Chap. 21

COST MINIMIZATION

Introduction

- A firm is a cost-minimizer if it produces any given output level y at smallest possible total cost.
- $c(y)$ denotes the firm's smallest possible total cost for producing y units of output.
- $c(y)$ is the firm's total cost function.
- When the firm faces given input prices $(w_1,w_2,\ldots,w_n)$, the total cost function will be written as

$$c(w_1,\ldots,w_n, y).$$

- How to find the cost function?

Cost minimization(2-inputs)

- Formulation

$$\min_{\{x_1,x_2\}} \; c = w_1x_1 + w_2x_2$$
$$s.t. \quad f\left(x_1,x_2\right) = y$$

- Lagrangian

$$L = w_1x_1 + w_2x_2 - \lambda\left(f\left(x_1,x_2\right) - y\right)$$

- F.O.C.

$$\frac{\partial L}{\partial \lambda} = y - f\left(x_1^*,x_2^*\right) = 0$$
$$\frac{\partial L}{\partial x_1} = w_1 - \lambda \frac{\partial f\left(x_1^*,x_2^*\right)}{\partial x_1} = 0$$
$$\frac{\partial L}{\partial x_2} = w_2 - \lambda \frac{\partial f\left(x_1^*,x_2^*\right)}{\partial x_2} = 0$$

- Conditional factor demand function

$$x_1^*\left(w_1,w_2,y\right)$$
$$x_2^*\left(w_1,w_2,y\right)$$

- Cost function

$$c\left(w_1,w_2,y\right) = w_1x_1^*\left(w_1,w_2,y\right) + w_2x_2^*\left(w_1,w_2,y\right)$$

Cost minimization(2-inputs)

- F.O.C.

$$\frac{\partial L}{\partial x_1} = w_1 - \lambda \frac{\partial f\left(x_1^*, x_2^*\right)}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial x_2} = w_2 - \lambda \frac{\partial f\left(x_1^*, x_2^*\right)}{\partial x_2} = 0$$

$$\Rightarrow \quad \lambda = \frac{w_1}{\partial f\left(x_1^*, x_2^*\right)/\partial x_1} = \frac{w_2}{\partial f\left(x_1^*, x_2^*\right)/\partial x_2}$$

- Optimality condition

$$\frac{w_1}{w_2} = \frac{\partial f\left(x_1^*, x_2^*\right)/\partial x_1}{\partial f\left(x_1^*, x_2^*\right)/\partial x_2} = \frac{MP_1}{MP_2} = TRS(x_1^*, x_2^*)$$

Economic rate of substitution

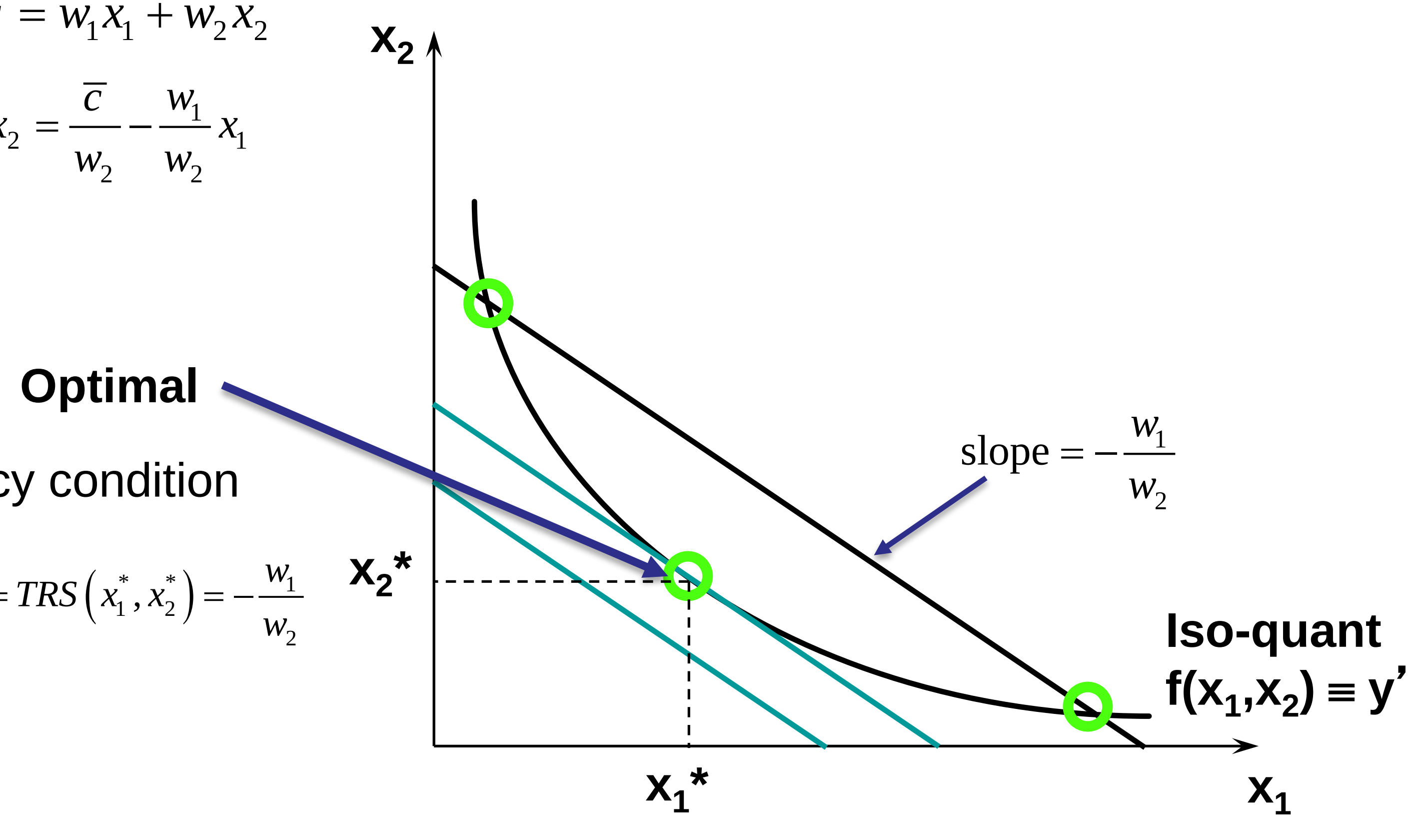
Technical rate of substitution

Cost minimization(2-inputs)

- Iso-cost
  - All combinations of inputs that have some given level of cost

$$\bar{c} = w_1 x_1 + w_2 x_2$$

$$x_2 = \frac{\bar{c}}{w_2} - \frac{w_1}{w_2} x_1$$



Tangency condition

$$-\frac{MP_1\left(x_1^*, x_2^*\right)}{MP_2\left(x_1^*, x_2^*\right)} = TRS\left(x_1^*, x_2^*\right) = -\frac{w_1}{w_2}$$

Cost minimization(n-inputs)

- **Formulation**

$$\min_{\tilde{x}} \ c = \tilde{w} \cdot \tilde{x}$$
$$s.t. \quad f\left(\tilde{x}\right) = y$$

- **F.O.C.**

$$\frac{\partial L}{\partial \lambda} = y - f\left(\tilde{x}\right) = 0$$
$$\frac{\partial L}{\partial x_i} = w_i - \lambda \frac{\partial f\left(\tilde{x}^*\right)}{\partial x_i} = 0, \quad i = 1, \ldots, n$$

- Optimality condition

$$\frac{w_i}{w_j} = \frac{\partial f\left(\tilde{x}^*\right) / \partial x_i}{\partial f\left(\tilde{x}^*\right) / \partial x_j}$$

- Lagrangian

$$L\left(\lambda, \tilde{x}\right) = \tilde{w} \cdot \tilde{x} - \lambda\left(f\left(\tilde{x}\right) - y\right)$$

- Conditional factor demand function

$$x_i^*\left(\tilde{w}, y\right), \quad i = 1, \ldots, n$$

- Cost function

$$c\left(\tilde{w}, y\right) = \sum_{i=1}^{n} w_i x_i^*\left(\tilde{w}, y\right)$$

Cost minimization(n-inputs)

- Second-order condition
  - Bordered Hessian matrix should be PD
- Definiteness of a Bordered Hessian matrix
  - Let n=# of variables, m=# of constraints

$$\bar{\mathbf{H}} = \begin{pmatrix} 0 & B \\ B^T & A \end{pmatrix}$$

  - If the determinant of bordered Hessian has the same sign as $(-1)^n$ and if (n-m) leading principal minors alternate in sign, then ND (negative definite)
  - If the determinant of bordered Hessian and (n-m) leading principal minors have the same sign as $(-1)^m$, then PD (positive definite)

Cost minimization(n-inputs)

- Second-order condition
  - 2-inputs case (n=2, m=1)

$$L(\lambda, x_1, x_2) = w_1 x_1 + w_2 x_2 - \lambda\left(f(x_1, x_2) - y\right)$$

$$\text{F.O.C.} = DL(\lambda, \tilde{x}) = \begin{cases} \dfrac{\partial L}{\partial \lambda} = y - f(x_1, x_2) \\ \dfrac{\partial L}{\partial x_1} = w_1 - \lambda f_1 \\ \dfrac{\partial L}{\partial x_2} = w_2 - \lambda f_2 \end{cases}$$

$$\bar{\mathbf{H}} = D^2 L(\lambda, \tilde{x}) = \begin{pmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{12} \\ -f_2 & -\lambda f_{21} & -\lambda f_{22} \end{pmatrix} \xrightarrow{\text{S.O.C.}} \begin{array}{l} \left|\bar{\mathbf{H}}\right| < 0 \\ \begin{vmatrix} 0 & -f_1 \\ -f_1 & -\lambda f_{11} \end{vmatrix} < 0 \end{array}$$

Cost minimization(n-inputs)

- Second-order condition
  - 3-inputs case (n=3, m=1)

$$\bar{\mathbf{H}} = \begin{pmatrix} 0 & -f_1 & -f_2 & -f_3 \\ -f_1 & -\lambda f_{11} & -\lambda f_{12} & -\lambda f_{13} \\ -f_2 & -\lambda f_{21} & -\lambda f_{22} & -\lambda f_{23} \\ -f_3 & -\lambda f_{31} & -\lambda f_{32} & -\lambda f_{33} \end{pmatrix}$$

S.O.C.

$$\left|\bar{\mathbf{H}}\right| < 0 \qquad \begin{vmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{12} \\ -f_2 & -\lambda f_{21} & -\lambda f_{22} \end{vmatrix} < 0$$

Cost minimization

- Example: Leontief technology

$$f(x_1, x_2) = \min\{ax_1, bx_2\} \quad \Rightarrow \quad y = \begin{cases} ax_1, \text{if } ax_1 \le bx_2 \\ bx_2, \text{if } ax_1 \ge bx_2 \end{cases}$$

  - Thus if we want to produce y, we only need

$$x_1^* = \frac{y}{a}, \; x_2^* = \frac{y}{b}, \quad \text{for any } w_1, w_2.$$

  - Cost function

$$c(w_1, w_2, y) = w_1 x_1^* + w_2 x_2^* = \left(\frac{w_1}{a} + \frac{w_2}{b} \right) y$$

Cost minimization

- Example: Linear technology

$$f\left(x_1, x_2\right) = ax_1 + bx_2$$

  - Since the inputs are Perfect substitutes, the firm will use whichever is cheaper.

$$\begin{cases} x_1^* = \dfrac{y}{a} \\ x_2^* = 0 \end{cases}, \text{ if } w_1 \frac{y}{a} < w_2 \frac{y}{b} \qquad \begin{cases} x_1^* = 0 \\ x_2^* = \dfrac{y}{b} \end{cases}, \text{ if } w_1 \frac{y}{a} > w_2 \frac{y}{b}$$

$$\text{Any } \left(x_1^*, x_2^*\right) \text{ such that } ax_1^* + bx_2^* = y, \text{ if } w_1 \frac{y}{a} = w_2 \frac{y}{b}$$

  - Cost function

$$c\left(w_1, w_2, y\right) = \min\left\{w_1 \frac{y}{a}, w_2 \frac{y}{b}\right\}$$

Cost minimization

- Example: Linear technology

$$\min w_1x_1 + w_2x_2$$
$$s.t.\ \ ax_1 + bx_2 = y$$
$$x_1 \geq 0, x_2 \geq 0$$

  - Lagrangian

$$L(\lambda, \mu_1, \mu_2, x_1, x_2) = w_1x_1 + w_2x_2 - \lambda(ax_1 + bx_2 - y) - \mu_1x_1 - \mu_2x_2$$

  - K-T condition

$$w_1 - a\lambda - \mu_1 = 0$$
$$w_2 - b\lambda - \mu_2 = 0$$
$$ax_1 + bx_2 = y$$
$$x_1 \geq 0, \mu_1 \geq 0, \mu_1x_1 = 0$$
$$x_2 \geq 0, \mu_2 \geq 0, \mu_2x_2 = 0$$

Cost minimization

- Linear technology(Another approach)

(i) $x_1 = 0, x_2 = 0$

No solution unless y=0

(ii) $x_1 = 0, x_2 > 0$

$$\mu_1 \geq 0,\ \mu_2 = 0 \Rightarrow \begin{aligned} w_1 &= \lambda a + \mu_1 \\ w_2 &= \lambda b \end{aligned} \Rightarrow \frac{w_1}{a} > \frac{w_2}{b}$$

From $ax_1 + bx_2 = y,\ x_2^* = \dfrac{y}{b}$

(iii) $x_1 > 0, x_2 = 0$

Similarly, $x_1^* = \dfrac{y}{a}$ when $\dfrac{w_1}{a} < \dfrac{w_2}{b}$

(iv) $x_1 > 0, x_2 > 0$

$$\mu_1 = 0,\ \mu_2 = 0 \Rightarrow \begin{aligned} w_1 &= \lambda a \\ w_2 &= \lambda b \end{aligned} \Rightarrow \frac{w_1}{a} = \frac{w_2}{b}$$

Any $\left(x_1^*, x_2^*\right)$ such that $ax_1^* + bx_2^* = y$

Cost minimization

- Example: C-D technology

$$\min \; w_1x_1 + w_2x_2 \qquad s.t. \quad Ax_1^a x_2^b = y$$

$$x_2 = A^{-1/b} y^{1/b} x_1^{-a/b} \Longrightarrow \min \; w_1x_1 + w_2\left(A^{-1/b} y^{1/b} x_1^{-a/b}\right)$$

- F.O.C.

$$\partial c/\partial x_1 = w_1 - \frac{a}{b} w_2 A^{-1/b} y^{1/b} x_1^{-(a+b)/b} = 0$$

- Conditional factor demand function

$$x_1^*\left(w_1, w_2, y\right) = A^{-1/(a+b)} \left[\frac{aw_2}{bw_1}\right]^{b/(a+b)} y^{1/(a+b)}$$

$$x_2^*\left(w_1, w_2, y\right) = A^{-1/(a+b)} \left[\frac{aw_2}{bw_1}\right]^{-a/(a+b)} y^{1/(a+b)}$$

- Cost function

$$c\left(w_1, w_2, y\right) = w_1x_1^* + w_2x_2^* = A^{-1/(a+b)} \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{a}{b}\right)^{\frac{-a}{a+b}}\right] w_1^{a/(a+b)} w_2^{b/(a+b)} y^{1/(a+b)}$$

Cost minimization

- Example: CES technology $f(x_1, x_2) = \left(x_1^\rho + x_2^\rho\right)^{1/\rho}$

$$\begin{aligned} &\min \ w_1 x_1 + w_2 x_2 \\ &s.t. \ \ x_1^\rho + x_2^\rho = y^\rho \end{aligned} \quad \Rightarrow \quad L(\lambda, x_1, x_2) = w_1 x_1 + w_2 x_2 - \lambda\left(x_1^\rho + x_2^\rho - y^\rho\right)$$

  - F.O.C.

$$\begin{aligned} &w_1 - \lambda\rho x_1^{\rho-1} = 0 \quad \cdots(1) \\ &w_2 - \lambda\rho x_2^{\rho-1} = 0 \quad \cdots(2) \\ &x_1^\rho + x_2^\rho = y^\rho \qquad \cdots(3) \end{aligned}$$

  - Solving (1) & (2), and inserting them into (3)

$$\begin{aligned} x_1^\rho &= w_1^{\frac{\rho}{\rho-1}}(\lambda\rho)^{\frac{-\rho}{\rho-1}} \\ x_2^\rho &= w_2^{\frac{\rho}{\rho-1}}(\lambda\rho)^{\frac{-\rho}{\rho-1}} \end{aligned} \quad \Rightarrow \quad w_1^{\frac{\rho}{\rho-1}}(\lambda\rho)^{\frac{-\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}(\lambda\rho)^{\frac{-\rho}{\rho-1}} = y^\rho$$

$$\Rightarrow \quad (\lambda\rho)^{\frac{-\rho}{\rho-1}}\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right] = y^\rho$$

$$\Rightarrow \quad (\lambda\rho)^{\frac{-\rho}{\rho-1}} = \left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]^{-1} y^\rho$$

Cost minimization

- Conditional factor demand function

$$x_1^*(w_1,w_2,y) = w_1^{\frac{1}{\rho-1}}\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]^{-1/\rho} y$$

$$x_2^*(w_1,w_2,y) = w_2^{\frac{1}{\rho-1}}\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]^{-1/\rho} y$$

- Cost function

$$\begin{aligned} c(w_1,w_2,y) &= w_1x_1^* + w_2x_2^* \\ &= y\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]^{-1/\rho} \\ &= y\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]^{\frac{\rho-1}{\rho}} \end{aligned}$$

- If we let $r = \dfrac{\rho}{\rho-1}$, $c(w_1,w_2,y) = y\left[w_1^r + w_2^r\right]^{\frac{1}{r}}$

Hoteling's lemma

- Profit max.

$$\underset{x_i}{\text{Max}} \quad \pi = p \cdot f(\tilde{x}) - \sum_{i=1}^{m} w_i x_i$$

$x_i^*(\tilde{w}, p)$: Factor demand function $\Rightarrow$ $y(p) = f(\tilde{x}^*(p, \tilde{w}))$: supply function

$\Rightarrow$ Profit function: $\pi(p, \tilde{w}) = p \cdot f(\tilde{x}^*) - \sum_{i=1}^{m} w_i x_i^*$

  - *Given profit function, how to get factor demand function or supply function?*

- Hoteling's lemma

$$x_i^*(p, \tilde{w}) = -\frac{\partial \pi(p, \tilde{w})}{\partial w_i} \qquad \text{and} \qquad y(p, \tilde{w}) = \frac{\partial \pi(p, \tilde{w})}{\partial p}$$

Hoteling's lemma (generalized version)

- Assuming that the derivative of profit function exists and $p_i > 0$, $y_i(\tilde{p}) = \dfrac{\partial \pi(\tilde{p})}{\partial p_i}$ for $i = 1, \ldots, n$
- Proof

Suppose $\tilde{y}^*$ is profit-max output at $\tilde{p}^*$.

Define the function of price vector $\tilde{p}$

$$g(\tilde{p}) = \pi(\tilde{p}) - \tilde{p} \cdot \tilde{y}^*$$

For any $\tilde{p} \neq \tilde{p}^*, g(\tilde{p}) \geq 0$ since $\pi(\tilde{p})$ is a maximized profit,

and if $\tilde{p} = \tilde{p}^*, g(\tilde{p}) = 0$

Thus $g(\tilde{p})$ reaches a min. value of 0 at $\tilde{p}^*$

F.O.C. $\dfrac{\partial g(\tilde{p}^*)}{\partial p_i} = \dfrac{\partial \pi(\tilde{p}^*)}{\partial p_i} - y_i^* = 0, \ i = 1, \ldots, n$

This is true for all choices of $\tilde{p}^*$

Shepherd's lemma

- Cost min. $\min_{\tilde{x}} \; c = \tilde{w} \cdot \tilde{x}$

$$s.t. \quad f\left(\tilde{x}\right) = y$$

$x_i^*(\tilde{w}, y)$:Conditional factor demand function

Cost function: $c\left(\tilde{w}, y\right) = \sum_{i=1}^{n} w_i x_i^* \left(\tilde{w}, y\right)$

  - *Given cost function, how to get conditional factor demand function?*

- Shepherd's lemma

$$x_i^*(\tilde{w}, y) = \frac{\partial c(\tilde{w}, y)}{\partial w_i}$$

Shepherd's lemma

- Let $x_i(\tilde{w}, y)$ be the conditional factor demand for input i.

  If the cost function is differentiable at $(\tilde{w}, y)$, and $w_i > 0$ for $i = 1, \ldots, n$, then

  $$x_i(\tilde{w}, y) = \frac{\partial c(\tilde{w}, y)}{\partial w_i}, i = 1, \ldots, n$$

- Proof

  Let $\tilde{x}^*$ be the cost-min. bundle that produce y at price $\tilde{w}^*$

  Define the function of $\tilde{w}$ such that $g(\tilde{w}) = c(\tilde{w}, y) - \tilde{w} \cdot \tilde{x}^*$

  Then $g(\tilde{w}) \leq 0$ since $c(\tilde{w}, y)$ is the cheapest cost,

  and $g(\tilde{w}^*) = 0$

  Since $g(\tilde{w})$ obtains the max. value of 0 when $\tilde{w} = \tilde{w}^*$

  $$\frac{\partial g(\tilde{w}^*)}{\partial w_i} = \frac{\partial c(\tilde{w}^*, y)}{\partial w_i} - x_i^* = 0, i = 1, \ldots, n$$

Profit max. vs. Cost min.

- Profit max.

$$\max_{\tilde{x}} \ \pi = py - \sum_{i=1}^{m} w_i x_i$$
$$s.t. \quad f(\tilde{x}) = y$$

$x_i^*(p, \tilde{w})$: Factor demand function

$y^*(p, \tilde{w})$: Supply function

$$\pi(p, \tilde{w})$$

Hoteling's lemma $y(p, \tilde{w}) = \dfrac{\partial \pi(p, \tilde{w})}{\partial p}, \ x_i^*(p, \tilde{w}) = -\dfrac{\partial \pi(p, \tilde{w})}{\partial w_i}$

- Cost min.

$$\min \ c = \tilde{w} \cdot \tilde{x}$$
$$s.t. \quad f(\tilde{x}) = y$$

$x_i^*(\tilde{w}, y)$: Conditional factor demand

$$c(\tilde{w}, y)$$

Shepherd's lemma $x_i^*(\tilde{w}, y) = \dfrac{\partial c(\tilde{w}, y)}{\partial w_i}$