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

After ten times of roulette games, multiplication of the transition matrix gives

$$T^{10} \approx \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.734 & 0.016 & 0 & 0.016 & 0.234 \\ 0.484 & 0 & 0.031 & 0 & 0.484 \\ 0.234 & 0.016 & 0 & 0.016 & 0.734 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

What this result is saying is that probability of losing all initial pocket money (\$20) as a result of 10 times of roulette games is about 48.4% under the given condition.

The idea of Markov Chain Monte Carlo (MCMC) is basically the same as the Markov model in that it defines posterior position of the sampling point based on the prior information of the sampled points. Two most important techniques can be employed in MCMC, the Metropolis-Hastings algorithm and Gibbs sampling.

Metropolis algorithm, which is the most simplified MCMC method can be performed by the following steps

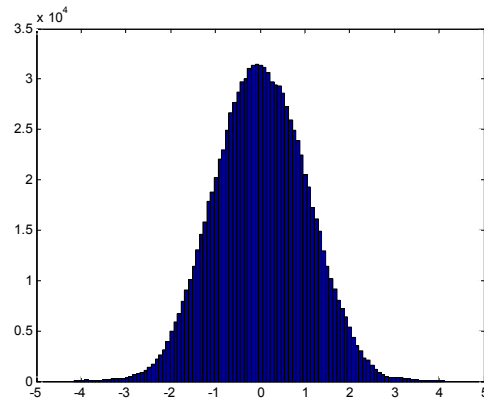
Step 1. Set a sample index  $i$  to 0 and initial sampling point  $x_0$   
 Step 2. Pick a random value  $u \sim U(0,1)$ , where  $u$  follows the uniform distribution  
 Step 3. Define a candidate of the next sampling point  $x^* \sim P(x^*|x_i)$ , where  $P$  is  
 ‘proposal distribution’  $\rightarrow$  generate ‘random walk’ using a proposal density  
 Step 4. If  $u < \min\left\{1, \frac{p(x^*)}{p(x^i)}\right\}$   
 $x^{i+1} = x^* \quad \rightarrow$  accept a proposal  
 else  
 $x^{i+1} = x^i \quad \rightarrow$  reject a proposal  
 end

In step 4, decision criterion is defined based on the ratio of probability of the candidate position and probability of the prior sampling point. Thus, the next position of the sampling point is defined in most likely direction.

For example, it is possible to design the sampling position for the Gaussian distribution with mean of zero and standard deviation of one using the Metropolis algorithm, where ‘proposal function’  $P$  follows Gaussian distribution ( $\text{norm}(x^i, 0.05)$ ).

```
>> n=1000000;
>> x=zeros(n,1);
>> x0=0.5; % Step 1
>> x(1)=x0;
>> for i=1:n-1
>> x_star=normrnd(x(i),0.05); % Step 2
>> u=rand; % Step 3
>> if u<min(1,normpdf(x_star)/normpdf(x(i))) % Step 4
>> x(i+1)=x_star;
>> else
>> x(i+1)=x(i);
>> end
>> end
>>
>> figure;
>> hist(x,100);
```





Others include **particle filtering, (extended) Karman filtering**, etc. An in-depth theoretical discussion of these techniques is beyond the scope of this book. Readers are recommended to refer to some Bayesian statistics books for detailed information.

### 3.2.4 How to Model Prior Distribution?

- Informative Prior Distribution

Generally we have two ways to handle known information ( $\mathbf{x}$ ):

1. Histogram
2. Select a prior density function with unknown parameters firstly, and then estimate the unknown parameters for the data.

- Non-informative Prior Distribution

Non-informative prior distribution means determining the prior distribution when no other information about the parameter  $\theta$  is available except its feasible field  $\Theta$ .

References for Bayesian statistics:

1. <http://en.wikipedia.org/wiki/Bayesian>
2. Singpurwalla, N.D., 2006, Reliability and Risk: A Bayesian Perspective, Wiley.
2. Andrew Gelman, John B. Carlin, Hal S. Stern and Donald B. Rubin, 2004, Bayesian Data Analysis, Second Edition, Chapman & Hall/CRC.
3. Bernardo, J.M., and Smith A.F.M., 1994, *Bayesian Theory*, John Wiley & Son Ltd..

## CHAPTER 4. PROBABILISTIC ENGINEERING ANALYSIS – TIME-INDEPENDENT PERFORMANCE

### 4.1 Motivation

Many system failures can be traced back to various difficulties in evaluating and designing complex systems under highly uncertain manufacturing and operational conditions and our limited understanding of physics-of-failures (PoFs). One of the greatest challenges in engineered systems design is how to evaluate the probability of an engineering event accurately before prototyping or actual testing. One way to evaluate the probability of an engineering event is known as *Monte Carlo simulation*, based on random sampling. Due to inefficiency of Monte Carlo when data is not given sufficiently, many “efficient” methods have been devised to alleviate the need for Monte Carlo simulation. These methods included the first and second-order reliability method (FORM and SORM), the response surface method (RSM), and the Bayesian inference.

### 4.2 Probabilistic Description of System Performance

Uncertainty affects the entire lifecycle of engineered systems from the impurity of the resources to the assembly of the finished goods. No matter manufacturer design the product perfectly, there is always errors or imperfection in manufacturing and operation. It is extremely difficult to predict engineering performances precisely due to substantial uncertainty in engineering design, manufacturing and operation. For example, engineers cannot predict how much engine mount bushing transmits engine noise and vibration to drivers and passengers; how much head/neck/chest/femur injury occurs during a car crash; what is a critical height for a drop test that breaks the display of a smartphone. Thus, we should define engineering performances as a function of uncertainty as shown below.

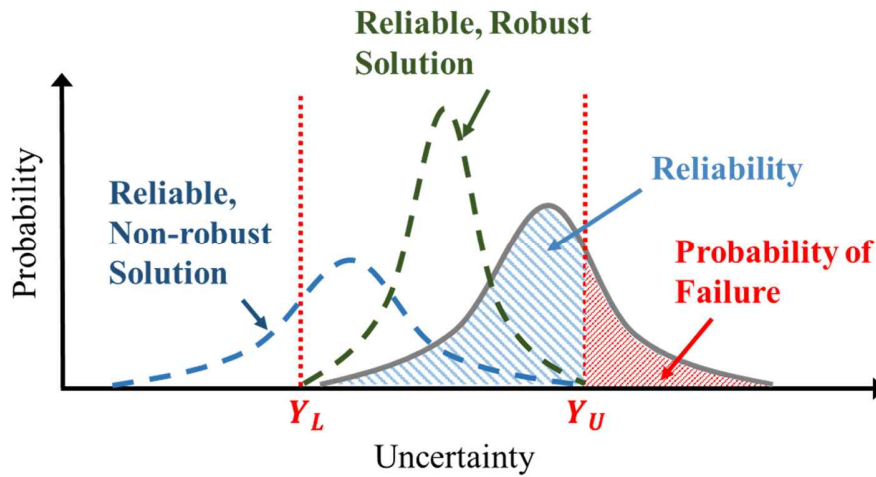
Probabilistic performance  
 $= Y(\mathbf{X})$ ;  $\mathbf{X}$  is a vector of uncertainty that affects system performance

Engineering systems have specifications in terms of systems’ performances. The specification can set a threshold in a quantitative scale. Therefore we can set a probability of safety which is under the pre-determined threshold, say  $Y_U$ .

Probability ( $Y(\mathbf{X}) \leq Y_U$ ) = Probability of safety (=success)  
 $=$  Reliability  
 $= 1 -$  Probability of failure

On the other hand, our system is now reliable—it meets our design goals or specifications—but it may not be robust. Operation of the system is affected by variabilities of the inputs. To be robust, a system performance must be insensitive to input variabilities. In other words, the performance thus possesses a narrow distribution subject to input variabilities as shown below

Probability ( $Y_L \leq Y(\mathbf{X}) \leq Y_U$ ) = Robustness



**Figure 4.1:** Fundamentals of Probabilistic Performance Analysis

### 4.3 Probabilistic Description of System Performance – Reliability

A system performance is defined in many different ways as practiced in different applications; say the electronics, civil structures, nuclear/chemical plants, and aerospace industries. In some instances, system performances can be treated time-independently due to their characteristics. Other instances situate the performances time-dependently.

#### 4.3.1 Time-Independent Performance:

The probability that the actual performance of a particular system will meet the required or specified design performance without considering the degradation of system performances over time. It is often found in mechanical and civil structural systems.

$$R(\mathbf{X}) = P(Y(\mathbf{X}) > Y_c) = 1 - P(Y(\mathbf{X}) \leq Y_c) \quad \text{for larger-the-better performances}$$

where the safety of the system is defined as  $Y > Y_c$  and  $Y_c$  is the critical value for  $Y$ .  $Y_c$  can be either deterministic or random. Examples include natural frequency, engine power, energy efficiency, etc.

$$R(\mathbf{X}) = P(Y(\mathbf{X}) < Y_c) = 1 - P(Y(\mathbf{X}) \geq Y_c) \quad \text{for smaller-the-better performances}$$

where the safety of the system is defined as  $Y < Y_c$ . Examples include stress, strain, crack size, etc.

#### 4.3.2 Time-Dependent Performance:

The probability that the actual life of a particular system will exceed the required or specified design life.

$$R_T(t) = P[T(\mathbf{X}) > t] = 1 - P[T(\mathbf{X}) \leq t] = 1 - F_T(t)$$

where the time-to-failure (TTF) of a system is defined as a time that a system health condition,  $G(\mathbf{X})$ , is worse than its critical value,  $G_c$ , and  $\mathbf{X}$  is the random vector representing engineering uncertain factors.

#### 4.3.3 Challenges:

1. Modeling random variables ( $\mathbf{X}$ ) for future loading, material property, and manufacturing tolerances (section 3).
2. Analyzing how input uncertainties propagate to those of system performances (section 4.4-4.7)
3. Extending the ideas of probabilistic analysis to the case with a lack of data (section 4.8)
4. Identification of the probability distribution for a reliability function (sections 5).
5. Predicting the failure time or performance failure when designing a system or component (section 5).
6. A long-time failure or lack of failure in test-based reliability assessment (section 5).
7. Consideration of performance degradation in time-dependent reliability (sections 5).

### 4.4 Probabilistic Description of Time-Independent Performance

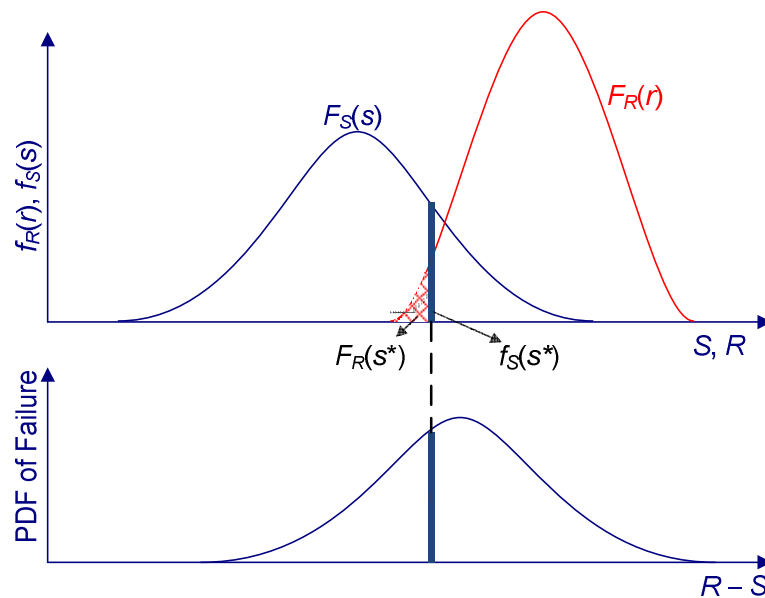
- Structural reliability is defined in many different ways as practiced in different applications; say the electronics, civil structures, nuclear/chemical plants, and aero-space industries.
- Most electrical, electronic and mechanical components and systems deteriorate during use as a result of elevated operating temperatures, chemical changes, mechanical wear, fatigue, overloading, and for a number of other reasons. Failure of a particular component may eventually occur for one of these reasons, or it may be caused indirectly as a result of the deterioration of some other parts of the system. However, it is very difficult to estimate TTF distribution precisely.
- In contrast to electronic/mechanical systems, structural systems tend not to deteriorate, except by the mechanical corrosion and fatigue, and in some cases may even get stronger, for example, the increase in the strength of concrete with time, and the increase in the strength of soils as a result of consolidation.
- In other cases, engineers are interested in initial performances.

For a simple structural member, the strength  $R$  and load  $S$  of the structure can describe the probability of failure or reliability. Suppose the strength  $R$  and load  $S$  to be random with the known distributions,  $F_R(r)$  and  $F_S(s)$ . The probability of failure is defined as

$$P_f = P(R - S \leq 0) = \int_{-\infty}^{\infty} F_R(s) f_S(s) ds \quad (30)$$

Then, the reliability can be defined as

$$R = 1 - P_f = 1 - \int_{-\infty}^{\infty} F_R(s) f_S(s) ds \quad (31)$$



**Figure 4.2:** A Simple Case of Reliability ( $= 1 - P_f$ ): Strength-Load

#### 4.5 General Description of Time-Independent Performance

The reliability is defined as the probability that the performance of a system exceeds the required or specified design limit over operating time  $t$ .

$$R(t) = P(Y(\mathbf{X}, t) \geq Y_c) = 1 - P(Y(\mathbf{X}, t) < Y_c) \quad \text{for Larger-the-better type}$$

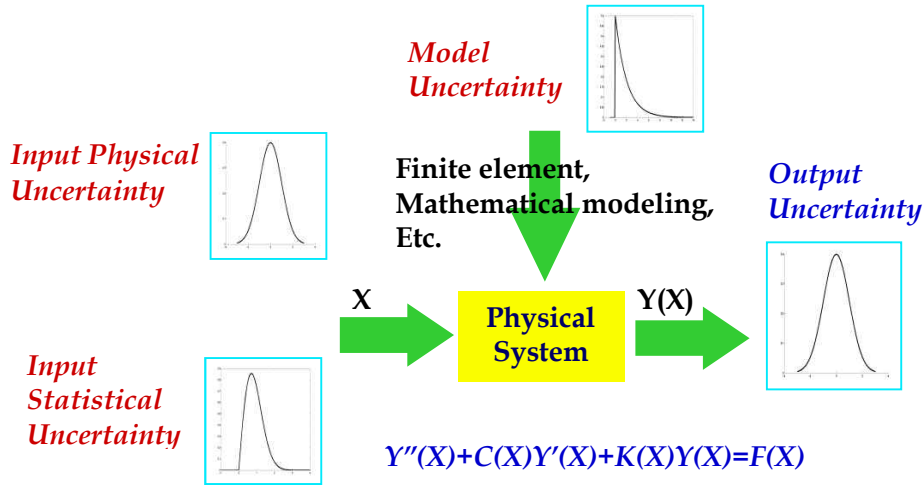
$$R(t) = P(Y(\mathbf{X}, t) \leq Y_c) = 1 - P(Y(\mathbf{X}, t) > Y_c) \quad \text{for Smaller-the-better type}$$

where the failure of the system is defined as  $Y \geq Y_c$  for L-Type (or  $Y \leq Y_c$ ) and  $Y_c$  is the required design limit for  $Y$ .  $Y_c$  can be either deterministic or random.

#### 4.6 Probabilistic Engineering Analysis Using Simulation Models

For probabilistic engineering analysis, uncertainty in engineered system performances (or outputs) must be understood by taking into account various uncertainties in engineered system inputs. As shown Fig. 4.3, input uncertainties are propagated through the system to those in outputs (e.g., natural frequency, fuel

consumption, energy conversion efficiency, vibration, transmission error, temperature distribution, head injury).



**Figure 4.3:** Uncertainty Propagation through Physical System

Then, the probability of safety (L-Type) can be estimated by integrating the PDFs of the system performances over the safety region.

$$R = P\{Y(\mathbf{X}, t) \geq Y_c\} = \int_{Y_c}^{\infty} f_Y(y) dy = \int_{Y(\mathbf{X}) \geq Y_c} \dots \int f_X(\mathbf{x}) d\mathbf{x} \quad (32)$$

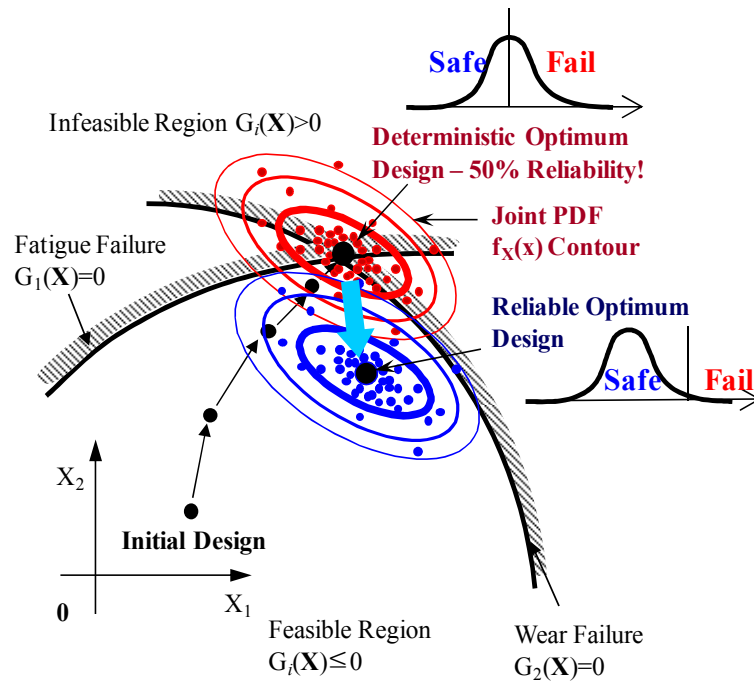
## 4.7 Methods for Probabilistic Performance Analysis (Frequentist)

### 4.7.1 General Model of Design under Uncertainty

The design under uncertainty can generally be defined as:

$$\begin{aligned} &\text{Minimize} && \text{Cost}(\mathbf{d}) \\ &\text{subject to} && P\{G_i\{\mathbf{X}; \mathbf{d}(\mathbf{X})\} > 0\} < P_{f_i}, \quad i = 1, \dots, nc \\ &&& \mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U, \quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr} \end{aligned} \quad (33)$$

where  $nc$  is the number of probabilistic constraints;  $nd$  is the number of design parameters;  $nr$  is the number of random variables;  $\mathbf{d} = [d_i]^T = \mu(\mathbf{X})$  is the design vector;  $\mathbf{X} = [X_i]^T$  is the random vector; and the probabilistic constraints are described by the performance function  $G_i\{\mathbf{X}; \mathbf{d}(\mathbf{X})\}$ , their probabilistic models, and the probability of failure. The probability of failure is defined as  $P_f \equiv \Phi(-\beta_i)$  with a target reliability index  $\beta_i$  where the failure is defined as  $G_i\{\mathbf{X}; \mathbf{d}(\mathbf{X})\} = Y_c - Y_i(\mathbf{X}; \mathbf{d}(\mathbf{X})) > 0$  for L-type. The design procedure under uncertainty is graphically illustrated in Fig. 4.9.



**Figure 4.4:** Design under Uncertainty

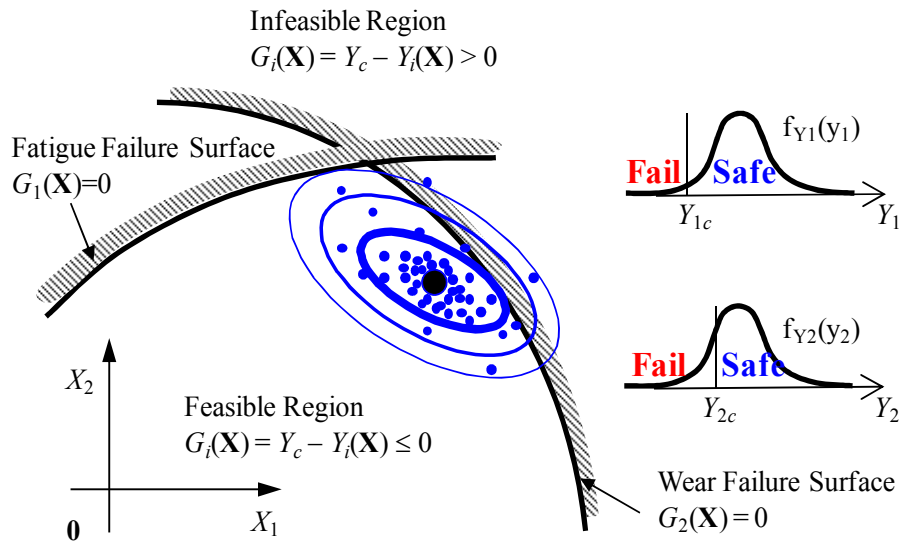
The probability of failure is defined as

$$\begin{aligned}
 P(G(\mathbf{X}) > 0) &= 1 - P(G(\mathbf{X}) \leq 0) \\
 &= 1 - F_G(0) \\
 &= \int_{G(\mathbf{X}) > 0} \cdots \int f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \quad \mathbf{X} \in R^{nr}
 \end{aligned}
 \tag{34}$$

The reliability (or the probability of safety) is inversely defined as

$$\begin{aligned}
 P(G(\mathbf{X}) \leq 0) &= F_G(0) \\
 &= \int_{G(\mathbf{X}) \leq 0} \cdots \int f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \quad \mathbf{X} \in R^{nr}
 \end{aligned}
 \tag{35}$$

Figure 4.5 explains both the probability of failure and reliability.



**Figure 4.5:** Reliability or Probability of Safety

The statistical description of the safety (or failure) of the constraint function  $G_i(\mathbf{X})$  requires a reliability analysis and is expressed by the CDF  $F_{G_i}(0)$  of the constraint as

$$\begin{aligned}
 P(G_i(\mathbf{X}) \leq 0) &= F_{G_i}(0) \geq \Phi(\beta_i) \text{ or } R_i \\
 \text{Time-dependent: } &G_i(\mathbf{X}, T) = T_d - T_i \leq 0 \text{ where } T_d \text{ is a designed life.} \\
 \text{Time-independent: } &G_i(\mathbf{X}) = P_i - P_c \leq 0 \text{ where } P_c \text{ is a critical buckling load.}
 \end{aligned} \tag{36}$$

where the probability of the safety constraint  $G_i(\mathbf{X}) \leq 0$  is described as

$$F_{G_i}(0) = \int_{-\infty}^0 f_{G_i}(g_i) dg_i = \int_{G_i(\mathbf{X}) \leq 0} \dots \int f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \quad i = 1, \dots, nc \text{ and } \mathbf{x} \in R^{nr} \tag{37}$$

In Eq. (37),  $f_{\mathbf{X}}(\mathbf{x})$  is the joint PDF of all random parameters and the evaluation of Eq. (37) involves multiple integration. Neither analytical multi-dimensional integration nor direct numerical integration is possible for large-scale engineering applications. Existing approximate methods for probability analysis can be categorized into four groups: 1) sampling method; 2) expansion method; 3) the most probable point (MPP)-based method; and 4) stochastic response surface method.

#### 4.7.2 Random sampling techniques (Monte Carlo simulation)

Let us recall the reliability or the probability of safety as



$$P\{G_i(\mathbf{X}; \mathbf{d}) \leq 0\} = F_{G_i}(0) = \int_{G_i(\mathbf{x}) \leq 0} \dots \int f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \approx \frac{\text{Number of safe trials}}{\text{Number of total trials}} \quad (38)$$

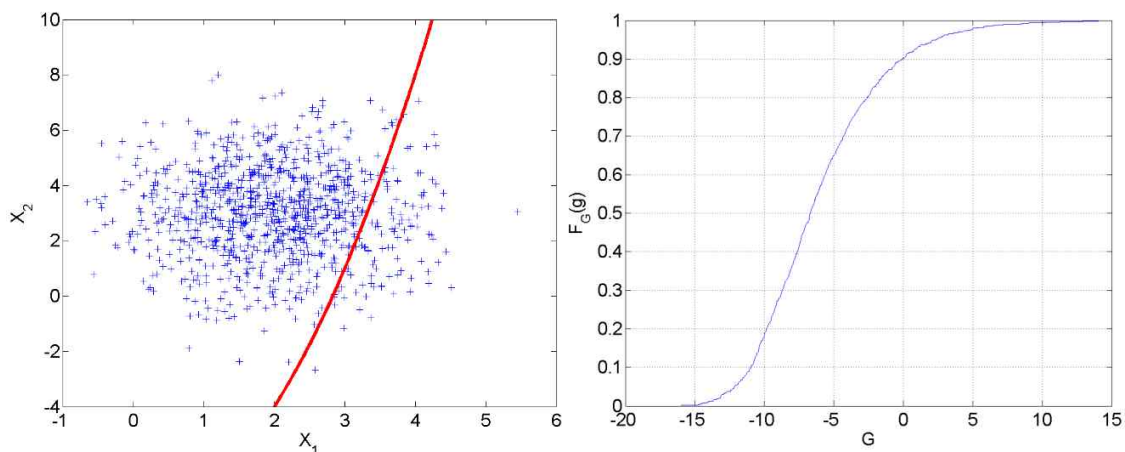
Or, inversely, the probability of failure can be obtained as

$$P\{G_i(\mathbf{X}; \mathbf{d}) > 0\} = 1 - F_{G_i}(0) = \int_{G_i(\mathbf{x}) > 0} \dots \int f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \approx \frac{\text{Number of failure trials}}{\text{Number of total trials}} \quad (39)$$

- Simple but extremely expensive
- Seldom used due to its computational intensiveness, but used for a benchmark study
- To estimate a failure rate,

$$p_f = \frac{N_f}{N}, \quad N_f : \text{Number of failed samples}$$

$$N : \text{Number of total samples}$$



% generate random samples

```
>> m=[2 3];
>> s=[1 0;0 3];
>> n=1000;
>> x=mvnrnd(m,s,n);
>> plot(x(:,1),x(:,2),'+')
```

% plot a failure surface

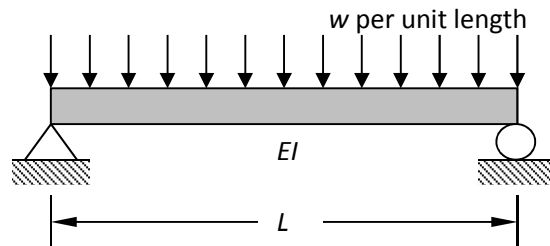
```
>> [x1,x2] = meshgrid(-1:1:6,-4:2:10);
>> gg=x1.^2-x2-8;
>> v=[0 0];
>> [C,h]=contour(x1,x2,gg,v)
```

% calculate reliability

```
>> ns=0;
>> for i = 1:1000
    g(i) = x(i,1)^2-x(i,2)-8;
    if g(i) <= 0
        ns = ns + 1
    end
end
>> rel = ns/n
>> cdfplot(g)
```

**Homework 12: Monte Carlo Simulation**

Consider the following simply supported beam subject to a uniform load, as illustrated in Figure below. Suppose  $L = 5$  m and  $w = 10$  kN/m.



**Figure 4.6:** Simply Supported Beam

Random Vector:

$$EI = X_1 \sim N(\mu_{X_1} = 3 \times 10^7, \sigma_{X_1} = 10^5)$$

$$w = X_2 \sim N(\mu_{X_2} = 10^4, \sigma_{X_2} = 10^3)$$

The maximum deflection of the beam is shown as

$$Y = g(X_1, X_2) = -\frac{5X_2L^4}{384X_1}$$

Determine the PDF (or CDF) of the maximum deflection and estimate its reliability using the MC simulation when the failure is defined as  $Y < y_c = -3 \times 10^{-3}$  m.

### 4.7.3 Expansion methods

#### First-order method

Any nonlinear function ( $Y$ ) can be linearized in terms of an input random vector

$\mathbf{X} = \{X_1, \dots, X_n\}^T$ , i.e.,

$$Y(\mathbf{X}) = Y(\boldsymbol{\mu}_{\mathbf{X}}) + \sum_{i=1}^n \frac{\partial Y(\boldsymbol{\mu}_{\mathbf{X}})}{\partial X_i} (X_i - \mu_{X_i}) + h.o.t.$$

$$\approx a_1 X_1 + \dots + a_n X_n + b$$

or  $Y \approx \mathbf{a}^T \mathbf{X} + b$

(40)

where  $\mathbf{a} = \{a_1, \dots, a_n\}^T$  is a sensitivity vector of  $Y$ .

- Mean of  $Y$

$$\begin{aligned}
 E[Y] &= \mu_Y \approx E[\mathbf{a}^T \mathbf{X} + b] \\
 &= E[\mathbf{a}^T \mathbf{X}] + E[b] \\
 &= \mathbf{a}^T E[\mathbf{X}] + b \\
 &= \mathbf{a}^T \boldsymbol{\mu}_X + b
 \end{aligned}$$

- Variance of  $Y$

$$\begin{aligned}
 \text{Var}[Y] &= \sigma_Y^2 = E[(Y - \mu_Y)^2] \\
 &= E[(Y - \mu_Y)(Y - \mu_Y)^T] \\
 &\approx E[(\mathbf{a}^T \mathbf{X} + b - \mathbf{a}^T \boldsymbol{\mu}_X - b)(\mathbf{a}^T \mathbf{X} + b - \mathbf{a}^T \boldsymbol{\mu}_X - b)^T] \\
 &= \mathbf{a}^T E[(\mathbf{X} - \boldsymbol{\mu}_X)(\mathbf{X} - \boldsymbol{\mu}_X)^T] \mathbf{a} \\
 &= \mathbf{a}^T \boldsymbol{\Sigma}_X \mathbf{a}
 \end{aligned}$$

- Generalization

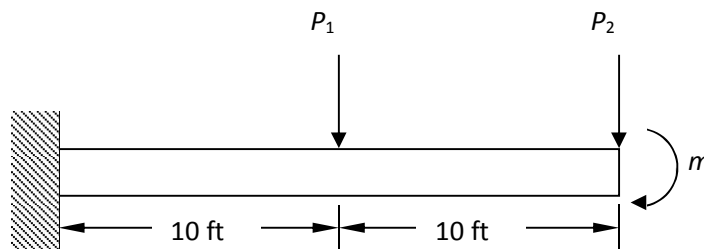
Let  $\mathbf{Y} \in R^m$  be a random response vector of interest, which is related to input  $\mathbf{X} \in R^n$ . The linear system is given in the following equation.

$$\mathbf{Y} \approx \mathbf{A}^T \mathbf{X} + \mathbf{B}$$

where  $\mathbf{A} \in R^n \times R^m$  and  $\mathbf{B} \in R^m$  are coefficient matrix and vector, respectively. Let  $\boldsymbol{\mu}_Y \in R^m$  and  $\boldsymbol{\Sigma}_Y \in R^m \times R^m$  be the mean vector and covariance matrix of output  $\mathbf{Y}$ . Then,

$$\begin{aligned}
 \boldsymbol{\mu}_Y &\approx \mathbf{A}^T \boldsymbol{\mu}_X + \mathbf{B} \\
 \boldsymbol{\Sigma}_Y &\approx \mathbf{A}^T \boldsymbol{\Sigma}_X \mathbf{A}
 \end{aligned} \tag{41}$$

### Exercise: Cantilever Beam



Given that:

$$P_1 \sim N[1000 \text{ lb}, 100^2 \text{ lb}^2], P_2 \sim N[500 \text{ lb}, 50^2 \text{ lb}^2], m = 10000 \text{ lb-ft, deterministic}$$

Assume  $P_1$  and  $P_2$  are uncorrelated. Calculate mean, standard deviation, and coefficient of variation (COV) of maximum moment at the fixed end. Estimate

the reliability when an allowable bending moment ( $M_a$ ) is 33,000 lb-ft.

**Solution:** At fixed end, the maximum moment is expressed as

$$M_{\max} = 10P_1 + 20P_2 + 10000 = \mathbf{a}^T \mathbf{X} + b$$

where  $\mathbf{a}^T = \{10 \ 20\}$ ,  $b = 10000$  lb-ft,

$$\mathbf{X} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \text{ lb}, \quad \mu_X = \begin{Bmatrix} 1000 \\ 500 \end{Bmatrix} \text{ lb}, \quad \Sigma_X = \begin{bmatrix} 100^2 & 0 \\ 0 & 50^2 \end{bmatrix} \text{ lb}^2$$

Mean of  $M_{\max}$

$$\begin{aligned} \mu_{M_{\max}} &= \mathbf{a}^T \mu_X + b = \{10 \ 20\} \begin{Bmatrix} 1000 \\ 500 \end{Bmatrix} + 10000 \\ &= 30,000 \text{ lb-ft} \end{aligned}$$

Variance of  $M_{\max}$

$$\begin{aligned} \sigma_{M_{\max}}^2 &= \mathbf{a}^T \Sigma_X \mathbf{a} = \{10 \ 20\} \begin{bmatrix} 100^2 & 0 \\ 0 & 50^2 \end{bmatrix} \begin{Bmatrix} 10 \\ 20 \end{Bmatrix} \\ &= 2 \times 10^6 \text{ [lb-ft]}^2 \\ \sigma_{M_{\max}} &= \sqrt{2 \times 10^6} = 1414.2 \text{ lb-ft} \end{aligned}$$

Coefficient of Variation (COV) of  $M_{\max}$

$$\text{COV} \equiv \frac{\text{standard deviation}}{\text{mean}} = \frac{1414.2}{30000} = 0.047 \text{ or } 4.7\%$$

Reliability with an allowable moment  $M_a = 33,000$  lb-ft

$$\begin{aligned} R &= P(M_{\max} \leq M_a) \\ &= P\left(\frac{M_{\max} - \mu_{M_{\max}}}{\sigma_{M_{\max}}} \leq \frac{M_a - \mu_{M_{\max}}}{\sigma_{M_{\max}}}\right) \\ &= P\left(z \leq \frac{33,000 - 30,000}{1414} = 2.12\right) \\ &= \Phi(2.12) = 98.3\% \end{aligned}$$

### **Homework 13: Expansion method**

Recall Homework 12. Estimate its reliability using the expansion method when the failure is defined as  $Y < y_c = -3 \times 10^{-3} \text{ m}$ .

Second-order method

Second-order approximation of any nonlinear function ( $Y$ ) can be used for the second-order method as

$$Y(\mathbf{X}) \approx Y(\boldsymbol{\mu}_X) + \sum_{i=1}^n \frac{\partial Y(\boldsymbol{\mu}_X)}{\partial X_i} (X_i - \mu_{X_i}) + \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 Y(\boldsymbol{\mu}_X)}{\partial X_i \partial X_j} (X_i - \mu_{X_i})(X_j - \mu_{X_j}) \quad (42)$$

- Mean of  $Y$

$$E[Y] = \mu_Y \approx Y(\mu_X) + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 Y(\mu_X)}{\partial X_i^2} \sigma_{X_i}^2$$

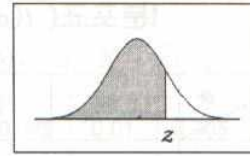
- Variance of  $Y$

$$Var[Y] = \sigma_Y^2 \approx \sum_{i=1}^n \left( \frac{\partial Y(\boldsymbol{\mu}_X)}{\partial X_i} \right)^2 \sigma_{X_i}^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 Y(\boldsymbol{\mu}_X)}{\partial X_i \partial X_j} \sigma_{X_i}^2 \sigma_{X_j}^2$$

### 3. 확률분포표

표준정규분포표

$$(\Pr(Z \leq z) = \Phi(z), Z \sim N(0, 1))$$

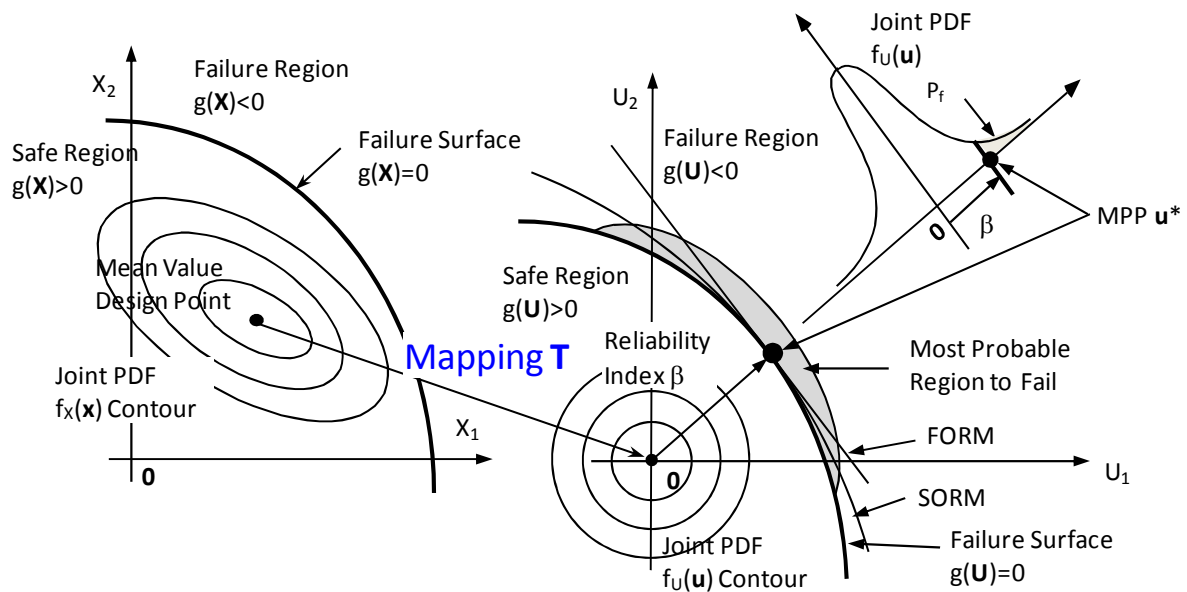


| z   | 0.00   | 0.01   | 0.02   | 0.03   | 0.04   | 0.05   | 0.06   | 0.07   | 0.08   | 0.09   |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |

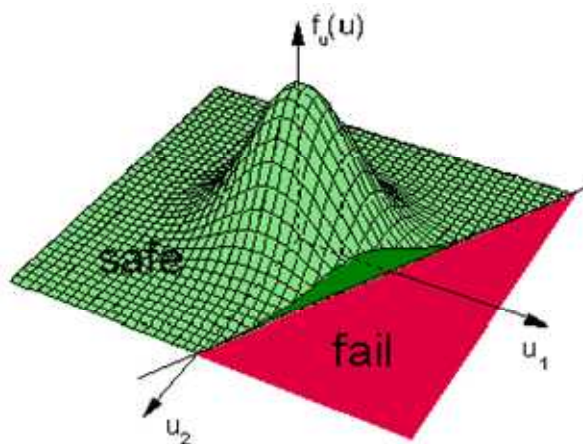
#### 4.7.4 Most Probable Point (MPP) based methods

Most probable point (MPP) based methods include the first order reliability method (FORM) and second order reliability method (SORM). Instead of approximating a response  $Y$  at the mean of  $\mathbf{X}$ , it approximates the function at the most probable point in either a linear or quadratic manner. This is illustrated in Figure 4.6. The MPP is a

pointwise representation of the failure surface and normally computed in a transformed space (or standard Gaussian space). In the MPP based methods, the reliability analysis requires a transformation  $\mathbf{T}$  from the original random parameter  $\mathbf{X}$  to the independent and standard normal parameter  $\mathbf{U}$ . The constraint function  $G(\mathbf{X})$  in  $X$ -space can then be mapped onto  $G(\mathbf{T}(\mathbf{X})) \equiv G(\mathbf{U})$  in  $U$ -space. Rosenblatt transformation is most widely used for transforming any non-normally distributed random vector to standard normal random vector.



( a ) Nonlinear Transformation of Non-normal Distributions



( b ) First-Order Reliability Method

**Figure 4.6: Nonlinear Transformation of Non-normal Distributions**

**Table 4.1: Nonlinear Transformation, T: X → U**

|           | Parameters  | PDF   | Transformation                                  |
|-----------|---|---|---|
| Normal    | $\mu = \text{mean},$<br>$\sigma = \text{standard deviation}$  | $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-0.5[(x-\mu)/\sigma]^2}, -\infty \leq x \leq \infty$     | $X = \mu + \sigma U$                            |
| Lognormal | $\mu = \text{mean}, \sigma = \text{standard deviation}$<br>$\bar{\sigma}^2 = \ln[1 + (\sigma/\mu)^2]$<br>$\bar{\mu} = \ln(\mu) - 0.5\bar{\sigma}^2$ | $f(x) = \frac{1}{\sqrt{2\pi}x\bar{\sigma}} e^{-0.5[(\ln x - \bar{\mu})/\bar{\sigma}]^2}, x > 0$ | $X = e^{\bar{\mu} + \bar{\sigma}U}$             |
| Weibull   | $k > 0, \mu = \Gamma(1 + 1/k)$<br>$\sigma^2 = \nu^2 [\Gamma(1 + 2/k) - \Gamma^2(1 + 1/k)]$  | $f(x) = \frac{k}{\nu} \left(\frac{x}{\nu}\right)^{k-1} e^{-(x/\nu)^k}, x > 0$                   | $X = \nu [-\ln(\Phi(-U))]^{1/k}$                |
| Gumbel    | $\mu = \nu + (0.577/\alpha), \sigma = \pi/\sqrt{6\alpha}$   | $f(x) = \alpha e^{-\alpha(x-\nu) - e^{-\alpha(x-\nu)}}, -\infty \leq x \leq \infty, \sqrt{2}$   | $X = \nu - \frac{1}{\alpha} \ln[-\ln(\Phi(U))]$ |
| Uniform   | $\mu = (a+b)/2, \sigma = (b-a)/\sqrt{12}$   | $f(x) = \frac{1}{b-a}, a \leq x \leq b$   | $X = a + (b-a)\Phi(U)$                          |

where  $\Phi(U) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^U e^{-u^2/2} du$ .

$$P(G_i(\mathbf{X}) \leq 0) = F_{G_i}(0) \geq \Phi(\beta_{t_i}) \text{ or } R_{t_i}$$

$$\text{Time-dependent: } G_i(\mathbf{X}, T) = T_d - T_i \leq 0 \text{ where } T_d \text{ is a designed life.} \quad (43)$$

$$\text{Time-independent: } G_i(\mathbf{X}) = P_i - P_c \leq 0 \text{ where } P_c \text{ is a critical buckling load.}$$

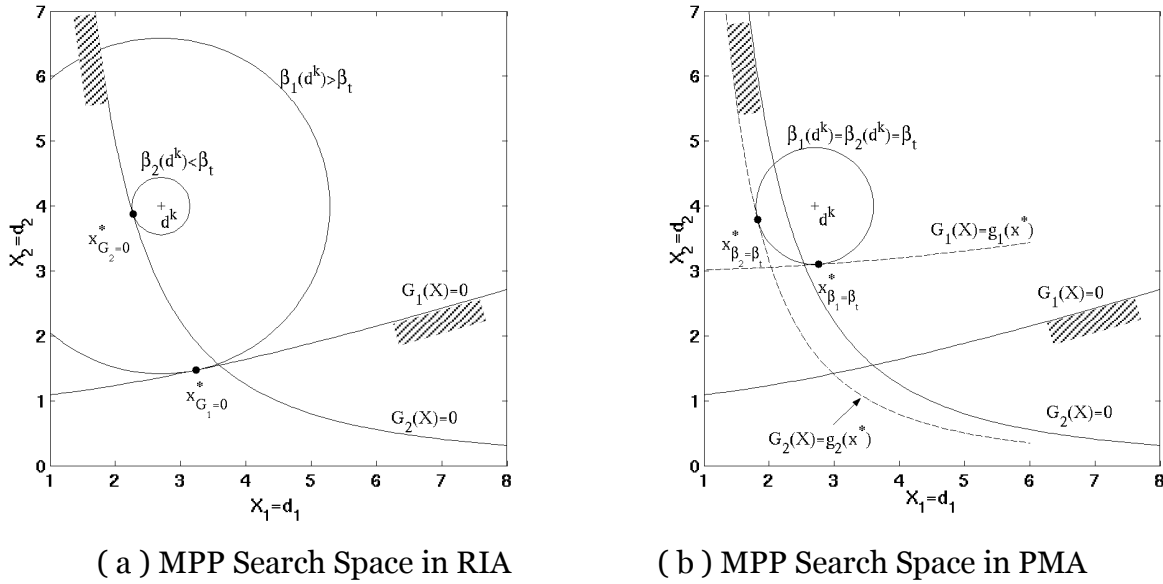
The probabilistic constraint in Eq. (36) can be further expressed in two different ways through inverse transformations as (see Fig. 4.7):

$$\text{RIA: } \beta_{s_i} = \Phi^{-1}\{F_{G_i}(0)\} \geq \beta_{t_i} \quad (44)$$

$$\text{PMA: } G_{p_i} = F_{G_i}^{-1}\{\Phi(\beta_{t_i})\} \leq 0 \quad (45)$$

where  $\beta_{s_i}$  and  $G_{p_i}$  are respectively called the safety reliability index and the probabilistic performance measure for the  $i^{\text{th}}$  probabilistic constraint. Equation (44) is employed to prescribe the probabilistic constraint in Eq. (33) using the reliability measure, i.e. the so-called Reliability Index Approach (RIA). Similarly, Eq. (45) can replace the same probabilistic constraint in Eq. (33) with the performance measure, which is referred to as the Performance Measure Approach (PMA).





**Figure 4.7:** Random Search Space in RIA and PMA

**Formulation for Reliability Index Approach (RIA)**

In RIA, the first-order safety reliability index  $\beta_{s,FORM}$  is obtained using FORM by formulating as an optimization problem with one equality constraint in  $U$ -space, which is defined as a limit state function:

$$\begin{aligned} &\text{minimize} && \| \mathbf{U} \| \\ &\text{subject to} && G(\mathbf{U}) = 0 \end{aligned} \tag{46}$$

where the optimum point on the failure surface is called the **Most Probable Failure Point (MPFP)**  $\mathbf{u}_{G(\mathbf{U})=0}^*$ , and thus  $\beta_{s,FORM} = \| \mathbf{u}_{G(\mathbf{U})=0}^* \|$ .

Either MPFP search algorithms specifically developed for the first-order reliability analysis, or general optimization algorithms can be used to solve Eq. (46). The HL-RF method is employed to perform reliability analyses in RIA due to its simplicity and efficiency.

**HL-RF Method**

The HL-RF method is formulated as follows

$$\begin{aligned} \mathbf{u}^{(k+1)} &= \left( \mathbf{u}^{(k)} \bullet \mathbf{n}^{(k)} - \frac{G(\mathbf{u}^{(k)})}{\| \nabla_U G(\mathbf{u}^{(k)}) \|} \right) \mathbf{n}^{(k)} \\ &= \left[ \nabla_U G(\mathbf{u}^{(k)}) \bullet \mathbf{u}^{(k)} - G(\mathbf{u}^{(k)}) \right] \frac{\nabla_U G(\mathbf{u}^{(k)})}{\| \nabla_U G(\mathbf{u}^{(k)}) \|^2} \end{aligned} \tag{47}$$

where the normalized steepest ascent direction of  $G(\mathbf{U})$  at  $\mathbf{u}^{(k)}$

$$\mathbf{n}^{(k)} = \mathbf{n}(\mathbf{u}^{(k)}) = \frac{\nabla_U G(\mathbf{u}^{(k)})}{\|\nabla_U G(\mathbf{u}^{(k)})\|}$$

and the second term in Eq. (47) is introduced to account for the fact that  $G(\mathbf{U})$  may be other than zero.

```
function [beta,dbeta]=HL_RF(x,kc)
    u=zeros(1,nd); iter=0; Dif=1;
    while Dif >= 1d-5 & iter < 20
        iter=iter + 1;
        [ceq,GCEq]=cons(u,x,kc);
        u=(GCEq*u'-ceq)/norm(GCEq)^2/GCEq;
        U(iter,:)=u/norm(u);
        if iter>1
            Dif=abs(U(iter-1,:)*U(iter,:)'-1);
        end
    end
    beta = norm(u);
    dbeta = -u./(beta*stdx);
end
```

### Formulation for Performance Measure Approach (PMA)

Reliability analysis in PMA can be formulated as the inverse of reliability analysis in RIA. The first-order probabilistic performance measure  $G_{p,FORM}$  is obtained from a nonlinear optimization problem in  $U$ -space defined as

$$\begin{aligned} & \text{maximize} && G(\mathbf{U}) \\ & \text{subject to} && \|\mathbf{U}\| = \beta_t \end{aligned} \quad (48)$$

where the optimum point on a target reliability surface is identified as the **Most Probable Point (MPP)**  $\mathbf{u}_{\beta=\beta_t}^*$  with a prescribed reliability  $\beta_t = \|\mathbf{u}_{\beta=\beta_t}^*\|$ , which will be referred to as MPP. Unlike RIA, only the direction vector  $\mathbf{u}_{\beta=\beta_t}^* / \|\mathbf{u}_{\beta=\beta_t}^*\|$  needs to be determined by exploring the explicit sphere constraint  $\|\mathbf{U}\| = \beta_t$ .

General optimization algorithms can be employed to solve the optimization problem in Eq. (48). However, the Advanced Mean Value (AMV) method is well suited for PMA due to its simplicity and efficiency.

### AMV method

Thus, the AMV method can be formulated as

$$\mathbf{u}_{AMV}^{(1)} = \mathbf{u}_{MV}^*, \quad \mathbf{u}_{AMV}^{(k+1)} = \beta_t \mathbf{n}(\mathbf{u}_{AMV}^{(k)}) \quad (49)$$

where

$$\mathbf{n}(\mathbf{u}_{AMV}^{(k)}) = \frac{\nabla_U G(\mathbf{u}_{AMV}^{(k)})}{\|\nabla_U G(\mathbf{u}_{AMV}^{(k)})\|} \quad (50)$$

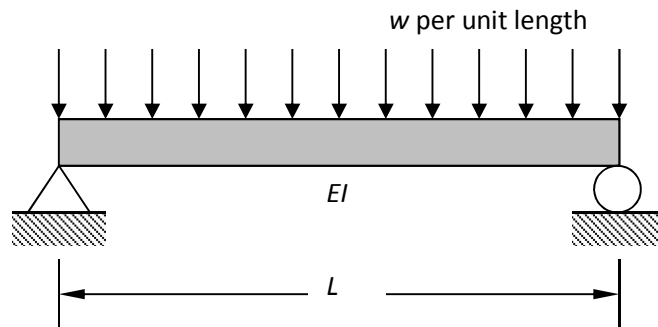
```
function [G,DG]=AMV(x,kc)
    u=zeros(1,nd); iter = 0; Dif=1;
    while Dif>1d-5 & iter<20
        iter=iter+1;
        if iter>1
            u=DG*bt/norm(DG);
        end
        [G,DG]=cons(u,x,kc);
        U(iter,:)=u/bt;
        if iter>1
            Dif=abs(U(iter,:)*U(iter-1,:)'-1);
        end
    end
end
```

**Table 4.2: Properties of the RIA and PMA**

|     | Properties  |
|-----|---|
| RIA | <ol style="list-style-type: none"> <li>1. Good for reliability analysis</li> <li>2. Expensive with sampling method and MPP-based method when reliability is high.</li> <li>3. MPP-based method could be unstable when reliability is high or a performance function is highly nonlinear.</li> </ol> |
| PMA | <ol style="list-style-type: none"> <li>1. Good for design optimization.</li> <li>2. Not suitable for assessing reliability.</li> <li>3. Efficient and stable for design optimization.</li> </ol>  |

**Homework 14: First-Order Reliability Method**

Consider the following simply supported beam subject to a uniform load, as illustrated in Fig. 4.13. Suppose  $L = 5 \text{ m}$  and  $w = 10 \text{ kN/m}$ .



**Figure 4.8:** Simply Supported Beam

Random Vector:

$$EI = X_1 \sim N(\mu_{X_1} = 3 \times 10^7, \sigma_{X_1} = 10^5)$$

$$w = X_2 \sim N(\mu_{X_2} = 10^4, \sigma_{X_2} = 10^3)$$

The maximum deflection of the beam is shown as

$$Y = g(X_1, X_2) = -\frac{5X_2L^4}{384X_1}$$

Estimate its reliability using the MPP-based method (HL-RF) when the failure is defined as  $Y < y_c = -3 \times 10^{-3} \text{ m}$ . Make your own discussion and conclusion.

#### 4.7.5 Stochastic response surface method

##### Dimension reduction family:

Dimension reduction (DR) method simplifies a single multi-dimensional integration to multiple one-dimensional integration or multiple one- and two-dimensional integration using additive decomposition. This section introduces univariate dimension reduction (UDR) method.

For the approximation of the multi-dimensional integration, consider an integration of two dimensional function which can be expressed by the Taylor series expansion by

$$\begin{aligned}
I(Y(x_1, x_2)) &= I(Y(0, 0)) \\
&+ \sum_{i=1}^2 \frac{\partial Y}{\partial x_i}(0, 0) I[x_i] \\
&+ \frac{1}{2!} \sum_{i=1}^2 \frac{\partial^2 Y}{\partial x_i^2}(0, 0) I[x_i^2] + \frac{\partial^2 Y}{\partial x_1 \partial x_2}(0, 0) I[x_1 x_2] \\
&+ \frac{1}{3!} \sum_{i=1}^2 \frac{\partial^3 Y}{\partial x_i^3}(0, 0) I[x_i^3] + \frac{1}{2!} \frac{\partial^3 Y}{\partial x_1^2 \partial x_2}(0, 0) I[x_1^2 x_2] + \frac{1}{2!} \frac{\partial^3 Y}{\partial x_1 \partial x_2^2}(0, 0) I[x_1 x_2^2] \\
&+ \frac{1}{4!} \sum_{i=1}^2 \frac{\partial^4 Y}{\partial x_i^4}(0, 0) I[x_i^4] \\
&+ \frac{1}{3!} \frac{\partial^4 Y}{\partial x_1^3 \partial x_2}(0, 0) I[x_1^3 x_2] + \frac{1}{3!} \frac{\partial^4 Y}{\partial x_1^2 \partial x_2^2}(0, 0) I[x_1^2 x_2^2] + \frac{1}{3!} \frac{\partial^4 Y}{\partial x_1 \partial x_2^3}(0, 0) I[x_1 x_2^3] + \dots
\end{aligned}$$

where integration term can be defined as

$$I[Y(x_1, x_2)] = \int_{-a}^{+a} \int_{-a}^{+a} Y(x_1, x_2) dx_1 dx_2$$

Because integrations of the odd functions are zero, the integration of Taylor series expansion of the target function ( $Y$ ) can be expressed as:

$$\begin{aligned}
I[Y(x)] &= I[Y(0)] + \frac{1}{2!} \sum_{i=1}^N \frac{\partial^2 Y}{\partial x_i^2}(0) I[x_i^2] \\
&+ \frac{1}{4!} \sum_{i=1}^N \frac{\partial^4 Y}{\partial x_i^4}(0) I[x_i^4] \\
&+ \frac{1}{2!2!} \sum_{i < j} \frac{\partial^4 Y}{\partial x_i^2 \partial x_j^2}(0) I[x_i^2 x_j^2] + \dots
\end{aligned}$$

where  $I(\bullet)$  calculates integration over the given space.

This is also computationally expensive because of the terms including multi-dimensional integration such as  $I[x_i^2 x_j^2]$ . To effectively remove the terms with multi-dimensional integration, additive decomposition,  $Y_a$ , is defined as:

$$\begin{aligned}
Y(X_1, \dots, X_N) &\cong Y_a(X_1, \dots, X_N) \\
&= \sum_{j=1}^N Y(\mu_1, \dots, \mu_{j-1}, X_j, \mu_{j+1}, \dots, \mu_N) - (N-1) Y(\mu_1, \dots, \mu_N)
\end{aligned}$$

Integration of Taylor series expansion of the additive decomposition ( $Y_a$ ) can be expressed as:

$$\begin{aligned}
I[Y_a(x)] &= I[Y(0)] + \frac{1}{2!} \sum_{i=1}^N \frac{\partial^2 Y}{\partial x_i^2}(0) I[x_i^2] \\
&+ \frac{1}{4!} \sum_{i=1}^N \frac{\partial^4 Y}{\partial x_i^4}(0) I[x_i^4] \\
&+ \frac{1}{6!} \sum_{i=1}^N \frac{\partial^6 Y}{\partial x_i^6}(0) I[x_i^6] + \dots
\end{aligned}$$

This results the largest error at the fourth even-order term, producing negligible error.

$$I[Y(x)] - I[Y_a(x)] = \frac{1}{2!2!} \sum_{i < j}^N \frac{\partial^4 Y}{\partial x_i^2 \partial x_j^2}(0) I[x_i^2 x_j^2] + \dots$$

For probabilistic engineering analysis, the  $m^{\text{th}}$  statistical moments for the responses are considered as

$$\begin{aligned} E[Y^m(X)] &\cong E[Y_a^m(X)] \\ &= E \left\{ \left[ \sum_{j=1}^N Y(\mu_1, \dots, \mu_{j-1}, X_j, \mu_{j+1}, \dots, \mu_N) - (N-1)Y(\mu_1, \dots, \mu_N) \right]^m \right\} \end{aligned}$$

Applying the Binomial formula on the right-hand side of the equation above gives

$$\begin{aligned} m_l &\cong \sum_{i=0}^l \binom{l}{i} \mathcal{E} \left\{ \sum_{j=1}^N Y(\mu_1, \dots, \mu_{j-1}, X_j, \mu_{j+1}, \dots, \mu_N) \right\}^i \\ &\quad [-(N-1)y(\mu_1, \dots, \mu_N)]^{l-i}. \end{aligned}$$

One-dimensional integration will be performed with integration weights  $w_{j,i}$  and points  $x_{j,i}$  as

$$\begin{aligned} E \left[ \sum_{j=1}^N Y^m(\mu_1, \dots, \mu_{j-1}, X_j, \mu_{j+1}, \dots, \mu_N) \right] \\ \cong \sum_{j=1}^N \sum_{i=1}^n w_{j,i} Y^m(\mu_1, \dots, \mu_{j-1}, x_{j,i}, \mu_{j+1}, \dots, \mu_N) \end{aligned}$$

where  $N$  is the number of input random parameters and  $n$  is the number integration point along each random variable. An empirical sample point distribution for the UDR when  $m = 3$  is shown in the Fig. 4-DR1. We can see that, compared to the full factorial sample points, the UDR achieves a significant reduction in the number of sample points.

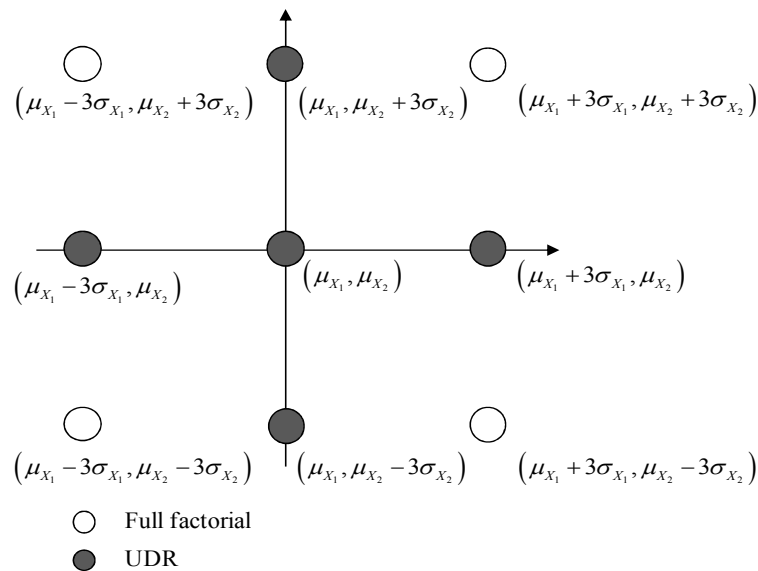


Fig. 4-DR1. Empirical sample point distribution for UDR ( $m=3$ )

- Refer to [http://www.sciencedirect.com/science?\\_ob=ArticleURL&\\_udi=B6V4M-4H74MBO-1&\\_user=961305&\\_rdoc=1&\\_fmt=&\\_orig=search&\\_sort=d&view=c&\\_acct=C00049425&\\_version=1&\\_urlVersion=0&\\_userid=961305&md5=6e56b71561720cf e918f32c3eaa2cf86](http://www.sciencedirect.com/science?_ob=ArticleURL&_udi=B6V4M-4H74MBO-1&_user=961305&_rdoc=1&_fmt=&_orig=search&_sort=d&view=c&_acct=C00049425&_version=1&_urlVersion=0&_userid=961305&md5=6e56b71561720cf e918f32c3eaa2cf86)
- Refer to <http://www.springerlink.com/content/416l79447313n8q1>

Polynomial Chaos Expansion (PCE) method  
Tensor-Product (or Stochastic Collocation) Method

#### 4.8 Bayesian Description of Time-Independent Performance

When modeling uncertainties with insufficient data, the probability of safety (or satisfying a specification), referred to as reliability, must be uncertain and subjective. Because the Bayes theory provides a systematic framework of aggregating and updating uncertain information, reliability analysis based on the Bayes theory, referred to as Bayesian reliability, is employed to deal with subjective and insufficient data sets.

##### 4.8.1 Bayesian binomial inference - reliability

###### • Bayesian binomial inference

If the probability of a safety event in each sample is  $r$  and the probability of failure is  $(1-r)$ , then the probability of  $x$  safety occurrences out of a total of  $N$  samples can be described by the probability mass function (PMF) of a Binomial distribution as

$$\Pr(X = x, N | r) = \binom{N}{x} r^x (1-r)^{N-x}, \quad x = 0, 1, 2, \dots, N \quad (51)$$

When  $r$  is an uncertain parameter and a prior distribution is provided, a Bayesian inference process can be employed to update  $r$  based on the outcomes of the sample tests. It is possible to obtain a posterior distribution with any type of a prior distribution. A Bayesian inference model is called a conjugate model if the conjugate prior distribution is used. For conjugate Bayesian inference models, the updating results are independent of the sequence of data sets.

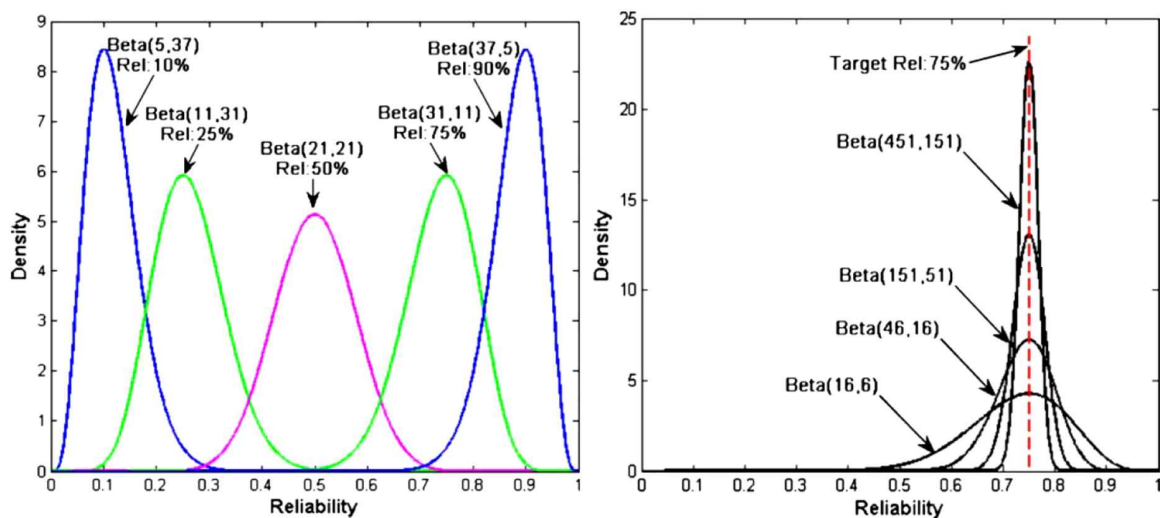
#### • Conjugate prior reliability distribution

For Bayesian reliability analysis, both prior reliability distribution ( $r$ ) and the number ( $x$ ) of safety occurrences out of the total number of test data set  $N$  must be known. If prior reliability distribution ( $r$ ) is unavailable, it will be simply modeled with a uniform distribution,  $r \sim U(a, b)$  where  $a < b$  and  $a, b \in [0, 1]$ . In all cases, reliability will be modeled with Beta distribution, the conjugate distribution of the Bayesian binomial inference, because the uniform distribution is a special case of the Beta distribution.

$$f(r|x) = \frac{1}{B(a,b)} r^{a-1} (1-r)^{b-1}, \quad B(a,b): \text{Beta function} \quad (52)$$

where  $a = x + 1$  and  $b = N - x + 1$ . The larger the number of safety occurrences for a given  $N$  samples, the greater the mean of reliability, as shown in Figure 4.9 (a). As the total number of samples is increased, the variation of reliability is decreased, as shown in Figure 4.9 (b).

In Bayesian inference model, the binomial distribution likelihood function is used for test data, whereas the conjugate prior distribution of this likelihood function is used for reliability ( $r$ ), which is a beta distribution. However, it is found that the Bayesian updating results often depend on the selection of a prior distribution in the conjugate models. Besides, the available conjugate Bayesian models are limited. To eliminate the dependency and the limitation, a non-conjugate Bayesian updating model can be developed using Markov chain Monte Carlo methods. This is, however, more computationally intensive.





(a) (b)  
**Figure 4.9:** Dependence of the PDF of reliability on the number of safety occurrences,  $x$  and the total number of samples,  $N$

#### 4.8.2 Definition of Bayesian reliability

Bayesian reliability must satisfy two requirements: (a) sufficiency and (b) uniqueness. The sufficiency requirement means that the Bayesian reliability must be no larger than an exact reliability, when it is realized with a sufficient amount of data for input uncertainties. The uniqueness requirement means that the Bayesian reliability must be uniquely defined for the purpose of design optimization. To meet these two requirements, Bayesian reliability is generally defined with a confidence level of reliability prediction where the confidence level  $C_L$  of Bayesian reliability is defined as

$$C_L = \Pr(R > R_B) = \int_{R_B}^1 f(r|\bar{x})dr = 1 - F_R(R_B) \quad (53)$$

With the predefined confidence level  $C_L$ , Bayesian reliability can be defined as

$$R_B = F_R^{-1}[1 - C_L] \quad (54)$$

Therefore, Bayesian reliability can be formulated as a function of a predefined confidence level. Bayesian reliability is desirable since it is defined from the reliability distribution with a corresponding confidence level and accounts for reliability modeling error due to the lack of data.

To guarantee the sufficiency requirement, extreme distribution theory for the smallest reliability value is employed. Based on the extreme distribution theory, the extreme distribution for the smallest reliability value is constructed from the reliability distribution, beta distribution. For random reliability  $R$ , which follows the beta distribution,  $F_R(r)$ , let  $R_1$  be the smallest value among  $N$  data points, the CDF of the smallest reliability value,  $R_1$ , can be expressed as

$$1 - F_{R_1}(r) = \Pr(R_1 > r) = \Pr(R_1 > r, R_2 > r, \dots, R_N > r) \quad (55)$$

Since the  $i$ th smallest reliability values,  $R_i(i=1, \dots, N)$ , are identically distributed and statistically independent, the CDF of the smallest reliability value becomes

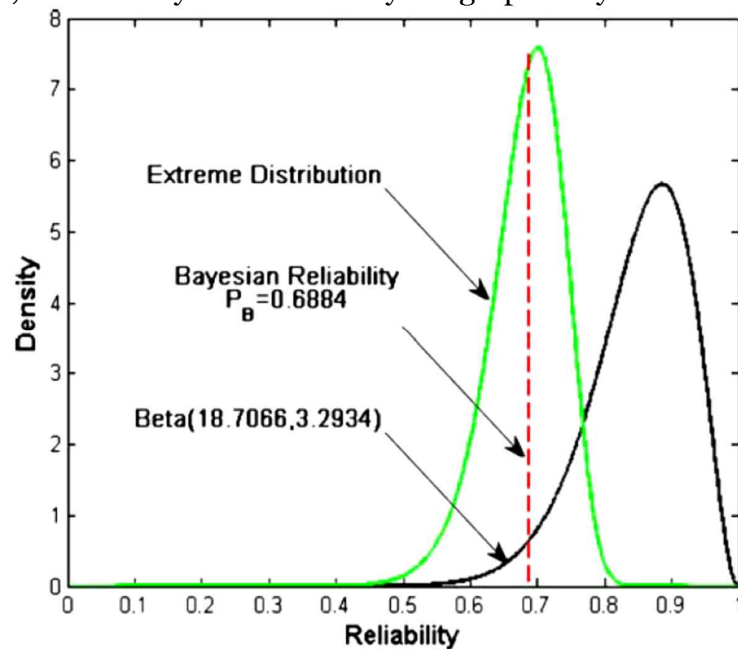
$$F_{R_1}(r) = 1 - [1 - F_R(r)]^N \quad (56)$$

Then Bayesian reliability,  $R_B$ , is uniquely determined as the median value of the extreme distribution. Based on this definition, Bayesian reliability and its confidence level can be respectively obtained as the solution of the nonlinear equation, by setting  $F_{R_1}(R_B) = 0.5$

$$R_B = F_R^{-1}\left[1 - \sqrt[N]{1 - F_{R_1}(R_B)}\right] = F_R^{-1}\left[1 - \sqrt[N]{0.5}\right] \quad (57)$$

$$C_L = 1 - F_R(R_B) = 1 - F_R\left(F_R^{-1}\left[1 - \sqrt[N]{0.5}\right]\right) = \sqrt[N]{0.5} \quad (58)$$

The Beta distribution for reliability, its extreme distribution for the smallest reliability value, and the Bayesian reliability are graphically shown as below.



**Figure 4.10:** Bayesian reliability

#### 4.8.3 Numerical procedure of Bayesian reliability analysis

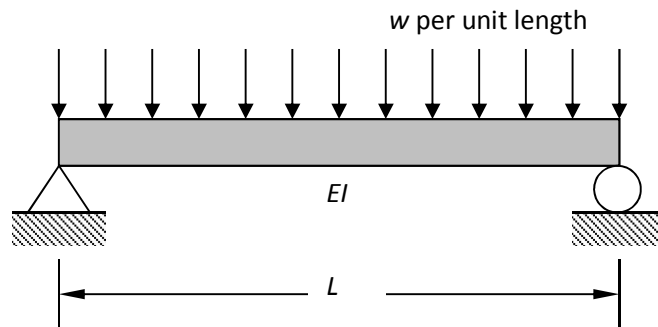
Bayesian reliability analysis can be conducted using a numerical procedure as follows.

- Step 1: collect a limited data set for epistemic uncertainties where the data size is  $N$ .
- Step 2: calculate reliabilities ( $R_k$ ) with consideration of aleatory uncertainties at all epistemic data points.
- Step 3: build a distribution of reliability using the beta distribution with aleatory and/or epistemic uncertainties.
- Step 4: select an appropriate confidence level,  $C_L$ , of Bayesian reliability.
- Step 5: determine the Bayesian reliability.

Refer to <http://www.springerlink.com/content/u1185070336p4116/fulltext.pdf>.

**Homework 15: Bayesian Reliability Analysis**

Consider the following simply supported beam subject to a uniform load, as illustrated in Fig. 4.14. Suppose  $L = 5$  m and  $w = 10$  kN/m.



**Figure 4.11:** Simply Supported Beam

Random Vector:

$$EI = X_1 \sim N(\mu_{X_1} = 3 \times 10^7, \sigma_{X_1} = 10^5)$$

$$w = X_2 \sim \text{epistemic}$$

The maximum deflection of the beam is shown as

$$Y = g(X_1, X_2) = -\frac{5X_2L^4}{384X_1}$$

The  $X_2$  is an epistemic uncertainty. For  $X_2$ , it is assumed that 10 data sets are gradually obtained at different times. Using MPP-based method (HL-RF), determine the reliability of the maximum deflection constraint,  $P(Y(X_1) \geq y_c = -3 \times 10^{-3} \text{m})$ , at all individual  $X_2$  points in the table. Predict the PDF of reliability in a Bayesian sense using the first 10 data set and gradually update the PDFs of reliability using the second and third data sets. Make your own discussion and conclusion, and attach your code used for Bayesian reliability analysis.

**Table 4.3** Three sets of 10 data for  $X_2$  ( $\times 10^4$ )

|      |        |        |        |        |        |        |        |        |        |        |
|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Set1 | 1.0000 | 0.8126 | 1.0731 | 1.0677 | 0.9623 | 0.9766 | 1.1444 | 1.0799 | 1.0212 | 0.9258 |
| Set2 | 0.9682 | 1.0428 | 1.0578 | 1.0569 | 0.9704 | 1.0118 | 0.9649 | 1.0941 | 1.0238 | 1.1082 |
| Set3 | 1.1095 | 1.0896 | 1.0040 | 0.9744 | 0.8525 | 1.0315 | 1.0623 | 0.9008 | 0.8992 | 0.9869 |

## CHAPTER 5. PROBABILISTIC ENGINEERING ANALYSIS – TIME-DEPENDENT PERFORMANCE

In the previous chapter, methods were examined for obtaining the system's reliability function analytically or computationally. In the calculation of probability values, we consider time independent safety events. However, in many practical cases, system's performance degrade over time. In this chapter, time dependency in the probability of safety occurrence (or reliability) will be introduced. We will develop the reliability models necessary to observe the reliability over the life of the system, instead of at just one point in time. In addition, performance measure such as MTTF and failure rate are presented and also its related distributions are introduced. An accelerated life test will be discussed to acquire time dependent data in an efficient manner. Lastly, we take a glance at overview of PHM in the end of the chapter.

### 5.1 Reliability Function (Time-Dependent)

#### 5.1.1 Reliability Function

The **Reliability Function**  $R(t)$ , also known as the **Survival Function**  $S(t)$ , is defined by:

$R(t) = S(t)$  = the probability a unit survives beyond a designed life  $t$ .

Since a unit either fails or survives, one of these two mutually exclusive alternatives must occur as

$$\begin{aligned} R(t) &= P(T > t) = 1 - P(T \leq t) = 1 - F_T(t) \\ &= 1 - \int_0^t f_T(\tau) d\tau = \int_t^{\infty} f_T(\tau) d\tau \end{aligned} \quad (59)$$

where  $F_T(t)$  is the probability distribution function or CDF of an actual life and  $f_T(t)$  is the PDF of an actual life.

#### 5.1.2 Expected Life or Mean Time-To-Failure (MTTF):

$$\begin{aligned} E[T] &= \int_0^{\infty} \tau f_T(\tau) d\tau = - \int_0^{\infty} \tau \frac{\partial R(\tau)}{\partial \tau} d\tau \\ &= - [tR(t)]_0^{\infty} + \int_0^{\infty} R(\tau) d\tau \\ &= \int_0^{\infty} R(\tau) d\tau \end{aligned} \quad (60)$$

#### 5.1.3 Failure Rate (or Hazard Function):

Insight is normally gained into failure mechanisms by examining the behavior of the failure rate. The failure rate,  $h(t)$ , may be defined in terms of the reliability or the PDF of the time-to-failure (TTF). Let  $h(t)\Delta t$  be the probability that the system will fail at some time  $T < t + \Delta t$  given that it has not yet failed at  $T = t$ . Thus, it is the conditional probability as

$$\begin{aligned}
 h(t)\Delta t &= P\{T < t + \Delta t \mid T > t\} = \frac{P\{(T > t) \cap (T < t + \Delta t)\}}{P\{T > t\}} \\
 &= \frac{P\{t < T < t + \Delta t\}}{R(t)} = \frac{f_T(t)\Delta t}{R(t)}
 \end{aligned} \tag{61}$$

or

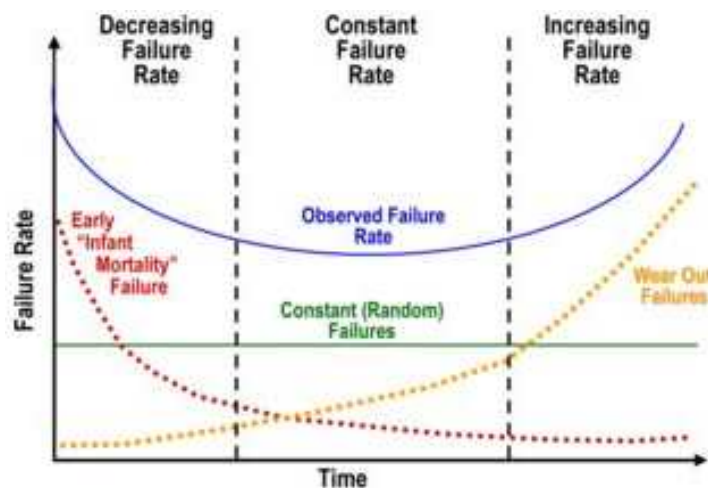
$$h(t) = \frac{f_T(t)}{R(t)}$$

There are a handful of parametric models that have successfully served as population models for failure times (TTF) arising from a wide range of products and failure mechanisms. Sometimes there are probabilistic arguments based on the physics of the failure mechanics that tend to justify the choice of model. Other times the model is used solely because of its empirical success in fitting actual failure data.

#### 5.1.4 Bathtub Curve:

The **bathtub curve** is widely used in reliability engineering, although the general concept is also applicable to humans. It describes a particular form of the **hazard function** which comprises three parts:

- The first part is a **decreasing failure rate**, known as early failures or **infant mortality**.
- The second part is a **constant failure rate**, known as **random failures**.
- The third part is an **increasing failure rate**, known as **wear-out failures**.



**Figure 5.1:** Bathtub Curve for Hazard Function (or Failure Rate)

#### **Homework 16:** *Failure testing*

Perform the failure testing of a paper clip as instructed.

## 5.2 Parametric Distribution for Life Data

Some parametric models will be described in this section. There are two classes to describe a failure rate: (1) constant failure rate (section 4.4.1) and (2) time-dependent failure rate (sections 4.4.2-4.4.4).

### 5.2.1 Exponential Distribution (Constant Failure Rate)

The exponential model, with only one unknown parameter, is the simplest of all life distribution models.

PDF:

$$f_T(t; \lambda) = \begin{cases} \lambda e^{-\lambda t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

where  $\lambda > 0$  is a rate parameter of the distribution.

CDF:

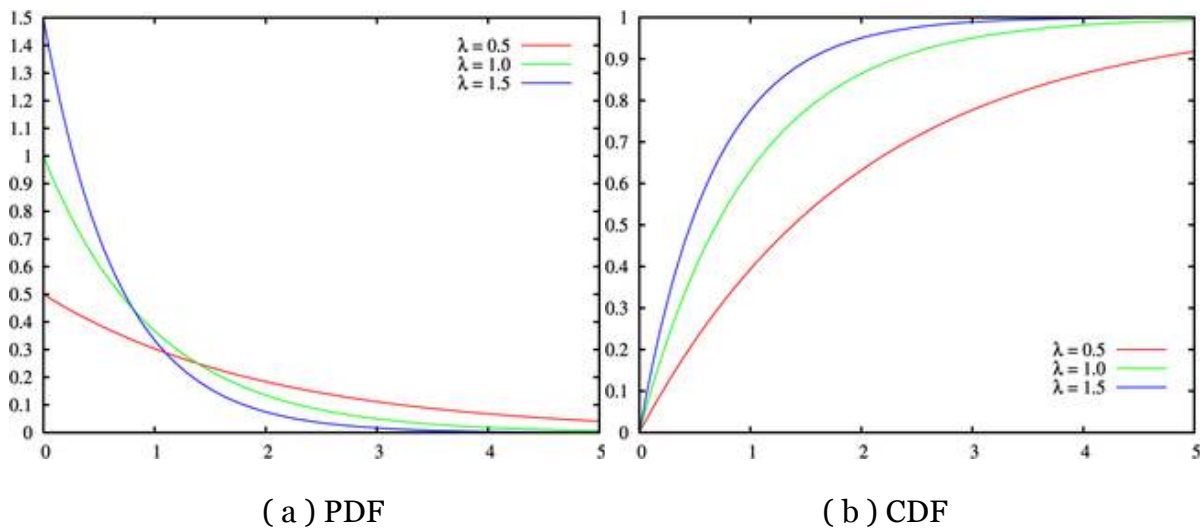
$$F_T(t; \lambda) = \begin{cases} 1 - e^{-\lambda t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

Reliability and Hazard Functions:

$$R(t) = 1 - \int_0^t f_T(\tau) d\tau = e^{-\lambda t} \quad \text{and} \quad h(t) = \frac{f_T(t)}{R(t)} = \lambda$$

MTTF:

$$\text{MTTF} = \mu_T = \int_0^{\infty} \tau f_T(\tau) d\tau = 1/\lambda \quad \text{and} \quad \sigma_T^2 = 1/\lambda^2$$



**Figure 5.2:** Exponential Distribution

**5.2.2 Weibull Distribution**

PDF:

$$f_T(t; k, \lambda, a) = \frac{k-a}{\lambda-a} \left( \frac{t-a}{\lambda-a} \right)^{k-1} e^{-\left(\frac{t-a}{\lambda-a}\right)^k}$$

where  $t \geq a$ ,  $a$  is a waiting time parameter,  $k > 0$  is a shape parameter, and  $\lambda > 0$  is the scale parameter of the distribution.

CDF:

$$F_T(t; \lambda) = 1 - \exp\left[-(t/\lambda)^k\right]$$

Reliability and Hazard Functions:

$$R(t) = 1 - \int_0^t f_T(\tau) d\tau = \exp\left[-(t/\lambda)^k\right] \quad \text{and} \quad h(t) = \frac{k}{\lambda} \left(\frac{t}{\lambda}\right)^{k-1}$$

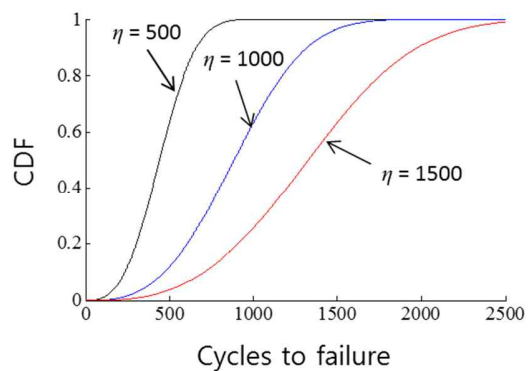
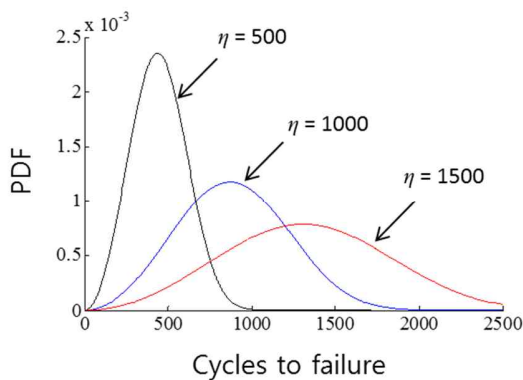
MTTF:

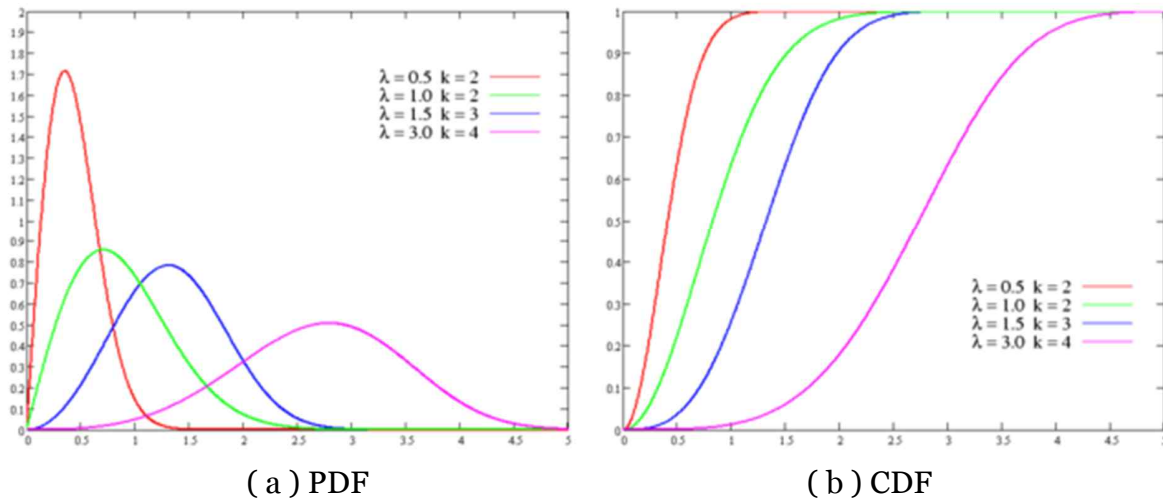
$$\text{MTTF} = \mu_T = \int_0^\infty \tau f_T(\tau) d\tau = \lambda \Gamma\left(1 + \frac{1}{k}\right) \quad \text{and} \quad \sigma_T^2 = \lambda^2 \Gamma\left(1 + \frac{2}{k}\right) - \mu_T^2$$

▪ **척도 모수  $\eta$ 의 영향**

- ✓  $f_T(t; \beta, \eta) = \frac{\beta}{\eta^\beta} t^{\beta-1} e^{-(t/\eta)^\beta}$
- ✓  $\eta$ : 특성 수명 = 63.2% 고장
- ✓  $\eta = 40,000$  hrs (Ball bearing)

|                  |     |       |       |
|------------------|-----|-------|-------|
| 형상모수 ( $\beta$ ) | 3   |       |       |
| 척도모수 ( $\eta$ )  | 500 | 1,000 | 1,500 |





**Figure 5.3:** Weibull Distribution

The Weibull is a very flexible life distribution model with two parameters.

- When  $k = 1$ , the Weibull reduces to the exponential model with  $\mu_T = 1/\lambda$ .
- For  $k < 1$ , failure rates are typical of infant mortality and decrease.
- For  $k > 1$ , failure rates are typical of aging effects and increase.
- For  $k = 2$ , the Weibull becomes the Rayleigh distribution.
- For  $k > 4$ , the Weibull becomes closer to a normal.

### 5.2.3 Normal Distribution

PDF:

$$f_T(t; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right\}$$

where  $\mu$  and  $\sigma$  are the mean and standard deviation of the distribution.

CDF:

$$F_T(t; \lambda) = \int_{-\infty}^t \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\tau-\mu}{\sigma}\right)^2\right\} d\tau = \Phi\left(\frac{t-\mu}{\sigma}\right)$$

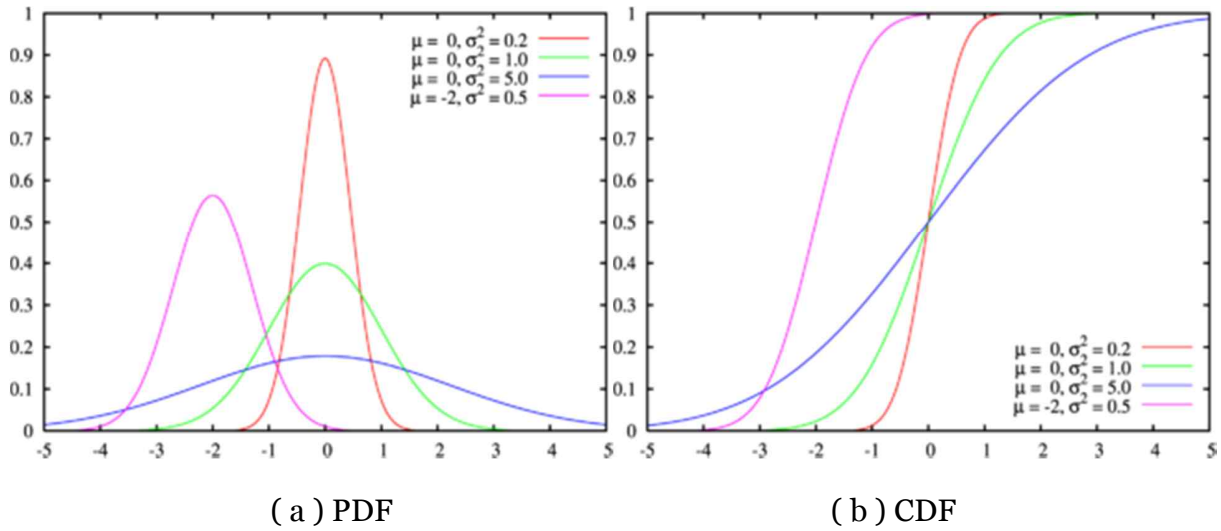
Reliability and Hazard Functions:

$$R(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right) \quad \text{and} \quad h(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right\} \left\{1 - \Phi\left(\frac{t-\mu}{\sigma}\right)\right\}^{-1}$$

MTTF:

$$\text{MTTF} = \mu_T = \mu \quad \text{and} \quad \sigma_T^2 = \sigma^2$$





**Figure 5.4:** Normal Distribution

The normal distribution describes the life of a tread on a tire or the cutting edge on a machine tool where the wearout time ( $\mu$ ) is reasonably well-defined.

**5.2.4 Other Distributions**

Lognormal, Gamma, and others are available at <http://www.itl.nist.gov/div898/handbook/apr/section1/apr16.htm>.

**Homework 17: Reliability Function**

Suppose it is desired to estimate the failure rate of an electronic component. A test can be performed to estimate its failure rate. A target life is set to 2000 minutes.  $R(t) = P(T > 2000 \text{ minutes})$  Answer the following questions:

- (1) Construct a histogram of TTF.
- (2) Find out the best probability distribution model and its parameters,  $f_T(t)$ , for the TTF data.
- (3) Construct a reliability function.
- (4) Determine MTTF, standard deviation of TTF, and hazard function.
- (5) Compare the reliability,  $n_f/N$ , from the TTF data with the reliability from the reliability function when  $t = 2000$  where  $n_f$  is the number of failed components and  $N (= 100)$  is the total components.

**Table 5.1:** Data for 100 Electronics Time-To-Failure (TTF) [minute]

|        |        |        |        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1703.2 | 1071.4 | 2225.8 | 1826.5 | 1131   | 2068.9 | 1573.5 | 1522.1 | 1490.7 | 2226.6 |
| 1481.1 | 2065.1 | 1880.9 | 2290.9 | 1786.4 | 1867.2 | 1859.1 | 1907.5 | 1791.8 | 1871   |
| 1990.4 | 2024.1 | 1688.6 | 1962.7 | 2191.7 | 1841   | 1814.1 | 1918.1 | 2237.5 | 1396.8 |

|        |        |        |        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1692.8 | 707.2  | 2101.3 | 2165.4 | 1975.2 | 1961.6 | 2116.7 | 1373   | 1798.8 | 2248.4 |
| 1872.3 | 1597.8 | 1865.1 | 742.8  | 1436.7 | 1380.8 | 2258.2 | 1960   | 2182.8 | 1772.7 |
| 2003.6 | 1589.4 | 1988.3 | 1874.9 | 1859   | 2051.9 | 1763   | 1854.6 | 1974.7 | 2289.9 |
| 1945.7 | 1774.8 | 1579.6 | 1430.5 | 1855   | 1757.9 | 1029.3 | 1707.2 | 1864.7 | 1964.8 |
| 1719.4 | 1565.2 | 1736.8 | 1759.4 | 1939.4 | 2065.7 | 2258.5 | 2292.8 | 1452.5 | 1692.2 |
| 2120.7 | 1934.8 | 999.4  | 1919.9 | 2162.4 | 2094.9 | 2158.2 | 1884.2 | 1748.7 | 2260.3 |
| 1040.8 | 1535   | 1283.4 | 2267.7 | 2100.3 | 2007.9 | 2499.8 | 1902.9 | 1599.6 | 1567.5 |

### 5.3 Time-Dependent Reliability Analysis: (Physical) Accelerated Tests

The product life test would require a long-time test (e.g.,  $10^4 \sim 10^5$  hours) under normal stress condition. The questions then arise of how to collect information about the corresponding life distributions under normal use conditions and how to make a product design reliable. There are two closely related problems that are typical with reliability data:

- **Censoring** (when the observation period ends, not all units have failed - some are survivors): Censored Type I (observe  $r$  for a fixed time,  $T$ ) and Type II (observe  $T$  for a fixed number of failures,  $r$ ).
- **Lack of Failures** (even if there is too much censoring, say a large number of units under observation, the information in the data can be limited due to the lack of actual failures).  
→ How to deal with **suspension data** and to **design life testing**

These problems cause practical difficulty when planning reliability assessment tests and analyzing failure data. A common way of tackling this problem is an **Accelerated Life Testing (ALT)**.

- **Compressed-time testing**

Many products experience on-off operation cycles instead of continuous operation. Reliability tests are performed in which appliance doors are more frequently opened and closed, consumer electronics is more frequently turned on and off, or pumps or motors are more frequently started and stopped to reach a designed life. These are referred to as compressed-time tests. The tests are used more steadily or frequently than in normal use, but the loads and environmental stresses are maintained at the level expected in normal use.

If the cycle is accelerated too much, however, the conditions of operation may change and thus artificially generate different failure mechanisms. In other words, compressed-time testing (e.g., door open/close) may introduce

different failure mechanisms instead of a primary failure mechanism under normal field operation.

- Advanced stress testing (or physical acceleration testing)

Failure mechanisms may not be accelerated using the forgoing time-compressed testing. Advanced stress testing, however, may be employed to accelerate failures, since as increased loads or harsher environments are applied to a device, an increased failure rate may be observed. If a decrease in reliability can be quantitatively related to an increase in stress level, the life tests can be performed at high stress levels, and the reliability at normal levels inferred. Both random failures and aging effects may be the subject of advanced stress tests.

Some engineering instances include:

- In the electronics industry, components are tested at elevated temperatures to increase the incidence of random failure.
- In the nuclear industry, pressure vessel steels are exposed to extreme levels of neutron irradiation to increase the rate of failure.

### 5.3.1 Physical Acceleration (or True Acceleration)

Physical acceleration means that operating a unit at high stress (i.e., higher temperature or voltage or humidity or duty cycle, etc.) produces the same failures that would occur at normal-use stresses, except that they happen much quicker. Failure may be due to mechanical fatigue, corrosion, chemical reaction, diffusion, migration, etc. These are the same causes of failure under normal stress; the time scale is simply different.

#### Exercise: Non-parametric process

Accelerated life tests are run on four sets of 12 flashlight bulbs and the failure times in minutes are found in Table 5.2. Estimate the MTTF at each voltage and extrapolate the results to the normal operating voltage of 6.0 volts.

**Table 5.2:** Life Data for Flashlight Bulbs (TTF) [minute]

| Voltage | 9.4  | 12.6 | 14.3 | 16 |
|---------|------|------|------|----|
| 1       | 63   | 87   | 9    | 7  |
| 2       | 3542 | 111  | 13   | 9  |
| 3       | 3782 | 117  | 23   | 9  |
| 4       | 4172 | 118  | 25   | 9  |
| 5       | 4412 | 121  | 28   | 9  |
| 6       | 4647 | 121  | 30   | 9  |

|    |      |     |    |    |
|----|------|-----|----|----|
| 7  | 5610 | 124 | 32 | 10 |
| 8  | 5670 | 125 | 34 | 11 |
| 9  | 5902 | 128 | 37 | 12 |
| 10 | 6159 | 140 | 37 | 12 |
| 11 | 6202 | 148 | 39 | 13 |
| 12 | 6764 | 177 | 41 | 14 |

Solution:

The MTTFs can be obtained as

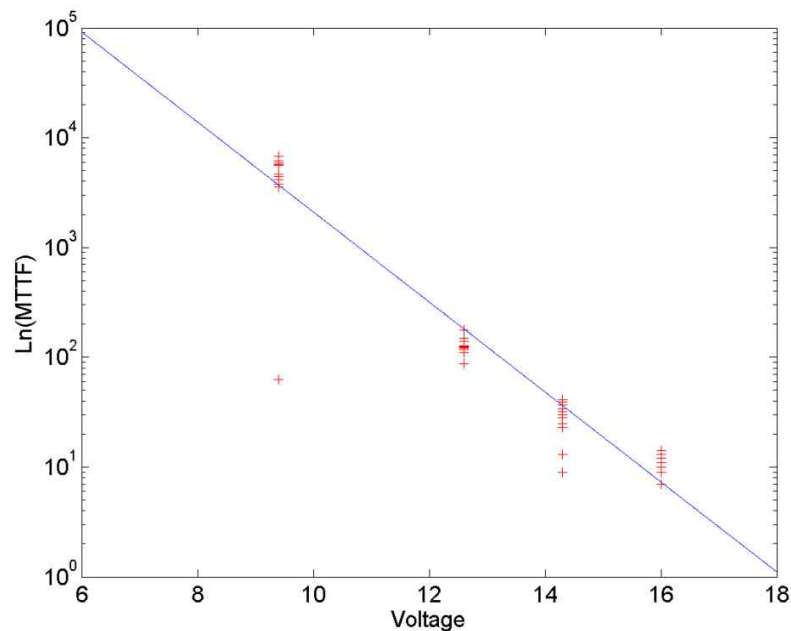
MTTF(9.4 voltage) = 4,744 mim.

MTTF(12.6 voltage) = 126 mim.

MTTF(14.3 voltage) = 29.0 mim.

MTTF(16.0 voltage) = 10.3 mim.

```
>> y = load TTF.dat
>> m(1)=log(mean(y(:,1)));m(2)=log(mean(y(:,2)));...
>> m(3)=log(mean(y(:,3)));m(4)=log(mean(y(:,4)));
>> p=polyfit(x,m,1)
p =
-0.9438 17.0917
>> hold on; x=[6:0.01:18]; y=exp(p(1)*x+p(2)); plot(x,y)
```



In the figure above, MTTF versus voltage is plotted in a logarithmic scale:

The least-square fit indicates

$$\text{Ln}(\text{MTTF}) = -0.9438 \times v + 17.0917$$

Hence,

$$\begin{aligned} \text{MTTF} &= \exp(17.0917 - 0.9438 \times v) = 241 \times 10^6 \exp(-0.9438 \times v) \text{ [min]} \\ &= 1.8385 \times 10^4 \exp(-0.9438 \times v) \text{ days} \end{aligned}$$

At 6 volts:

$$\begin{aligned} \text{MTTF} &= 1.8385 \times 10^4 \exp(-0.9438 \times 6) \text{ days} = 64 \text{ days} = 2.13 \text{ months} \\ &= 9.184 \times 10^4 \text{ minutes} \end{aligned}$$

The previous approach is a non-parametric process, while straightforward. It has several drawbacks relative to the parametric methods.

1. It requires that a complete set of life data be available at each stress level in order to use the sample mean to calculate the MTTF.
2. Without attempting to fit the data to a distribution, one has no indication whether the shape, as well as the time scale of the distribution, is changing. Since the changes in distribution shape are usually indications that a new failure mechanism is being activated by the higher-stress levels, there is a greater danger that the non-parametric estimate will be inappropriately extrapolated.

We use the following notation:

$t_s$  = TTF at stress

$t_u$  = corresponding TTF at use

$F_s(t)$  = CDF at stress

$F_u(t)$  = CDF at use

When there is a true acceleration, changing stress is equivalent to transforming the time scale used to record when failures occur. The transformations commonly used are *linear*, which means that TTF at high stress just has to be multiplied by a constant (the **acceleration factor**) to obtain the equivalent TTF at use.

$$F_u(t_u) \rightarrow F_s(t_s) = F_s(t_s = t_u/AF)$$

The Weibull and lognormal distributions are particularly well suited for the analysis of advanced-stress tests, for in each case there is a scale parameter that is inversely proportional to the acceleration factor and a shape parameter that should be unaffected by acceleration.

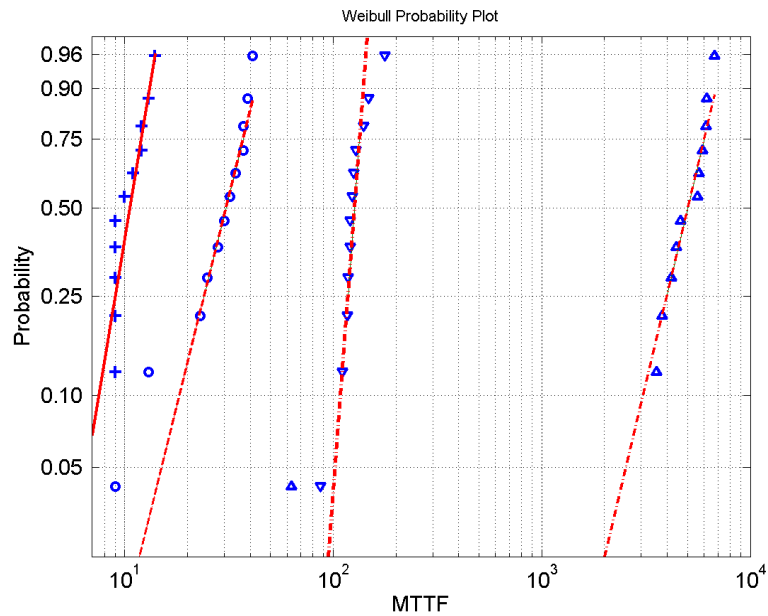
### Exercise: Parametric process

Let us consider the Weibull distribution as

$$F_T(t; \lambda) = 1 - \exp\left[-(t/\lambda)^k\right]$$

```
>> close
```

```
>> wblplot(y(:,1)),hold on; wblplot(y(:,2)); wblplot(y(:,3)); wblplot(y(:,4));
```



```

>> wblfit(y(:,1))
ans =
    1.0e+003 *
    5.0904  0.0022 → first parameter for scale ( $\lambda$ ) and second for shape ( $k$ )
>> wblfit(y(:,2))
ans =
    135.4589  5.9462
>> wblfit(y(:,3))
ans =
    32.2150  3.6187
>> wblfit(y(:,4))
ans =
    11.1620  5.6660
>> wblfit(y(2:12,1))    → excluding one outlier
ans =
    1.0e+003 *
    5.5959  0.0058

```

Assume that the shape parameter ( $k$ : second value) be unchanged but the scale parameter ( $\lambda$ : first value) be changed.

We use the following notation:

$f_s(t)$  = PDF at stress

$f_u(t)$  = PDF at use

$h_s(t)$  = failure rate at stress

$h_u(t)$  = failure rate at use

Then, an acceleration factor  $AF$  between stress and use means the following relationships hold:

### Linear Acceleration Relationships

|                            |  |
|----------------------------|--|
| <b>MTTF</b>                | $t_u = AF \times t_s$                          |
| <b>Failure Probability</b> | $F_u(t_u) \rightarrow F_s(t_u/AF)$             |
| <b>Reliability</b>         | $R_u(t_u) \rightarrow R_s(t_u/AF)$             |
| <b>PDF</b>                 | $f_u(t) \rightarrow (1/AF) \times f_s(t_u/AF)$ |
| <b>Failure Rate</b>        | $h_u(t) \rightarrow (1/AF) \times h_s(t_u/AF)$ |

#### 5.3.2 Common Acceleration Models

- Arrhenius

One of the earliest and most successful acceleration models predicts how TTF varies with temperature. This empirical model is known as the Arrhenius equation as

$$TTF = A \exp\left\{\frac{\Delta H}{k\theta}\right\} \quad \text{or} \quad TTF = A \exp\left\{\frac{B}{\theta}\right\} \quad (62)$$

with  $\theta$  denoting temperature measured in degrees Kelvin ( $273.16 +$  degrees Celsius) at the point when the failure process takes place and  $k$  is Boltzmann's constant ( $8.617 \times 10^{-5}$  in eV/K). The constant  $A$  is a scaling factor that drops out when calculating acceleration factors, with  $\Delta H$  denoting the activation energy, which is the critical parameter in the model.

The acceleration factor between a high temperature  $\theta_2$  and a low temperature  $\theta_1$  is given by

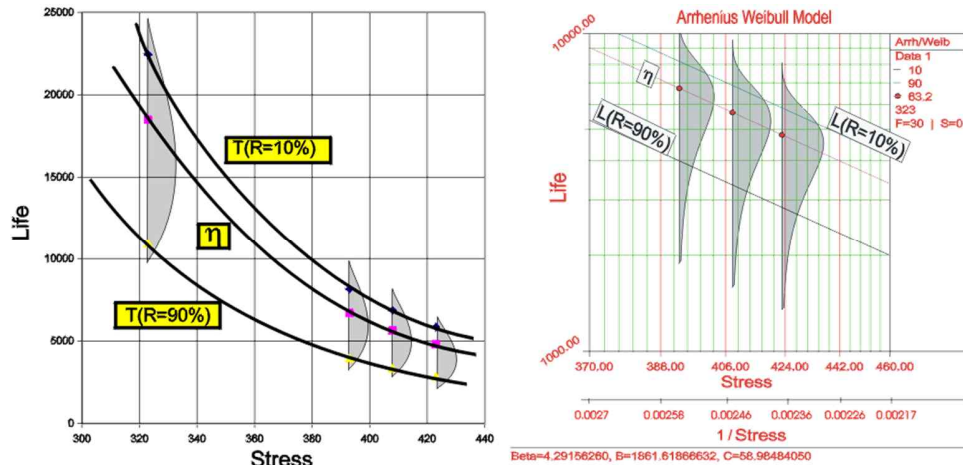
$$AF = \frac{t_1}{t_2} = \exp\left\{\frac{\Delta H}{k} \left[\frac{1}{\theta_1} - \frac{1}{\theta_2}\right]\right\} \quad (63)$$

The value of  $\Delta H$  depends on the failure mechanism and the materials involved, and typically ranges from 0.3 to 1.5, or even higher. Acceleration factors between two temperatures increase exponentially as  $\Delta H$  increases.

Using the value of  $k$  given above, this can be written in terms of  $\theta$  in degrees Celsius as

$$AF = \exp\left\{\Delta H \times 11605 \times \left[\frac{1}{(\theta_1 + 273.16)} - \frac{1}{(\theta_2 + 273.16)}\right]\right\} \quad (64)$$

Note that the only unknown parameter in this formula is  $\Delta H$ .



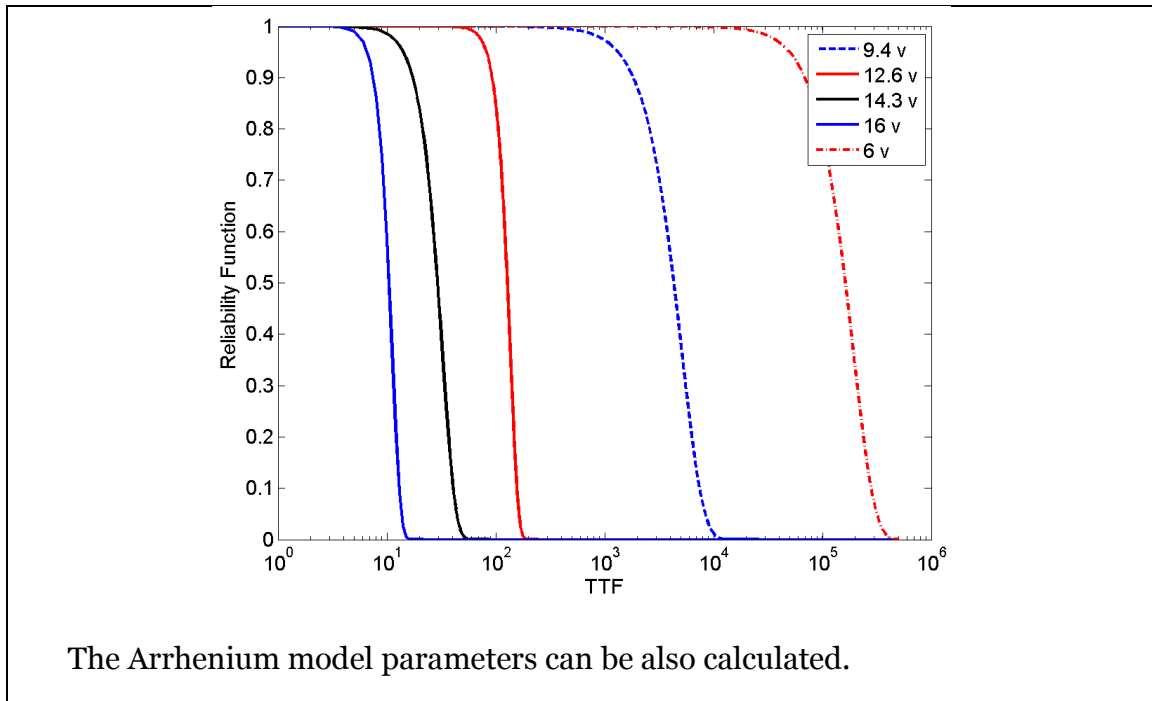
**Figure 5.5 Arrhenius plot for Weibull life distribution**  
[http://www.weibull.com/AccelTestWeb/arrhenius\\_relationship\\_chap\\_.htm](http://www.weibull.com/AccelTestWeb/arrhenius_relationship_chap_.htm)

### Exercise: Parametric process

Consider the accelerated life tests for the four sets of 12 flashlight bulbs and the failure times in minutes are found in the Table 4.2. Estimate the MTTF at normal operating 6.0 voltage using Arrhenius model. Assume  $v_1 = 9.4$  and  $v_2 = 12.6$ . Accordingly,  $t_1 = 4744$  and  $t_2 = 126$ . Hence,  $AF = 4744/126 = 37.65$ . Reliability function can be calculated as

```
>> t=[0:10:500000];r1=exp(-(t./5090.4).^2.2); plot(t,r1)
>> hold on;
>> t=[0:1:500000];r2=exp(-(t./135.5).^5.9); plot(t,r2)
>> t=[0:1:500000];r3=exp(-(t./32.22).^3.6); plot(t,r3)
>> t=[0:1:500000];r4=exp(-(t./11.16).^5.7); plot(t,r4)
>> t=[0:10:500000];r5=exp(-(t./5090.4/37.65).^2.2); plot(t,r5)
>> R = @(t) exp(-(t./5090.4/37.65).^2.2);
>> MTTF = quad(R,0,10^6)
MTTF =
1.6973e+005
>> R = @(t) exp(-(t./5595.9/37.65).^5.8);
>> MTTF = quad(R,0,10^6)
MTTF =
1.9509e+005
```





The Arrhenius model has been used successfully for failure mechanisms that depend on chemical reactions, diffusion processes or migration processes. This covers many of the thermally-induced mechanical failure modes that cause electronic equipment failure.

- Eyring

Henry Eyring's contributions to chemical reaction rate theory have led to a very general and powerful model for acceleration known as the Eyring Model. This model has several key features:

- ✓ It has a theoretical basis from chemistry and quantum mechanics.
- ✓ If a chemical process (chemical reaction, diffusion, corrosion, migration, etc.) is causing degradation leading to failure, the Eyring model describes how the rate of degradation varies with stress or, equivalently, how TTF varies with stress.
- ✓ The model includes temperature and can be expanded to include other relevant stresses.
- ✓ The temperature term by itself is very similar to the Arrhenius empirical model, explaining why that model has been so successful in establishing the connection between the  $\Delta H$  parameter and the quantum theory concept of "activation energy needed to cross an energy barrier and initiate a reaction".

The model for temperature and one additional stress takes the general form:

$$TTF = A\theta^\alpha \exp\left\{\frac{\Delta H}{k\theta} + \left(B + \frac{C}{\theta}\right)S_1\right\} \quad (65)$$

for which  $S_i$  could be some function of voltage or current or any other relevant stress and the parameters  $k$ ,  $\Delta H$ ,  $B$ , and  $C$  determine acceleration between stress combinations. As with the Arrhenius Model,  $k$  is Boltzmann's constant and temperature is in degrees Kelvin. If we want to add an additional non-thermal stress term, the model becomes

$$t_f = A\theta^\alpha \exp\left\{\frac{\Delta H}{k\theta} + \left(B_1 + \frac{C_1}{\theta}\right)S_1 + \left(B_2 + \frac{C_2}{\theta}\right)S_2\right\} \quad (66)$$

and as many stresses as are relevant can be included by adding similar terms.

#### Advantages of the Eyring Model

- ✓ Can handle many stresses.
- ✓ Can be used to model degradation data as well as failure data.
- ✓ The  $\Delta H$  parameter has a physical meaning and has been studied and estimated for many well known failure mechanisms and materials.

#### Disadvantages of the Eyring Model

- ✓ Even with just two stresses, there are 5 parameters to estimate. Each additional stress adds 2 more unknown parameters.
- ✓ Many of the parameters may have only a second-order effect. For example, setting  $\alpha = 0$  works quite well since the temperature term then becomes the same as in the Arrhenius model. Also, the constants  $C_1$  and  $C_2$  are only needed if there is a significant temperature interaction effect with respect to the other stresses.

- Other models
  - a. (Inverse) Power Rule for Voltage
  - b. Exponential Voltage Model
  - c. Two Temperature/Voltage Models
  - d. Electromigration Model
  - e. Three-Stress Models (Temperature, Voltage, and Humidity)
  - f. Coffin-Manson Mechanical Crack Growth Model

Refer to <http://www.itl.nist.gov/div898/handbook/apr/section1/apr153.htm>

#### **Homework 18: Failure analysis of a paper clip twisting**

Answer the following questions:

- (a) Identify data outlier(s) and justify it.
- (b) Develop a probability density function model for TTF data under twisting moment. Use a Weibull distribution.
- (c) Calculate the MTTF and develop reliability function and failure rate models for the TTF data under a twisting condition.

**Homework 19: *Life analysis of a paper clip bending***

Answer the following questions:

- (a) Develop probability density function models for TTF data under four bending conditions,  $180^\circ$ ,  $135^\circ$ ,  $90^\circ$ , and  $45^\circ$ . Use a Weibull distribution and report the statistical parameters in table.
- (b) Discuss the result above.
- (c) Use the Arrhenius model with the TTF data ( $180^\circ$ ,  $135^\circ$ ,  $90^\circ$ ) to calculate the Accelerating Factor (AF) and plot  $\text{Log}(\text{Life})$  vs  $\text{Stress}(\text{bending angle})$ .
- (d) Predict a TTF under a bending angle ( $45^\circ$ ) using the Arrhenius model obtained in (c) and compare the predicted TTF with the observed TTF from (a).

**5.4 Degradation-based Simulations****5.4.1 Fatigue**

See the handout, [fatigue\\_wiki.pdf](#)

**5.4.2 Wear**

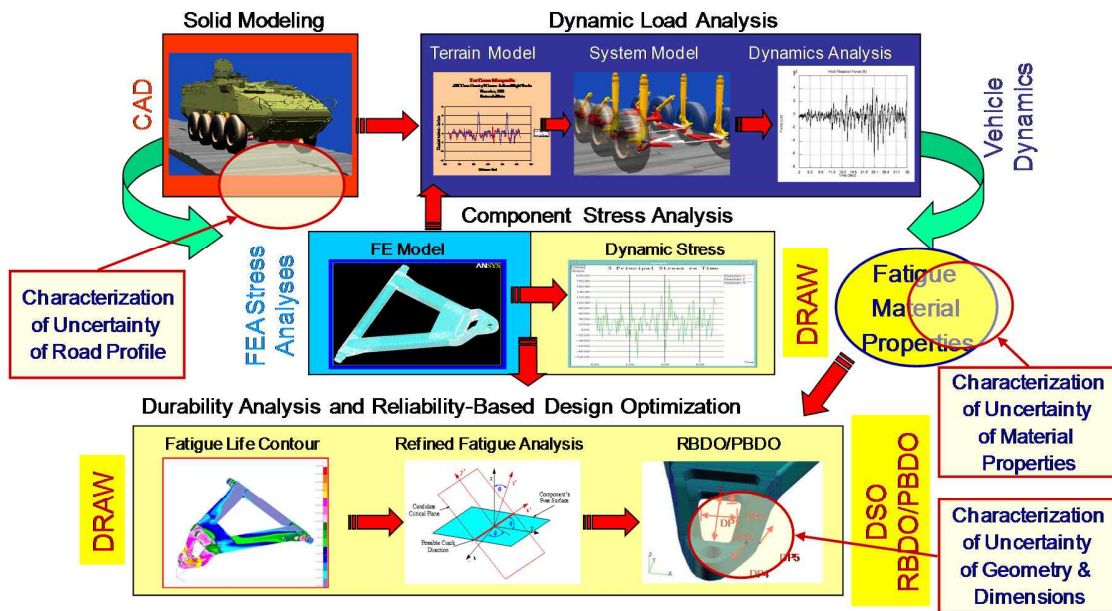
See the handout, [wear\\_wiki.pdf](#)

**5.4.3 Corrosion**

See the handout, [corrosion\\_wiki.pdf](#)

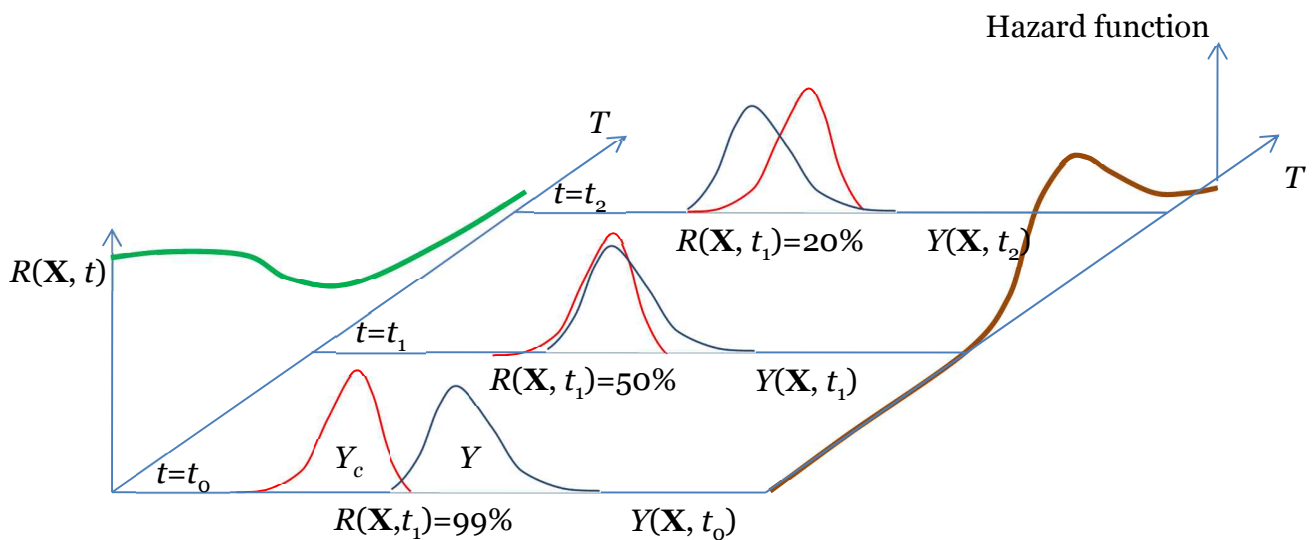
**5.4.4 Creep**

See the handout, [creep\\_wiki.pdf](#)



- How to Optimize the Vehicle Design to Minimize/Reduce the Weight?
- Under These Uncertainties, How to Achieve the Component Level Reliability?
- Under These Uncertainties, How to Achieve the System Level Reliability?

**Figure 5.6: Fatigue Simulation Model** – Fatigue Life =  $Y(X)$



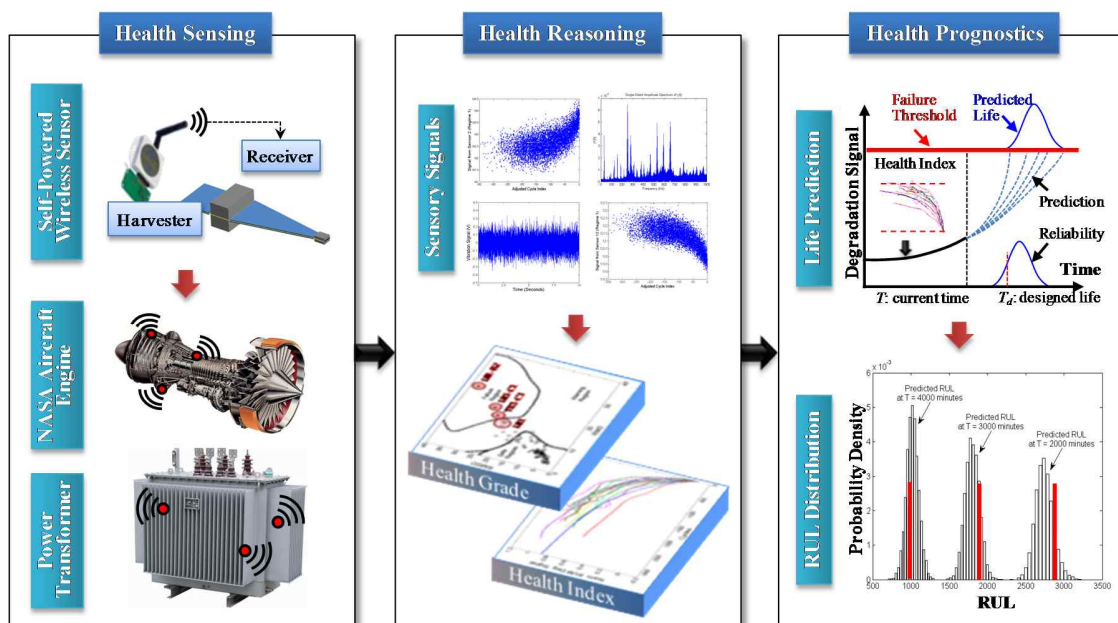
**Figure 5.7: General Description of Reliability (L-Type)**

### 5.5 Health monitoring and prognostics

Accelerated life testing (ALT) is capable of providing an instantaneous reliability estimate for an engineered system based on degradation characteristics of historical units. We refer to this approach as the **classical reliability approach, which incorporates**

population characteristics into reliability estimation by modeling a life distribution. However, this classical reliability approach only provides an overall reliability estimate that takes the same value for the whole population of units. In engineering practice, we are more interested in investigating the specific reliability information of a particular unit under its actual life cycle conditions to determine the advent of a failure and mitigate potential risk.

To overcome the limitation of the classical reliability approach, prognostics and health management (PHM) has recently emerged as a key technology to evaluate the current health condition (health monitoring) and predict the future degradation behavior (health prognostics) of an engineered system throughout its lifecycle. In general, PHM consists of four basic functions: health sensing function, health reasoning function, health prognostics function and health management functions (see Fig. 5.8 for the first three functions).



**Figure 5.8:** Basic PHM Functions

- Health Sensing Function: To acquire sensory signal with in-situ monitoring techniques and to ensure high damage detectability by designing an optimal wireless sensor network (WSN);
- Health Reasoning Function: To extract system health relevant information in real-time with feature extraction techniques and to classify system health condition with health classification techniques;
- Health Prognostics Function: To predict the time remaining before an engineered system no longer performs the required function(s) or the remaining useful life (RUL) in real-time with advanced machine learning techniques;

- Health Management Function: To enable optimal decision making on maintenance of engineered systems based on RUL predictions from health prognostics function with trade-off analysis and random process modeling techniques.

In recent years, prognostics and health management (PHM) has been successfully applied to many engineered systems to assess their health conditions in real-time under actual operation conditions and adaptively enhance life cycle reliabilities with condition-based maintenance that will effectively avoid unexpected failures. Figure 5.8 exemplifies several engineered systems that capitalize on PHM to enable an early anticipation of failure, to develop cost-effective maintenance strategies and to seek opportunities for life extensions.



Figure 5.9: Engineered Systems Capitalizing on PHM

An example is provided in Fig. 5.10 to demonstrate the three main PHM functions.

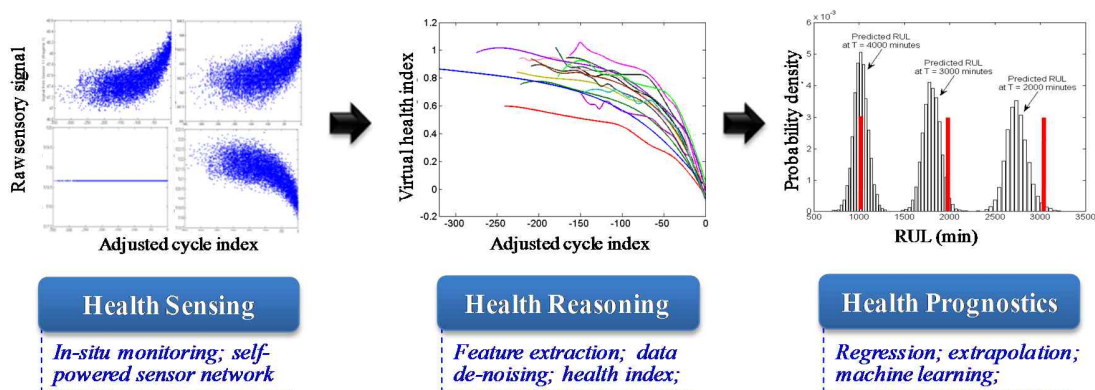


Figure 5.10: An Example Illustrating Three Main PHM Functions.

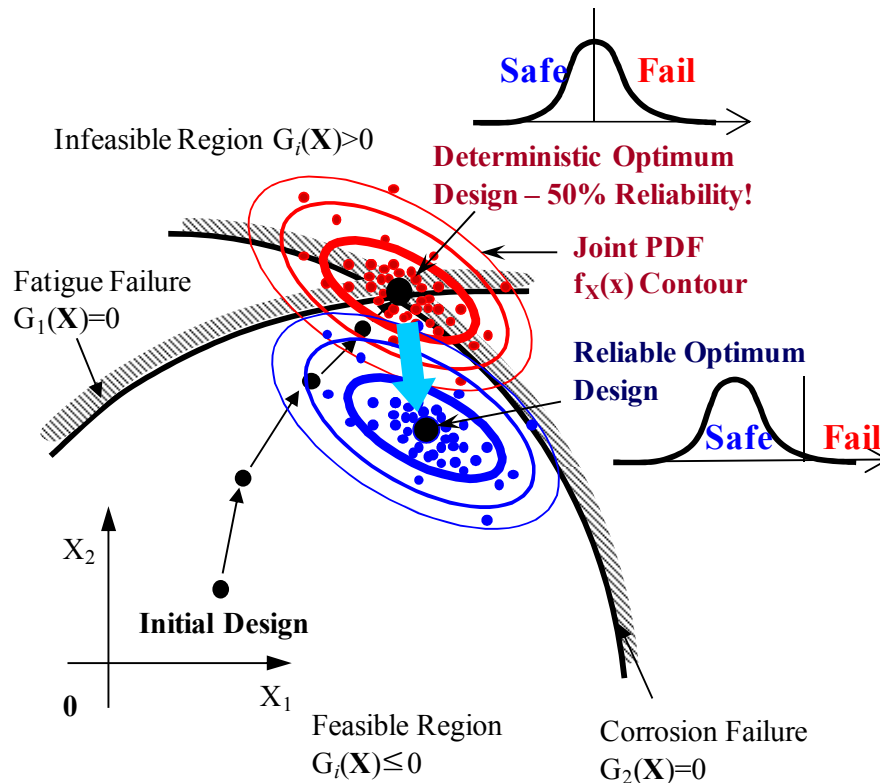
**CHAPTER 6. DESIGN OPTIMIZATION**

**6.1 General Model of Design under Uncertainty**

The design under uncertainty can generally be defined as:

$$\begin{aligned}
 &\text{Minimize} && \text{Cost}(\mathbf{d}) \text{ or Risk}(\mathbf{d}) \\
 &\text{subject to} && P\{G_i\{\mathbf{X}; \mathbf{d}(\mathbf{X})\} > 0\} \leq P_{f_i}, \quad i = 1, \dots, nc \\
 &&& \mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U, \quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr}
 \end{aligned}
 \tag{67}$$

where  $nc$  is the number of probabilistic constraints;  $nd$  is the number of design parameters;  $nr$  is the number of random variables;  $\mathbf{d} = [d_i]^T = \mu(\mathbf{X})$  is the design vector;  $\mathbf{X} = [X_i]^T$  is the random vector; and the probabilistic constraints are described by the performance function  $G_i\{\mathbf{X}; \mathbf{d}(\mathbf{X})\}$ , their probabilistic models, and the probability of failure. The probability of failure is defined as  $P_f \equiv \Phi(-\beta_i)$  with a target reliability index  $\beta_i$  where the failure is defined as  $G_i\{\mathbf{X}; \mathbf{d}(\mathbf{X})\} > 0$ .



**Figure 6.1: Design under Uncertainty**

## 6.2 A General Formulation of Design Optimization

In general, design optimization can be formulated as

$$\begin{aligned}
 &\text{Minimize (or Maximize)} && f(\mathbf{x}) \\
 &\text{Subject to} && h_i(\mathbf{x}) = 0, \quad i = 1, \dots, p \\
 & && g_j(\mathbf{x}) \leq 0, \quad j = 1, \dots, m \\
 & && \mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U, \quad \mathbf{x} \in R^n
 \end{aligned} \tag{68}$$

where  $m$ : no of inequality constraints, feasible where  $g_j(\mathbf{x}) \leq 0$   
 $p$ : no of equality constraints, feasible where  $h_i(\mathbf{x}) = 0$

## 6.3 Optimality Condition

Refer to Section 6.1 (Arora, 2004): First-order necessary KKT condition.

- **Lagrangian function:**

$$L = f(\mathbf{x}) + \sum_{i=1}^p v_i h_i(\mathbf{x}) + \sum_{j=1}^m u_j g_j(\mathbf{x}) \tag{69}$$

- **Gradient conditions**

$$\frac{\partial L}{\partial v_i} = 0 \Rightarrow h_i(\mathbf{x}^*) = 0; \quad i = 1 \sim p \tag{70}$$

$$\frac{\partial L}{\partial x_k} = 0 \Rightarrow \frac{\partial f}{\partial x_k} + \sum_{i=1}^p v_i^* \frac{\partial h_i}{\partial x_k} + \sum_{j=1}^m u_j^* \frac{\partial g_j}{\partial x_k} = 0; \quad k = 1 \sim nd \tag{71}$$

- **Feasibility check**

$$g_j(\mathbf{x}^*) \leq 0; \quad j = 1 \sim m \tag{72}$$

- **Switching conditions**

$$u_j^* g_j(\mathbf{x}^*) = 0; \quad j = 1 \sim m \tag{73}$$

- **Nonnegativity of Lagrange multipliers for inequalities**

$$u_j^* \geq 0; \quad j = 1 \sim m \tag{74}$$

- **Regularity check**

Gradients of active constraints must be linearly independent. In such a case, the Lagrangian multipliers for the constraints are unique.



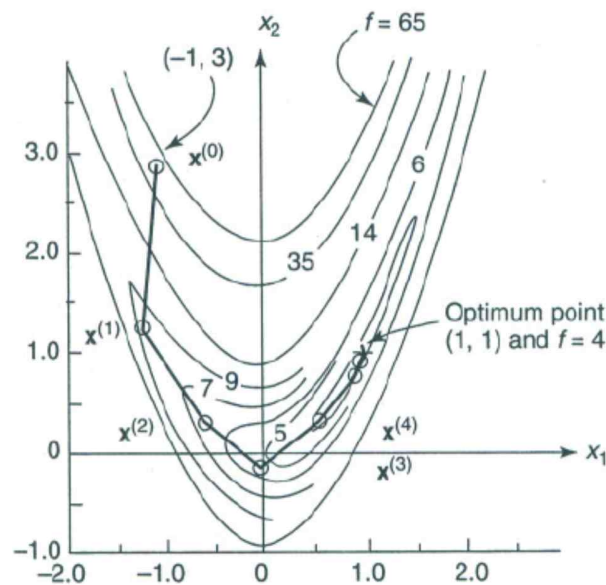
**Exercise:** Check for KKT necessary conditions

Minimize  $f(x, y) = (x - 10)^2 + (y - 8)^2$  subject to  $g_1 = x + y - 12 \leq 0$ ,  $g_2 = x - 8 \leq 0$

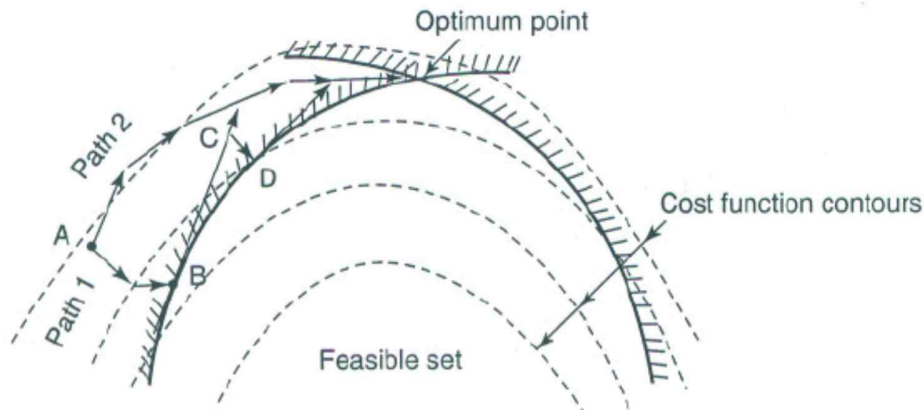
Refer to Example 5.1 (Arora, 2004).

Arora, J.S. Introduction to Optimum Design, Second Edition, Elsevier, 2004

The ordinary optimization task is where many constraints are imposed. In the process of finding a usable-feasible search direction, we are able to detect if the KKT conditions are satisfied. If they are, the optimization process must be terminated.

**6.4 Concept of Numerical Algorithms in Design Optimization**

**Figure 6.2:** Conceptual steps of unconstrained optimization algorithm



**Figure 6.3:** Conceptual steps of **constrained optimization algorithm**

Iterative numerical search methods are employed for the optimization. Two basic calculations are involved in the numerical search methods for optimum design: (1) calculation of a search direction and (2) calculation of a step size in the search direction. It can be generally expressed as

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \Delta \mathbf{x}^{(k)} \quad \text{where } \Delta \mathbf{x}^{(k)} = \alpha_k \mathbf{d}^{(k)} \quad (75)$$

So, finding  $\alpha_k$  is a line search and  $\mathbf{d}^{(k)}$  is the direction search.

#### 6.4.1 Line Search

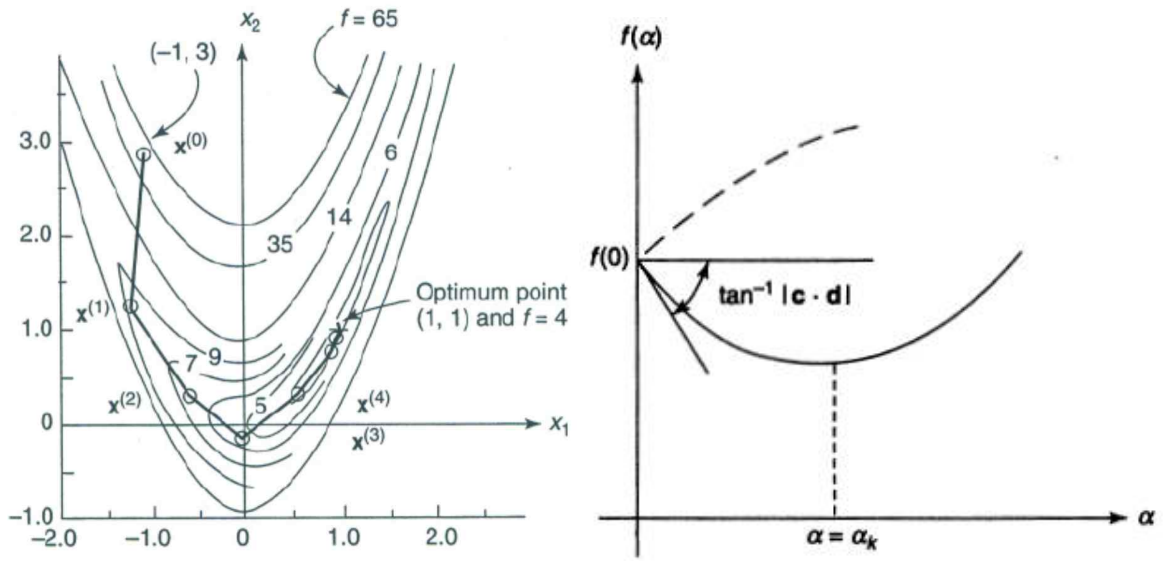
The cost function  $f(\mathbf{x})$  is given as

$$f(\mathbf{x}^{(k+1)}) = f(\mathbf{x}^{(k)} + \alpha \mathbf{d}^{(k)}) = f(\alpha) \quad (76)$$

It is important to understand this reduction of a function of  $n$  variables to a function of one variable. The descent condition for the cost function can be expressed as the inequality:

$$f(\alpha) < f(0) \quad (77)$$

To satisfy the inequality (77), the curve  $f(\alpha)$  must have a negative slope when  $\alpha=0$ .



**Figure 6.4:** Descent condition for the cost function

Let  $\nabla f(\mathbf{x})$  be  $\mathbf{c}(\mathbf{x})$ . In fact, the slope of the curve  $f(\alpha)$  at  $\alpha=0$  is calculated as  $f'(0) = \mathbf{c}^{(k)} \cdot \mathbf{d}^{(k)} < 0$ . If  $\mathbf{d}^{(k)}$  is a descent direction, then  $\alpha$  must always be a positive scalar. Thus, the one-dimensional minimization problem is to find  $\alpha_k = \alpha$  such that  $f(\alpha)$  is minimized.

The necessary condition for the optimal step size is  $df(\alpha)/d\alpha = 0$ , and the sufficient condition is  $d^2f(\alpha)/d\alpha^2 > 0$ . Note that differentiation of  $f(\mathbf{x}^{(k+1)})$  with respect to  $\alpha$  gives

$$\left. \frac{df(\mathbf{x})}{d\alpha} \right|_{\mathbf{x}^{(k+1)}} = \frac{df^T(\mathbf{x}^{(k+1)})}{d\mathbf{x}} \frac{d(\mathbf{x}^{(k+1)})}{d\alpha} = \nabla f(\mathbf{x}^{(k+1)}) \cdot \mathbf{d}^{(k)} = \mathbf{c}^{(k+1)} \cdot \mathbf{d}^{(k)} = 0 \tag{78}$$

### Analytical Step Size Determination

Let a direction of change for the function

$$f(\mathbf{x}) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + 7 \tag{a}$$

at the point  $(1, 2)$  be given as  $(-1, -1)$ . Compute the step size  $\alpha_k$  to minimize  $f(\mathbf{x})$  in the given direction.

**Solution.** For the given point  $\mathbf{x}^{(k)} = (1, 2)$ ,  $f(\mathbf{x}^{(k)}) = 22$ , and  $\mathbf{d}^{(k)} = (-1, -1)$ . We first check to see if  $\mathbf{d}^{(k)}$  is a direction of descent using Inequality (8.8). The gradient of the function at  $(1, 2)$  is given as  $\mathbf{c}^{(k)} = (10, 10)$  and  $\mathbf{c}^{(k)} \cdot \mathbf{d}^{(k)} = 10(-1) + 10(-1) = -20 < 0$ . Therefore,  $(-1, -1)$  is a direction of descent. The new point  $\mathbf{x}^{(k+1)}$  using Eq. (8.9a) is given as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{(k+1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \text{or} \quad x_1^{(k+1)} = 1 - \alpha; \quad x_2^{(k+1)} = 2 - \alpha \quad (\text{b})$$

Substituting these equations into the cost function of Eq. (a), we get

$$f(\mathbf{x}^{(k+1)}) = 3(1 - \alpha)^2 + 2(1 - \alpha)(2 - \alpha) + 2(2 - \alpha)^2 + 7 = 7\alpha^2 - 20\alpha + 22 = f(\alpha) \quad (\text{c})$$

Therefore, along the given direction  $(-1, -1)$ ,  $f(\mathbf{x})$  becomes a function of the single variable  $\alpha$ . Note from Eq. (c) that  $f(0) = 22$ , which is the cost function value at the current point, and that  $f'(0) = -20 < 0$ , which is the slope of  $f(\alpha)$  at  $\alpha = 0$  (also recall that  $f'(0) = \mathbf{c}^{(k)} \cdot \mathbf{d}^{(k)}$ ). Now using the necessary and sufficient conditions of optimality for  $f(\alpha)$ , we obtain

$$\frac{df}{d\alpha} = 14\alpha_k - 20 = 0; \quad \alpha_k = \frac{10}{7}; \quad \frac{d^2f}{d\alpha^2} = 14 > 0 \quad (\text{d})$$

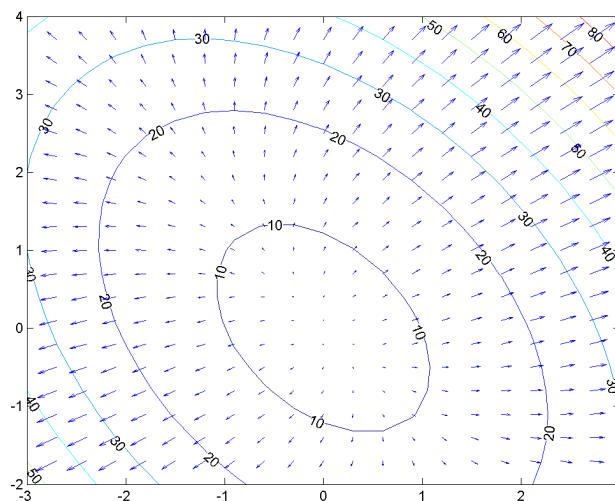
Therefore,  $\alpha_k = \frac{10}{7}$  minimizes  $f(\mathbf{x})$  in the direction  $(-1, -1)$ . The new point is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{(k+1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left(\frac{10}{7}\right) \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} \\ \frac{4}{7} \end{bmatrix} \quad (\text{e})$$

Substituting the new design  $(-\frac{3}{7}, \frac{4}{7})$  into the cost function  $f(\mathbf{x})$  we find the new value of the cost function as  $\frac{54}{7}$ . This is a substantial reduction from the cost function value of 22 at the previous point. Note that Eq. (d) for calculation of step size  $\alpha$  can also be obtained by directly using the condition given in Eq. (8.11). Using Eq. (b), the gradient of  $f$  at the new design point in terms of  $\alpha$  is given as

$$\mathbf{c}^{(k+1)} = (6x_1 + 2x_2, 2x_1 + 4x_2) = (10 - 8\alpha, 10 - 6\alpha) \quad (\text{f})$$

Using the condition of Eq. (8.11), we get  $14\alpha - 20 = 0$  which is same as Eq. (d).



```
>> [X,Y] = meshgrid(-3:.3:3,-2:.3:4);
>> f=3*X.^2+2*X.*Y+2*Y.^2+7;
>> [C,h]=contour(X,Y,f); clabel(C,h); hold on
>> [U,V] = gradient(f,2,2); quiver(X,Y,U,V)
```

### Line Search Methods

1. Equal interval search
2. Golden section search
3. Quadratic interpolation method

With the assumption that the function  $f(\alpha)$  is sufficiently smooth and unimodal,  $f(\alpha)$  is approximated using a quadratic function with respect to  $\alpha$  as

$$f(\alpha) \approx q(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 \quad (79)$$

The minimum point  $\bar{\alpha}$  of the quadratic curve is calculated by solving the necessary condition  $dq/d\alpha = 0$ .

### One-dimensional Minimization with Quadratic Interpolation

Find the minimum point of  $f(\alpha) = 2 - 4\alpha + e^\alpha$  of Example 8.3 by polynomial interpolation. Use the golden section search with  $\delta = 0.5$  to bracket the minimum point initially.

*Iteration 1.* From Example 8.3 the following information is known.

$$\alpha_l = 0.5, \quad \alpha_i = 1.309017, \quad \alpha_u = 2.618034$$

$$f(\alpha_l) = 1.648721, \quad f(\alpha_i) = 0.466464, \quad f(\alpha_u) = 5.236610$$

$$a_2 = \frac{1}{1.30902} \left( \frac{3.5879}{2.1180} - \frac{-1.1823}{0.80902} \right) = 2.410$$

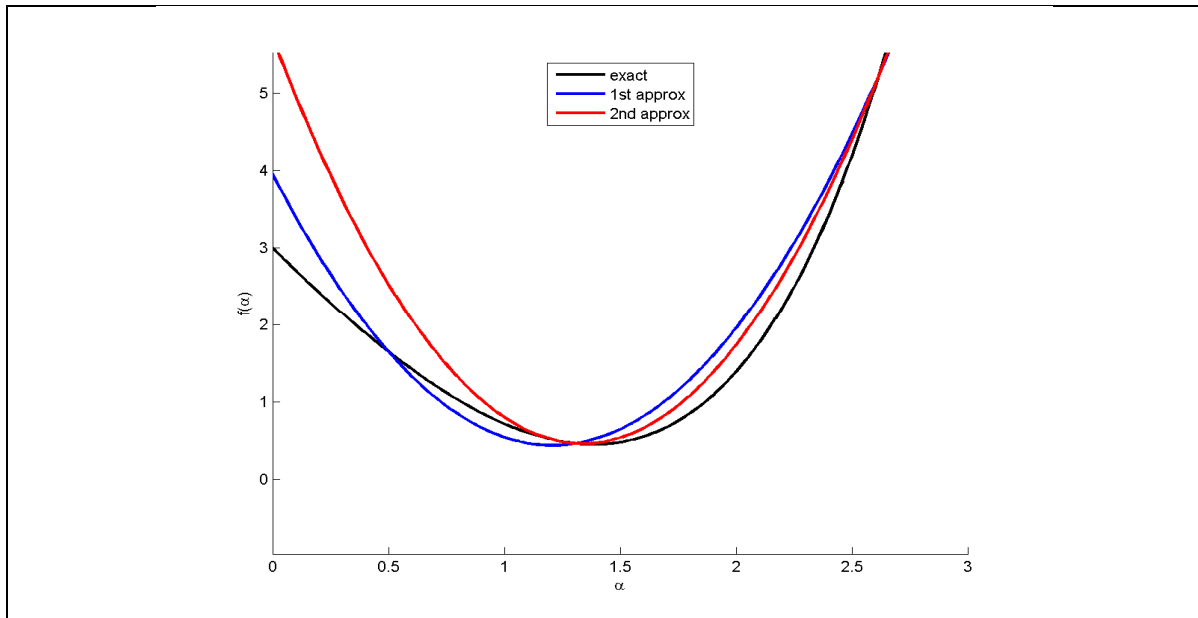
$$a_1 = \frac{-1.1823}{0.80902} - (2.41)(1.80902) = -5.821$$

$$a_0 = 1.648271 - (-5.821)(0.50) - 2.41(0.25) = 3.957$$

Therefore,  $\bar{\alpha} = 1.2077$  from Eq. (9.3), and  $f(\bar{\alpha}) = 0.5149$ . Note that  $\bar{\alpha} < \alpha_i$  and  $f(\alpha_i) < f(\bar{\alpha})$ . Thus, new limits of the reduced interval of uncertainty are  $\alpha'_l = \bar{\alpha} = 1.2077$ ,  $\alpha'_u = \alpha_u = 2.618034$ , and  $\alpha'_i = \alpha_i = 1.309017$ .

$$\alpha_l = 1.2077, \quad \alpha_i = 1.309017, \quad \alpha_u = 2.618034$$

The coefficients  $a_0$ ,  $a_1$ , and  $a_2$  are calculated as before,  $a_0 = 5.7129$ ,  $a_1 = -7.8339$ , and  $a_2 = 2.9228$ . Thus,  $\bar{\alpha} = 1.34014$  and  $f(\bar{\alpha}) = 0.4590$ .



### **Homework 21: Optimization Reading 1**

Chapters 4.3-4.5

Chapters 5.1-5.2

Chapters 8.1, 8.2

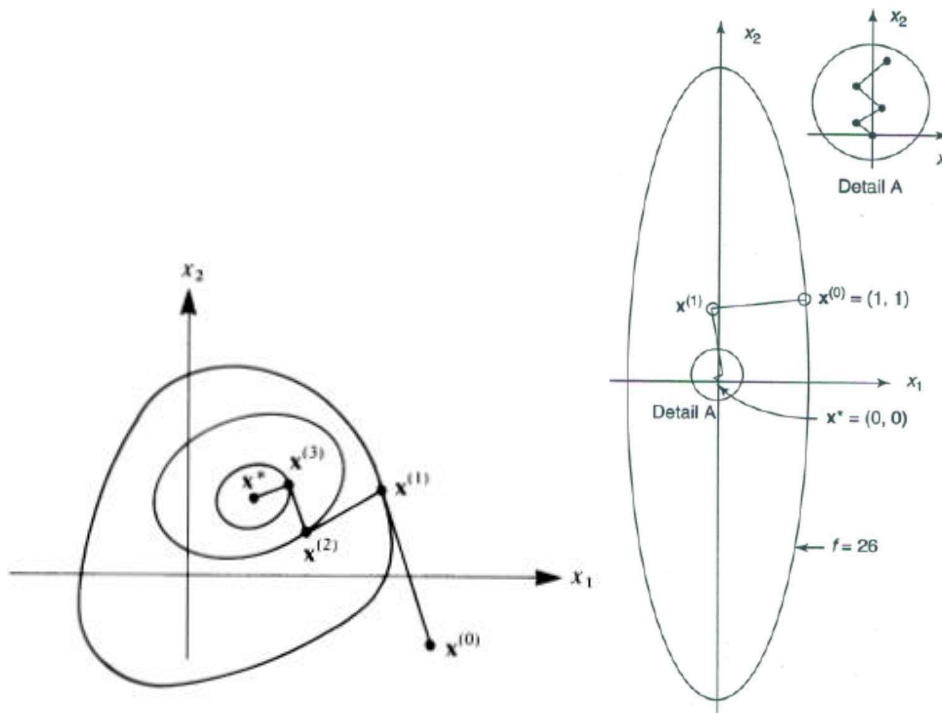
#### 6.4.2 Direction Search

The basic requirement for  $\mathbf{d}$  is that the cost function be reduced if we make a small move along  $\mathbf{d}$ ; that is, the descent condition ( $f'(0) = \mathbf{c}^{(k)} \cdot \mathbf{d}^{(k)} < 0$ ) be satisfied. This is called the descent direction.

##### Search Direction Methods

1. Steepest descent method
2. Conjugate gradient method
3. Newton's Method
4. Quasi-Newton's Method
5. Sequential linear programming (SLP)
6. Sequential quadratic programming (SQP)

The first four are used for an unconstrained optimization problem whereas the last two are often used for a constrained optimization problem.



( a ) Steepest descent method

( b ) Conjugate gradient method

**Figure 6.5:** Search direction methods using gradient method

## a. Steepest descent method

**Step 1.** Estimate a starting design  $\mathbf{x}^{(0)}$  and set the iteration counter  $k = 0$ . Select a convergence parameter  $\varepsilon > 0$ .

**Step 2.** Calculate the gradient of  $f(\mathbf{x})$  at the point  $\mathbf{x}^{(k)}$  as  $\mathbf{c}^{(k)} = \nabla f(\mathbf{x}^{(k)})$ .

**Step 3.** Calculate  $\|\mathbf{c}^{(k)}\|$ . If  $\|\mathbf{c}^{(k)}\| < \varepsilon$ , then stop the iterative process because  $\mathbf{x}^* = \mathbf{x}^{(k)}$  is a minimum point. Otherwise, continue.

**Step 4.** Let the search direction at the current point  $\mathbf{x}^{(k)}$  be  $\mathbf{d}^{(k)} = -\mathbf{c}^{(k)}$ .

**Step 4.** Let the search direction at the current point  $\mathbf{x}^{(k)}$  be  $\mathbf{d}^{(k)} = -\mathbf{c}^{(k)}$ .

**Step 5.** Calculate a step size  $\alpha_k$  that minimizes  $f(\mathbf{x}^{(k)} + \alpha \mathbf{d}^{(k)})$ . Any one-dimensional search algorithm may be used to determine  $\alpha_k$ .

**Step 6.** Update the design as  $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k \mathbf{d}^{(k)}$ . Set  $k = k + 1$ , and go to Step 2.

**EXAMPLE 8.4 Use of Steepest Descent Algorithm**

Minimize  $f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1x_2$  using the steepest descent method starting from the point (1, 0).

1. The starting design is given as  $\mathbf{x}^{(0)} = (1, 0)$ .
2.  $\mathbf{c}^{(0)} = (2x_1 - 2x_2, 2x_2 - 2x_1) = (2, -2)$ .
3.  $\|\mathbf{c}^{(0)}\| = 2\sqrt{2} \neq 0$
4. Set  $\mathbf{d}^{(0)} = -\mathbf{c}^{(0)} = (-2, 2)$ .
5. Calculate  $\alpha$  to minimize  $f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)})$  where  $\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (1 - 2\alpha, 2\alpha)$ :

$$\begin{aligned} f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)}) &= (1 - 2\alpha)^2 + (2\alpha)^2 + (2\alpha)^2 - 2(1 - 2\alpha)(2\alpha) \\ &= 16\alpha^2 - 8\alpha + 1 = f(\alpha) \end{aligned}$$

Using the analytic approach

$$\frac{df(\alpha)}{d\alpha} = 0; \quad 32\alpha - 8 = 0 \quad \text{or} \quad \alpha_0 = 0.25 \quad \frac{d^2f(\alpha)}{d\alpha^2} = 32 > 0.$$

6. Updating the design ( $\mathbf{x}^{(0)} + \alpha_0\mathbf{d}^{(0)}$ ):  $x_1^{(1)} = 1 - 0.25(2) = 0.5$ ,  $x_2^{(1)} = 0 + 0.25(2) = 0.5$ . Solving for  $\mathbf{c}^{(1)}$  from the expression in Step 2, we see that  $\mathbf{c}^{(1)} = (0, 0)$ , which satisfies the stopping criterion. Therefore, (0.5, 0.5) is a minimum point for  $f(\mathbf{x})$  and  $f^* = 0$ .

## b. Conjugate gradient method

Actually, the *conjugate gradient directions*  $\mathbf{d}^{(i)}$  are orthogonal with respect to a symmetric and positive definite matrix  $\mathbf{A}$ , i.e.,  $\mathbf{d}^{(i)\top} \mathbf{A} \mathbf{d}^{(j)} = 0$  for all  $i$  and  $j$ ,  $i \neq j$ . The *conjugate gradient algorithm* is stated as follows:

*Step 1.* Estimate a starting design as  $\mathbf{x}^{(0)}$ . Set the iteration counter  $k = 0$ . Select the convergence parameter  $\epsilon$ . Calculate

$$\mathbf{d}^{(0)} = -\mathbf{c}^{(0)} = -\nabla f(\mathbf{x}^{(0)}) \quad (8.21a)$$

*Check stopping criterion.* If  $\|\mathbf{c}^{(0)}\| < \epsilon$ , then stop. Otherwise, go to Step 4 (note that Step 1 of the conjugate gradient and the steepest descent methods is the same).

*Step 2.* Compute the gradient of the cost function as  $\mathbf{c}^{(k)} = \nabla f(\mathbf{x}^{(k)})$ .

*Step 3.* Calculate  $\|\mathbf{c}^{(k)}\|$ . If  $\|\mathbf{c}^{(k)}\| < \epsilon$ , then stop; otherwise continue.

*Step 4.* Calculate the new conjugate direction as

$$\mathbf{d}^{(k)} = -\mathbf{c}^{(k)} + \beta_k \mathbf{d}^{(k-1)}; \quad \beta_k = (\|\mathbf{c}^{(k)}\| / \|\mathbf{c}^{(k-1)}\|)^2 \quad (8.21b)$$

*Step 5.* Compute a step size  $\alpha_k = \alpha$  to minimize  $f(\mathbf{x}^{(k)} + \alpha\mathbf{d}^{(k)})$ .

*Step 6.* Change the design as follows, set  $k = k + 1$  and go to Step 2.

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k \mathbf{d}^{(k)} \quad (8.22)$$



**EXAMPLE 8.6 Use of Conjugate Gradient Algorithm**

Consider the problem solved in Example 8.5: minimize  $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$

Carry out two iterations of the conjugate gradient method starting from the design (2, 4, 10).

The first iteration  $\mathbf{c}^{(0)} = (12, 40, 48)$ ;  $\|\mathbf{c}^{(0)}\| = 63.6$ ,  $f(\mathbf{x}^{(0)}) = 332.0$

$$\mathbf{x}^{(1)} = (0.0956, -2.348, 2.381)$$

The second iteration 2.  $\mathbf{c}^{(1)} = (-4.5, -4.438, 4.828)$ ,  $f(\mathbf{x}^{(1)}) = 10.75$

3.  $\|\mathbf{c}^{(1)}\| = 7.952 > \varepsilon$ , so continue.

$$4. \beta_1 = \|\mathbf{c}^{(1)}\|/\|\mathbf{c}^{(0)}\| = (7.952/63.3)^2 = 0.015633$$

$$\mathbf{d}^{(1)} = -\mathbf{c}^{(1)} + \beta_1 \mathbf{d}^{(0)} = \begin{bmatrix} 4.500 \\ 4.438 \\ -4.828 \end{bmatrix} + (0.015633) \begin{bmatrix} -12 \\ -40 \\ -48 \end{bmatrix} = \begin{bmatrix} 4.31241 \\ 3.81268 \\ -5.57838 \end{bmatrix}$$

5. Step size in the direction  $\mathbf{d}^{(1)}$  is calculated as  $\alpha = 0.3156$ .

$$6. \text{ The design is updated as } \mathbf{x}^{(2)} = \begin{bmatrix} 0.0956 \\ -2.348 \\ 2.381 \end{bmatrix} + \alpha \begin{bmatrix} 4.31241 \\ 3.81268 \\ -5.57838 \end{bmatrix} = \begin{bmatrix} 1.4566 \\ -1.1447 \\ 0.6205 \end{bmatrix}$$

Calculating the gradient at this point, we get  $\mathbf{c}^{(2)} = (0.6238, -0.4246, 0.1926)$ .  $\|\mathbf{c}^{(2)}\| = 0.7788 > \varepsilon$ , so we need to continue the iterations. Note that  $\mathbf{c}^{(2)} \cdot \mathbf{d}^{(1)} = 0$ .

**TABLE 8-3** Optimum Solution for Example 8.6 with the Conjugate Gradient Method:  
 $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$

|                                      |                                       |
|--------------------------------------|---------------------------------------|
| Starting values of design variables: | 2, 4, 10                              |
| Optimum design variables:            | -6.4550E-10, -5.8410E-10, 1.3150E-10. |
| Optimum cost function value:         | 6.8520E-20.                           |
| Norm of the gradient at optimum:     | 3.0512E-05.                           |
| Number of iterations:                | 4                                     |
| Number of function evaluations:      | 10                                    |

**Homework 21: Optimization Reading 2**

Chapters 8.3, 8.4

## c. Newton's method

The basic idea of the Newton's method is to use a second-order Taylor's expansion of the function about the current design point.

$$f(\mathbf{x} + \Delta\mathbf{x}) = f(\mathbf{x}) + \mathbf{c}^T \Delta\mathbf{x} + 0.5 \Delta\mathbf{x}^T \mathbf{H} \Delta\mathbf{x} \quad (80)$$

The optimality conditions ( $\partial f / \partial(\Delta\mathbf{x}) = 0$ ) for the function above

$$\mathbf{c} + \mathbf{H} \Delta\mathbf{x} = 0 \quad \Delta\mathbf{x} = -\mathbf{H}^{-1} \mathbf{c} \quad (81)$$

The optimal step size must be calculated for design optimization.

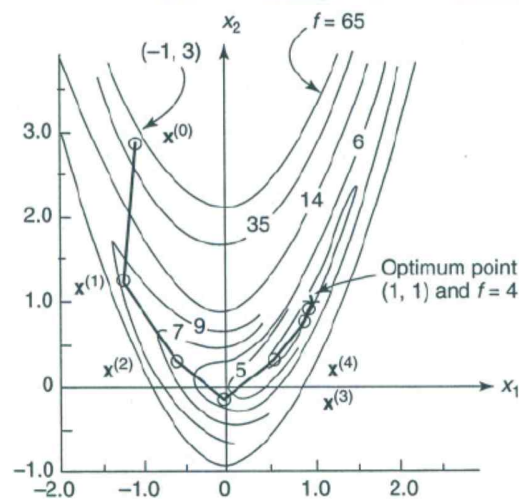
## Use of Modified Newton's Method

$$\text{Minimize } f(\mathbf{x}) = 10x_1^4 - 20x_1^2x_2 + 10x_2^2 + x_1^2 - 2x_1 + 5 \quad (\text{a})$$

using the computer program for the modified Newton's method given in Appendix D from the point  $(-1, 3)$ . Golden section search may be used for step size determination with  $\delta = 0.05$  and line search accuracy equal to 0.0001. For the stopping criterion, use  $\varepsilon = 0.005$ .

$$\mathbf{c} = \nabla f(\mathbf{x}) = (40x_1^3 - 40x_1x_2 + 2x_1 - 2, -20x_1^2 + 20x_2) \quad (\text{b})$$

$$\mathbf{H} = \nabla^2 f(\mathbf{x}) = \begin{bmatrix} 120x_1^2 - 40x_2 + 2 & -40x_1 \\ -40x_1 & 20 \end{bmatrix} \quad (\text{c})$$



### Optimum Solution with Modified Newton's Method

|                                 |                          |
|---------------------------------|--------------------------|
| Starting point:                 | -1, 3                    |
| Optimum design variables:       | 9.99880E-01, 9.99681E-01 |
| Optimum cost function value:    | 4.0                      |
| Norm of gradient at optimum:    | 3.26883E-03              |
| Number of iterations:           | 8                        |
| Number of function evaluations: | 198                      |

- M-file

```
function opt()
x0 = [-1;3];
x = x0;
[x,fval,exitflag,output,grad]= fminunc(@obj,x0);
end
```

```
function f = obj(x)
```

```
f = 10*x(1)^4-20*x(1)^2*x(2)+10*x(2)^2+x(1)^2-2*x(1)+5
end
```

The *drawbacks* of the modified Newton's method for general applications are:

1. It requires calculations of second-order derivatives at each iteration, which is usually quite time consuming. In some applications it may not even be possible to calculate such derivatives. Also, a linear system of equations in Eq. (9.11) needs to be solved. Therefore, each iteration of the method requires substantially more calculations compared with the steepest descent or conjugate gradient method.
2. The Hessian of the cost function may be singular at some iterations. Thus, Eq. (9.11) cannot be used to compute the search direction. Also, unless the Hessian is positive definite, the search direction cannot be guaranteed to be that of descent for the cost function, as discussed earlier.
3. The method is not convergent unless the Hessian remains positive definite and a step size is calculated along the search direction to update design. However, the method has a quadratic rate of convergence when it converges. For a strictly convex quadratic function, the method converges in just one iteration from any starting design.

### Comparison of Steepest Descent, Conjugate Gradient, and Modified Newton Methods

Minimize  $f(\mathbf{x}) = 50(x_2 - x_1^2)^2 + (2 - x_1)^2$  starting from the point (5, -5). Use the steepest descent, Newton, and conjugate gradient methods, and compare their performance.

**TABLE 9-3** Evaluation of Three Methods for Example 9.8:  $f(\mathbf{x}) = 50(x_2 - x_1^2)^2 + (2 - x_1)^2$

|                             | <i>Steepest descent</i> | <i>Conjugate gradient</i> | <i>Modified Newton</i> |
|-----------------------------|-------------------------|---------------------------|------------------------|
| $x_1$                       | 1.9941                  | 2.0000                    | 2.0000                 |
| $x_2$                       | 3.9765                  | 3.9998                    | 3.9999                 |
| $f$                         | 3.4564E-05              | 1.0239E-08                | 2.5054E-10             |
| $\ l\ $                     | 3.3236E-03              | 1.2860E-04                | 9.0357E-04             |
| No. of function evaluations | 138,236                 | 65                        | 349                    |
| No. of iterations           | 9670                    | 22                        | 13                     |

#### d. Quasi-Newton Method

Only the first derivatives of the function are used to generate these Newton approximations. Therefore the methods have desirable features of both the

conjugate gradient and the Newton's methods. They are called *quasi-Newton methods*.

$$\mathbf{c} + \mathbf{H}\Delta\mathbf{x} = 0 \quad \Delta\mathbf{x} = -\mathbf{H}^{-1}\mathbf{c} \quad (82)$$

There are several ways to approximate the Hessian or its inverse. The basic idea is to update the current approximation of the Hessian using two pieces of information: the gradient vectors and their changes in between two successive iterations. While updating, the properties of symmetry and positive definiteness are preserved. Positive definiteness is essential because the search direction may not be a descent direction for the cost function with the property.

### Hessian Updating: BFGS (Broyden-Fletcher-Goldfarb-Shanno) Method

*Step 1.* Estimate an initial design  $\mathbf{x}^{(0)}$ . Choose a symmetric positive definite  $n \times n$  matrix  $\mathbf{H}^{(0)}$  as an estimate for the Hessian of the cost function. In the absence of more information, let  $\mathbf{H}^{(0)} = \mathbf{I}$ . Choose a convergence parameter  $\varepsilon$ . Set  $k = 0$ , and compute the gradient vector as  $\mathbf{c}^{(0)} = \nabla f(\mathbf{x}^{(0)})$ .

*Step 2.* Calculate the norm of the gradient vector as  $\|\mathbf{c}^{(k)}\|$ . If  $\|\mathbf{c}^{(k)}\| < \varepsilon$  then stop the iterative process; otherwise continue.

*Step 3.* Solve the linear system of equations  $\mathbf{H}^{(k)}\mathbf{d}^{(k)} = -\mathbf{c}^{(k)}$  to obtain the search direction.

*Step 4.* Compute optimum step size  $\alpha_k = \alpha$  to minimize  $f(\mathbf{x}^{(k)} + \alpha\mathbf{d}^{(k)})$ .

*Step 5.* Update the design as  $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k\mathbf{d}^{(k)}$

*Step 6.* Update the Hessian approximation for the cost function as

$$\mathbf{H}^{(k+1)} = \mathbf{H}^{(k)} + \mathbf{D}^{(k)} + \mathbf{E}^{(k)} \quad (a)$$

where the correction matrices  $\mathbf{D}^{(k)}$  and  $\mathbf{E}^{(k)}$  are given as

$$\mathbf{D}^{(k)} = \frac{\mathbf{y}^{(k)}\mathbf{y}^{(k)T}}{(\mathbf{y}^{(k)} \cdot \mathbf{s}^{(k)})}; \quad \mathbf{E}^{(k)} = \frac{\mathbf{c}^{(k)}\mathbf{c}^{(k)T}}{(\mathbf{c}^{(k)} \cdot \mathbf{d}^{(k)})} \quad (b)$$

$$\begin{aligned} \mathbf{s}^{(k)} &= \alpha_k\mathbf{d}^{(k)} \text{ (change in design);} & \mathbf{y}^{(k)} &= \mathbf{c}^{(k+1)} - \mathbf{c}^{(k)} \text{ (change in gradient);} \\ \mathbf{c}^{(k+1)} &= \nabla f(\mathbf{x}^{(k+1)}) \end{aligned} \quad (c)$$

*Step 7.* Set  $k = k + 1$  and go to Step 2.

### Example of BFGS Method

Execute two iterations of the BFGS method for the problem:

minimize  $f(\mathbf{x}) = 5x_1^2 + 2x_1x_2 + x_2^2 + 7$  starting from the point (1, 2).

**Solution.** We shall follow steps of the algorithm. Note that the first iteration gives steepest descent step for the cost function.

**Iteration 1 ( $k = 0$ ).**

1.  $\mathbf{x}^{(0)}(1, 2)$ ,  $\mathbf{H}^{(0)} = \mathbf{I}$ ,  $\varepsilon = 0.001$ ,  $k = 0$   
 $\mathbf{c}^{(0)} = (10x_1 + 2x_2, 2x_1 + 2x_2) = (14, 6)$
2.  $\|\mathbf{c}^{(0)}\| = \sqrt{14^2 + 6^2} = 15.232 > \varepsilon$ , so continue
3.  $\mathbf{d}^{(0)} = -\mathbf{c}^{(0)} = (-14, -6)$ ; since  $\mathbf{H}^{(0)} = \mathbf{I}$
4. Step size determination (same as Example 9.9):  $\alpha_0 = 0.099$
5.  $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha_0 \mathbf{d}^{(0)} = (-0.386, 1.407)$
6.  $\mathbf{s}^{(0)} = \alpha_0 \mathbf{d}^{(0)} = (-1.386, 0.593)$ ;  $\mathbf{c}^{(1)} = (-1.046, 2.042)$

$$\mathbf{y}^{(0)} = \mathbf{c}^{(1)} - \mathbf{c}^{(0)} = (-15.046, -3.958); \quad \mathbf{y}^{(0)} \cdot \mathbf{s}^{(0)} = 23.20; \quad \mathbf{c}^{(0)} \cdot \mathbf{d}^{(0)} = -232.0 \quad (\text{a})$$

$$\mathbf{y}^{(0)} \mathbf{y}^{(0)T} = \begin{bmatrix} 226.40 & 59.55 \\ 59.55 & 15.67 \end{bmatrix}; \quad \mathbf{D}^{(0)} = \frac{\mathbf{y}^{(0)} \mathbf{y}^{(0)T}}{\mathbf{y}^{(0)} \cdot \mathbf{s}^{(0)}} = \begin{bmatrix} 9.760 & 2.567 \\ 2.567 & 0.675 \end{bmatrix} \quad (\text{b})$$

$$\mathbf{c}^{(0)} \mathbf{c}^{(0)T} = \begin{bmatrix} 196 & 84 \\ 84 & 36 \end{bmatrix}; \quad \mathbf{E}^{(0)} = \frac{\mathbf{c}^{(0)} \mathbf{c}^{(0)T}}{\mathbf{c}^{(0)} \cdot \mathbf{d}^{(0)}} = \begin{bmatrix} -0.845 & -0.362 \\ -0.362 & -0.155 \end{bmatrix} \quad (\text{c})$$

$$\mathbf{H}^{(1)} = \mathbf{H}^{(0)} + \mathbf{D}^{(0)} + \mathbf{E}^{(0)} = \begin{bmatrix} 9.915 & 2.205 \\ 2.205 & 0.520 \end{bmatrix} \quad (\text{d})$$

**Iteration 2 ( $k = 1$ ).**

2.  $\|\mathbf{c}^{(1)}\| = 2.29 > \varepsilon$ , so continue
3.  $\mathbf{H}^{(1)} \mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$ ; or,  $\mathbf{d}^{(1)} = (17.20, -76.77)$
4. Step size determination:  $\alpha_1 = 0.018455$
5.  $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha_1 \mathbf{d}^{(1)} = (-0.0686, -0.0098)$
6.  $\mathbf{s}^{(1)} = \alpha_1 \mathbf{d}^{(1)} = (0.317, -1.417)$ ;  $\mathbf{c}^{(2)} = (-0.706, -0.157)$  (e)

$$\mathbf{y}^{(1)} = \mathbf{c}^{(2)} - \mathbf{c}^{(1)} = (0.317, -2.199); \quad \mathbf{y}^{(1)} \cdot \mathbf{s}^{(1)} = 3.224; \quad \mathbf{c}^{(1)} \cdot \mathbf{d}^{(1)} = -174.76 \quad (\text{f})$$

$$\mathbf{y}^{(1)} \mathbf{y}^{(1)T} = \begin{bmatrix} 0.1156 & -0.748 \\ -0.748 & 4.836 \end{bmatrix}; \quad \mathbf{D}^{(1)} = \frac{\mathbf{y}^{(1)} \mathbf{y}^{(1)T}}{\mathbf{y}^{(1)} \cdot \mathbf{s}^{(1)}} = \begin{bmatrix} 0.036 & -0.232 \\ -0.232 & 1.500 \end{bmatrix} \quad (\text{g})$$

$$\mathbf{c}^{(1)} \mathbf{c}^{(1)T} = \begin{bmatrix} 1.094 & -2.136 \\ -2.136 & 4.170 \end{bmatrix}; \quad \mathbf{E}^{(1)} = \frac{\mathbf{c}^{(1)} \mathbf{c}^{(1)T}}{\mathbf{c}^{(1)} \cdot \mathbf{d}^{(1)}} = \begin{bmatrix} -0.0063 & 0.0122 \\ 0.0122 & -0.0239 \end{bmatrix} \quad (\text{h})$$

$$\mathbf{H}^{(2)} = \mathbf{H}^{(1)} + \mathbf{D}^{(1)} + \mathbf{E}^{(1)} = \begin{bmatrix} 9.945 & 1.985 \\ 1.985 & 1.996 \end{bmatrix} \quad (\text{i})$$

It can be verified that  $\mathbf{H}^{(2)}$  is quite close to the Hessian of the given cost function. One more iteration of the BFGS method will yield the optimum solution of (0, 0).

**Table 6.1:** Summary of Numerical Aspects in Unconstrained Optimization Algorithms

| Methods      | Steepest           | Conjugate          | Newton                      | Quasi Newton       |
|--------------|--------------------|--------------------|-----------------------------|--------------------|
| Requirements | Function, Gradient | Function, Gradient | Function, Gradient, Hessian | Function, Gradient |
| Stability    | Good               | Good               | Good                        | Good               |
| Efficiency   | Bad                | Good               | Bad                         | Good               |
| Speed        | Bad                | Good               | Good                        | Good               |

Let us recall a constrained design optimization formulated as

$$\begin{aligned}
 &\text{Minimize } f(\mathbf{x}) \\
 &\text{Subject to } h_i(\mathbf{x}) = 0, \quad i = 1, \dots, p \\
 &\quad \quad \quad g_j(\mathbf{x}) \leq 0, \quad j = 1, \dots, m \\
 &\quad \quad \quad \mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U, \quad \mathbf{x} \in R^n
 \end{aligned} \tag{83}$$

where  $m$ : no of inequality constraints, feasible where  $g_j(\mathbf{x}) \leq 0$   
 $p$ : no of equality constraints, feasible where  $h_i(\mathbf{x}) = 0$

### **Homework 22: Optimization Reading 3**

Chapters 9.1, 9.3, 9.4.1, 9.4.2, 9.5

## 6.5 Sequential Linear Programmng (SLP)

At each iteration, most numerical methods for constrained optimization compute design change by solving an approximate subproblem that is obtained by writing linear Taylor's expansions for the cost and constraint functions.

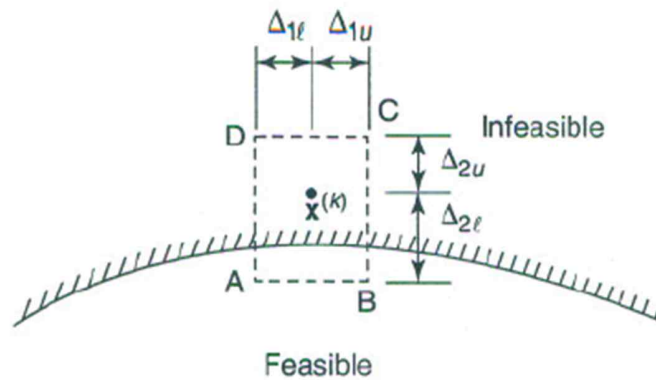
$$\begin{aligned}
 &\text{Minimize } f(\mathbf{x}^{(k)} + \Delta\mathbf{x}^{(k)}) \cong f(\mathbf{x}^{(k)}) + \nabla f^T(\mathbf{x}^{(k)})\Delta\mathbf{x}^{(k)} \\
 &\text{Subject to } h(\mathbf{x}^{(k)} + \Delta\mathbf{x}^{(k)}) \cong h(\mathbf{x}^{(k)}) + \nabla h^T(\mathbf{x}^{(k)})\Delta\mathbf{x}^{(k)} = 0, \quad i = 1, \dots, p \\
 &\quad \quad \quad g(\mathbf{x}^{(k)} + \Delta\mathbf{x}^{(k)}) \cong g(\mathbf{x}^{(k)}) + \nabla g^T(\mathbf{x}^{(k)})\Delta\mathbf{x}^{(k)} \leq 0, \quad j = 1, \dots, m \\
 &\quad \quad \quad \mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U, \quad \mathbf{x} \in R^n
 \end{aligned} \tag{84}$$

The linearization of the problem can be rewritten in a simple form as

$$\begin{aligned}
 \text{Minimize } f &\cong \bar{f} = \sum_{i=1}^{nd} c_i d_i = \mathbf{c}^T \mathbf{d} \\
 \text{Subject to } h &\cong \bar{h} = \sum_{i=1}^{nd} n_{ij} d_i = e_j, \quad i=1, \dots, p \Rightarrow \mathbf{N}^T \mathbf{d} = \mathbf{e} \\
 g &\cong \bar{g} = \sum_{i=1}^{nd} a_{ij} d_i \leq b_j, \quad j=1, \dots, m \Rightarrow \mathbf{A}^T \mathbf{d} \leq \mathbf{b} \\
 \mathbf{x}_L &\leq \mathbf{x} \leq \mathbf{x}_U, \quad \mathbf{x} \in R^n
 \end{aligned} \tag{85}$$

It must be noted that **the problem may not have a bounded solution**, or the changes in design may become too large. Therefore, limits must be imposed on changes in design. Such constraints are usually called “move limits”, expressed as

$$-\Delta_{il}^{(k)} \leq d_i \leq \Delta_{iu}^{(k)} \quad i = 1 \text{ to } n \tag{86}$$



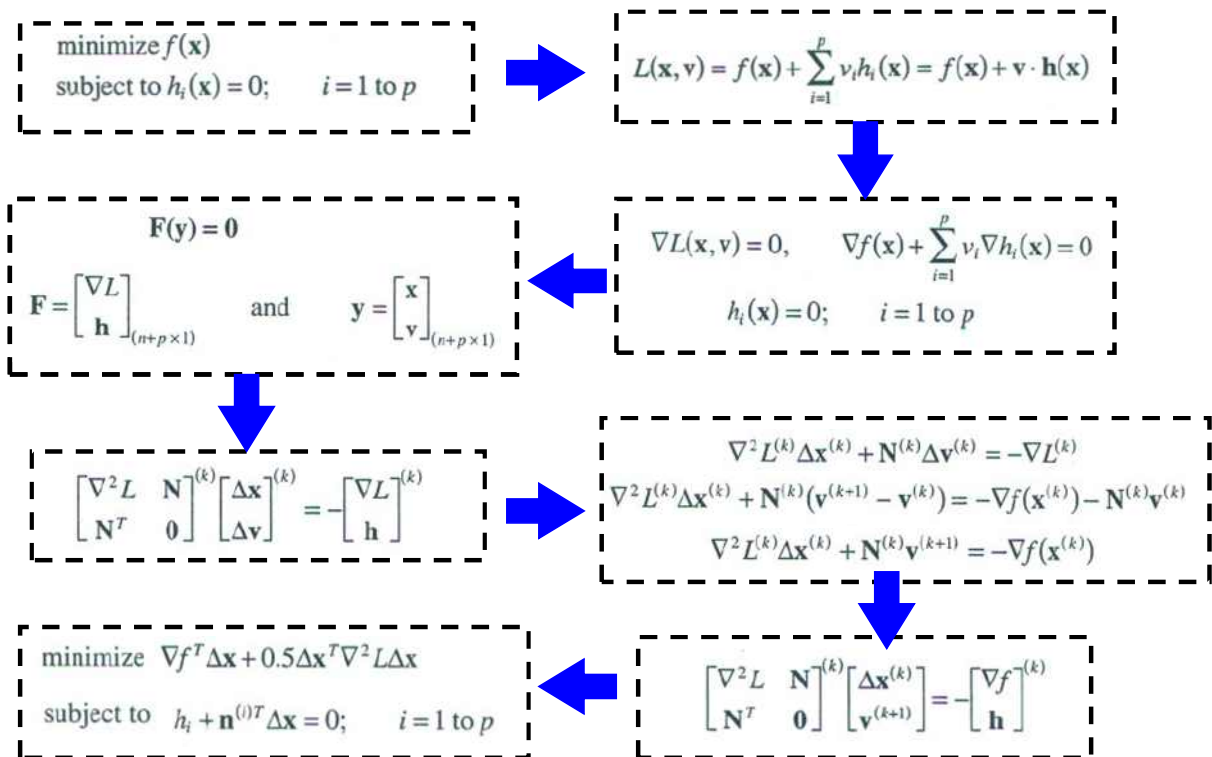
**Figure 6.6:** Linear move limits on design changes

1. *The method should not be used as a black box approach for engineering design problems.* The selection of move limits is a trial and error process and can be best achieved in an interactive mode. The move limits can be too restrictive resulting in no solution for the LP subproblem. Move limits that are too large can cause oscillations in the design point during iterations. Thus performance of the method depends heavily on selection of move limits.
2. *The method may not converge to the precise minimum* since no descent function is defined, and line search is not performed along the search direction to compute a step size. Thus progress toward the solution point cannot be monitored.
3. *The method can cycle between two points* if the optimum solution is not a vertex of the feasible set.
4. The method is quite simple conceptually as well as numerically. Although it may not be possible to reach the precise optimum with the method, it can be used to obtain improved designs in practice.

### 6.6 Sequential Quadratic Programmng (SQP)

There are several ways to derive the quadratic programming (QP) subproblem that has to be solved at each optimization iteration. The QP subproblem can be defined as

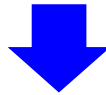
$$\begin{aligned}
 &\text{Minimize } f \cong \bar{f} = \mathbf{c}^T \mathbf{d} + \frac{1}{2} \mathbf{d}^T \mathbf{H} \mathbf{d} \\
 &\text{Subject to } \mathbf{N}^T \mathbf{d} = \mathbf{e} \\
 &\quad \mathbf{A}^T \mathbf{d} \leq \mathbf{b} \\
 &\quad \mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U, \quad \mathbf{d} \in R^n
 \end{aligned} \tag{87}$$



The Hessian matrix can be updated using the quasi-Newton method. The optimization with the equality constraints can be extended to that with both equality and inequality constraints. There is no need to define a move limit unlike SLP.



$$\begin{aligned} & \text{minimize } \nabla f^T \Delta \mathbf{x} + 0.5 \Delta \mathbf{x}^T \nabla^2 L \Delta \mathbf{x} \\ & \text{subject to } h_i + \mathbf{n}^{(i)T} \Delta \mathbf{x} = 0; \quad i = 1 \text{ to } p \end{aligned}$$



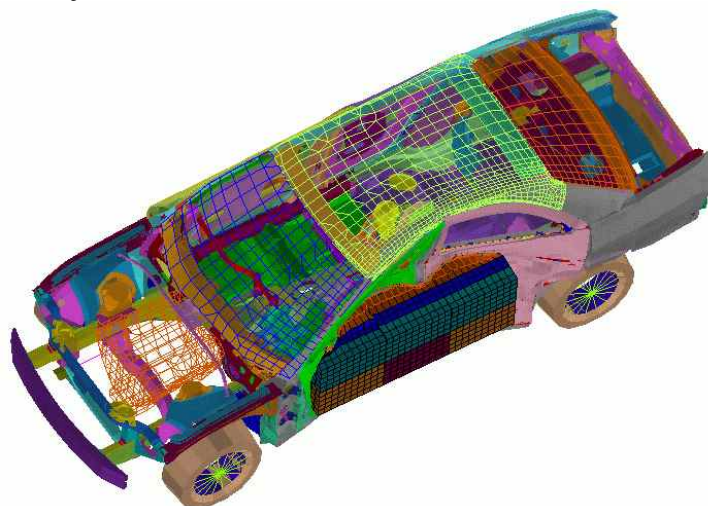
$$\begin{aligned} & \text{minimize } \bar{f} = \mathbf{c}^T \mathbf{d} + 0.5 \mathbf{d}^T \mathbf{H} \mathbf{d} \\ & \text{subject to } \quad \mathbf{n}^{(i)T} \mathbf{d} = e_i; \quad i = 1 \text{ to } p \\ & \quad \quad \quad \mathbf{a}^{(i)T} \mathbf{d} \leq b_i; \quad i = 1 \text{ to } m \end{aligned}$$

### **Homework 22:** *Optimization Reading 4*

Chapters 9.1, 9.3, 9.4.1, 9.4.2, 9.5

### **Homework 23:** *Design Optimization of Crashworthiness Problem*

A vehicle side impact problem is considered for design optimization. All the design variables are shown in Table 6.2. In this example, the weight of the vehicle is treated as an objective function with ten constraints defined in Table 5.3.



**Table 6.2:** Properties of design variables  
( $X_{10}$  and  $X_{11}$  have “o” value)

| Random Variables | $d^L$  | $d$   | $d^U$ |
|------------------|--|-------|-------|
| $X_1$            | 0.500  | 1.000 | 1.500 |
| $X_2$            | 0.500  | 1.000 | 1.500 |
| $X_3$            | 0.500  | 1.000 | 1.500 |
| $X_4$            | 0.500  | 1.000 | 1.500 |
| $X_5$            | 0.500  | 1.000 | 1.500 |
| $X_6$            | 0.500  | 1.000 | 1.500 |
| $X_7$            | 0.500  | 1.000 | 1.500 |
| $X_8$            | 0.192  | 0.300 | 0.345 |
| $X_9$            | 0.192  | 0.300 | 0.345 |
| $X_{10}$         | $X_{10}$ and $X_{11}$ are not design variables |       |       |
| $X_{11}$         |  |       |       |

**Table 6.3:** Design variables and their bounds

| Constraints<br>$G_j$                    | Safety Criteria<br>$G_j^c$ |
|---|----------------------------|
| G1: Abdomen load (kN)                   | $\leq 1$                   |
| G2-G4: Rib deflection (mm)              | $\leq 32$                  |
| Upper                                   |                            |
| Middle                                  |                            |
|   | Lower                      |
| G5-G7: VC (m/s)                         | $\leq 0.32$                |
| Upper                                   |                            |
| Middle                                  |                            |
|   | Lower                      |
| G8: Pubic symphysis force (kN)          | $\leq 4$                   |
| G9: Velocity of B-pillar                | $\leq 9.9$                 |
| G10: Velocity of front door at B-pillar | $\leq 15.7$                |

Responses:

$$\text{Cost}(\text{weight}) = 1.98 + 4.90 \cdot x(1) + 6.67 \cdot x(2) + 6.98 \cdot x(3) + 4.01 \cdot x(4) + 1.78 \cdot x(5) + 2.73 \cdot x(7)$$

$$G1 = (1.16 - 0.3717 \cdot x(2) \cdot x(4) - 0.00931 \cdot x(2) \cdot x(10) - 0.484 \cdot x(3) \cdot x(9) + 0.01343 \cdot x(6) \cdot x(10)) - 1;$$

$$G2 = (28.98 + 3.818 \cdot x(3) - 4.2 \cdot x(1) \cdot x(2) + 0.0207 \cdot x(5) \cdot x(10) + 6.63 \cdot x(6) \cdot x(9) - 7.7 \cdot x(7) \cdot x(8) + 0.32 \cdot x(9) \cdot x(10)) - 32;$$

$$G3 = (33.86 + 2.95 \cdot x(3) + 0.1792 \cdot x(10) - 5.057 \cdot x(1) \cdot x(2) - 11 \cdot x(2) \cdot x(8) - 0.0215 \cdot x(5) \cdot x(10) - 9.98 \cdot x(7) \cdot x(8) + 22 \cdot x(8) \cdot x(9)) - 32;$$

$$G4 = (46.36 - 9.9 \cdot x(2) - 12.9 \cdot x(1) \cdot x(8) + 0.1107 \cdot x(3) \cdot x(10)) - 32;$$

$$G5 = (0.261 - 0.0159 \cdot x(1) \cdot x(2) - 0.188 \cdot x(1) \cdot x(8) -$$

$$0.019*x(2)*x(7)+0.0144*x(3)*x(5)+0.0008757*x(5)*x(10)+0.08045*x(6)*x(9)+0.00139*x(8)*x(11)+0.00001575*x(10)*x(11))-0.32;$$

$$G6 = (0.214+0.00817*x(5)-0.131*x(1)*x(8)-0.0704*x(1)*x(9)+ 0.03099*x(2)*x(6)-0.018*x(2)*x(7)+0.0208*x(3)*x(8)+ 0.121*x(3)*x(9)-0.00364*x(5)*x(6)+0.0007715*x(5)*x(10)-0.0005354*x(6)*x(10)+0.00121*x(8)*x(11)+0.00184*x(9)*x(10)- 0.018*x(2).^2)-0.32;$$

$$G7 = (0.74-0.61*x(2)-0.163*x(3)*x(8)+0.001232*x(3)*x(10)-0.166*x(7)*x(9)+0.227*x(2).^2)-0.32;$$

$$G8 = (4.72-0.5*x(4)-0.19*x(2)*x(3)-0.0122*x(4)*x(10)+0.009325*x(6)*x(10)+0.000191*x(11).^2)-4;$$

$$G9 = (10.58-0.674*x(1)*x(2)-1.95*x(2)*x(8)+0.02054*x(3)*x(10)-0.0198*x(4)*x(10)+0.028*x(6)*x(10))-9.9;$$

$$G10 = (16.45-0.489*x(3)*x(7)-0.843*x(5)*x(6)+0.0432*x(9)*x(10)-0.0556*x(9)*x(11)-0.000786*x(11).^2)-15.7;$$

The Design Optimization is formulated as

$$\text{Minimize } f(\mathbf{x})$$

$$\text{Subject to } g_j(\mathbf{x}) = G_j(\mathbf{x}) - G_j^c \leq 0, \quad j = 1, \dots, 9$$

$$\mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U, \quad \mathbf{x} \in R^9$$

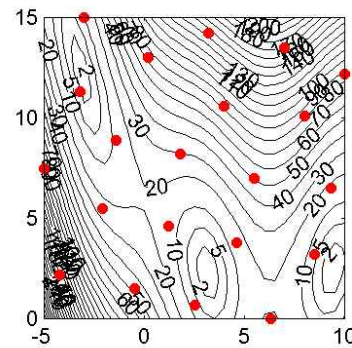
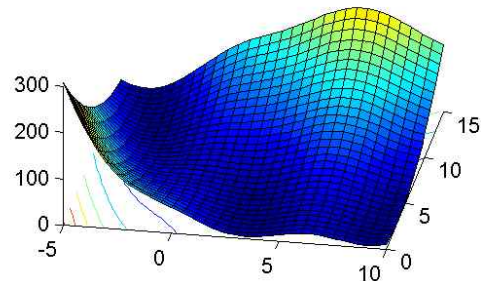
Solve this optimization problem using the sequential quadratic programming (use the matlab function, 'fmincon', in Matlab). Make your own discussion and conclusion.

## CHAPTER 7. SURROGATE MODELING (OR RESPONSE SURFACE METHODOLOGY)

### 7.1 Introduction

Response surface methodology (RSM) is a collection of statistical and mathematical techniques useful for developing, improving, and optimizing processes. The most extensive applications of RSM are in the industrial world, particularly in situations where several input variables potentially influence some responses (e.g., performance measure or quality characteristic of the product or process). The input variables are called “independent variables”, and they are subject to the control of the engineer or scientist, at least for purposes of a test or an experiment.

Figure in the side shows graphically the relationship between the response ( $y$ ) and the two design variables (or independent control variables). To construct the response surface, there must be a systematic way of gathering response data in the design space. Two primary procedures are involved with collecting the response information: (1) Design of Experiment (DOE) and (2) Response Approximation (or Surrogate Modeling). The general procedure can be summarized as



Step 1. Choose design variables and response model(s) to be considered.

Step 2. Plan “Design of Experiments (DOE)” over a design space.

Step 3. Perform “experiments” or “simulation” at the DOE points.

Step 4. Construct a response surface over the design space.

Step 5. Determine a confidence interval of the response surface.

Step 6. Check the model adequacy over the design space.

Step 7. If not adequate, then go to step 1 and refine the model.

### 7.2 Design of Experiments (DOEs)

Design of experiments is the design of all information-gathering exercises where variation is present, whether under the full control of the experimenter or not. Often the experimenter is interested in the effect of some product or process parameters on some relevant responses, which may be product performances or process quality attributes. Design of experiments is thus a discipline that has very broad application across all the natural and social sciences, and various engineering.

In basic, it is concerned about how to gather the information as effective as possible. Thus, the objective of the DOE is to collect the information with minimal experimental cost and maximum model accuracy. The existing DOEs include:

1. (Two-level) Full factorial designs
2. (Two-level) Fractional factorial designs

### 3. Orthogonal designs (or arrays)

- 3.a Box-Behnken designs
- 3.b Koshal design
- 3.c Hybrid design
- 3.d Design optimality

### 7.3 Response Surface Methods (RSMs)

In general, suppose that the scientist or engineer is concerned with a product, process, or system involving a response  $y$  that depends on the controllable input variables  $(x_1, x_2, \dots, x_n)$ . The relationship is

$$\hat{g} = f(x_1, x_2, \dots, x_n) + \varepsilon(x_1, x_2, \dots, x_n) \quad (88)$$

where the form of the true response function  $f$  is unknown and perhaps very complicated, and  $\varepsilon$  is a term that represents other sources of variability not accounted for in  $f$ . Thus,  $\varepsilon$  includes errors in measurement, regression (or interpolation), numerical noise, etc.

#### 7.3.1 Least Squares (LS) Method

The LS approximation can be formulated as

$$\hat{g}(\mathbf{x}) = \sum_{i=1}^{NB} h_i(\mathbf{x})a_i \equiv \mathbf{h}^T(\mathbf{x})\mathbf{a}, \quad \mathbf{x} \in R^{ND} \quad (89)$$

where  $NB$  is the number of terms in the basis,  $ND$  is the number of elements in the union set of both design and random parameters,  $\mathbf{h}$  is the basis functions, and  $\mathbf{a}$  is the LS coefficient vector. Mutually independent functions must be used in a basis. A global LS approximation at  $\mathbf{x}_I$  can be expressed as

$$\hat{g}(\mathbf{x}_I) = \sum_{i=1}^{NB} h_i(\mathbf{x}_I)a_i = \mathbf{h}^T(\mathbf{x}_I)\mathbf{a}, \quad I = 1, \dots, NS \quad (90)$$

where  $NS$  is the number of sample points and  $\mathbf{x}_I$  is a given sample point. The coefficients  $a_i$  are obtained by performing a least squares fit for the global approximation, which is obtained by minimizing the difference between the global approximation and exact response at the set of given sample points. This yields the quadratic form

$$\begin{aligned} E &= \sum_{I=1}^{NS} [\hat{g}(\mathbf{x}_I) - g(\mathbf{x}_I)]^2 \\ &= \sum_{I=1}^{NS} \left[ \sum_{i=1}^{NB} h_i(\mathbf{x}_I)a_i - g(\mathbf{x}_I) \right]^2 \end{aligned} \quad (91)$$

Equation above can be rewritten in a matrix form as

$$E = [\mathbf{H}\mathbf{a} - \mathbf{g}]^T [\mathbf{H}\mathbf{a} - \mathbf{g}] \quad (92)$$

where

$$\begin{aligned}
\mathbf{g} &= [g(\mathbf{x}_1) \quad g(\mathbf{x}_2) \quad \cdots \quad g(\mathbf{x}_{NS})]^T, \\
\mathbf{a} &= [a_1 \quad a_2 \quad \cdots \quad a_{NS}]^T, \quad \text{and} \\
\mathbf{H} &= \begin{bmatrix} h_1(\mathbf{x}_1) & h_2(\mathbf{x}_1) & \cdots & h_{NB}(\mathbf{x}_1) \\ h_1(\mathbf{x}_2) & h_2(\mathbf{x}_2) & \cdots & h_{NB}(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ h_1(\mathbf{x}_{NS}) & h_2(\mathbf{x}_{NS}) & \cdots & h_{NB}(\mathbf{x}_{NS}) \end{bmatrix}
\end{aligned} \tag{93}$$

To find the coefficients  $\mathbf{a}$ , the extreme of the square error  $E(\mathbf{x})$  can be obtained by

$$\frac{\partial E}{\partial \mathbf{a}} = \mathbf{H}^T \mathbf{H} \mathbf{a} - \mathbf{H}^T \mathbf{g} = 0 \tag{94}$$

where  $\mathbf{H}$  is referred to as the basis matrix. The coefficient vector in Eq. (89) is represented by

$$\mathbf{a} = (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \mathbf{g} \tag{95}$$

By substituting Eq. (95) into Eq. (89), the approximation  $\hat{g}(\mathbf{x})$  can then be expressed as

$$\begin{aligned}
\hat{g}(\mathbf{x}) &= \mathbf{h}^T(\mathbf{x}) \mathbf{a} \\
&= \mathbf{h}^T(\mathbf{x}) (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \mathbf{g}
\end{aligned} \tag{96}$$

Read Chapter 2 in the reference book, Response Surface Methodology, written by Raymond H. Myers and Douglas C. Montgomery.

### 7.3.2 Moving Least Squares (MLS) Method

The MLS approximation can be formulated as

$$\hat{g}(\mathbf{x}) = \sum_{i=1}^{NB} h_i(\mathbf{x}) a_i(\mathbf{x}) \equiv \mathbf{h}^T(\mathbf{x}) \mathbf{a}(\mathbf{x}), \quad \mathbf{x} \in R^{ND} \tag{97}$$

where  $NB$  is the number of terms in the basis,  $ND$  is the number of elements in the union set of both design and random parameters,  $\mathbf{h}$  is the basis functions, and  $\mathbf{a}(\mathbf{x})$  is the MLS coefficient vector, which as indicated, is a function of the design parameter  $\mathbf{x}$ . Mutually independent functions must be used in a basis. Any function included in the basis can be exactly reproduced using MLS approximation, which is characterized as a consistency.

Lancaster and Salkauskas (1986) defined a local approximation at  $\mathbf{x}_I$  by

$$\hat{g}(\mathbf{x}, \mathbf{x}_I) = \sum_{i=1}^{NB} h_i(\mathbf{x}_I) a_i(\mathbf{x}) = \mathbf{h}^T(\mathbf{x}_I) \mathbf{a}(\mathbf{x}), \quad I = 1, \dots, NS \tag{98}$$

where  $NS$  is the number of sample points and  $\mathbf{d}_I$  is a given sample point. The coefficients  $a_i(\mathbf{x})$  are obtained by performing a weighted least squares fit for the local approximation, which is obtained by minimizing the difference between the local approximation and exact response at the set of given sample points. This yields the quadratic form

$$\begin{aligned}
 E(\mathbf{x}) &= \sum_{I=1}^{NS} w(\mathbf{x} - \mathbf{x}_I) [\hat{g}(\mathbf{x}, \mathbf{x}_I) - g(\mathbf{x}_I)]^2 \\
 &= \sum_{I=1}^{NS} w(\mathbf{x} - \mathbf{x}_I) \left[ \sum_{i=1}^{NB} h_i(\mathbf{x}_I) a_i(\mathbf{x}) - g(\mathbf{x}_I) \right]^2
 \end{aligned} \tag{99}$$

where  $w(\mathbf{x} - \mathbf{x}_I)$  is a weight function with a compact support. An appropriate support size for the weight function at any data point  $\mathbf{x}_I$  must be selected so that a large enough number of neighboring data points is included to avoid a singularity. A variable weight over the compact support furnishes a local averaging property of the response.

Equation (99) can be rewritten in a matrix form as

$$E(\mathbf{x}) = [\mathbf{H}\mathbf{a}(\mathbf{x}) - \mathbf{g}]^T \mathbf{W}(\mathbf{x}) [\mathbf{H}\mathbf{a}(\mathbf{x}) - \mathbf{g}] \tag{100}$$

where

$$\begin{aligned}
 \mathbf{g} &= [g(\mathbf{x}_1) \quad g(\mathbf{x}_2) \quad \cdots \quad g(\mathbf{x}_{NS})]^T, \\
 \mathbf{a}(\mathbf{x}) &= [a_1 \quad a_2 \quad \cdots \quad a_{NS}]^T, \text{ and} \\
 \mathbf{H} &= \begin{bmatrix} h_1(\mathbf{x}_1) & h_2(\mathbf{x}_1) & \cdots & h_{NB}(\mathbf{x}_1) \\ h_1(\mathbf{x}_2) & h_2(\mathbf{x}_2) & \cdots & h_{NB}(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ h_1(\mathbf{x}_{NS}) & h_2(\mathbf{x}_{NS}) & \cdots & h_{NB}(\mathbf{x}_{NS}) \end{bmatrix}
 \end{aligned} \tag{101}$$

and

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} w(D_1 = |\mathbf{x} - \mathbf{x}_1|) & 0 & \cdots & 0 \\ 0 & w(D_2 = |\mathbf{x} - \mathbf{x}_2|) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(D_{NS} = |\mathbf{x} - \mathbf{x}_{NS}|) \end{bmatrix} \tag{102}$$

To find the coefficients  $\mathbf{a}(\mathbf{x})$ , the extreme of the weighted square error  $E(\mathbf{x})$  can be obtained by

$$\frac{\partial E(\mathbf{x})}{\partial \mathbf{a}(\mathbf{x})} = \mathbf{M}(\mathbf{x})\mathbf{a}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{g} = 0 \tag{103}$$

where  $\mathbf{M}(\mathbf{x})$  is referred to as the moment matrix, and is given by

$$\mathbf{M}(\mathbf{x}) = \mathbf{H}^T \mathbf{W}(\mathbf{x}) \mathbf{H} \quad \text{and} \quad \mathbf{B}(\mathbf{x}) = \mathbf{H}^T \mathbf{W}(\mathbf{x}) \tag{104}$$

The coefficient vector in Eq. (97) is represented by

$$\mathbf{a}(\mathbf{x}) = \mathbf{M}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \mathbf{g} \tag{105}$$

By substituting Eq. (105) into Eq. (97), the approximation  $\hat{g}(\mathbf{x})$  can then be expressed as

$$\begin{aligned}\widehat{\mathbf{g}}(\mathbf{x}) &= \mathbf{h}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \\ &= \mathbf{h}^T(\mathbf{x})\mathbf{M}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{g}\end{aligned}\quad (106)$$

In this study, the modified form of the original **exponential weight function**,  $w(D_I) = e^{-\alpha D_I^2}/2$ , is proposed to yield the form

$$w(D_I(\mathbf{x})) = \begin{cases} \frac{1}{2} \frac{(e^{-\alpha D_I^2} - e^{-\alpha})^2}{1 - e^{-\alpha}} & \text{for } D_I \leq 1 \\ 0 & \text{for } D_I > 1 \end{cases}$$

$$D_I(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_I|$$

in order to possess  $C^1$  continuity. The parameter  $\alpha$  determines the localizing magnitude of the weight function. As parameter  $\alpha$  decreases, exponential weighting possesses more localizing characteristics, since it approaches the Dirac delta function with an unchanged support. The appropriate localizing parameter is selectively suggested in the range  $\alpha=8\sim 12$ .

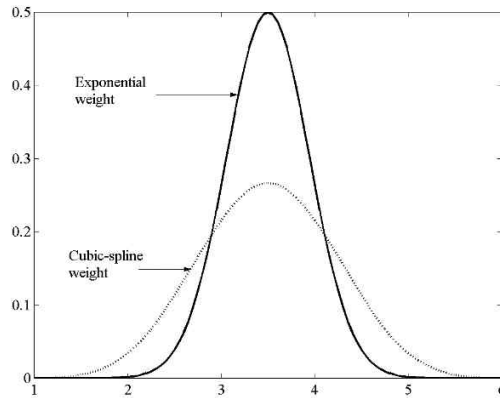
The **cubic-spline weight function**, which is a  $C^2$ -function, is expressed in the form

$$w(D_I) = \begin{cases} \frac{2}{3} - 4D_I^2 + 4D_I^3 & \text{for } 0 \leq D_I \leq \frac{1}{2} \\ \frac{4}{3} - 4D_I + 4D_I^2 - \frac{4}{3}D_I^3 & \text{for } \frac{1}{2} \leq D_I \leq 1 \\ 0 & \text{for } D_I \geq 1 \end{cases}$$

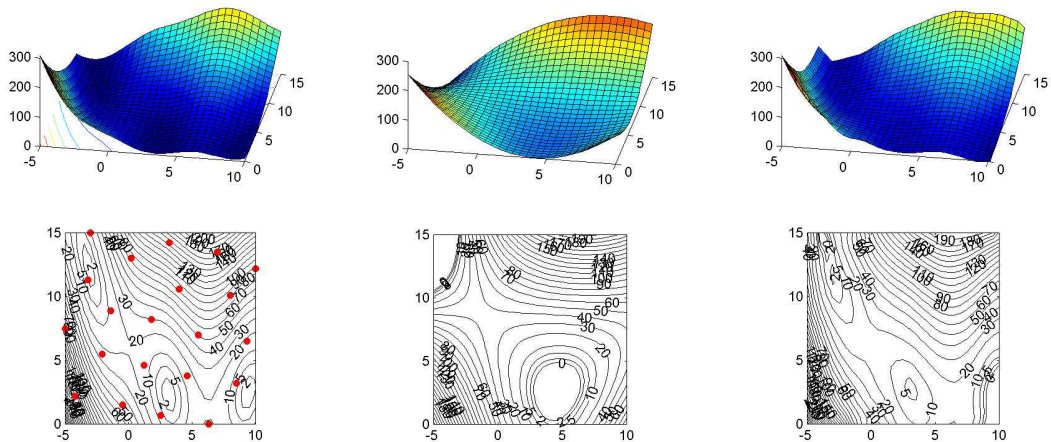
$$D_I(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_I|$$

Exponential and cubic-spline weight functions are comparatively plotted in Figure below. In comparison, the cubic-spline function possesses less localizing and greater averaging features over the compactly supported subdomain. Hence, the approximation is evenly accurate.



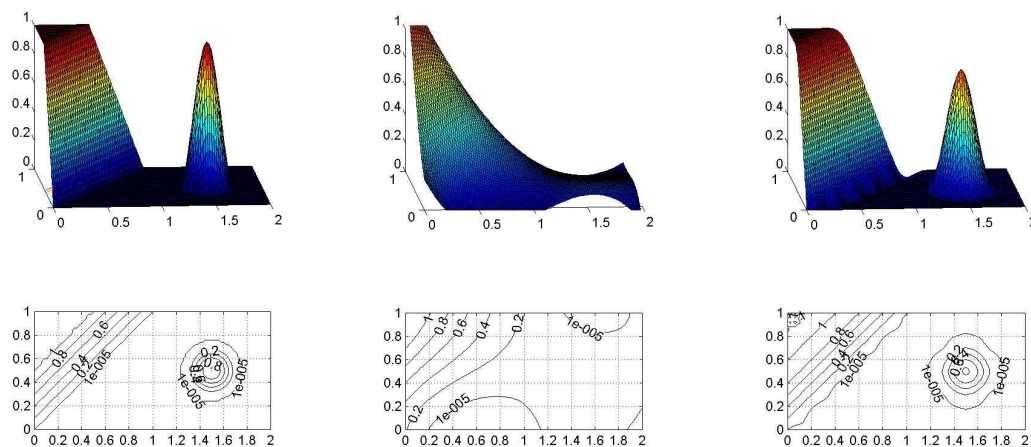


Exponential Weight ( $\alpha=8$ ) and Cubic-Spline Weight ( $s=2.5$ ) Functions



( a ) Exact Response      ( b ) Using LS Method      ( c ) Using MLS Method

Response Surface Approximation for Branin Response



( a ) Exact Response      ( b ) Using LS Method      ( c ) Using MLS Method

Response Surface Approximation for Severe Response

### 7.3.3 Kriging Method

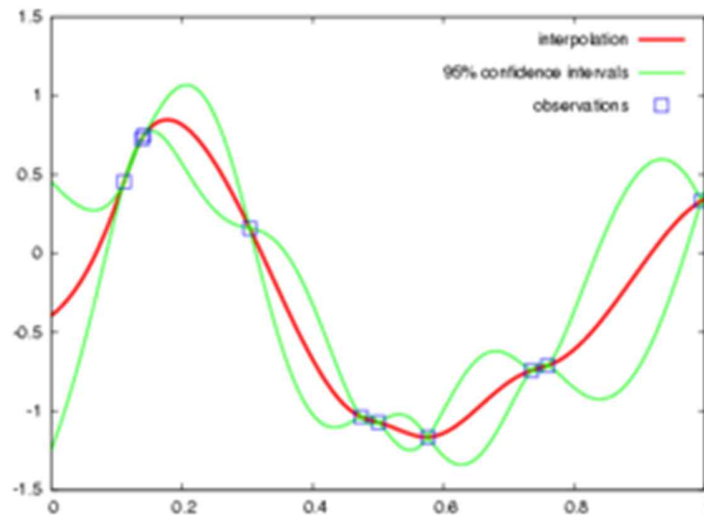
Kriging belongs to the family of nonlinear least squares estimation algorithms. As illustrated in Figure below, the aim of kriging is to estimate the value of an unknown real-valued function,  $g$ , at a point,  $\mathbf{x}^*$ , given the values of the function at some other points,  $x_1, \dots, x_{NS}$ . A kriging estimator is said to be linear because the predicted value  $\hat{g}(\mathbf{x})$  is a linear combination that may be written as

$$\hat{g}(\mathbf{x}) = \sum_{I=1}^{NS} w(D_I(\mathbf{x}))g(\mathbf{x}_I) \quad \text{where } D_I(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_I\|$$

The weights  $w_i(D)$  are solutions of a system of linear equations which is obtained by assuming that  $g$  is a sample-path of a random process  $G(\mathbf{x})$ , and that the error of prediction

$$\varepsilon(\mathbf{x}) = G(\mathbf{x}) - \sum_{I=1}^{NS} w(D_I(\mathbf{x}))g(\mathbf{x}_I)$$

is to be minimized in some sense. For instance, the so-called *simple kriging* assumption is that the mean and the covariance of  $G(\mathbf{x})$  is known and then, the kriging predictor is the one that minimizes the variance of the prediction error.



Other advanced RSM techniques include “radial basis function”, “support vector machine”, “relevance vector machine”, “polynomial chaos expansion”, “stochastic collocation method”, etc.

#### Advantages of RSM:

1. Sensitivity (or derivative) of system performances with respect to design variables can be obtained based on the approximate responses.

2. No need to construct an interface between design optimization and engineering analysis.
3. Graphical method can be used for design optimization.

**Disadvantages of RSM:**

1. A curse of dimensionality
2. Accuracy of an approximate response surface and its derivative is of greatest concern.
3. It may lead to a wrong solution although it is more convenient.

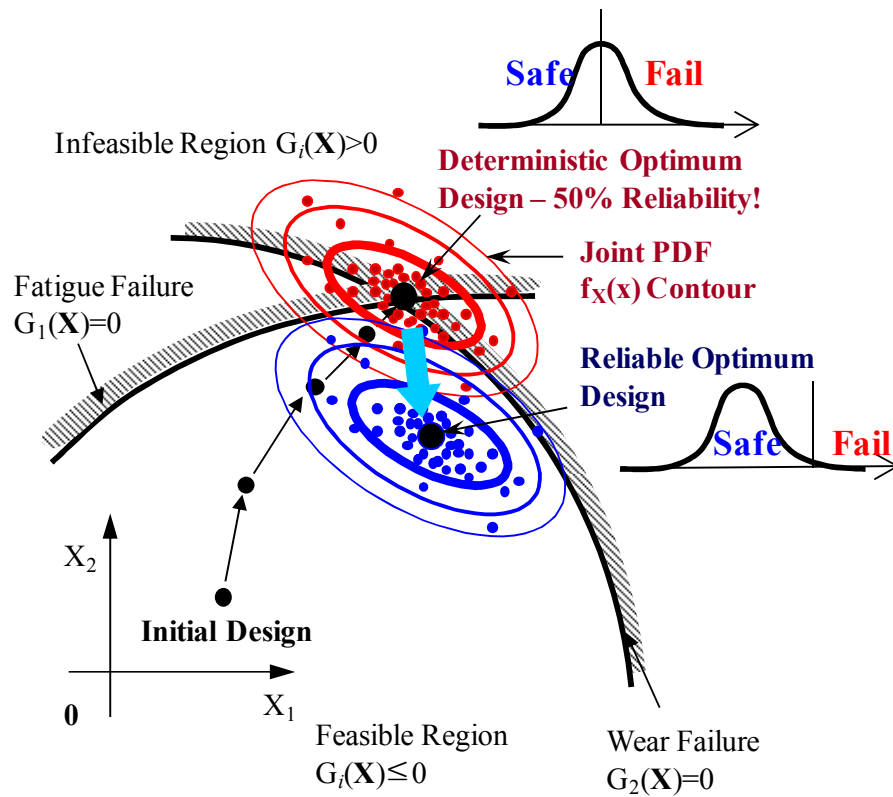
**CHAPTER 8. DESIGN UNDER UNCERTAINTY**

**8.1 Formulation of Design under Uncertainty**

The design under uncertainty, so-called Reliability-Based Design Optimization (RBDO) model, can generally be defined as:

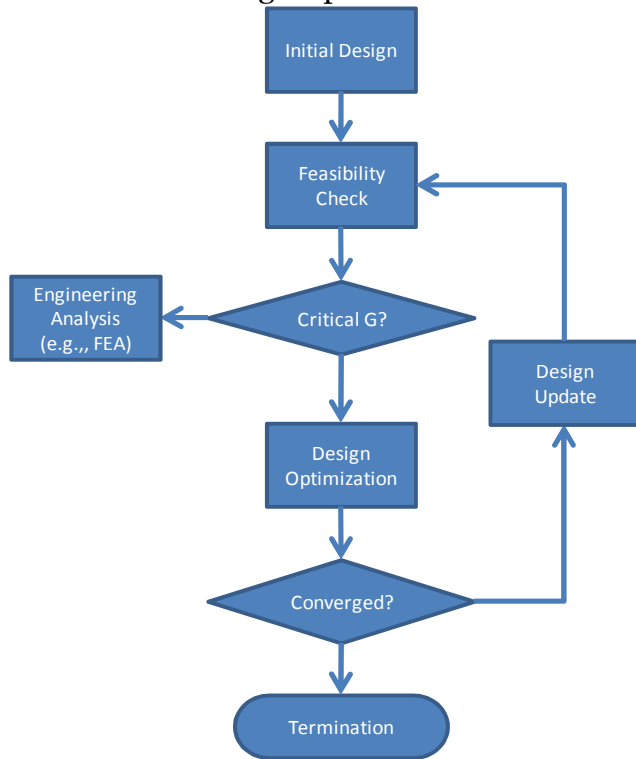
$$\begin{aligned} &\text{Minimize} && \text{Cost}(\mathbf{d}) \\ &\text{subject to} && P\{G_i\{\mathbf{X}; \mathbf{d}(\mathbf{X})\} > 0\} < P_f, \quad i = 1, \dots, nc \\ &&& \mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U, \quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr} \end{aligned}$$

where  $nc$  is the number of probabilistic constraints;  $nd$  is the number of design parameters;  $nr$  is the number of random variables;  $\mathbf{d} = [d_i]^T = \mu(\mathbf{X})$  is the design vector;  $\mathbf{X} = [X_i]^T$  is the random vector; and the probabilistic constraints are described by the performance function  $G_i\{\mathbf{X}; \mathbf{d}(\mathbf{X})\}$ , their probabilistic models, and the probability of failure. The probability of failure is defined as  $P_f \equiv \Phi(-\beta_i)$  with a target reliability index  $\beta_i$  where **the failure is defined as  $G_i\{\mathbf{X}; \mathbf{d}(\mathbf{X})\} = Y_c - Y_i(\mathbf{X}; \mathbf{d}(\mathbf{X})) > 0$** . The RBDO procedure is graphically illustrated in Fig. 7.1.

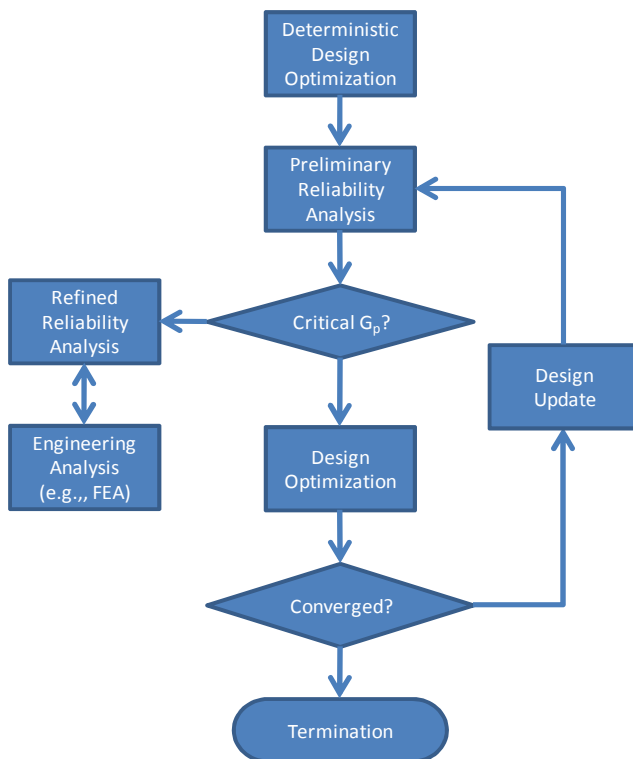


**Figure 8.1:** Reliability-Based Design Optimization (RBDO)

### 8.2 Deterministic Design Optimization vs. RBDO Procedure



**Figure 8.2** Deterministic Design Procedure



**Figure 8.3** RBDO Procedures

### 8.3 Approaches

The statistical description of the failure of the performance function  $G_i(\mathbf{X})$  requires a reliability analysis and is expressed by the CDF  $F_{G_i}(0)$  of the performance function as

$$P(G_i(\mathbf{X}) \leq 0) = F_{G_i}(0) \geq \Phi(\beta_{t_i})$$

where the CDF of the performance function is  $F_{G_i}(g_i)$  and its reliability is described as

$$F_{G_i}(0) = \int_{-\infty}^0 f_{G_i}(g_i) dg_i = \int_{G_i(\mathbf{X}) \leq 0} \dots \int f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \quad i = 1, \dots, nc \text{ and } \mathbf{x} \in R^{nr}$$

The probabilistic constraint can be further expressed in two different ways through inverse transformations:

$$\text{RIA: } \beta_{s_i} = \Phi^{-1}\{F_{G_i}(0)\} \geq \beta_{t_i}$$

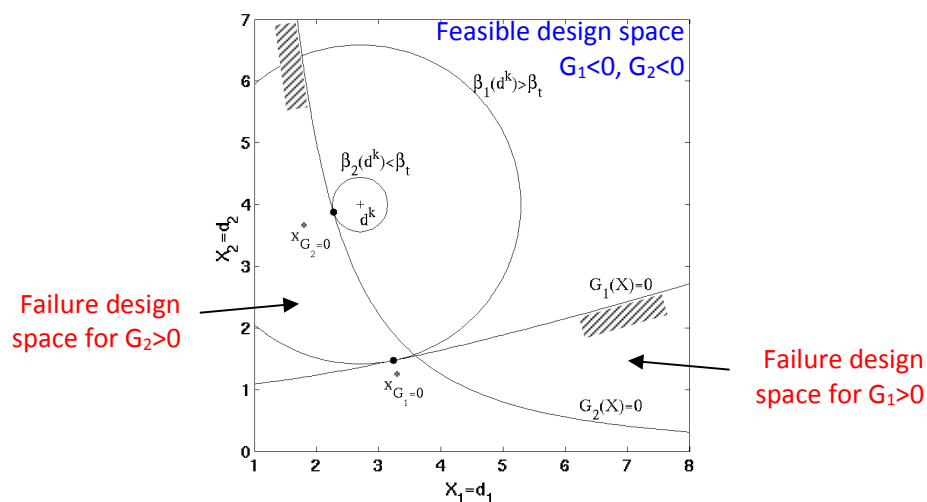
$$\text{PMA: } G_{p_i} = F_{G_i}^{-1}\{\Phi(\beta_{t_i})\} \leq 0$$

where  $\beta_{s_i}$  and  $G_{p_i}$  are respectively called the safety reliability index and the probabilistic performance measure for the  $i^{\text{th}}$  probabilistic constraint.

RIA-RBDO:

$$\begin{aligned} &\text{Minimize} && \text{Cost}(\mathbf{X}; \mathbf{d}) \\ &\text{subject to} && G_{RIA}(\mathbf{X}; \mathbf{d}) = \beta_{t_i} - \beta_{s_i}(\mathbf{X}; \mathbf{d}) \leq 0, \quad i = 1, \dots, nc \\ &&& \mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U, \quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr} \end{aligned}$$

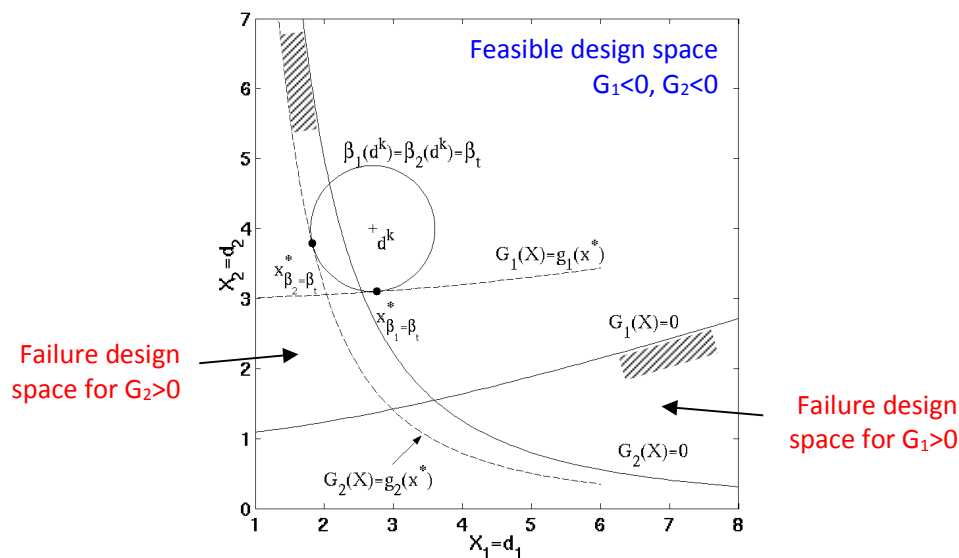
$$\begin{aligned} &\text{minimize} && \|\mathbf{U}\| \\ &\text{subject to} && G(\mathbf{U}) = 0 \end{aligned}$$



**Figure 8.4: Random Search Space in RIA**

PMA-RBDO:

$$\begin{aligned}
 &\text{Minimize} && \text{Cost}(\mathbf{X}; \mathbf{d}) \\
 &\text{subject to} && G_{RIA}(\mathbf{X}; \mathbf{d}) = G_{p_i}(\mathbf{X}; \mathbf{d}) \leq 0, \quad i = 1, \dots, nc \\
 &&& \mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U, \quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr} \\
 &&& \text{maximize} && G(\mathbf{U}) \\
 &&& \text{subject to} && \|\mathbf{U}\| = \beta_t
 \end{aligned}$$



**Figure 7.5:** Random Search Space in PMA

**Table 7.1:** Summary of the RIA and PMA

| Summary |  |
|---------|--|
| RIA     | 1. Good for reliability analysis<br>2. Expensive for sampling method and MPP-based method when reliability is high.<br>3. MPP-based method could be unstable when reliability is high or a performance function is highly nonlinear. |
| PMA     | 1. Good for design optimization.<br>2. Efficient and stable when reliability is high and/or a performance function is highly nonlinear.  |

Neither analytical multi-dimensional integration nor direct numerical integration is possible for large-scale engineering applications. Existing approximate methods for

probability analysis can be categorized into five groups: 1) sampling method; 2) expansion method; 3) the most probable point (MPP)-based method; and 4) stochastic response surface approximate method.

## 8.4 Probabilistic Design Sensitivity Analysis for Design under Uncertainty

In the process of design under uncertainty, an important component is a sensitivity analysis of the estimated failure probability with respect to both random and deterministic design parameters, which is known as the Reliability-Based Design Sensitivity Analysis. The sensitivity information is useful since it quantifies the effect of variations in design parameters on the structural failure probability. The reliability-based design sensitivity analysis of the probabilistic constraint in RIA and PMA with respect to design parameters are presented.

### 8.4.1 Probabilistic Design Sensitivity Analysis in RIA

In RIA, probabilistic constraints are expressed in terms of the safety reliability index. The derivative of safety reliability with respect to design parameter  $d_i$ ,  $i=1, \dots, n$  can be obtained by using a chain rule as

$$\begin{aligned}
 \frac{\partial \beta_{s, \text{FORM}}}{\partial d_i} &= \left. \frac{\partial (\mathbf{U}^t \mathbf{U})^{1/2}}{\partial d_i} \right|_{\mathbf{U}=\mathbf{u}_{G(\mathbf{U})=0}^*} \\
 &= \frac{\partial (\mathbf{U}^t \mathbf{U})^{1/2}}{\partial \mathbf{U}} \cdot \left. \frac{\partial \mathbf{U}}{\partial d_i} \right|_{\mathbf{U}=\mathbf{u}_{G(\mathbf{U})=0}^*} \\
 &= \frac{1}{2} (\mathbf{U}^t \mathbf{U})^{-1/2} \cdot (2\mathbf{U}^t) \cdot \left. \frac{\partial \mathbf{U}}{\partial d_i} \right|_{\mathbf{U}=\mathbf{u}_{G(\mathbf{U})=0}^*} \\
 &= \frac{\mathbf{U}^t}{\beta_{s, \text{FORM}}} \cdot \left. \frac{\partial \mathbf{U}}{\partial d_i} \right|_{\mathbf{U}=\mathbf{u}_{G(\mathbf{U})=0}^*}
 \end{aligned} \tag{107}$$

where  $\mathbf{u}_{G(\mathbf{U})=0}^*$  is the MPFP in  $U$ -space. Using the transformation  $\mathbf{U}=\mathbf{T}(\mathbf{X}; \mathbf{d})$ , Eq. (107) can be rewritten as

$$\frac{\partial \beta_{s, \text{FORM}}}{\partial d_i} = \frac{\mathbf{T}(\mathbf{X}; \mathbf{d})^t}{\beta_{s, \text{FORM}}} \left. \frac{\partial \mathbf{T}(\mathbf{X}; \mathbf{d})}{\partial d_i} \right|_{\mathbf{X}=\mathbf{x}_{G(\mathbf{X})=0}^*} \tag{108}$$



### 8.4.2 Probabilistic Design Sensitivity Analysis in PMA

In PMA, probabilistic constraints are described in terms of the probabilistic performance measure, which in PMA is nothing but the performance measure evaluated at the MPP. The derivative of the estimated probabilistic performance measure with respect to design parameter  $d_i$ ,  $i=1, \dots, n$  can be obtained as

$$\frac{\partial G_{p, \text{FORM}}}{\partial d_i} = \frac{\partial G(\mathbf{U})}{\partial d_i} \bigg|_{\mathbf{U}=\mathbf{u}_{\beta=\beta_i}^*} \quad (109)$$

where  $\mathbf{u}_{\beta=\beta_i}^*$  is the MPP in  $U$ -space. Using the transformation  $\mathbf{U}=\mathbf{T}(\mathbf{X}; \mathbf{d})$ , Eq. (109) can be rewritten as

$$\frac{\partial G_{p, \text{FORM}}}{\partial d_i} = \frac{\partial G(\mathbf{T}(\mathbf{X}; \mathbf{d}))}{\partial d_i} \bigg|_{\mathbf{X}=\mathbf{x}_{\beta=\beta_i}^*} \quad (110)$$

**Table 8.2: Nonlinear Transformation,  $T: X \rightarrow U$**

|           | Parameters  | PDF   | Transformation                                |
|-----------|---|---|---|
| Normal    | $\mu = \text{mean},$<br>$\sigma = \text{standard deviation}$  | $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-0.5[(x-\mu)/\sigma]^2}, -\infty \leq x \leq \infty$     | $X = \mu + \sigma U$                          |
| Lognormal | $\mu = \text{mean}, \sigma = \text{standard deviation}$<br>$\bar{\sigma}^2 = \ln[1 + (\sigma/\mu)^2]$<br>$\bar{\mu} = \ln(\mu) - 0.5\bar{\sigma}^2$ | $f(x) = \frac{1}{\sqrt{2\pi}x\bar{\sigma}} e^{-0.5[(\ln x - \bar{\mu})/\bar{\sigma}]^2}, x > 0$ | $X = e^{\bar{\mu} + \bar{\sigma}U}$           |
| Weibull   | $k > 0, \mu = \Gamma(1 + 1/k)$<br>$\sigma^2 = v^2[\Gamma(1 + 2/k) - \Gamma^2(1 + 1/k)]$   | $f(x) = \frac{k}{v} \left(\frac{x}{v}\right)^{k-1} e^{-(x/v)^k}, x > 0$                         | $X = v[-\ln(\Phi(-U))]^{1/k}$                 |
| Gumbel    | $\mu = v + (0.577/\alpha), \sigma = \pi/\sqrt{6\alpha}$   | $f(x) = \alpha e^{-\alpha(x-v)} e^{-e^{-\alpha(x-v)}}, -\infty \leq x \leq \infty$              | $X = v - \frac{1}{\alpha} \ln[-\ln(\Phi(U))]$ |
| Uniform   | $\mu = (a+b)/2, \sigma = (b-a)/\sqrt{12}$   | $f(x) = \frac{1}{b-a}, a \leq x \leq b$   | $X = a + (b-a)\Phi(U)$                        |

where  $\Phi(U) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^U e^{-u^2/2} du$ .

### 8.5 Code for Design under Uncertainty

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% A 99 LINE RBDO CODE WRITTEN BY WANG P.F. & YOUN B.D. %%%%%%%%%%
function RBDO(nc, nd, x0, dist, lb, ub, rt)
clear all; close all; clc;
global nc nd nm bt stdx Iters Cost
nm=2; nc=3; nd=2; bt=norminv(0.99, 0, 1);
x0=[5, 5]; stdx=[0.6, 0.6]; lb=[0, 0]; ub=[10, 10];
xp=x0; Iters=0;
options = optimset('GradConstr', 'on', 'GradObj', 'on', 'LargeScale', 'off');
[x, fval]=fmincon(@Costfun, x0, [], [], [], [], lb, ub, @frelcon, options)
%===== Obj. Function =====
function [f, g]= Costfun(x) → Interface must be implemented here.

```

```

f=x(1)+x(2);
g=[1 1];
Cost=f;
end
%===== Define Constraints and Gradients =====%
function [c,ceq,GC,GCEq] = frelcon(x)
ceq=[]; GCEq=[];
for j = 1:nc
    if nm==1
        [G,DG]=AMV(x,j);
        beta(j)=G;
        dbeta(:,j)=DG./stdx;
    elseif nm==2
        [G,DG]=HL_RF(x,j);
        beta(j)=bt-G;
        dbeta(:,j)=-DG;
    end
end
c=beta; GC=dbeta;
dx=norm(x-xp);
if dx>1d-5 || ITERS == 0
    ITERS=ITERS+1;
    SHOW(ITERS,x,c,GC);
end
xp = x;
end
%===== PMA Approach with AMV Algorithm =====%
function [G,DG]=AMV(x,kc)
u=zeros(1,nd); iter = 0; Dif=1;
while Dif>1d-5 & iter<20
    iter=iter+1;
    if iter>1
        u=DG*bt/norm(DG);
    end
    [G,DG]=cons(u,x,kc);
    U(iter,:)=u/bt;
    if iter>1
        Dif=abs(U(iter,:)*U(iter-1,:)'-1);
    end
end
end
%===== RIA Approach with HL_RF Algorithm =====%
function [beta,dbeta]=HL_RF(x,kc)
u=zeros(1,nd); iter=0; Dif=1; sign = 1;
while Dif >= 1d-5 & iter < 20
    iter=iter + 1;
    [ceq,GCEq]=cons(u,x,kc);
    u=(GCEq*u'-ceq)/norm(GCEq)^2*GCEq;
    U(iter,:)=u/norm(u);
    if iter ==1
        sign = -ceq/abs(ceq);
    elseif iter>1
        Dif=abs(U(iter-1,:)*U(iter,:)' - 1);
    end
end
beta = sign*norm(u);
dbeta = -u./(beta*stdx);

```

```

end
%===== Constraint Fun. =====%
function [ceq,GCEq]=cons(u,d,kc) → Interface must be implemented here.
    x = u.*stdx+d;
    if kc == 1
        ceq=1-x(1)^2*x(2)/20;
        GCEq(1)=-x(1)*x(2)/10*stdx(1);
        GCEq(2)=-x(1)^2/20*stdx(2);
    elseif kc == 2
        ceq=1-(x(1)+x(2)-5)^2/30-(x(1)-x(2)-12)^2/120;
        GCEq(1)=-(x(1)+x(2)-5)/15-(x(1)-x(2)-12)/60*stdx(1);
        GCEq(2)=-(x(1)+x(2)-5)/15+(x(1)-x(2)-12)/60*stdx(2);
    elseif kc == 3
        ceq=1-80/(x(1)^2+8*x(2)+5);
        GCEq(1)=x(1)*160*stdx(1)/((x(1)^2+8*x(2)+5))^2;
        GCEq(2)=80*8*stdx(2)/((x(1)^2+8*x(2)+5))^2;
    end
end
function SHOW(Iter,x,c,GC)%===== Display the Iteration Information=====
    fprintf(1,'\n***** Iter.%d *****\n',Iter);
    disp(['Des.: ' sprintf('%6.4f ',x)]);
    disp(['Obj.: ' sprintf('%6.4f',Cost)]);
    if nm==1
        disp(['Cons.: ' sprintf('%6.4f ',c)]);
    elseif nm==2
        disp(['Index.: ' sprintf('%6.4f ',bt-c)]);
    end
    disp(['Sens.: ' sprintf('%6.4f ',GC)]);
    fprintf('\n\n')
end
end

```

### Homework 24: RBDO of a Crashworthiness Problem

A vehicle side impact problem is considered for design optimization. All the design variables are shown in Table A. In this example, the abdomen load is treated as an objective function with nine constraints defined in Table B.

**Table A:** Properties of random and design variables  
( $X_{10}$  and  $X_{11}$  have “o” value)

| Random Variables | Distr. Type | Std Dev. | $d^L$ | $d$   | $d^U$ |
|------------------|-------------|----------|-------|-------|-------|
| $X_1$            | Normal      | 0.050    | 0.500 | 1.000 | 1.500 |
| $X_2$            | Normal      | 0.050    | 0.500 | 1.000 | 1.500 |
| $X_3$            | Normal      | 0.050    | 0.500 | 1.000 | 1.500 |
| $X_4$            | Normal      | 0.050    | 0.500 | 1.000 | 1.500 |
| $X_5$            | Normal      | 0.050    | 0.500 | 1.000 | 1.500 |
| $X_6$            | Normal      | 0.050    | 0.500 | 1.000 | 1.500 |
| $X_7$            | Normal      | 0.050    | 0.500 | 1.000 | 1.500 |
| $X_8$            | Lognorm     | 0.006    | 0.192 | 0.300 | 0.345 |

|          |         |       |                               |       |       |
|----------|---------|-------|-------------------------------|-------|-------|
| $X_9$    | Lognorm | 0.006 | 0.192                         | 0.300 | 0.345 |
| $X_{10}$ | Normal  | 10.0  | $X_{10}$ and $X_{11}$ are not |       |       |
| $X_{11}$ | Normal  | 10.0  | design variables              |       |       |

**Table B:** Design variables and their bounds

| Constraints                             |        | Safety Criteria |
|---|--------|-----------------|
| G1: Abdomen load (kN)                   |        | $\leq 1$        |
| G2-G4: Rib deflection (mm)              | Upper  | $\leq 32$       |
|   | Middle |                 |
|   | Lower  |                 |
| G5-G7: VC (m/s)                         | Upper  | $\leq 0.32$     |
|   | Middle |                 |
|   | Lower  |                 |
| G8: Pubic symphysis force (kN)          |        | $\leq 4$        |
| G9: Velocity of B-pillar                |        | $\leq 9.9$      |
| G10: Velocity of front door at B-pillar |        | $\leq 15.7$     |

$$\text{Cost}(\text{weight}) = 1.98 + 4.90 \cdot x(1) + 6.67 \cdot x(2) + 6.98 \cdot x(3) + 4.01 \cdot x(4) + 1.78 \cdot x(5) + 2.73 \cdot x(7)$$

$$G1 = (1.16 - 0.3717 \cdot x(2) \cdot x(4) - 0.00931 \cdot x(2) \cdot x(10) - 0.484 \cdot x(3) \cdot x(9) + 0.01343 \cdot x(6) \cdot x(10)) - 1;$$

$$G2 = (28.98 + 3.818 \cdot x(3) - 4.2 \cdot x(1) \cdot x(2) + 0.0207 \cdot x(5) \cdot x(10) + 6.63 \cdot x(6) \cdot x(9) - 7.7 \cdot x(7) \cdot x(8) + 0.32 \cdot x(9) \cdot x(10)) - 32;$$

$$G3 = (33.86 + 2.95 \cdot x(3) + 0.1792 \cdot x(10) - 5.057 \cdot x(1) \cdot x(2) - 11 \cdot x(2) \cdot x(8) - 0.0215 \cdot x(5) \cdot x(10) - 9.98 \cdot x(7) \cdot x(8) + 22 \cdot x(8) \cdot x(9)) - 32;$$

$$G4 = (46.36 - 9.9 \cdot x(2) - 12.9 \cdot x(1) \cdot x(8) + 0.1107 \cdot x(3) \cdot x(10)) - 32;$$

$$G5 = (0.261 - 0.0159 \cdot x(1) \cdot x(2) - 0.188 \cdot x(1) \cdot x(8) - 0.019 \cdot x(2) \cdot x(7) + 0.0144 \cdot x(3) \cdot x(5) + 0.0008757 \cdot x(5) \cdot x(10) + 0.08045 \cdot x(6) \cdot x(9) + 0.00139 \cdot x(8) \cdot x(11) + 0.00001575 \cdot x(10) \cdot x(11)) - 0.32;$$

$$G6 = (0.214 + 0.00817 \cdot x(5) - 0.131 \cdot x(1) \cdot x(8) - 0.0704 \cdot x(1) \cdot x(9) + 0.03099 \cdot x(2) \cdot x(6) - 0.018 \cdot x(2) \cdot x(7) + 0.0208 \cdot x(3) \cdot x(8) + 0.121 \cdot x(3) \cdot x(9) - 0.00364 \cdot x(5) \cdot x(6) + 0.0007715 \cdot x(5) \cdot x(10) - 0.0005354 \cdot x(6) \cdot x(10) + 0.00121 \cdot x(8) \cdot x(11) + 0.00184 \cdot x(9) \cdot x(10) - 0.018 \cdot x(2)^2) - 0.32;$$

$$G7 = (0.74 - 0.61 \cdot x(2) - 0.163 \cdot x(3) \cdot x(8) + 0.001232 \cdot x(3) \cdot x(10) - 0.166 \cdot x(7) \cdot x(9) + 0.227 \cdot x(2)^2) - 0.32;$$

$$G8 = (4.72 - 0.5 \cdot x(4) - 0.19 \cdot x(2) \cdot x(3) - 0.0122 \cdot x(4) \cdot x(10) + 0.009325 \cdot x(6) \cdot x(10) + 0.000191 \cdot x(11)^2) - 4;$$

$$G9 = (10.58 - 0.674 \cdot x(1) \cdot x(2) - 1.95 \cdot x(2) \cdot x(8) + 0.02054 \cdot x(3) \cdot x(10) - 0.0198 \cdot x(4) \cdot x(10) + 0.028 \cdot x(6) \cdot x(10)) - 9.9;$$

$$G10 = (16.45 - 0.489 \cdot x(3) \cdot x(7) - 0.843 \cdot x(5) \cdot x(6) + 0.0432 \cdot x(9) \cdot x(10) - 0.0556 \cdot x(9) \cdot x(11) - 0.000786 \cdot x(11)^2) - 15.7;$$

The Design Optimization is formulated as

$$\text{Minimize } f(\mathbf{x})$$

$$\text{Subject to } P(G_j(\mathbf{x}) \leq 0) \geq 90\%, \quad j = 1, \dots, 10$$

$$\mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U, \quad \mathbf{x} \in R^9$$

Solve the RBDO optimization problem using the matlab function, 'fmincon', in Matlab) starting at the initial design ( $d_1$  to  $d_7 = 1.000$ ,  $d_8 = d_9 = 0.300$ ) and deterministic optimum design. Make your own discussion and conclusion.

## 8.6 Case Studies

### Example 1: Reliability Analysis of Durability Model

A roadarm in a military tracked vehicle shown in Fig. E1 is employed to demonstrate the effectiveness of the HMV method for a large-scale problem. Reliability analysis for this example involves the crack initiation fatigue life performance measure. A 17-body dynamics model is created to drive the tracked vehicle on the Aberdeen proving ground 4 at a constant speed of 20 miles per hour forward (positive  $X_2$ ). A 20-second dynamic simulation is performed with a maximum integration time step of 0.05-second using the dynamic analysis package DADS.

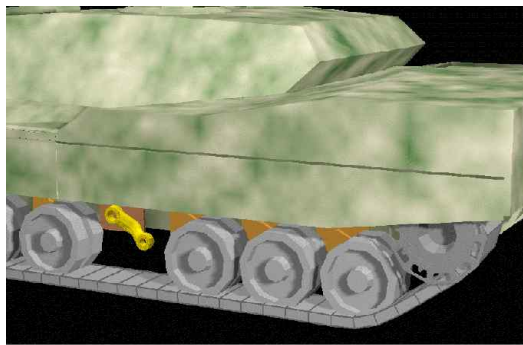
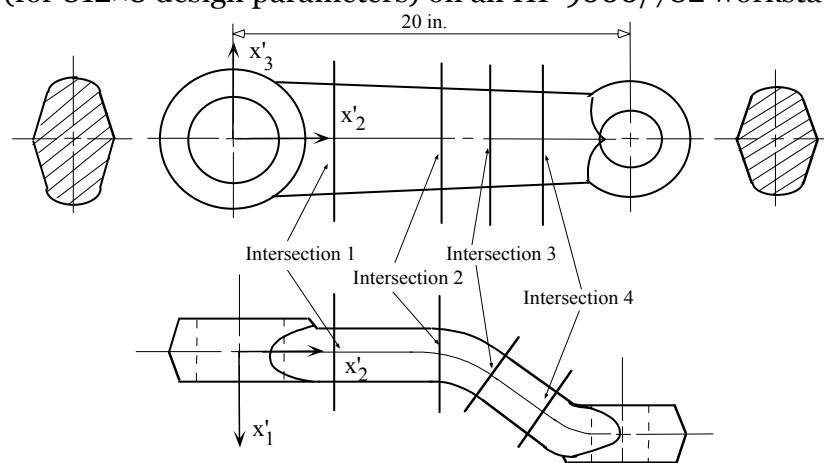
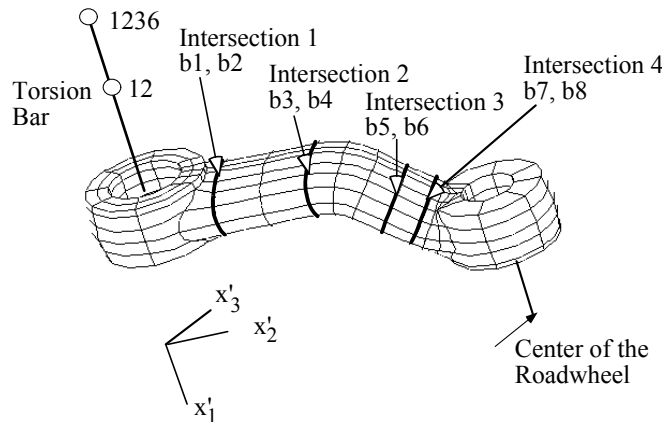


Figure E1. Military Tracked Vehicle

Three hundred and ten 20-node isoparametric finite elements, STIF95, and four beam elements, STIF4, of ANSYS are used for the roadarm finite element model shown in Fig. E2. The roadarm is made of S4340 steel with material properties of Young's modulus  $E=3.0 \times 10^7$  psi and Poisson's ratio  $\nu=0.3$ . Finite element analysis is performed to obtain the stress influence coefficient of the roadarm using ANSYS by applying 18 quasi-static loads. To compute the multiaxial crack initiation life of the roadarm, the equivalent von Mises strain approach is employed. The fatigue life contour in Fig. E3 shows critical nodes and the shortest life is listed in Table E1. The computation for fatigue life prediction and for design sensitivity require, respectively, 6950 and 6496 CPU seconds (for  $812 \times 8$  design parameters) on an HP 9000/782 workstation.



( a ) Geometry of Roadarm Model



( b ) Finite Element Model of Roadarm Model

Figure E2. Geometry and Finite Element Model for Roadarm Model

Table E1. Critical Nodes for Crack Initiation Fatigue Life

| Node ID | Life [Load Cycle] | Life [Year] |
|---------|-------------------|-------------|
| 885     | .9998E+07         | 6.34        |
| 889     | .1134E+08         | 7.19        |
| 990     | .5618E+08         | 35.63       |
| 994     | .6204E+08         | 39.35       |

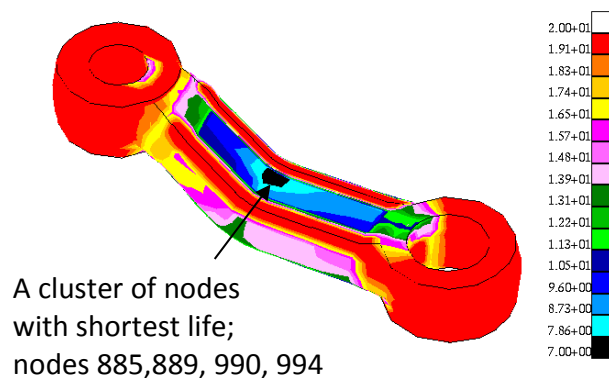


Figure E3. Contour for Crack Initiation Fatigue Life

The random variables and their statistical properties for the crack initiation life prediction are listed in Table E2. Eight tolerance random parameters characterize four cross sectional shapes of the roadarm. The contour of a cross sectional shape consists of four straight lines and four cubic curves, as shown in Fig. E4. Side variations ( $x_1'$  - direction) of the cross sectional shapes are defined as the random parameters  $b_1, b_3, b_5,$  and  $b_7$  for intersections 1 to 4, respectively, and vertical variations ( $x_3'$  -direction) of the cross sectional shapes are defined using the remaining four random variables.

For reliability analysis, a failure function is defined as

$$G(\mathbf{X}) = 1 - \frac{L(\mathbf{X})}{L_t} \quad (\text{E1})$$

where  $L(\mathbf{X})$  is the number of service blocks to initiate crack at node 885 and  $L_t$  is the number of target service blocks to initiate crack in the structural component. The number of blocks at node 885 for the current design is  $9.998\text{E}+6$  (20 seconds per block), which constitutes the shortest life of the component. The target crack initiation fatigue life is set as 0.1 years (i.e.,  $1.577\text{E}+5$  cycles) to illustrate the concave performance function.

Table E2. Definition of Random Variables for Crack Initiation Fatigue Life Prediction

| Random Variables | Mean Value | Standard Deviation | Distribution Type |
|------------------|------------|--------------------|-------------------|
| Tolerance b1     | 1.8776     | 0.094              | Normal            |
| Tolerance b2     | 3.0934     | 0.155              | Normal            |
| Tolerance b3     | 1.8581     | 0.093              | Normal            |
| Tolerance b4     | 3.0091     | 0.150              | Normal            |
| Tolerance b5     | 2.5178     | 0.126              | Normal            |
| Tolerance b6     | 2.9237     | 0.146              | Normal            |
| Tolerance b7     | 4.7926     | 0.246              | Normal            |
| Tolerance b8     | 2.8385     | 0.142              | Normal            |

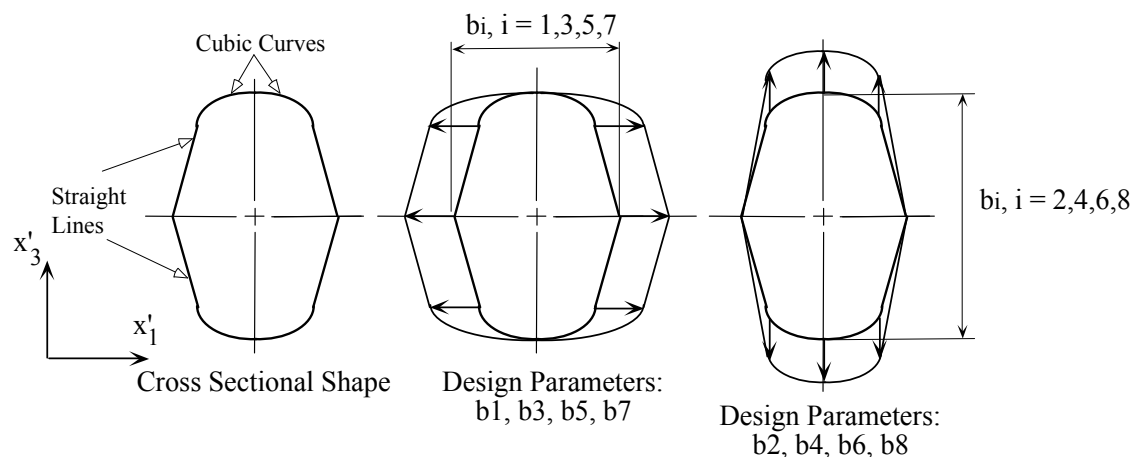


Figure E4. Definition of Random Parameters in Roadarm Model

The conventional AMV and proposed HMV method are used to calculate the reliability of the crack initiation life. Beginning at the mean point, the HMV method has converged to MPP at  $\mathbf{x}^* = [1.872, 3.093, 1.708, 2.830, 2.218, 2.755, 4.758, 2.836]^T$  with a target reliability index  $\beta_t = 3.325$ , as obtained from RIA. In contrast, the AMV method has diverged due to oscillation. Consistent with the previous concave function examples, the HMV method has converged while the AMV method has diverged.

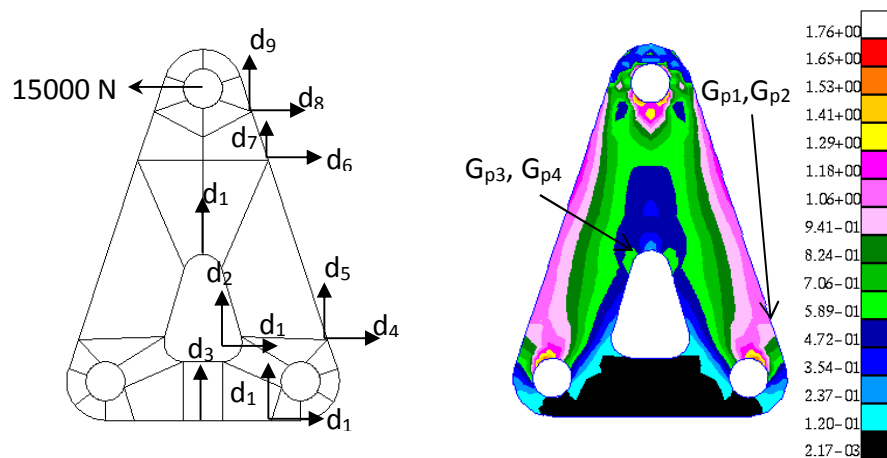


Table E3. MPP Search History in Roadarm Durability Model

| Iteration | AMV      |         | HMV       |         |         |
|-----------|----------|---------|-----------|---------|---------|
|           | $G(X)$   | $\beta$ | $G(X)$    | $\beta$ | $\zeta$ |
| 0         | 62.404   | 0.0     | 62.404    | 0.0     | N.A.    |
| 1         | 0.014    | 3.325   | 0.014     | 3.325   | N.A.    |
| 2         | 0.004    | 3.325   | 0.004     | 3.325   | N.A.    |
| 3         | -0.001   | 3.325   | 0.001     | 3.325   | -0.0038 |
| 4         | 0.002    | 3.325   | 0.000     | 3.325   | -0.0042 |
| 5         | -0.001   | 3.325   |           |         |         |
| 6         | 0.002    | 3.325   |           |         |         |
| 7         | -0.001   | 3.325   |           |         |         |
| ...       | ...      | ...     |           |         |         |
| 19        | -0.001   | 3.325   |           |         |         |
| 20        | 0.002    | 3.325   |           |         |         |
|           | Diverged |         | Converged |         |         |

**Example 2: Bracket Problem in RBDO Model**

Figure E5 shows design parameterization and stress analysis result of a bracket at the initial design. A total of 12 design parameters are selected to define the inner and outer boundary shapes of the bracket model while maintaining symmetry. Design parameterization is performed by selecting the control points of the parametric curves. The bracket is modeled as a plane stress problem using 769 nodes, 214 elements, and 552 DOF with the thickness of 1.0 cm. The boundary condition is imposed to fix two lower holes. Using FEM, stress analysis required 18.23 CPU sec., while a design sensitivity analysis required 35.44/12=2.95 sec. per design variable. The bracket is made of steel with  $E = 207 \text{ GPa}$ ,  $\nu = 0.3$ , and the yield stress of  $\sigma = 400 \text{ MPa}$ . Probabilistic constraints are defined on two critical regions using the von Mises stress as shown in Fig. E5(b). Random parameters are defined in Table E4 and a sequential quadratic programming optimizer is used with a target reliability index of  $\beta = 3.0$  in the RBDO model.



(a) Design Parameterization (b) Stress Contour at Initial Design  
Figure E5. Initial Bracket Design

Table E4. Random Variables in Bracket Model

| Random Variables | $d_L$  | Mean (Design) | $d_U$  | Standard Deviation | Distrib. Type |
|------------------|--------|---------------|--------|--------------------|---------------|
| 1                | 0.800  | 1.006         | 3.000  | 0.2                | Normal        |
| 2                | 1.600  | 3.004         | 3.500  | 0.2                | Normal        |
| 3                | 0.000  | 0.000         | 1.500  | 0.2                | Normal        |
| 4                | 4.470  | 6.388         | 7.000  | 0.2                | Normal        |
| 5                | 3.850  | 4.139         | 4.500  | 0.2                | Normal        |
| 6                | 2.690  | 3.332         | 3.800  | 0.2                | Normal        |
| 7                | 13.030 | 13.32         | 14.000 | 0.2                | Normal        |
| 8                | 1.850  | 2.493         | 2.800  | 0.2                | Normal        |
| 9                | 15.550 | 15.84         | 16.500 | 0.2                | Normal        |
| 10               | 2.500  | 3.509         | 3.800  | 0.2                | Normal        |
| 11               | 0.000  | 0.000         | 1.200  | 0.2                | Normal        |
| 12               | 6.000  | 7.776         | 14.000 | 0.2                | Normal        |

Figure E6 shows several design iterations throughout the RBDO process. At the optimum design, the overall area is substantially reduced at the inner boundary and slightly at the outer boundary. Figure E7 (a) shows the stress contour at the MPP of the initial design where all probabilistic constraints are largely inactive. Figure E7 (b) shows the stress contour at the MPP of the optimum design.

Design histories are shown in Fig. E8. The area of the reliability-based optimum design is reduced by 47% of the original area. The first probabilistic constraint becomes active while other probabilistic constraints inactive at the optimum design with 99.9% reliability as shown in Fig. E8 (b). The significantly changed shape design parameters are 12<sup>th</sup>, 1<sup>st</sup>, and 2<sup>nd</sup> parameters. In Table E5, the PMA with both HMV and AMV methods is compared to RIA in terms of computational efficiency and robustness. As in the roadarm model, the RIA fails to converge in reliability analysis, whereas PMA successfully obtains an optimal design for the bracket model. In addition, PMA with the HMV method performs better than with the conventional AMV method in terms of numerical efficiency (195 analyses vs. 295 analyses).

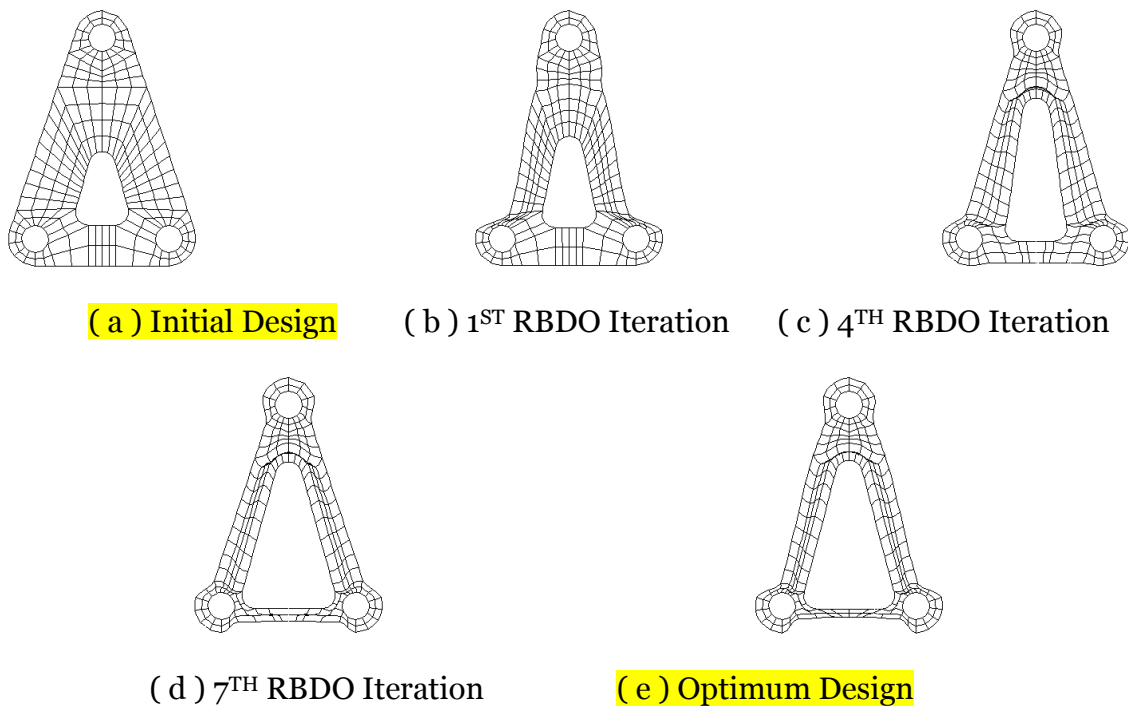
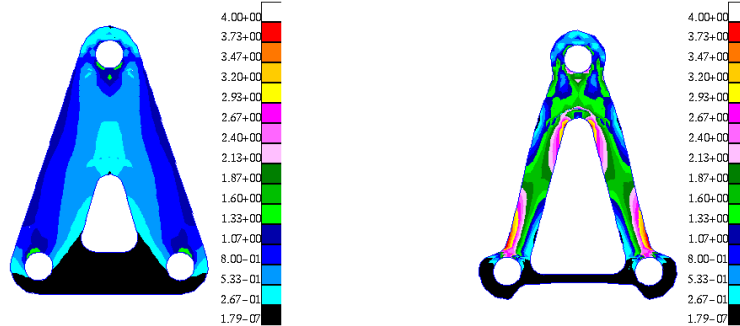
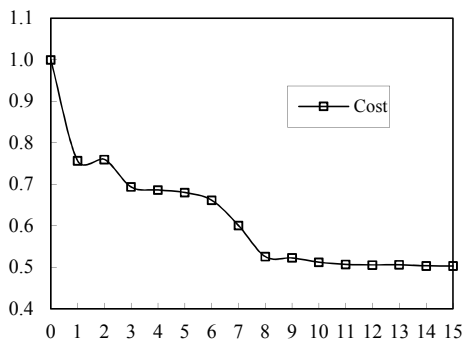


Figure E6. Shape Design History in RBDO Process

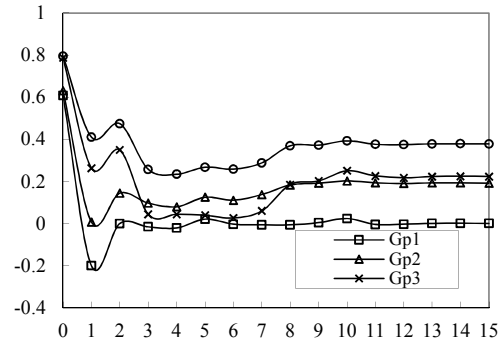


( a ) Stress Contour At Initial Design ( b ) Stress Contour At Optimum Design

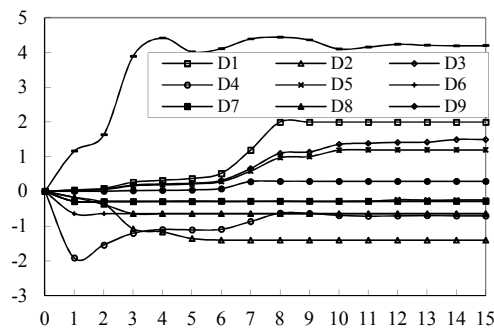
Figure E7. Analysis Results Comparison



( a ) Volume History



( b ) Probabilistic Constraint History



( c ) Design Parameter History

Figure E8. Design Optimization with 99.9 % Reliability

Table E5. Computational Efficiency and Robustness in RIA and PMA

| Opt. Iter. | PMA         |          |             |          | RIA                 |          |
|------------|-------------|----------|-------------|----------|---------------------|----------|
|            | HMV         |          | AMV         |          | HL-RF               |          |
|            | Line Search | Analysis | Line Search | Analysis | Line Search         | Analysis |
| 0          | 1           | 5        | 1           | 5        | 1                   | 40       |
| 1          | 1           | 7        | 1           | 7        | 3                   | 120      |
| 2          | 2           | 14       | 2           | 14       | 1                   | N.A.     |
| 3          | 1           | 5        | 1           | 5        |                     |          |
| 4          | 3           | 25       | 3           | 46       |                     |          |
| 5          | 2           | 16       | 2           | 16       |                     |          |
| 6          | 1           | 10       | 1           | 12       |                     |          |
| 7          | 1           | 10       | 1           | 16       |                     |          |
| 8          | 1           | 9        | 1           | 17       |                     |          |
| 9          | 1           | 10       | 1           | 8        |                     |          |
| 10         | 1           | 14       | 1           | 8        |                     |          |
| 11         | 1           | 16       | 1           | 9        |                     |          |
| 12         | 1           | 9        | 1           | 18       |                     |          |
| 13         | 2           | 25       | 1           | 19       |                     |          |
| 14         | 1           | 10       | 1           | 19       |                     |          |
| 15         | 1           | 10       | 2           | 25       |                     |          |
| 16         |             |          | 3           | 51       |                     |          |
| Optimum    | 21          | 195      | 24          | 295      | Failure to Converge |          |

### Example 3: Reliability-Based Robust Design Optimization for Gasket Sealing Performance

Reliability-based robust design optimization is applied to an engine rubber gasket design problem. An engine gasket is used to prevent oil leakage. The design objective is to determine the shape of the gasket so that robustness of the sealing performance and reliability of other performances like the contact force ( $F_i^{CT}$ ) and stress ( $\sigma_i$ ) are improved when installed and during the operation. Parametric spline curves are used to represent the gasket boundary, and the shapes of these curves are defined as design parameters. Nine shape design parameters are defined, as shown in Fig. E9. In order to maintain the symmetrical shape, four design parameters are linked. Both design and random parameters are defined in Table E6. All random parameters are assumed to be statistically independent.

Table E6. Properties of Design and Random Variables

| IDV,IRV | $\mathbf{d}_L$ | $\mathbf{d}$ , mean | $\mathbf{d}_U$ | Standard Dev. | Distribution |
|---------|----------------|---------------------|----------------|---------------|--------------|
| 1       | -0.01          | 0.00                | 0.01           | 0.001         | normal       |
| 2       | -0.01          | 0.00                | 0.01           | 0.001         | normal       |
| 3       | -0.4           | 0.00                | 0.4            | 0.01          | normal       |
| 4       | -0.4           | 0.00                | 0.4            | 0.01          | normal       |
| 5       | -0.4           | 0.00                | 0.4            | 0.01          | normal       |
| 6       | -0.4           | 0.00                | 0.4            | 0.01          | normal       |
| 7       | -0.4           | 0.00                | 0.4            | 0.01          | normal       |
| 8       | 0.00           | 0.00                | 0.01           | 0.001         | normal       |
| 9       | 0.00           | 0.00                | 0.01           | 0.001         | normal       |

Figure E9 shows the initial gasket geometry before installation. Since the engine block is much stiffer than the rubber gasket, only the gasket is modeled, using a meshfree method with 325 particles; it is assumed that all other parts are rigid.

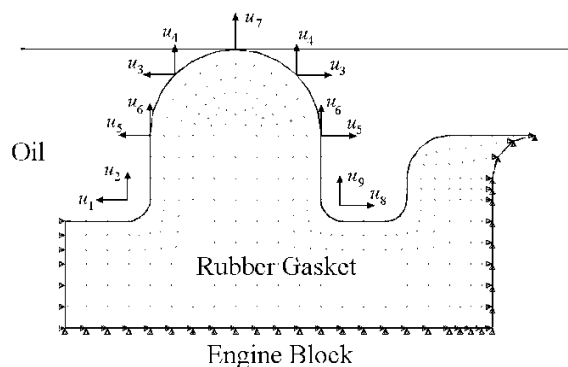


Figure E9. Design Parameterization of the Gasket

Although the sealing performance can be enhanced by increasing the gasket size

or by applying a large amount of installation force, such methods can cause stress concentration, and thus, short service life. The improvement of the sealing performance is defined as the minimization of the gap. Thus, the reliability-based robust design optimization is formulated as an S-Type as

$$\begin{aligned} \min \quad & \text{sgn}(\mu_{Gap}) \cdot \left( \frac{\mu_{Gap}}{\mu_{Gap_0}} \right)^2 + \left( \frac{\sigma_{Gap}}{\sigma_{Gap_0}} \right)^2 \\ \text{s.t.} \quad & P(G_i(\mathbf{X}; \mathbf{d}) \leq 0) \geq \Phi(\beta_i), \quad i = 1, \dots, np \\ & \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U \end{aligned} \quad (\text{E1})$$

where  $G_i(\mathbf{X}; \mathbf{d}) = F_i^{ct} - F_{i,threshold}^{ct} \leq 0, \quad i = 1, \dots, m$   
 $G_i(\mathbf{X}; \mathbf{d}) = \sigma_i - \sigma_{bound} \leq 0, \quad i = m + 1, \dots, np$

Since the contact region cannot be defined before analysis is carried out, a possible contact region is initially defined, and the square sum of the gap ( $\sum \text{gap}^2$ ) along the region is then measured at the specified points, as shown in Fig. E10.

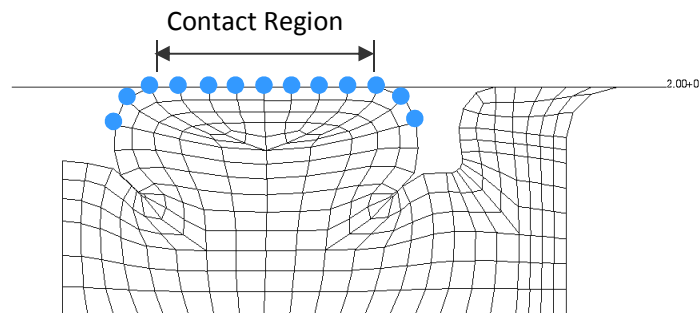


Figure E10. Sealing Performance

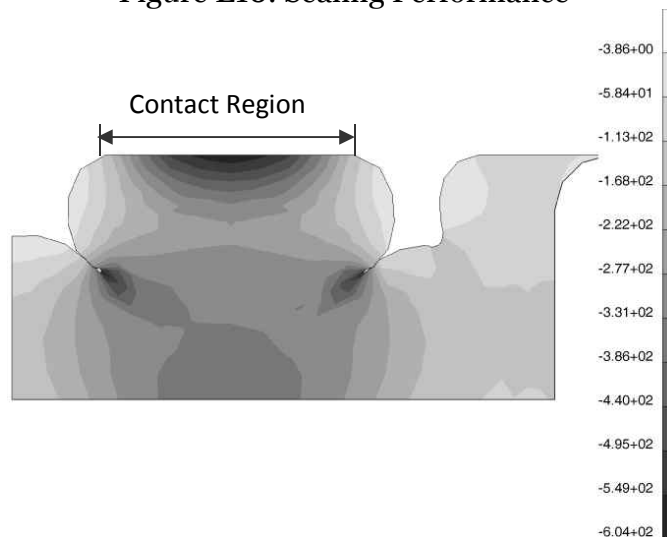


Figure E11. Stress Result at Initial Design

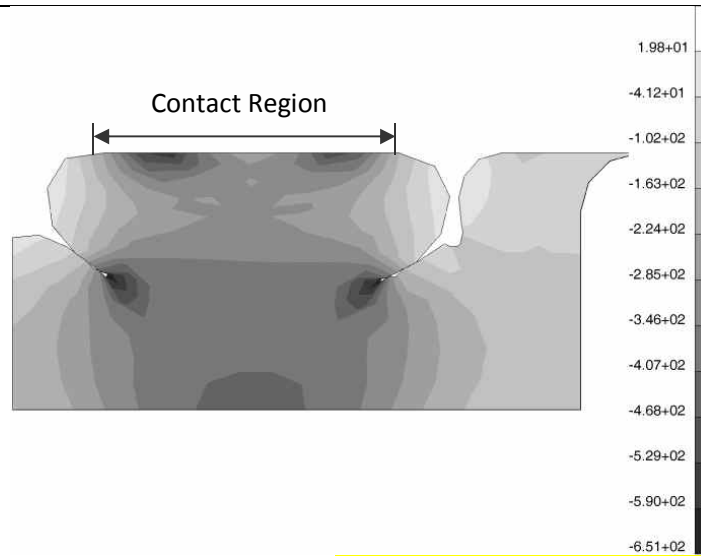


Figure E12. Stress Result at **Deterministic Optimum Design**

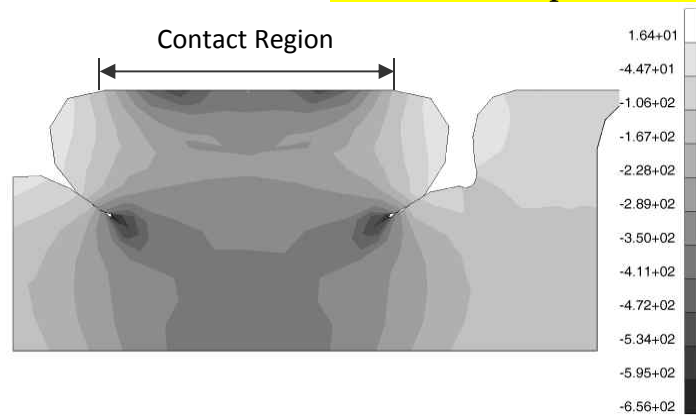
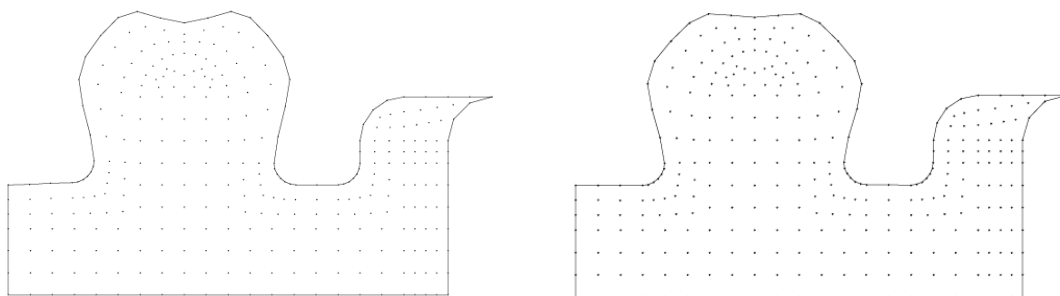


Figure E13. Stress Result at **Reliability-Based Robust Optimum Design**



( a ) **Deterministic Optimum Design**      ( b ) **Reliability-Based Robust Optimum Design**

Figure E14. Deterministic Optimum Design and Reliability-Based Robust Optimum Design

A reliability-based robust optimum design is successfully obtained after seven iterations of reliability-based robust design optimization. As shown in Figs. E12-14, the reliability-based robust optimum design is compared to the initial and deterministic optimum designs. The reliability-based robust optimum design has higher reliability



and quality while it has slightly smaller contact region than the deterministic optimum design. It is interesting to note that the initial circular region of the gasket top changes to an V-shape at the optimum design in order to reduce the concentration of stress, while increasing the contact region. The reliability-based robust optimum design has smaller dip than the deterministic optimum design, as shown in Fig. E14. Table E7 summarizes the result of sealing performance in terms of the gap, showing product quality improvement at the initial and optimum design. The quality of sealing performance  $(1 - C_{ql}(\mathbf{x}_{opt})/C_{ql}(\mathbf{x}_{int}))$  is improved by 39.2%.

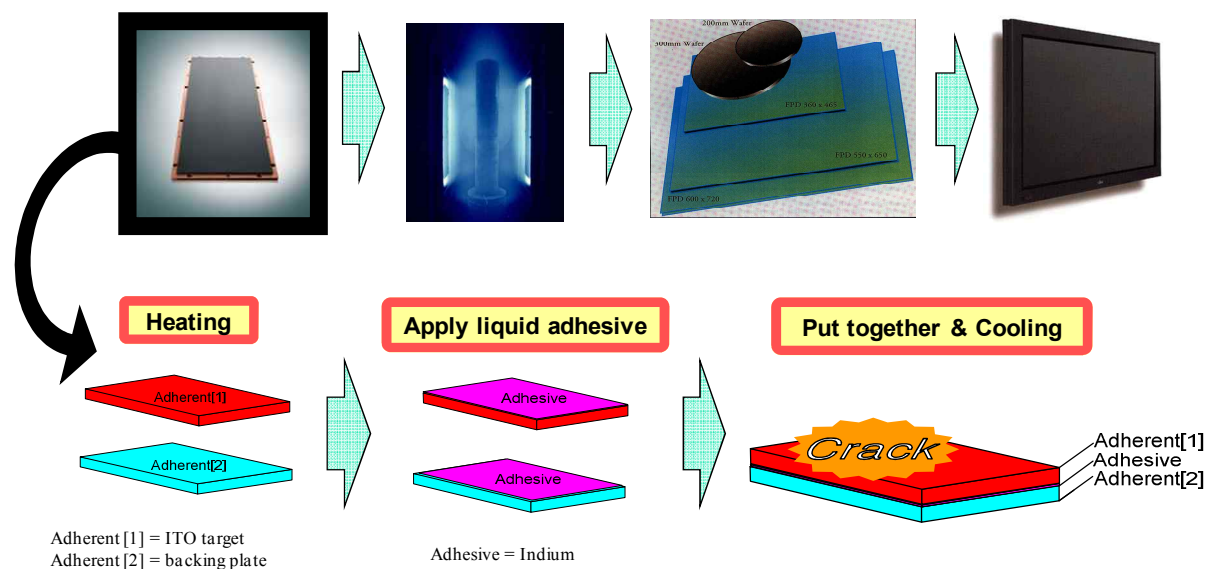
**Table E7. Gasket Sealing Quality Improvement**

| Gap       | Initial | Reliability-Based Robust Optimum |
|-----------|---------|----------------------------------|
| Mean      | 0.8692  | 0.3894                           |
| Std. Dev. | 0.0131  | 0.0132                           |

### Example 4: LCD Manufacturing Process

The bonding process of layered plates (called an Indium-Tin-Oxide (ITO) sputtering target process) is very popular in the manufacturing of semi-conductor or electronic display components. During this process, two plates (glass and copper) are bonded together by a suitable adhesive to form laminated stacks, which can be further processed in the following 4 steps:

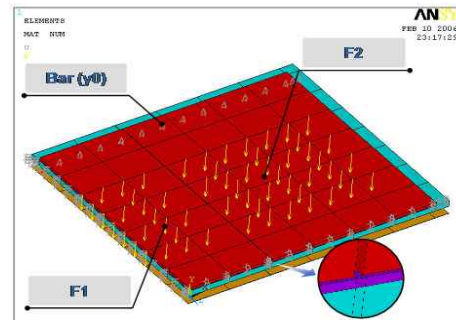
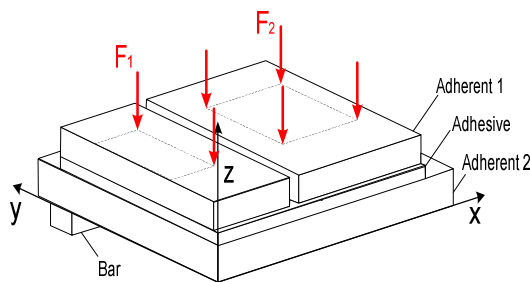
- 1) heating the two plates above the melting temperature of the adhesive;
- 2) applying the adhesive at each surface of the plate;
- 3) putting them in contact with each other;
- 4) cooling them down to a room temperature.



In this process, residual stress due to the mismatch of the thermal expansion coefficients of two layered plates could result in failures of the component such as crack, distortion, and interfacial delamination. Therefore, it is very important to accurately estimate the stress in order to improve the product quality. Herein, a transient thermal Finite Element (FE) analysis was used to predict the stress and deformation of plates. The model for the layered bonding plates is shown in Fig. E15. Considering the symmetry of the problem, a quarter of the model is used, as shown in Fig. E15(a). Due to the brittle property and high stress at the adherent 1, cracks and distortion could occur. To reduce such defects, weights are applied on top of the adherent 1, as shown in Fig. E15(a) from the beginning of the process, and are removed at the end of the cooling process. The bonding assembly is placed on a pair of supporting bars, as shown in Fig. E15(a). Three design variables, weight at the edge ( $X_1$  or  $F_2$ ), weight at the center ( $X_2$  or  $F_1$ ), and height of the bar ( $X_3$  or  $y_0$ ), are considered in this problem. Their statistical information is shown in Table E8. The objective function is to minimize the mean and standard deviation of residual stress. Two constraints are maximum stress during the process ( $< 130\text{MPa}$ ) and center displacement ( $< 3\text{mm}$ ).

**Table E8** Design/random properties of layered plate bonding model

| Design Variables | Distr.Type | Mean | Std. Dev. |
|------------------|------------|------|-----------|
| $X_1$            | Normal     | 4000 | 400       |
| $X_2$            | Normal     | 2000 | 200       |
| $X_3$            | Normal     | 1    | 0.1       |



(a) Isometric view of the quarter model

(b) FM model

**Figure E15** Target bonding process and FE model

The RBRDO is formulated as

$$\begin{aligned} &\text{Minimize } Q = \mu_r + \sigma_r \\ &\text{Subject to } R_i = P(G_i(X) \leq 0) \leq \Phi(-\beta_i), \quad i = 1, 2 \\ &2000 \leq X_1 \leq 10000; 1000 \leq X_2 \leq 5000; 1 \leq X_3 \leq 5; \end{aligned}$$

where  $\mu_r$  and  $\sigma_r$  are the mean and standard deviation of residual stress;  $G_1(X)$  is the instantaneous stress;  $G_2(X)$  is the edge displacement;  $\beta_{ti}=3$ .

The EDR method is carried out to evaluate the quality (= mean + standard deviation) of residual stress and the reliabilities of two constraints. The eigenvector sampling scheme for the EDR method is adaptively chosen in the RBRDO process, since the system responses are highly nonlinear. First, RBRDO starts with  $2N+1$  sampling scheme for the EDR method. Then when satisfying a relaxed convergence criteria ( $\varepsilon \leq 0.1$ ), the RBRDO process turns the  $4N+1$  sampling on. In this example, the standard deviation at the fourth design iteration is quite small but this estimation is not accurate enough because of highly nonlinear responses. Therefore, after the fourth design iteration, RBRDO is performed with the  $4N+1$  sampling scheme to enhance accuracy of the quality and reliability estimates. The SQP is used for a design optimizer in RBRDO. After eight design iterations, the optimum design is found where  $X_2$  is close to the upper bound, as shown in Table E9. The EDR method requires totally 87 function evaluations for RBRDO. MCS with 1000 random samples is used to confirm the EDR results at optimum design. It is found that the EDR estimates for the mean and standard deviation of the residual stress at the optimum design are very close to those using MCS. The overall quality is drastically improved by 38%.

**Table E9** Design history of layered bonding plates model

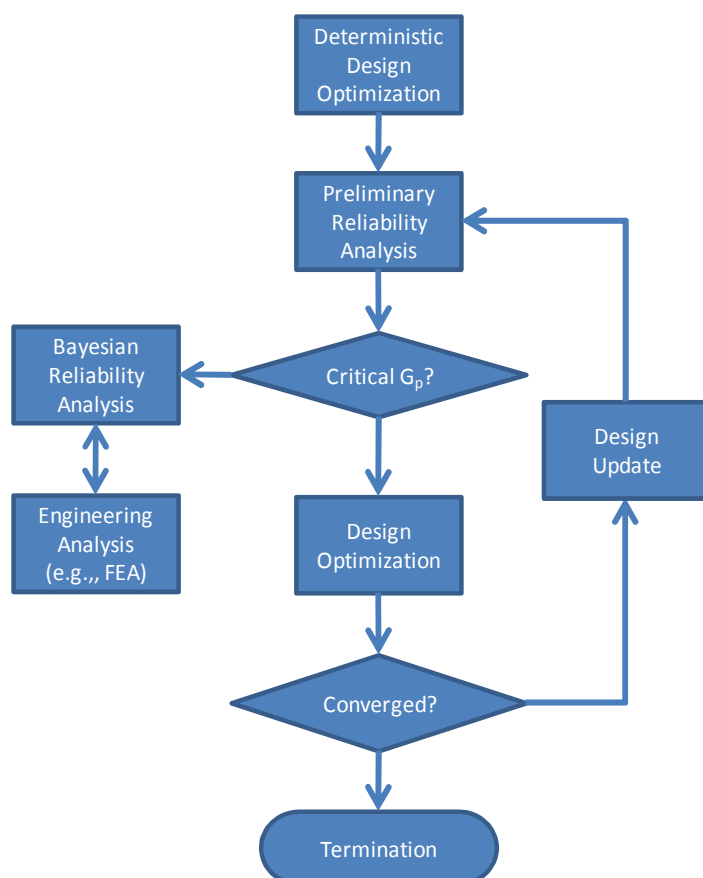
| <b>Iteration</b> | <b>Obj</b> | <b>Mean</b> | <b>Std.<br/>Dev.</b> | <b>X<sub>1</sub></b> | <b>X<sub>2</sub></b> | <b>X<sub>3</sub></b> | <b>G<sub>1</sub></b> | <b>G<sub>2</sub></b> | <b># of<br/>analysis</b> |
|------------------|------------|-------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|--------------------------|
| 0                | 23.322     | 23.020      | 0.302                | 4000.000             | 2000.000             | 1.000                | -94.876              | 1.051                | 7                        |
| 1                | 21.437     | 21.350      | 0.087                | 4579.841             | 3633.035             | 2.317                | -85.742              | 0.108                | 7                        |
| 2                | 21.358     | 21.215      | 0.143                | 4659.514             | 4704.467             | 3.356                | -79.354              | -0.467               | 7                        |
| 3                | 21.177     | 21.040      | 0.137                | 4316.124             | 5000.000             | 3.734                | -77.240              | -0.631               | 7                        |
| 4                | 20.884     | 20.808      | 0.075                | 3121.245             | 5000.000             | 3.772                | -77.371              | -0.567               | 7                        |
| 5                | 20.976     | 20.862      | 0.115                | 3121.245             | 5000.000             | 3.772                | -77.342              | -0.563               | 13                       |
| 6                | 20.909     | 20.802      | 0.110                | 2752.275             | 4996.178             | 3.024                | -80.775              | -0.207               | 13                       |
| 7                | 20.900     | 20.798      | 0.102                | 2554.780             | 4998.089             | 2.862                | -81.861              | -0.122               | 13                       |
| 8                | 20.898     | 20.795      | 0.103                | 2520.106             | 4998.208             | 2.849                | -82.046              | -0.114               | 13                       |
| Optimum          | 20.898     | 20.795      | 0.103                | 2520.106             | 4998.208             | 2.849                | Inactive             | Inactive             | 87                       |
| MCS              | 20.891     | 20.786      | 0.105                | 2520.106             | 4998.208             | 2.849                | Inactive             | Inactive             | 1000                     |

## 8.5 Bayesian RBDO

Knowing that both aleatory and epistemic uncertainties exist in the system of interest, Bayesian RBDO can be formulated as

$$\begin{aligned} & \text{minimize } C(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d}) \\ & \text{subject to } P_B(G_i(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d}) \leq 0) \geq \Phi(\beta_{ti}), \quad i = 1, \dots, np \\ & \quad \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X}_a \in R^{na}, \mathbf{X}_e \in R^{ne} \end{aligned}$$

where  $P_B(G_i(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d}) \leq 0) = R_{B,i}$  is Bayesian reliability where  $G_i(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d}) \leq 0$  is defined as a safety event;  $C(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d})$  is the objective function;  $\mathbf{d} = \boldsymbol{\mu}(\mathbf{X})$  is the design vector;  $\mathbf{X}_a$  and  $\mathbf{X}_e$  are the aleatory and epistemic random vectors, respectively;  $\beta_{ti}$  is a prescribed target Bayesian reliability index; and  $np$ ,  $nd$ ,  $na$ , and  $ne$  are the numbers of probabilistic constraints, design variables, aleatory random variables, and epistemic random variables, respectively.



**Figure 8.6:** Bayesian RBDO Procedures

**Example 5: Bayesian RBDO**

For Bayesian RBDO, different reliability analysis methods are used to compare numerical accuracy and efficiency: the FORM and the EDR method. Consider the following mathematical problem with three random variables. Two of them are aleatory with  $X_i \sim \text{Normal}(\mu_i, 0.6)$ ,  $i = 1, 2$ , and  $X_3$  is epistemic with  $N$  samples. In this paper, aleatory random variables are considered as design variables,  $\mathbf{d} = [d_1, d_2]^T = [\mu_1 = \mu(X_1), \mu_2 = \mu(X_2)]^T$ . Epistemic random variable,  $X_3$ , is not considered as a design variable, since none of its statistical properties are known. Here, the RBDO problem is defined as

$$\begin{aligned} &\text{Minimize} && d_1 + d_2 \\ &\text{Subject to} && P_B(G_i(X) \leq 0) = F_{G_i}^B(0) \geq R_{B,i}^t, \quad i = 1, 2, 3 \\ &&& 0 \leq d_1 \text{ \& } d_2 \leq 10 \end{aligned}$$

where

$$\begin{aligned} G_1 &= X_1^2 X_2 X_3 / 20 - 1 \\ G_2 &= \frac{1}{30} (X_1 + X_2 + X_3 - 6)^2 + \frac{1}{120} (X_1 - X_2 - X_3 - 11)^2 - 1 \\ G_3 &= 80 / (X_1^2 + 8X_2 X_3 + 5) - 1 \end{aligned}$$

In this study, the target reliability is set to  $R_{B,i}^t = 90\%$ .

Before performing Bayesian RBDO, the EDR method is compared to FORM, in terms of numerical accuracy of Frequentist RBDO. The optimum design using the EDR method is quite different from that from RBDO using FORM. These two optimum designs are verified using MCS with one million samples and the results are summarized in the Table E10. It is found that FORM yields an error in reliability estimates due to a linearization of the failure surface at the MPP. Since  $G_1$  and  $G_2$  at the optimum design are concave and convex, respectively, the reliability for  $G_1$  is underestimated, while  $G_2$  is overestimated. Whereas the optimum design using the EDR method precisely satisfies reliability constraints.

Bayesian RBDO is carried out for different sample sizes with  $N = 50, 100, \text{ and } 500$ . These samples are randomly generated during the design optimization by assuming  $X_3 \sim \text{Normal}(1.0, 0.1)$ . Different optimum designs will be obtained whenever Bayesian RBDO is performed, even with the same sample size,  $N$ . This is mainly because an insufficient data size leads to a subjective decision. To understand the subjective

decision due to the dearth of data, Bayesian RBDOs for each sample size are performed 20 times using both FORM and the EDR Method. Moreover, these results are compared to the Frequentist reliability-based optimum design by assuming  $X_3$  as aleatory with the statistical properties given above. As expected, both Bayesian RBDO results using FORM and the EDR method asymptotically approach that from the Frequentist results, when increases the sample size of the epistemic variables, as shown in Fig. E16. Bayesian RBDO with the smaller sample size ( $N = 50$ ) leads to more subjective decisions. In other words, the optimum designs are more widely scattered. Because of the sufficiency requirement given in Section 2.3, Bayesian RBDO with a smaller sample size yields the optimum designs with greater reliability compared with Frequentist RBDO results. When more than 500 samples are engaged into Bayesian RBDO, it produces the optimum design quite close to that from Frequentist RBDO. The Pareto frontier of the optimum designs can be constructed along the optimum design trajectory as the data size increases, as shown in Fig. E16.

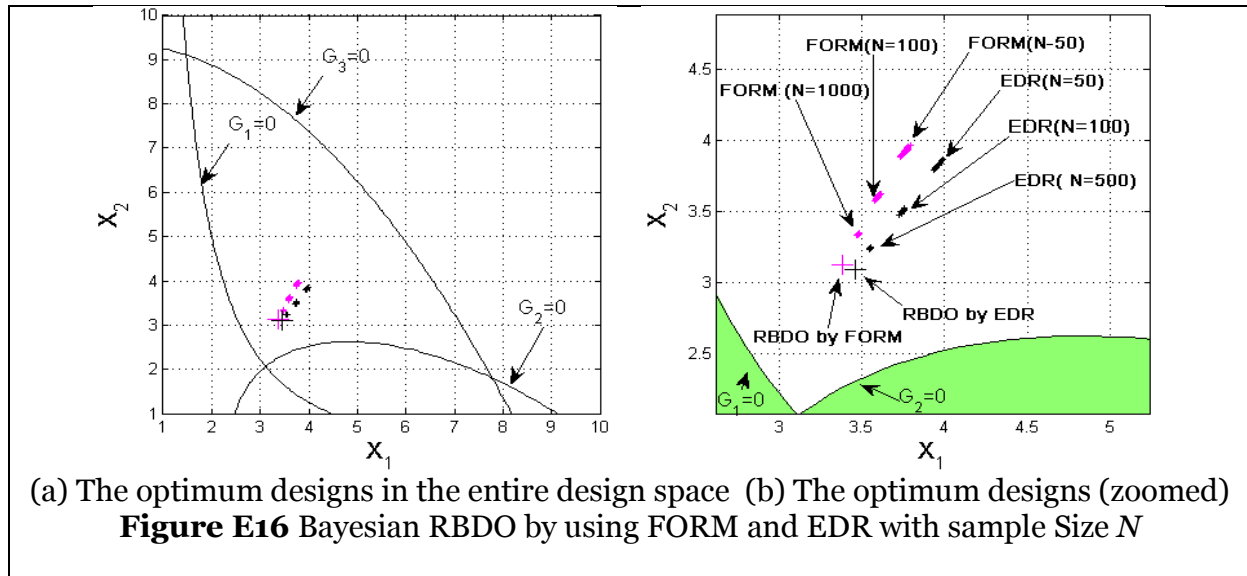
Table E11 shows the total number of function and sensitivity evaluations using FORM and the EDR method in Bayesian RBDO. This example employs 50 data samples for epistemic variables. It is found that the EDR method is much more efficient than FORM. This is because one EDR execution evaluates reliabilities for all constraints unlike FORM. From this example, it is apparent that the EDR method is much more efficient and accurate than FORM for Bayesian RBDO.

**Table E10** Verification of optimum designs (MCS with one million samples)

| Method | Optimum Points |        | Reliability ( By MCS ) |        |        |
|--------|----------------|--------|------------------------|--------|--------|
|        | $X_1$          | $X_2$  | $G_1$                  | $G_2$  | $G_3$  |
| FORM   | 3.3786         | 3.1238 | 0.8833                 | 0.9170 | 1.0000 |
| EDR    | 3.4576         | 3.0898 | 0.9000                 | 0.9001 | 1.0000 |

**Table E11** Efficiency comparison between EDR and FORM

| Methods                               | EDR | FORM |
|---------------------------------------|-----|------|
| Total times of function evaluation    | 250 | 1052 |
| Total times of sensitivity evaluation | 50  | 1052 |



### Example 6: Bayesian RBDO for A-Arm

The EDR method is more efficient and accurate than FORM and SORM. Since Bayesian RBDO is computationally intensive, it is integrated with the EDR method that evaluates Bayesian reliabilities efficiently and accurately. In this section, Bayesian reliability analysis using the EDR method is performed, considering a lower control arm for the High Mobility Multipurpose Wheeled Vehicle (HMMWV).

Vehicle suspension systems experience intense loading conditions throughout their service lives. Control arms act as the back-bone of suspension systems, where the majority of the loads are transmitted through. Therefore, it is crucial that control arms be highly reliable, while remaining cost effective. For the purpose of validating the Bayesian RBDO method, a HMMWV lower control arm is presented as a case study. The following example incorporates Bayesian reliability analysis, where a later section shows the use of Bayesian RBDO.

The lower control arm is modeled with plane stress elements using 54,666 nodes, 53,589 elements, and 327,961 DOFs, where all welds are modeled using rigid beam elements. For FE and design modeling, HyperWorks 7.0 is used. The loading and boundary conditions for this case study are shown in Fig. E17, where loading is applied at the ball-joint (Point D) in 3 directions, and the boundary conditions are applied at the bushings (Points A and B) and the shock-absorber/Spring Assemble (Point C). Due to a lack of data, the loads are considered as epistemic random variables. The design



variables for this problem are the thicknesses of the seven major components of the control arm, as shown in Fig. E17. The statistical information of these components, shown in Table E12, is well known, and these random parameters are therefore considered as aleatory variables in the Bayesian RBDO.

To determine the hot spots (high stress concentrations) in the model, which are used to determine the constraints, a worst case scenario analysis of the control arm is performed. For this worst case scenario, all the design variables are set at their lower bounds as shown in Table E12, and all the loads are set at their highest values attained from the epistemic data points.

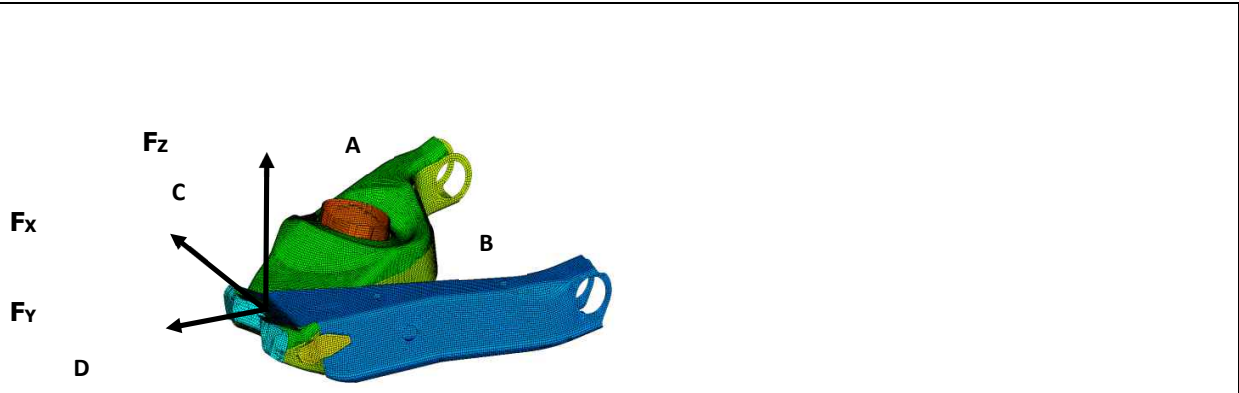
From the worst case scenario, thirty nine constraints ( $G_1$  to  $G_{39}$ ) are defined on several critical regions using the von Mises stress in Fig. E18. For those constraints, Bayesian reliabilities are defined as

$$R_i^B(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d}) = P_B(G_i(\mathbf{X}) = \frac{s_i(\mathbf{X})}{s_U} - 1 \leq 0) \quad (111)$$

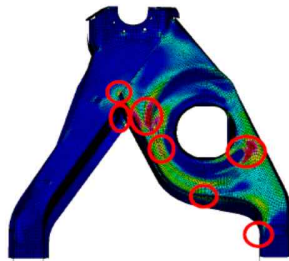
The PDFs for reliabilities at the critical spots are estimated using Bayesian inference. Four representative PDFs ( $G_1$ ,  $G_{24}$ ,  $G_{35}$  and  $G_{38}$ ) are plotted in the dotted curve in Fig. E19. The extreme distributions (solid curves) of the reliability PDFs are presented in the figures. The median values of the extreme distribution are then defined as the Bayesian reliabilities for different constraints which are also plotted in Fig. E19 as vertical dash lines. As illustrated in Figs. E19,  $G_1$  and  $G_{35}$  (the most critical spots at the current design point) are much less reliable than  $G_{24}$  and  $G_{38}$ . This observation is consistent with a stress contour in Fig. E18, since the stresses in  $G_1$  and  $G_{35}$  are extremely high. When a target Bayesian reliability is set to 90%,  $G_1$  and  $G_{35}$  are violated but others are inactive.

**Table E12** Random properties in lower control A Arm model

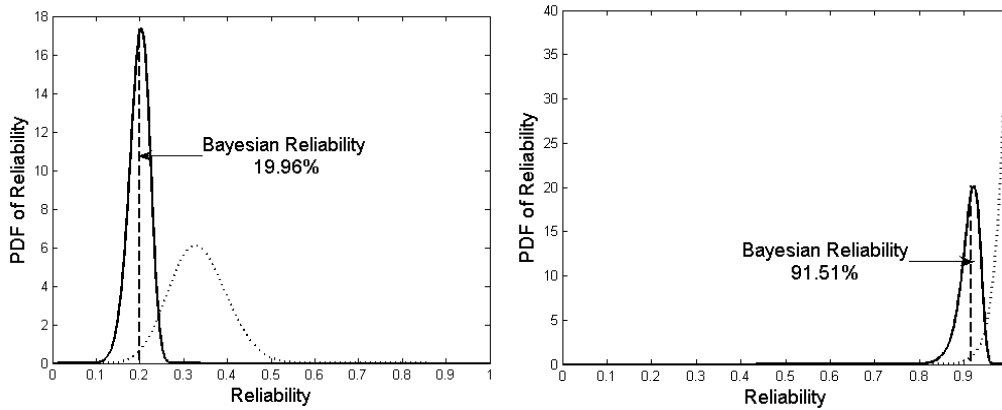
| Random Variable | Lower Bound of Mean | Mean  | Std. Dev. | Dist. Type |
|-----------------|---------------------|-------|-----------|------------|
| $X_1$           | 0.1                 | 0.12  | 0.006     | Normal     |
| $X_2$           | 0.1                 | 0.12  | 0.006     | Normal     |
| $X_3$           | 0.1                 | 0.18  | 0.009     | Normal     |
| $X_4$           | 0.1                 | 0.135 | 0.00675   | Normal     |
| $X_5$           | 0.15                | 0.25  | 0.0125    | Normal     |
| $X_6$           | 0.1                 | 0.18  | 0.009     | Normal     |
| $X_7$           | 0.1                 | 0.135 | 0.00675   | Normal     |

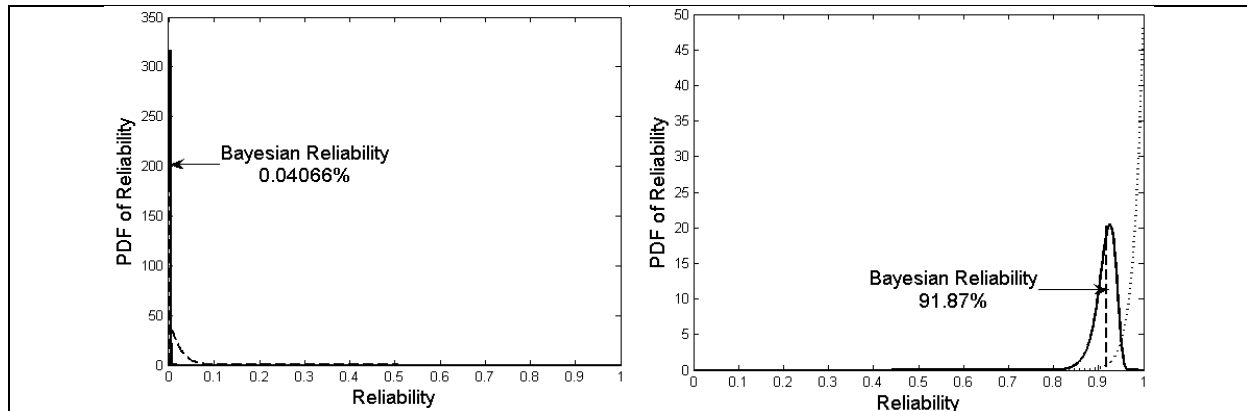


**Figure E17** Three loading variables (Epistemic) and Seven thickness variables (Aleatory)



**Figure E18** 39 Critical constraints of the lower control A-Arm model





**Figure E19** Bayesian reliability for  $G_1$ ,  $G_{24}$ ,  $G_{35}$ , and  $G_{38}$

The control arm used in Section 4 is used for Bayesian RBDO. In this example, seven thickness design variables are considered as aleatory random variables, whereas three load variables (not design variables) are considered as epistemic random variables. 50 data sets are employed for the epistemic loads during Bayesian RBDO. These samples are randomly generated using the assumed distributions shown in Table E13. The properties of the design and random variables are shown in Table E14.

**Table E13** Assumed random properties for epistemic uncertainties

| Epistemic Variable | Distribution             |
|--------------------|--------------------------|
| $F_x$              | $\sim Normal(1900, 95)$  |
| $F_y$              | $\sim Normal(95, 4.75)$  |
| $F_z$              | $\sim Normal(950, 47.5)$ |

**Table E14** Random properties in lower control A-Arm model

| Random Variable | $d_L$ | $\mu_{x=d}$<br>(Mean) | $d_U$ | Std. Dev. | Dist. Type |
|-----------------|-------|-----------------------|-------|-----------|------------|
| $X_1$           | 0.1   | 0.120                 | 0.5   | 0.00600   | Normal     |
| $X_2$           | 0.1   | 0.120                 | 0.5   | 0.00600   | Normal     |
| $X_3$           | 0.1   | 0.180                 | 0.5   | 0.00900   | Normal     |
| $X_4$           | 0.1   | 0.135                 | 0.5   | 0.00675   | Normal     |
| $X_5$           | 0.15  | 0.250                 | 0.5   | 0.01250   | Normal     |
| $X_6$           | 0.1   | 0.180                 | 0.5   | 0.00900   | Normal     |
| $X_7$           | 0.1   | 0.135                 | 0.5   | 0.00675   | Normal     |

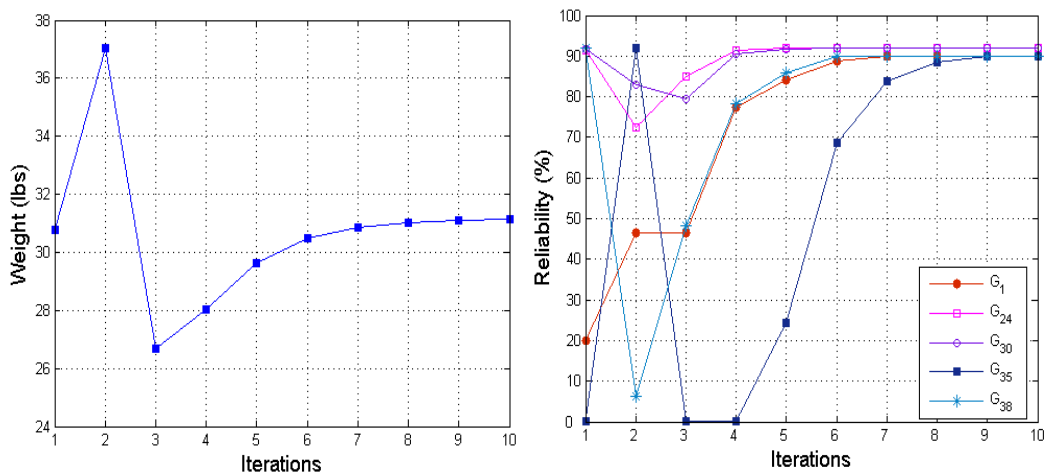
With 39 constraints, Bayesian RBDO is formulated as

$$\begin{aligned} & \text{Minimize } Mass \\ & \text{Subject to } P_B(G_i(\mathbf{X}) = \frac{s_i(\mathbf{X})}{s_U} - 1 \leq 0) = F_{G_i}^B(0) \geq \Phi(\beta_i), \quad i = 1, \dots, 39 \end{aligned} \tag{112}$$

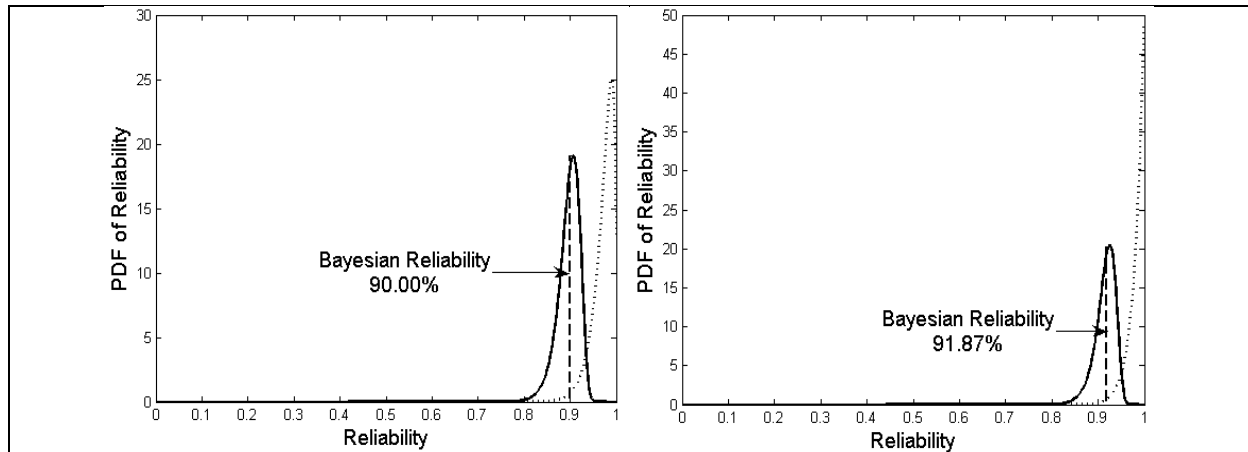
In this study, target reliability is set to  $R_{B,i}^t = 90\%$ . Ten design iterations reach the Bayesian reliability-based optimum design. The histories of the design parameters, objective function, and the Bayesian reliabilities for significant constraints are shown in Table E15, and Fig. E20. At the optimum design, three constraints,  $G_1$ ,  $G_{35}$  and  $G_{38}$ , become active and others are feasible. Figure E21 illustrates the reliability PDFs and Bayesian reliabilities at the optimum design for  $G_1$ ,  $G_{24}$ ,  $G_{35}$  and  $G_{38}$ , of which the PDFs at the initial design are shown in Fig. E19.

**Table E15** Bayesian RBDO design history

| Iter.           | Design |       |        |        |       |       |       | Mass   |
|-----------------|--------|-------|--------|--------|-------|-------|-------|--------|
|                 | $X_1$  | $X_2$ | $X_3$  | $X_4$  | $X_5$ | $X_6$ | $X_7$ |        |
| 1               | 0.12   | 0.120 | 0.180  | 0.135  | 0.25  | 0.180 | 0.135 | 30.76  |
| 2               | 0.10   | 0.100 | 0.109  | 0.307  | 0.15  | 0.500 | 0.100 | 37.04  |
| 3               | 0.10   | 0.143 | 0.143  | 0.100  | 0.15  | 0.500 | 0.100 | 26.70  |
| 4               | 0.10   | 0.144 | 0.153  | 0.107  | 0.15  | 0.242 | 0.500 | 28.013 |
| 5               | 0.10   | 0.137 | 0.153  | 0.141  | 0.15  | 0.500 | 0.100 | 29.64  |
| 6               | 0.10   | 0.138 | 0.157  | 0.151  | 0.15  | 0.500 | 0.100 | 30.51  |
| 7               | 0.10   | 0.138 | 0.156  | 0.156  | 0.15  | 0.500 | 0.100 | 30.84  |
| 8               | 0.10   | 0.137 | 0.156  | 0.158  | 0.15  | 0.500 | 0.164 | 31.01  |
| 9               | 0.10   | 0.137 | 0.156  | 0.160  | 0.15  | 0.500 | 0.156 | 31.11  |
| 10<br>(optimum) | 0.10   | 0.137 | 0.1559 | 0.1598 | 0.15  | 0.500 | 0.177 | 31.13  |



**Figure E20** Objective function and Bayesian reliability history



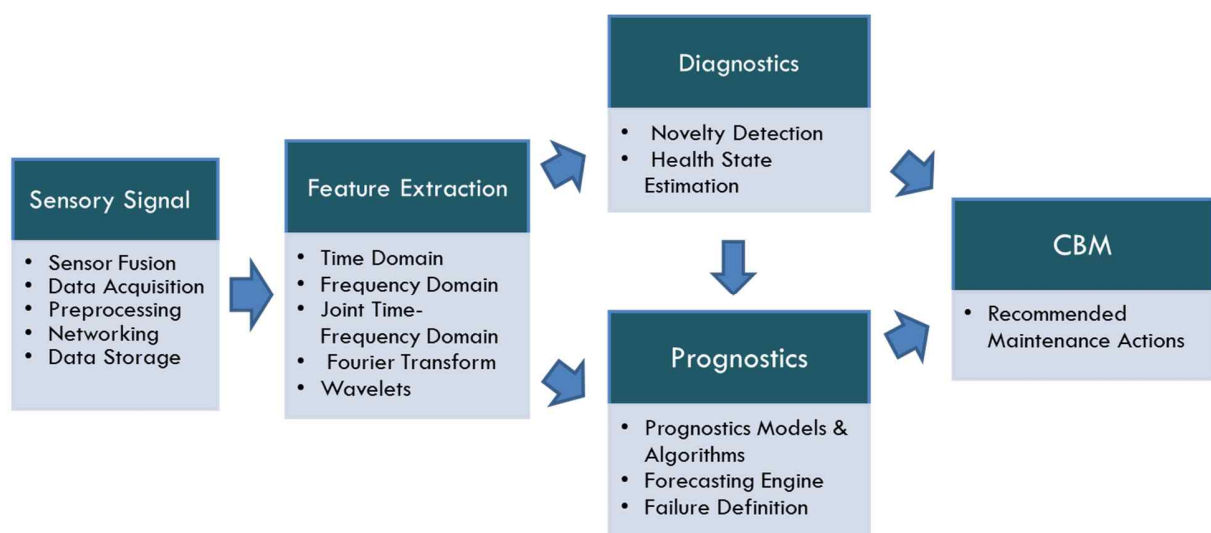
**Figure E21** Bayesian reliability for  $G_1$ ,  $G_{35}$  and  $G_{38}$  (left) and  $G_{24}$  (right) at the optimum design

Finally, the Bayesian reliability-based optimum design is verified by MCS with 10,000 samples. In this verification, three epistemic load variables are assumed to follow the distributions in Table E13. At the optimum design, reliabilities for  $G_1$ ,  $G_{35}$  and  $G_{38}$  are 98.85%, 99.15%, and 98.6%. The sufficiency requirement assures higher reliability than the target reliability, 90%.

## CHAPTER 9. HEALTH DIAGNOSTICS AND PROGNOSTICS

### 9.1 Introduction

Last several decades, tremendous advance has been made on the physics-based analysis and design under uncertainties. However, it is still difficult for the physics-based analysis and design to deal with system failures with multiple failure mechanisms, complex physics-of-failure, and/or complicated joint behaviors. To overcome the difficulties of physics-based approaches, **sensor-based approach** has been emerged and actively engaged to promote life prediction, reliability analysis, and maintenance. Basic elements of sensor-based approach are shown in Figure 29.



**Figure 29:** Basic Elements of Sensor-Based Risk Management

**Diagnostics** – The ability to detect and classify fault conditions.

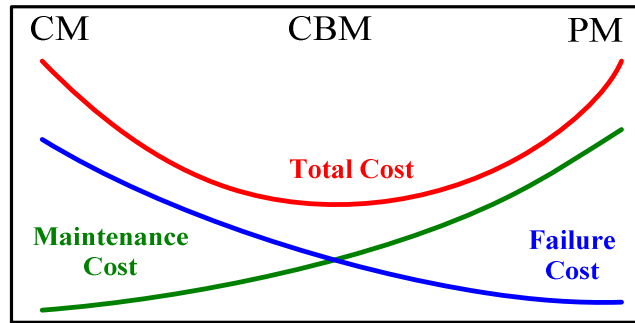
**Prognostics** – The capability to provide early detection of a possible failure condition and to manage and predict the progression of this fault condition to component failure.

**Maintenance:**

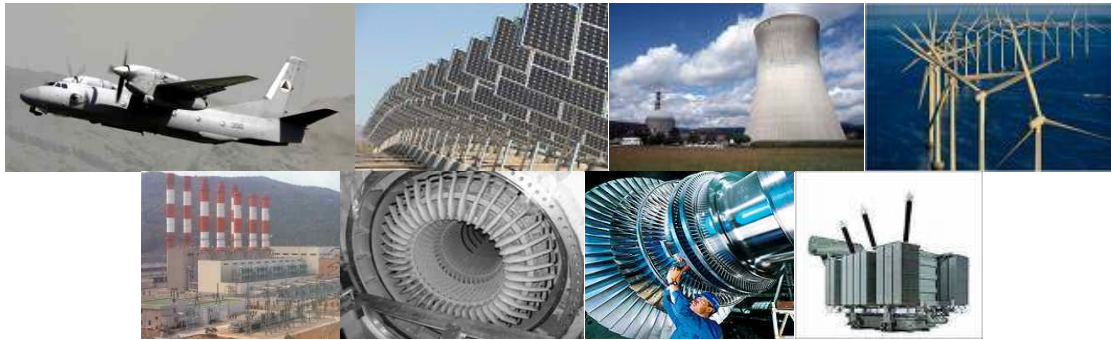
Corrective Maintenance (CM): Action after failure

Preventive Maintenance (PM): Time-based action

Condition-Based Maintenance (CBM): Action if needed



**Figure 30:** Cost Associated to Maintenance Strategies



Health diagnostics and prognostics are very useful to analyze health condition, to predict remaining useful life (RUL), and to make cost-effective maintenance action for large-scale systems, such as power plants, nuclear reactors, wind turbine generators, solar power systems, etc.





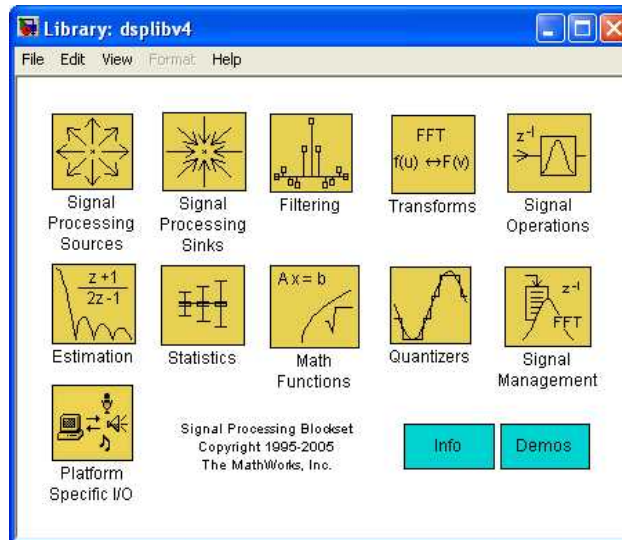
## 9.2 Signal Processing

Signal Processing can be any computer operation or series of operations performed on data to get insightful information. Usually, sensory data will be processed in either time domain, or frequency domain, and sometime in joint time-frequency domain to show extract the data feature.

### 9.2.1 Matlab Signal Processing Block-Set

The Signal Processing Blockset is a tool for digital signal processing algorithm simulation and code generation. It enables you to design and prototype signal processing systems using key signal processing algorithms and components in the Simulink® block format. Figure 31 shows the Library contained for signal processing block-set library (Type “*dsplib*” in the Matlab command window to open this library).





**Figure 31:** Matlab Signal Processing Block-Set Library

### 9.2.2 Time Domain Data Processing

Different statistics tools can be applied to time domain sensory signals to acquire data features, for example:

*Distribution Fitting:* find the best distribution fit of the input data

*Histogram:* generate histogram of input or sequence of inputs

*Autocorrelation:* compute autocorrelation of vector inputs

*Correlation:* compute cross-correlation of two inputs

*Max. /Min.:* find max. /Min. values in an input or sequence of inputs

*Mean:* find mean value of an input or sequence of inputs

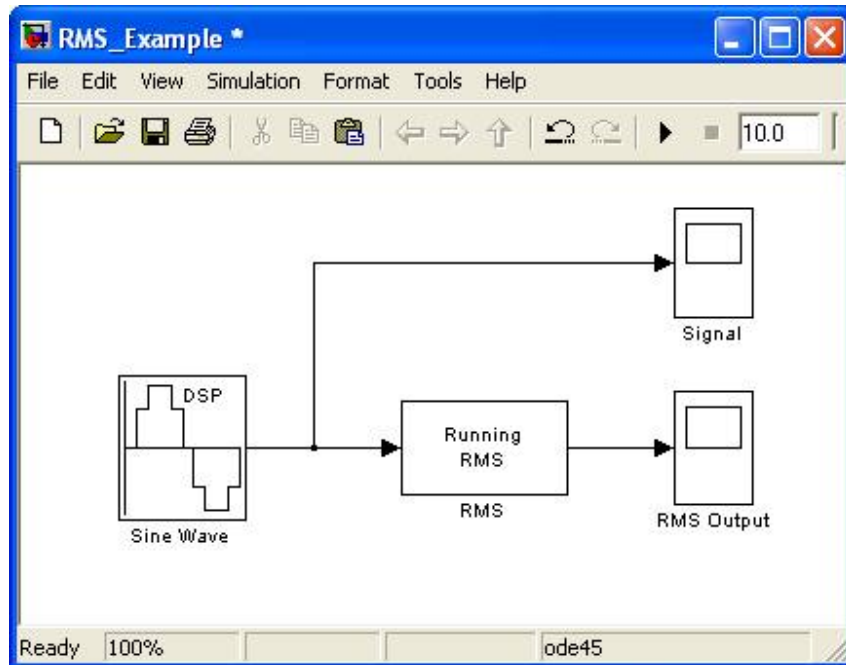
*RMS:* compute root-mean-square (RMS) value of an input or sequence of inputs

*Sort:* Sort input elements by value

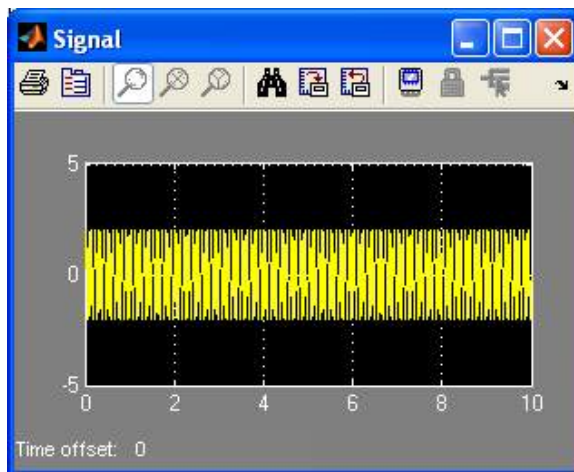
*Standard Deviation:* find standard deviation of an input or sequence of inputs

*Variance:* Compute variance of an input or sequence of inputs

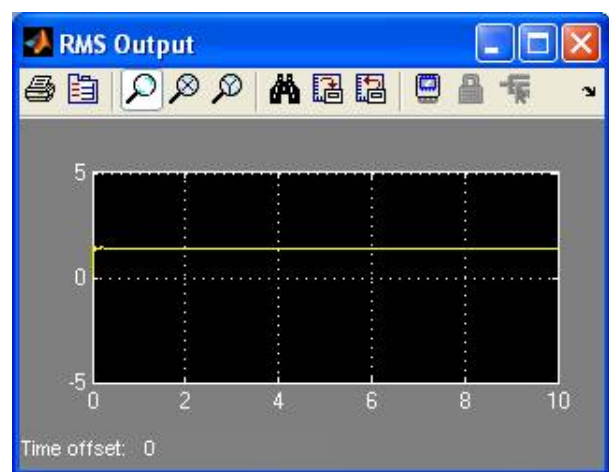
**Example:** Building the following data processing block diagram as shown in Fig. 32, when the sinusoid signal, as shown in Fig. 33(a) is being processed, the RMS signal in Fig. 33(b) can be obtained. RMS signals are usually used to detect the changes of machine vibrations.



**Figure 32:** Signal Processing Example: RMS block diagram



**(a)**



**(b)**

**Figure 33:** Sample Sinusoid Signal (a) and the RMS signal (b)

### 9.2.3 Frequency Domain Data Processing

When it is not clear of the data feature in time domain, usually we will transform the signal into frequency domain.

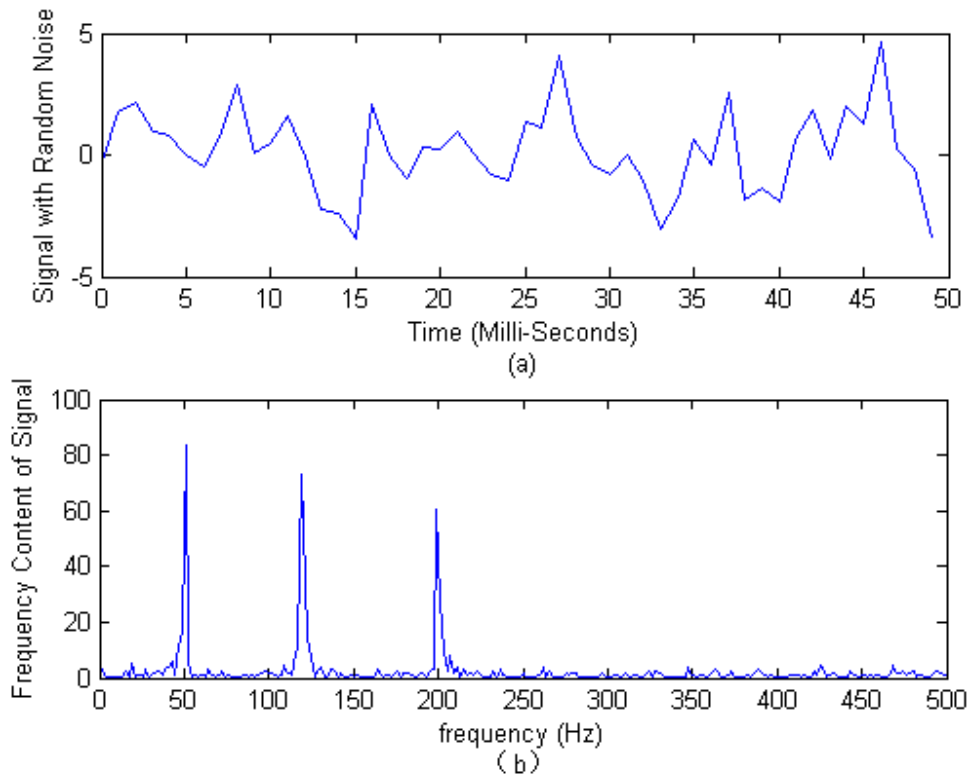
#### ***FFT***

The Fourier transform transforms a time domain signal into a frequency domain representation of that signal. This means that it generates a description of the distribution of the energy in the signal as a function of frequency. This is normally displayed as a plot of frequency (x-axis) against amplitude (y-axis) called a *spectrum*. In signal processing the Fourier transform is almost always performed using an algorithm called the ***Fast Fourier Transform (FFT)***.

**Example: FFT**

```
t = 0:0.001:0.6;
x = sin(2*pi*50*t)+sin(2*pi*120*t)+sin(2*pi*200*t);
y = x + randn(size(t));
figure(1)
subplot(2,1,1)
plot(1000*t(1:50),y(1:50))
xlabel('Time (Milli-Seconds)')
ylabel('Signal with Random Noise')

subplot(2, 1, 2)
Y = fft(y, 512);
Fy = Y.* conj(Y) / 512;
f = 1000*(0:256)/512;
plot(f, Fy(1:257))
xlabel('frequency (Hz)');
ylabel('Frequency Content of Signal');
```



**Figure 34:** Sample Time Domain Signal (a) and Frequency Domain Signal (b)

\* You can also build a FFT block Diagram to do this example.

#### 9.2.4 Joint Time-Frequency Domain Analysis

There is a tradeoff between resolution in frequency and resolution in time. Good frequency resolution implies poor time resolution and good time resolution implies poor frequency resolution. Although frequency-domain representations such as the power spectrum of a signal often show useful information, the representations don't show how the frequency content of a signal evolves over time.

*Joint Time-Frequency Analysis (JTFA) is a set of transforms that maps a one-dimensional time domain signal into a two-dimensional representation of energy versus time and frequency.* JTFA shows the frequency content of a signal and the change in frequency with time.

There are a number of different transforms available for JTFA. Each transform type shows a different time-frequency representation. The Short Time Fourier Transform (STFT) is the simplest JTFA transform. For the STFT, you apply FFT repeatedly to short segments of a signal at ever-later positions in time. You can display the result on a 3-D graph or a so-called 2-D 1/2 representation (the energy is mapped to light intensity or color values).

The STFT technique uses FFT and suffers from an inherent coupling between time resolution and frequency resolution as mentioned earlier (increasing the first decreases the second, and vice versa). Other JTFA methods and transforms can yield a more precise estimate of the energy in a given Frequency-Time domain. Some options include:

- Gabor spectrogram
- Wavelet transform
- Wigner distribution
- Cohen class transforms

### 9.3 Health Monitoring

The process of implementing a damage detection strategy for engineering structures is referred to as **Structural Health Monitoring (SHM)**. The SHM process involves the observation of a system over time using periodically sampled dynamic response measurements from an array of sensors, the extraction of damage-sensitive features from these measurements, and the statistical analysis of these features to determine the current state of system health. For long term SHM, the output of this process is periodically updated information regarding the ability of the structure to perform its intended function in light of the inevitable aging and degradation resulting from operational environments.

The SHM problem can be addressed in the context of a statistical pattern recognition paradigm, which includes four-step process: (i) Operational evaluation, (ii) Data acquisition, normalization and cleansing, (iii) Feature extraction and information condensation, and (iv) Statistical model development for feature discrimination.

- **Operational evaluation**

Operational evaluation attempts to answer the following four questions regarding the implementation of a damage identification capability.

- a) What are the life-safety and/or economic justification for performing SHM?
- b) How is damage defined for the system being investigated and, for multiple damage possibilities, which cases are of the most concern?
- c) What are the conditions, both operational and environmental, under which the system to be monitored functions?
- d) What are the limitations on acquiring data in the operational environment?

- **Data acquisition, normalization and cleansing**

The data acquisition portion of the SHM process involves selecting the excitation methods, the sensor types, number and locations, and the data acquisition/storage/transmittal hardware.

As data can be measured under varying conditions, the ability to normalize the data becomes very important to the damage identification process. One of the most common procedures is to normalize the measured responses by the measured inputs.

Data cleansing is the process of selectively choosing data to pass on to or reject from the feature extraction process. Signal processing techniques such as filtering and re-sampling can be used as data cleansing procedures.

- **Feature extraction and information condensation**

The area of the SHM process that receives the most attention in the technical literature is the **data feature extraction** that allows one to distinguish between the undamaged and damaged structure. Inherent in this feature selection process is the condensation of the data. The best features for damage identification are, again, application specific.

One of the most common feature extraction methods is based on correlating measured system response quantities, such a vibration amplitude or frequency, with the first-hand observations of the degrading system.

Another method of developing features for damage identification is to apply engineered flaws, similar to ones expected in actual operating conditions, to systems and develop an initial understanding of the parameters that are sensitive to the expected damage. The flawed system can also be used to validate that the diagnostic measurements are sensitive enough to distinguish between features identified from the undamaged and damaged system. The use of analytical tools such as experimentally-validated finite element models can be a great asset in this process. In many cases the analytical tools are used to perform numerical experiments where the flaws are introduced through computer simulation.

Damage accumulation testing, during which significant structural components of the system under study are degraded by subjecting them to realistic loading conditions, can also be used to identify appropriate features. This process may involve induced-damage testing, fatigue testing, corrosion growth, or temperature cycling to accumulate certain types of damage in an accelerated fashion. Insight into the appropriate features can be gained from several types of analytical and experimental studies as described above and is usually the result of information obtained from some combination of these studies.

The operational implementation and diagnostic measurement technologies needed to perform SHM produce more data than traditional uses of structural dynamics information. A condensation of the data is advantageous and necessary when comparisons of many feature sets obtained over the lifetime of the structure are envisioned. Also, because data will be acquired from a structure over an extended period of time and in an operational environment, robust data reduction techniques must be developed to retain feature sensitivity to the structural changes of interest in the presence of environmental and operational variability. To further aid in the extraction and recording of quality data needed to perform SHM, the statistical significance of the features should be characterized and used in the condensation process.

- **Statistical model development**

Statistical model development is concerned with the implementation of the algorithms that operate on the extracted features to quantify the damage state of the structure. The algorithms used in statistical model development usually fall into two categories: **supervised learning** and **unsupervised learning**. When data are available from

both the undamaged and damaged structure, the statistical pattern recognition algorithms fall into the general classification referred to as supervised learning. Group classification and regression analysis are categories of the supervised learning algorithms. Unsupervised learning refers to algorithms that are applied to data not containing examples from the damaged structure. Outlier or novelty detection is the primary class of algorithms applied in unsupervised learning applications. All of the algorithms analyze statistical distributions of the measured or derived features to enhance the damage identification process.

### 9.4 Health Prognostics

Real-time diagnosis and prognosis which interprets data acquired by smart sensors and distributed sensor networks, and utilizes these data streams in making critical decisions provides significant advancements across a wide range of application. Figure 35 shows a typical paradigm of the sensor-based life and reliability prognostics, which utilizes the sensory signal to produce the system degradation signal through the signal processing, and then the diagnostics of the system current health condition and further predict the system Residual Useful Life (RUL) and reliability will be carried out based on the system degradation signals. Uncertainties for sensory signal noise, data processing error and prediction variability are considered in this process. Technical approaches to system sensor-based life and reliability prognostics can be categorized broadly into model-based approaches and data-driven approaches.

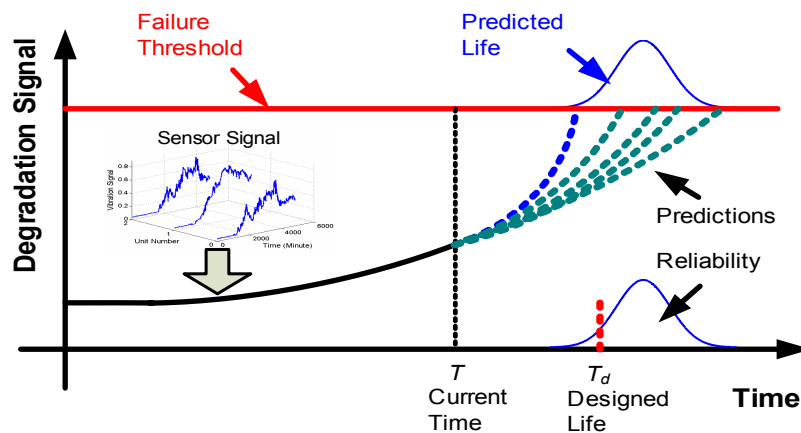
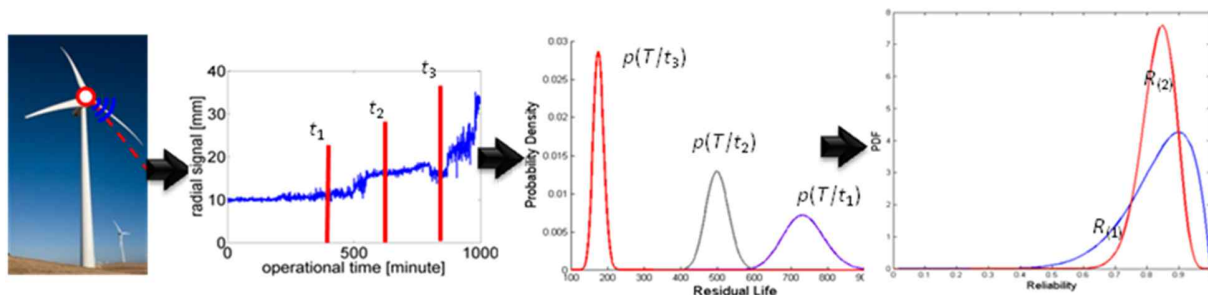


Figure 35: Sensor-Based Life and Reliability Prognostics



**Figure 36:** Procedures of Health Prognostics

- **Model-Based Prognostics**

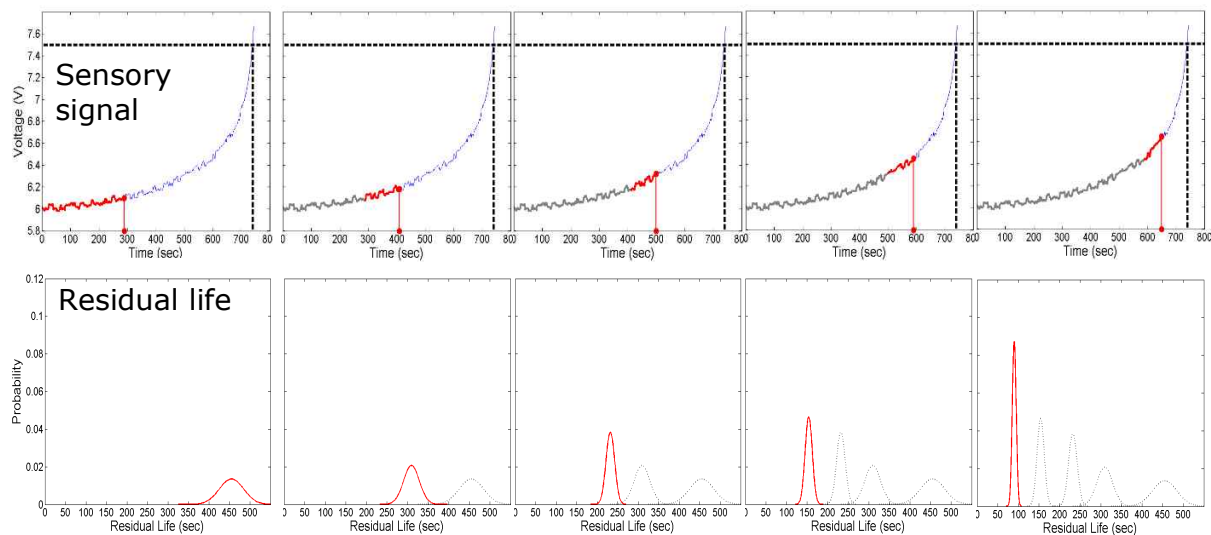
Model-based prognostic approaches attempt to incorporate physical understanding (physical models) of the system into the estimation of remaining useful life (RUL). Different stochastic degradation models have been investigated in the literature, to model various degradation phenomena of systems or components.

**Real-time degradation** model parameters **updating with evolving sensory signals** is a challenge of model-based prognostic approaches. Bayesian updating techniques are commonly used for this purpose. Table 1 describes a Markov-Chain Monte Carlo (MCMC) method for non-conjugate Bayesian updating.

**Example: An exponential degradation model**

$$S(t_i) = S_0 + \delta \cdot \exp(\alpha t_i^2 + \beta t_i + \varepsilon(t_i) - \frac{\sigma^2}{2})$$

where  $S(t_i)$  represents the degradation signal at time  $t_i$ ;  $S_0$  is a known constant;  $\delta$ ,  $\alpha$ , and  $\beta$  are stochastic model parameters and  $\varepsilon$  represents the random error term which follows normal distribution with zero mean and  $\sigma^2$  deviation. Figure 37 shows the updating of this model and corresponding RUL.

**Figure 37:** Model and RUL updating

- **Data-Driven Prognostics**

Data-driven prognostic techniques utilize monitored operational data related to system health. The major advantage of data driven approaches is that they can be deployed quicker and cheaper compared to other approaches, and that they can provide system-wide coverage. The principal disadvantage is that data driven approaches may have wider confidence intervals than other approaches and that they require a substantial amount of data for training.



Data and information updating schemes

- a) Numerical Methods
  - Linear Regression
  - Kalman Filters
  - Particle Filters

Machine learning techniques

- b) Artificial Intelligence Based Techniques
  - Artificial Neural Networks
  - Decision Tree Method
  - Novelty Detection Algorithms
  - Support Vector Machine (SVM)
  - Relevance Vector Machine (RVM)
  - Fuzzy Logic

**Homework 1: Sources of uncertainty in a vibration problem**

Let us consider an undamped system with a lumped mass and spring. The motion behavior of the system can be ideally modeled using a second-order ordinary differential equation as

$$my''(t) + ky(t) = 0; \quad y(0) = 15, \quad y'(0) = 0$$

where  $m$  and  $k$  are the mass and spring coefficient of the system, respectively. According to the manufacturer of the system, the mass and spring coefficient are believed to be 10 kg and 1000 N/m, respectively. At time  $t = 1$  second, ten experimental tests show a set of  $y$  data as (4.4456, 4.2094, 4.3348, 4.2441, 4.1768, 4.1756, 4.4057, 4.2448, 4.2303, 4.0952). Answer the following questions:

- (1) Identify all possible sources of uncertainties involved in this problem.
- (2) Please explain why experimentally measured  $y$  values are random.
- (3) Also, provide possible reasons for what causes the difference between experimental and analytical  $y$  values.

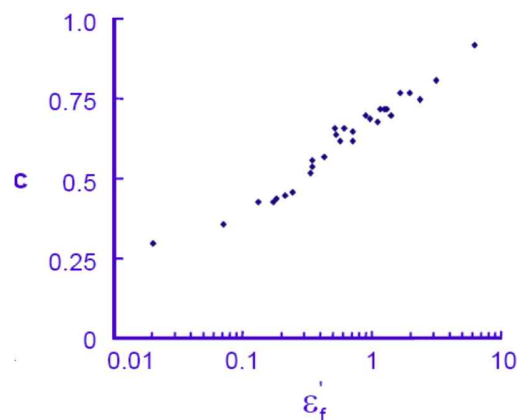
**Homework 2: Probability Distribution & Statistical Moments**

Let us recall the example of fatigue tests. The sample data can be obtained about the physical quantities in the damage model below.

$$\frac{\Delta \varepsilon}{2} = \frac{\sigma'_f}{E} (2N_f)^b + \varepsilon'_f (2N_f)^c$$

Let us consider a 30 data set (Table 5) for the fatigue ductility coefficient ( $\varepsilon'_f$ ) and exponent ( $c$ ) used in the strain-life formula shown above. Answer the following questions:

- (1) Construct the covariance matrix and find out the coefficient of correlation using the data set given in Table 5.
- (2) Use normal, weibull, and lognormal distributions. Find out the most suitable parameters of three distributions for the fatigue ductility coefficient ( $\varepsilon'_f$ ) and exponent ( $c$ ) using the MLE method.
- (3) Find out the most suitable distributions for the data set ( $\varepsilon'_f, c$ ) using the chi-square GOF.
- (4) Verify the results with the graphical methods (histogram and probability plots).



**Figure 8:** Statistical Correlation

**Table 5:** Data for the fatigue ductility coefficient and exponent

| $(\varepsilon'_f, c)$ |       | $(\varepsilon'_f, c)$ |       | $(\varepsilon'_f, c)$ |       | $(\varepsilon'_f, c)$ |       | $(\varepsilon'_f, c)$ |       |
|-----------------------|-------|-----------------------|-------|-----------------------|-------|-----------------------|-------|-----------------------|-------|
| 0.022                 | 0.289 | 0.253                 | 0.466 | 0.539                 | 0.630 | 0.989                 | 0.694 | 1.611                 | 0.702 |
| 0.071                 | 0.370 | 0.342                 | 0.531 | 0.590                 | 0.621 | 1.201                 | 0.690 | 1.845                 | 0.760 |
| 0.146                 | 0.450 | 0.353                 | 0.553 | 0.622                 | 0.653 | 1.304                 | 0.715 | 1.995                 | 0.759 |
| 0.185                 | 0.448 | 0.354                 | 0.580 | 0.727                 | 0.635 | 1.388                 | 0.717 | 2.342                 | 0.748 |
| 0.196                 | 0.452 | 0.431                 | 0.587 | 0.729                 | 0.645 | 1.392                 | 0.716 | 3.288                 | 0.821 |
| 0.215                 | 0.460 | 0.519                 | 0.655 | 0.906                 | 0.703 | 1.426                 | 0.703 | 6.241                 | 0.894 |

**Homework 3: Reliability Function**

Supposed it is desired to estimate the failure rate of an electronic component. A test can be performed to estimate its failure rate. A target life is set to 2000 minutes.  $R(t) = P(T > 2000 \text{ minutes})$  Answer the following questions:

- (1) Construct a histogram of TTF.
- (2) Find out a probability distribution model and its parameters,  $f_i(t)$ , for the TTF data.
- (3) Construct a reliability function.
- (4) Determine MTTF, standard deviation of TTF, and hazard function.
- (5) Compare the reliabilities from  $n_f/N$  from the TTF data and the reliability function when  $t = 2000$  where  $n_f$  is the number of failed components and  $N (= 100)$  is the total components.

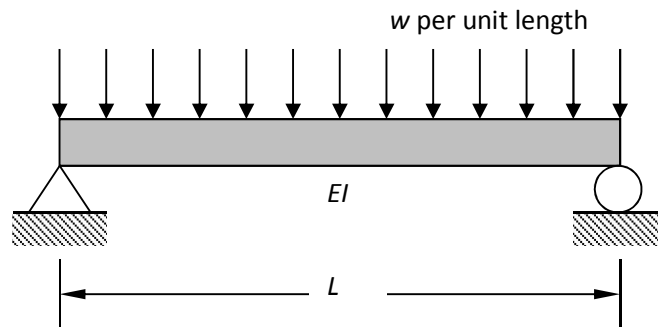
**Table 5:** Data for 100 Electronics Time-To-Failure (TTF) [minute]

|        |        |        |        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1703.2 | 1071.4 | 2225.8 | 1826.5 | 1131   | 2068.9 | 1573.5 | 1522.1 | 1490.7 | 2226.6 |
| 1481.1 | 2065.1 | 1880.9 | 2290.9 | 1786.4 | 1867.2 | 1859.1 | 1907.5 | 1791.8 | 1871   |
| 1990.4 | 2024.1 | 1688.6 | 1962.7 | 2191.7 | 1841   | 1814.1 | 1918.1 | 2237.5 | 1396.8 |
| 1692.8 | 707.2  | 2101.3 | 2165.4 | 1975.2 | 1961.6 | 2116.7 | 1373   | 1798.8 | 2248.4 |
| 1872.3 | 1597.8 | 1865.1 | 742.8  | 1436.7 | 1380.8 | 2258.2 | 1960   | 2182.8 | 1772.7 |
| 2003.6 | 1589.4 | 1988.3 | 1874.9 | 1859   | 2051.9 | 1763   | 1854.6 | 1974.7 | 2289.9 |
| 1945.7 | 1774.8 | 1579.6 | 1430.5 | 1855   | 1757.9 | 1029.3 | 1707.2 | 1864.7 | 1964.8 |
| 1719.4 | 1565.2 | 1736.8 | 1759.4 | 1939.4 | 2065.7 | 2258.5 | 2292.8 | 1452.5 | 1692.2 |
| 2120.7 | 1934.8 | 999.4  | 1919.9 | 2162.4 | 2094.9 | 2158.2 | 1884.2 | 1748.7 | 2260.3 |
| 1040.8 | 1535   | 1283.4 | 2267.7 | 2100.3 | 2007.9 | 2499.8 | 1902.9 | 1599.6 | 1567.5 |

- (6) Attempt to update the TTF mean value ( $\theta$ ) with aggregation of 100 TTF data using Bayesian inference. Assume that the TTF follows a normal distribution with the standard deviation ( $\sigma = 315.16$ ) and the prior distribution of  $\theta$  be  $P(\theta) = N(u = 1750.0, \tau^2 = 500^2)$ .

**Homework 4: Reliability Analysis**

Consider the following simply supported beam subject to a uniform load, as illustrated in Fig. 19. Suppose  $L = 5$  m and  $w = 10$  kN/m.



**Figure 19:** Simply Supported Beam

Random Vector:

$$EI = X_1 \square N(\mu_{X_1} = 3 \times 10^7, \sigma_{X_1} = 10^5)$$

$$w = X_2 \square N(\mu_{X_2} = 10^4, \sigma_{X_2} = 10^3)$$

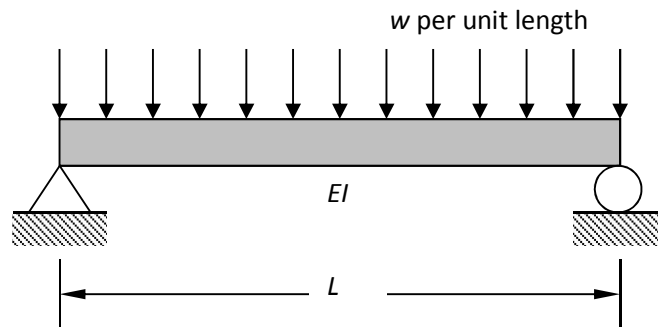
The maximum deflection of the beam is shown as

$$Y = g(X_1, X_2) = -\frac{5X_2L^4}{384X_1}$$

Using Monte Carlo simulation, first-order expansion method, MPP-based method (HL-RF) and Eigenvector Dimension Reduction (EDR) method, determine the PDF (or CDF) of the maximum deflection and estimate its reliability when the failure is defined as  $Y < y_c = -3 \times 10^{-3}$  m. Make your own discussion and conclusion.

**Homework 5: Bayesian Reliability Analysis**

Consider the following simply supported beam subject to a uniform load, as illustrated in Fig. 19. Suppose  $L = 5$  m and  $w=10$  kN/m.



**Figure 19:** Simply Supported Beam

Random Vector:

$$EI = X_1 \square N(\mu_{X_1} = 3 \times 10^7, \sigma_{X_1} = 10^5)$$

$$w = X_2 \square \text{epitemic}$$

The maximum deflection of the beam is shown as

$$Y = g(X_1, X_2) = -\frac{5X_2L^4}{384X_1}$$

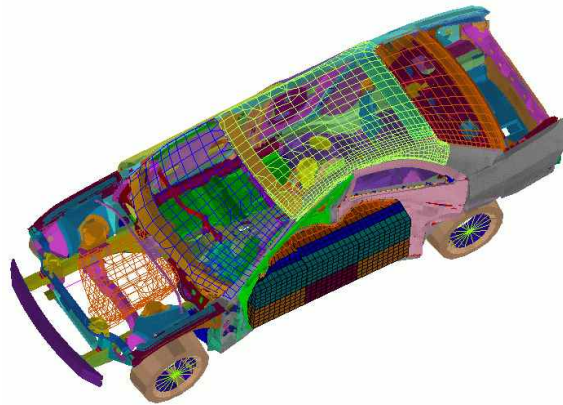
The  $X_2$  is an epistemic uncertainty. For  $X_2$ , it is assumed that 10 data sets are gradually obtained at different times. Using MPP-based method (HL-RF) and Eigenvector Dimension Reduction (EDR) method, determine the reliability of the maximum deflection constraint,  $P(Y(X_1) \geq y_c = -3 \times 10^{-3} \text{m})$ , at all individual  $X_2$  points in the table. Predict reliability in a Bayesian sense using the first 10 data set and gradually update the reliability using the second and third data sets. Make your own discussion and conclusion, and attach your code used for Bayesian reliability analysis.

**Table 9** Three sets of 10 data for  $X_2$  ( $\times 10^4$ )

|      |        |        |        |        |        |        |        |        |        |        |
|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Set1 | 1.0000 | 0.8126 | 1.0731 | 1.0677 | 0.9623 | 0.9766 | 1.1444 | 1.0799 | 1.0212 | 0.9258 |
| Set2 | 0.9682 | 1.0428 | 1.0578 | 1.0569 | 0.9704 | 1.0118 | 0.9649 | 1.0941 | 1.0238 | 1.1082 |
| Set3 | 1.1095 | 1.0896 | 1.0040 | 0.9744 | 0.8525 | 1.0315 | 1.0623 | 0.9008 | 0.8992 | 0.9869 |

**Homework 6: Design Optimization of Crashworthiness Problem**

A vehicle side impact problem is considered for design optimization. All the design variables are shown in Table 11. In this example, the abdomen load is treated as an objective function with nine constraints defined in Table 12.



**Table 11:** Properties of design variables  
( $X_{10}$  and  $X_{11}$  have “o” value)

| Design Variables | d <sup>L</sup>              | d     | d <sup>U</sup> |
|------------------|-----------------------------|-------|----------------|
| $X_1$            | 0.500                       | 1.000 | 1.500          |
| $X_2$            | 0.500                       | 1.000 | 1.500          |
| $X_3$            | 0.500                       | 1.000 | 1.500          |
| $X_4$            | 0.500                       | 1.000 | 1.500          |
| $X_5$            | 0.500                       | 1.000 | 1.500          |
| $X_6$            | 0.500                       | 1.000 | 1.500          |
| $X_7$            | 0.500                       | 1.000 | 1.500          |
| $X_8$            | 0.192                       | 0.300 | 0.345          |
| $X_9$            | 0.192                       | 0.300 | 0.345          |
| $X_{10}$         | 0.000 (not design variable) |       |                |
| $X_{11}$         | 0.000 (not design variable) |       |                |

**Table 12:** Design variables and their bounds

| Components                                 | Safety criteria                         |
|--|---|
| Objective: Abdomen load (kN)               | $\leq 1$                                |
| $G_1$ - $G_3$ : Rib Deflection             | Upper<br>Middle<br>Lower<br>$\leq 32$   |
| $G_4$ - $G_6$ : VC (m/s)                   | Upper<br>Middle<br>Lower<br>$\leq 0.32$ |
| $G_7$ : Pubic symphysis force (kN)         | $\leq 4$                                |
| $G_8$ : Velocity of B-pillar               | $\leq 9.9$                              |
| $G_9$ : Velocity of front door at B-pillar | $\leq 15.7$                             |

Responses:

$$\text{OBJ} = (1.16 - 0.3717x(2)x(4) - 0.00931x(2)x(10) - 0.484x(3)x(9) + 0.01343x(6)x(10));$$

$$G1 = (28.98 + 3.818x(3) - 4.2x(1)x(2) + 0.0207x(5)x(10) + 6.63x(6)x(9) - 7.7x(7)x(8) + 0.32x(9)x(10)) - 32;$$

$$G2 = (33.86 + 2.95x(3) + 0.1792x(10) - 5.057x(1)x(2) - 11x(2)x(8) - 0.0215x(5)x(10) - 9.98x(7)x(8) + 22x(8)x(9)) - 32;$$

$$G3 = (46.36 - 9.9x(2) - 12.9x(1)x(8) + 0.1107x(3)x(10)) - 32;$$

$$G4 = (0.261 - 0.0159x(1)x(2) - 0.188x(1)x(8) - 0.019x(2)x(7) + 0.0144x(3)x(5) + 0.0008757x(5)x(10) + 0.08045x(6)x(9) + 0.00139x(8)x(11) + 0.00001575x(10)x(11)) - 0.32;$$

$$G5 = (0.214 + 0.00817x(5) - 0.131x(1)x(8) - 0.0704x(1)x(9) + 0.03099x(2)x(6) - 0.018x(2)x(7) + 0.0208x(3)x(8) + 0.121x(3)x(9) - 0.00364x(5)x(6) + 0.0007715x(5)x(10) - 0.0005354x(6)x(10) + 0.00121x(8)x(11) + 0.00184x(9)x(10) - 0.018x(2)^2) - 0.32;$$

$$G6 = (0.74 - 0.61x(2) - 0.163x(3)x(8) + 0.001232x(3)x(10) - 0.166x(7)x(9) + 0.227x(2)^2) - 0.32;$$

$$G7 = (4.72 - 0.5x(4) - 0.19x(2)x(3) - 0.0122x(4)x(10) + 0.009325x(6)x(10) + 0.000191x(11)^2) - 4;$$

$$G8 = (10.58 - 0.674x(1)x(2) - 1.95x(2)x(8) + 0.02054x(3)x(10) - 0.0198x(4)x(10) + 0.028x(6)x(10)) - 9.9;$$

$$G9 = (16.45 - 0.489x(3)x(7) - 0.843x(5)x(6) + 0.0432x(9)x(10) - 0.0556x(9)x(11) - 0.000786x(11)^2) - 15.7;$$

The Design Optimization is formulated as

$$\text{Minimize } f(\mathbf{x})$$

$$\text{Subject to } g_j(\mathbf{x}) \leq 0, \quad j = 1, \dots, 9$$

$$\mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U, \quad \mathbf{x} \in R^9$$

Solve this optimization problem using the sequential quadratic programming (use the matlab function, 'fmincon', in Matlab). Make your own discussion and conclusion.



**Homework 7: RBDO of a Crashworthiness Problem**

A vehicle side impact problem is considered for design optimization. All the design variables are shown in Table A. In this example, the abdomen load is treated as an objective function with nine constraints defined in Table B.

**Table A:** Properties of random and design variables  
( $X_{10}$  and  $X_{11}$  have "o" value)

| Random Variables | Distr. Type | Std Dev. | $d^L$  | $d$   | $d^U$ |
|------------------|-------------|----------|--|-------|-------|
| $X_1$            | Normal      | 0.050    | 0.500  | 1.000 | 1.500 |
| $X_2$            | Normal      | 0.050    | 0.500  | 1.000 | 1.500 |
| $X_3$            | Normal      | 0.050    | 0.500  | 1.000 | 1.500 |
| $X_4$            | Normal      | 0.050    | 0.500  | 1.000 | 1.500 |
| $X_5$            | Normal      | 0.050    | 0.500  | 1.000 | 1.500 |
| $X_6$            | Normal      | 0.050    | 0.500  | 1.000 | 1.500 |
| $X_7$            | Normal      | 0.050    | 0.500  | 1.000 | 1.500 |
| $X_8$            | Lognorm     | 0.006    | 0.192  | 0.300 | 0.345 |
| $X_9$            | Lognorm     | 0.006    | 0.192  | 0.300 | 0.345 |
| $X_{10}$         | Normal      | 10.0     | $X_{10}$ and $X_{11}$ are not design variables |       |       |
| $X_{11}$         | Normal      | 10.0     |  |       |       |

**Table B:** Design variables and their bounds

| Components                                 | Safety criteria                         |
|--|---|
| Objective: Abdomen load (kN)               | $\leq 1$                                |
| $G_1$ - $G_3$ : Rib Deflection             | Upper<br>Middle<br>Lower<br>$\leq 32$   |
| $G_4$ - $G_6$ : VC (m/s)                   | Upper<br>Middle<br>Lower<br>$\leq 0.32$ |
| $G_7$ : Pubic symphysis force (kN)         | $\leq 4$                                |
| $G_8$ : Velocity of B-pillar               | $\leq 9.9$                              |
| $G_9$ : Velocity of front door at B-pillar | $\leq 15.7$                             |

The Design Optimization is formulated as

$$\text{Minimize } f(\mathbf{x})$$

$$\text{Subject to } P(G_j(\mathbf{x}) \leq 0) \geq 99\%, \quad j = 1, \dots, 9$$

$$\mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U, \quad \mathbf{x} \in R^9$$

Solve the RBDO optimization problem using the matlab function, 'fmincon',

**Mode 1:  
17.73 Hz**

in Matlab) starting at the initial design ( $d_1$  to  $d_7 = 1.000$ ,  $d_8 = d_9 = 0.300$ ) and deterministic optimum design (obtained in the previous homework). Make your own discussion and conclusion.