Assessment of CDS for C-D problems

- Transportiveness
  - CDS
    - Does not recognize the direction of the flow or the strength of convection relative to diffusion.
    - It does not possess the transportivenss property at high Pe.

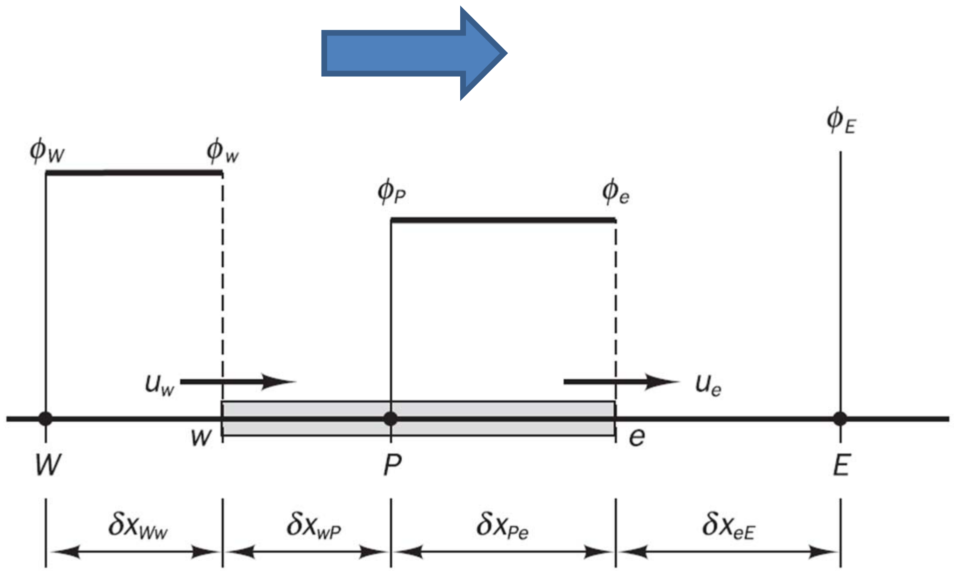
- Accuracy
  - Second-order accurate

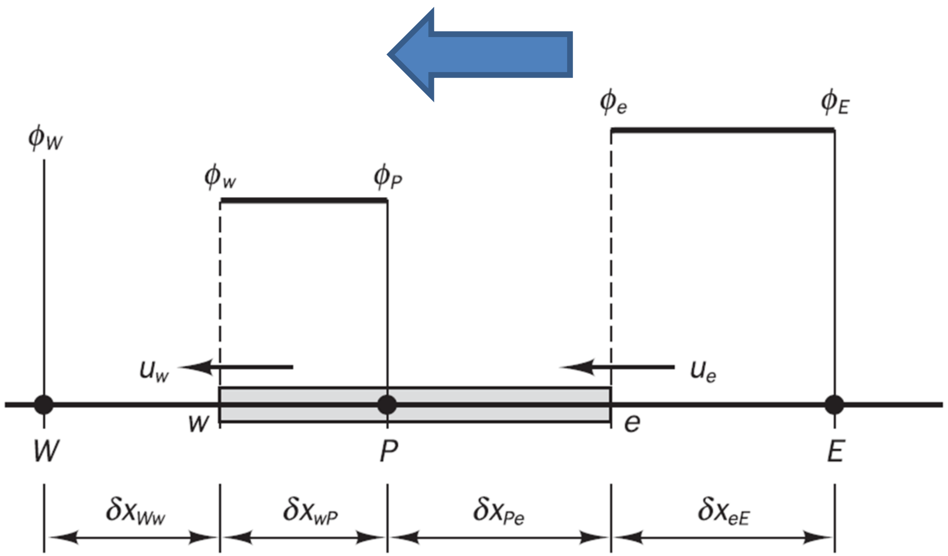
- Limitation
  - CDS is not a suitable discretization practice for general-purpose flow calculations.
  - Creates need for discretization schemes which possess more favorable properties

- Upwind, hybrid, power-law, QUICK, TVD

Contents

- UDS (Upwind difference scheme, 상류차분, 풍상차분)
  - One of the major inadequacies of CDS: inability to identify flow direction
  - In CDS
    - ϕ_w: influenced by both ϕ_W and ϕ_P
    - Unsuitable because the west cell face should receive much stronger influencing from node W thant from node P
  - In UDS or 'donor cell' differencing scheme
    - Takes into account the flow direction when determining the value at a cell face
    - Convected value of ϕ at a cell face = the value at the upstream values





UDS

❖ UDS (Upwind difference scheme, 상류차분, 풍상차분)

$$u_w > 0 \qquad \phi_w = \phi_W \qquad u_e > 0 \qquad \phi_e = \phi_P$$

For positive flow direction

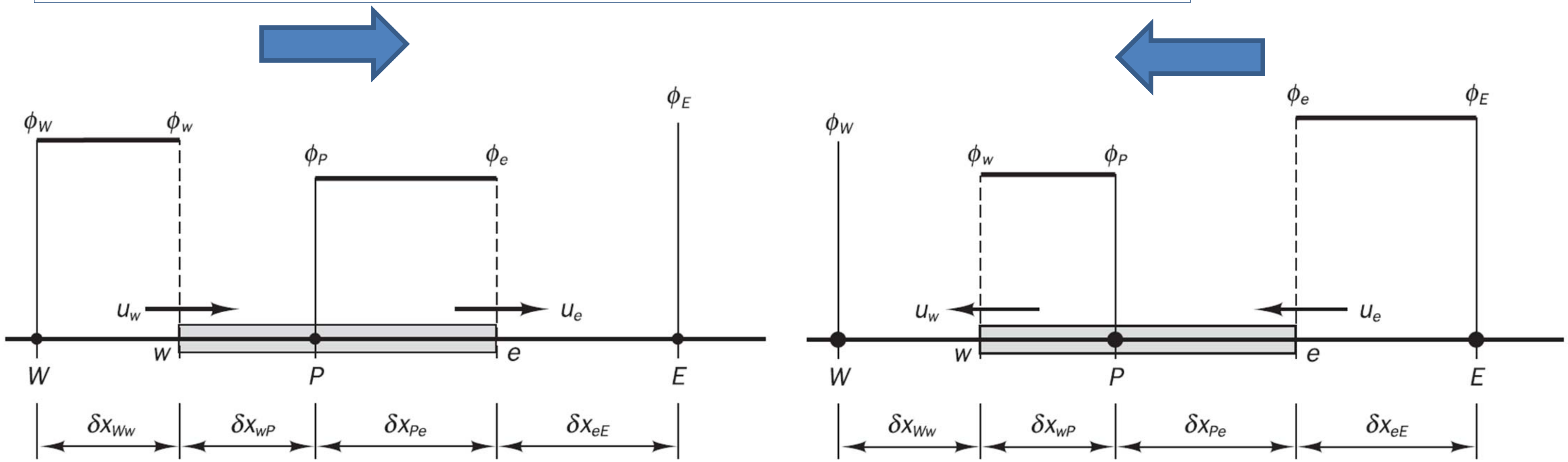
$$F_e\phi_e - F_w\phi_w = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

Only for the convection terms

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u$$

$$a_P = a_W + a_E + (F_e - F_w) - S_P$$

$$(D_w + D_e + F_e)\phi_P = (D_w + F_w)\phi_W + D_e\phi_E$$



UDS

❖ UDS (Upwind difference scheme, 상류차분, 풍상차분)

$$u_w < 0,\ u_e < 0 \qquad \phi_w = \phi_P \quad \text{and} \quad \phi_e = \phi_E$$

For negative flow direction

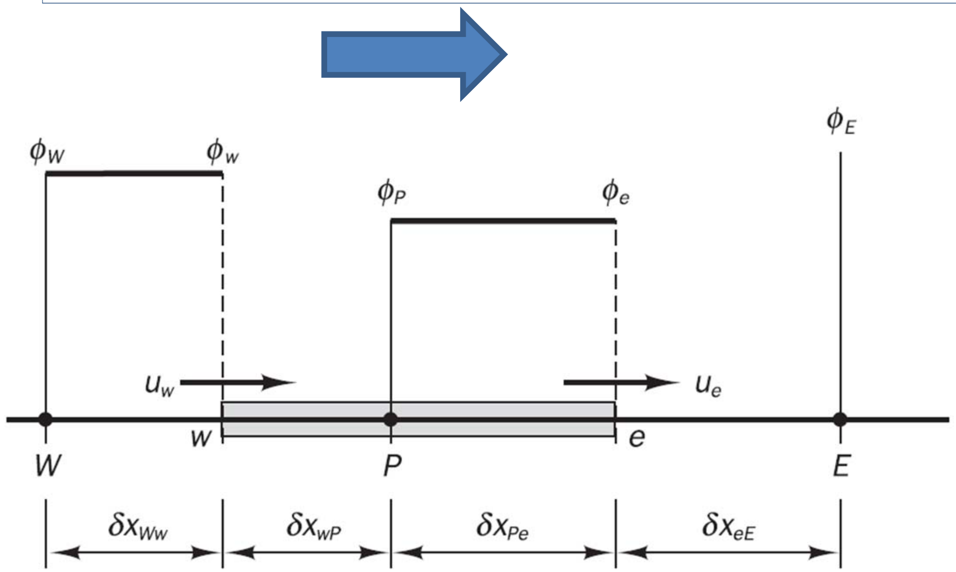
$$F_e\phi_e - F_w\phi_w = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

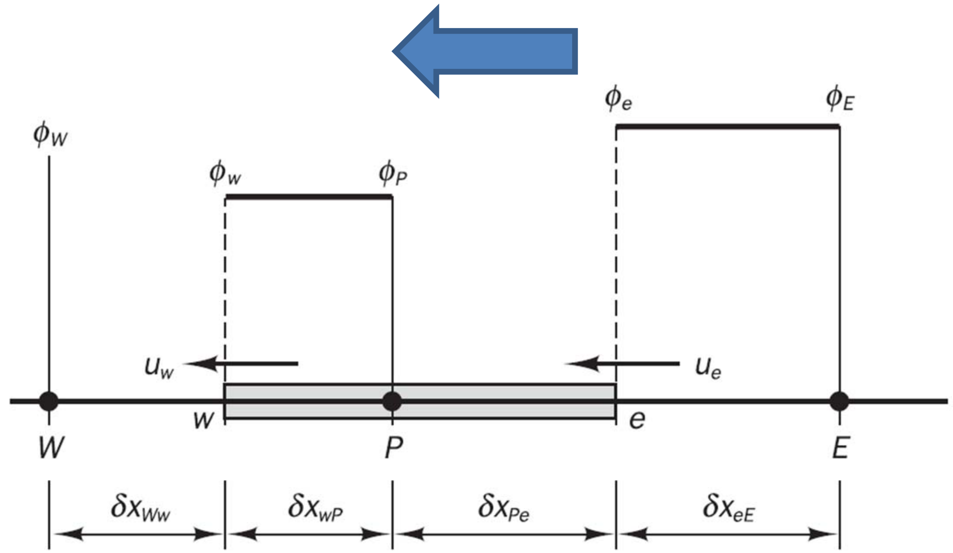
Only for the convection terms

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u$$

$$a_P = a_W + a_E + (F_e - F_w) - S_P$$

$$(D_w + D_e + F_e)\phi_P = (D_w + F_w)\phi_W + D_e\phi_E$$





UDS

- UDS (Upwind difference scheme, 상류차분, 풍상차분)
  - For positive flow direction

$$[(D_w + F_w) + D_e + (F_e - F_w)]\phi_P = (D_w + F_w)\phi_W + D_e\phi_E$$

  - For negative flow direction

$$[D_w + (D_e - F_e) + (F_e - F_w)]\phi_P = D_w\phi_W + (D_e - F_e)\phi_E$$

  - General form

$$a_P\phi_P = a_W\phi_W + a_E\phi_E \qquad a_P = a_W + a_E + (F_e - F_w)$$

	a_W	a_E
$F_w > 0, F_e > 0$	$D_w + F_w$	D_e
$F_w < 0, F_e < 0$	D_w	$D_e - F_e$

a_W	a_E
$D_w + \max(F_w, 0)$	$D_e + \max(0, -F_e)$

Example 5.2

- Same problem as Example 5.1 using UDS
  - Case 1: u = 0.1 m/s (use 5 CV's)
  - Case 2: *u* = 2.5 m/s (use 5 CV's)

$$F_e\phi_e - F_w\phi_w = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

$$F_e\phi_P - F_w\phi_W = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

- For cell #1

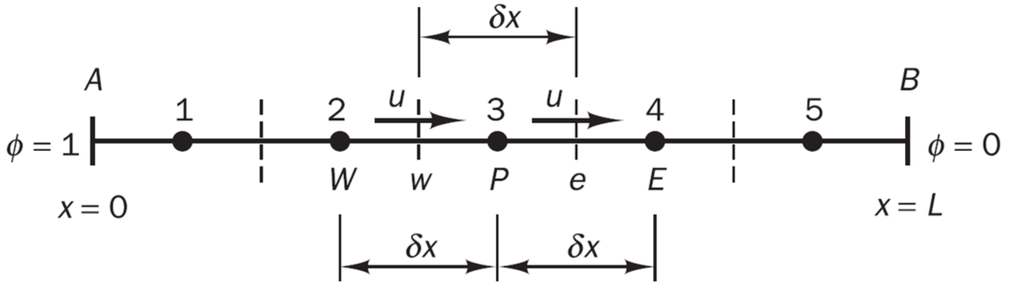
$$F_e\phi_P - F_A\phi_A = D_e(\phi_E - \phi_P) - D_A(\phi_P - \phi_A)$$

- For cell #5

$$F_B\phi_P - F_w\phi_W = D_B(\phi_B - \phi_P) - D_w(\phi_P - \phi_W)$$

$$D_A = D_B = 2\Gamma/\delta x = 2D$$

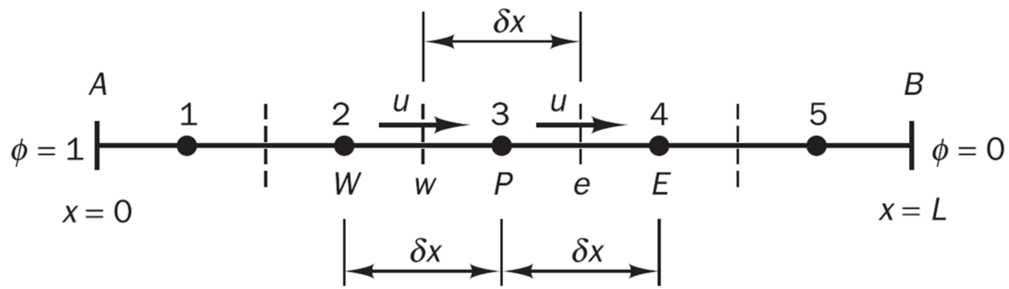
$$F_A = F_B = F$$



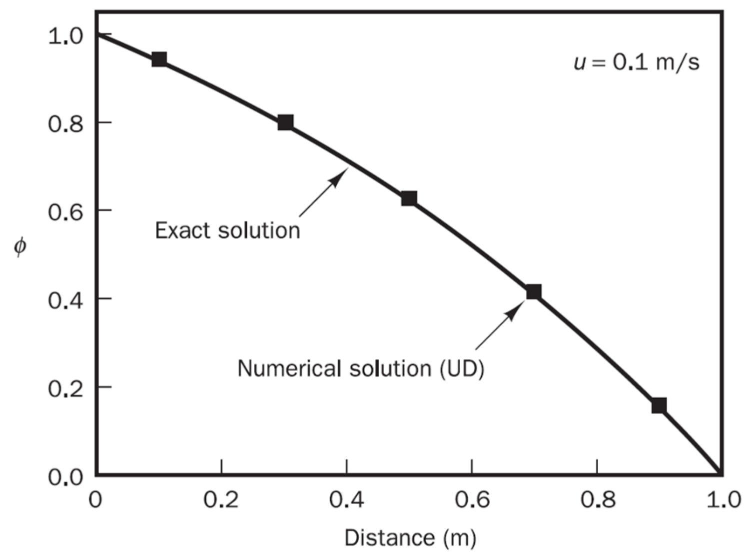
- Example 5.2
  - General form

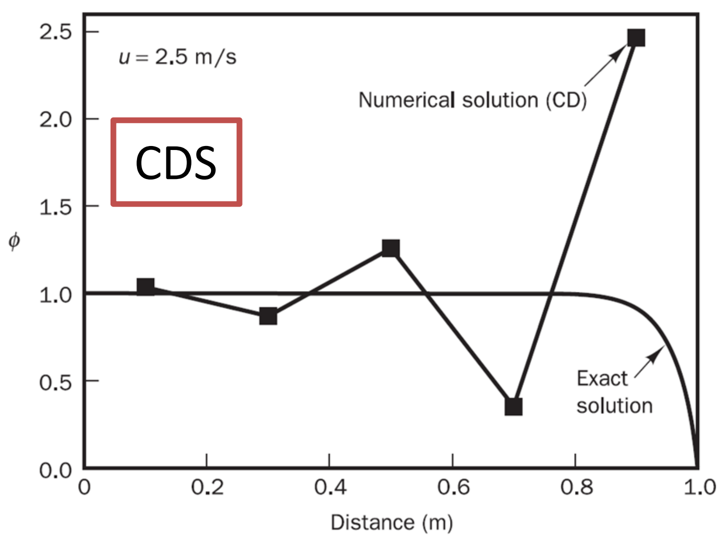
$$a_P \phi_P = a_W \phi_W + a_E \phi_E + S_u \qquad a_P = a_W + a_E + (F_e - F_w) - S_P$$

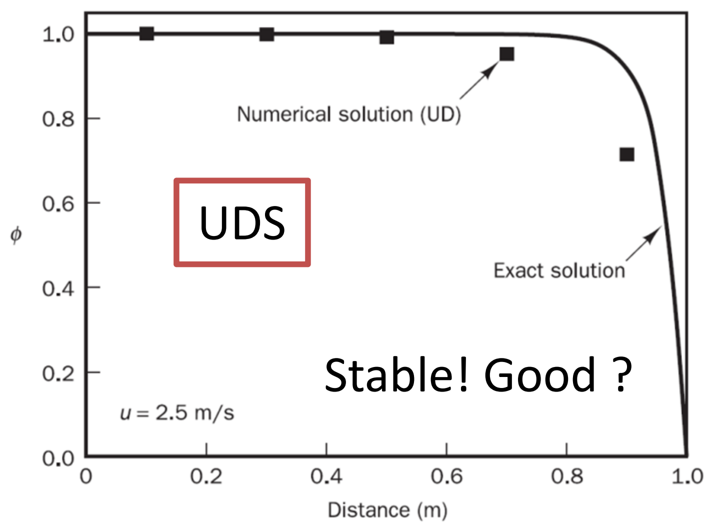
Node	a_W	a_E	S_P	S_u
1	0	D	$-(2D + F)$	$(2D + F)\phi_A$
2, 3, 4	$D + F$	D	0	0
5	$D + F$	0	$-2D$	$2D\phi_B$



- Example 5.2
  - Case-1 $Pe = F/D = 0.2$
    - Max. error ≈ 1.7 %
  - Case-2 $Pe = 5$
    - Max. error≈ 22.18 %







UDS

- Assessment
  - Conservativeness
    - Consistent evaluation for the cell faces $\Rightarrow$ satisfied
  - Boundedness
    - $S_p = -(2D + F) < 0$
    - All coefficients are positive.
    - Coefficient matrix is diagonally dominant.
  - Transportiveness
    - UDS accounts for the direction of the flow.
  - No wiggle in the solution
  - Accuracy: 1st order
  - Because of its simplicity, UDS has been widely applied in early CFD calculations.
  - Drawbacks
    - False diffusion: smeared property

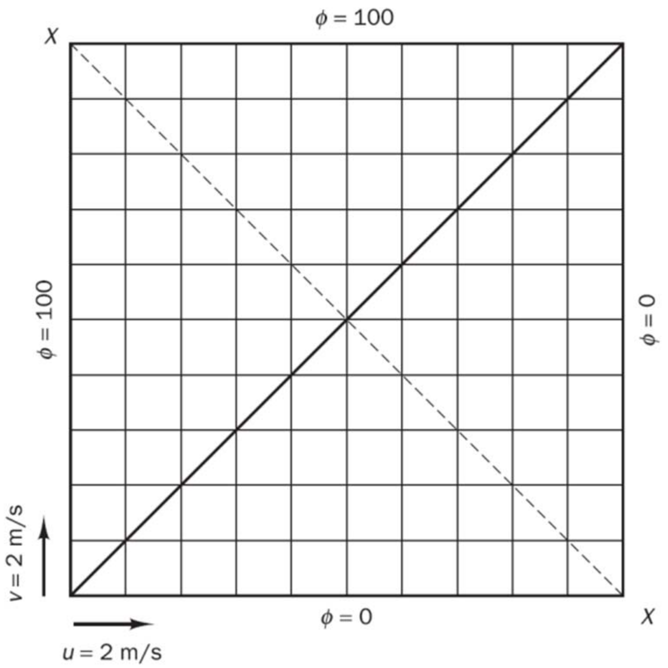
Node	a_W	a_E	S_P	S_u
1	0	D	$-(2D+F)$	$(2D+F)\phi_A$
2, 3, 4	$D+F$	D	0	0
5	$D+F$	0	$-2D$	$2D\phi_B$

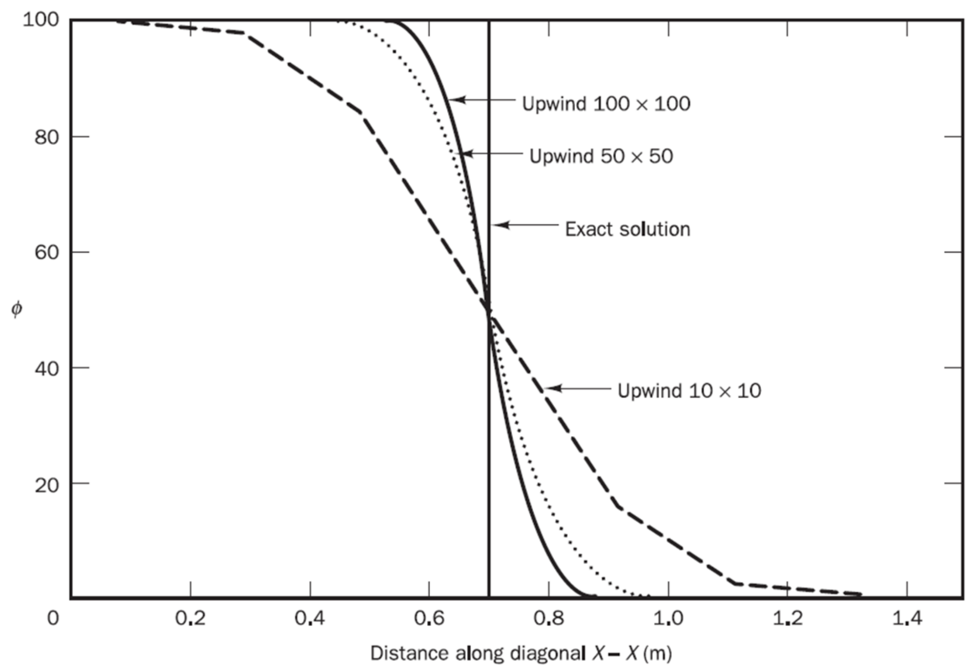
$$a_P = a_W + a_E + (F_e - F_w)$$

$$\frac{\sum |a_{nb}|}{|a'_P|} \begin{cases} \leq 1 \text{ at all nodes} \\ < 1 \text{ at one node at least} \end{cases}$$

- Assessment
  - False diffusion or numerical diffusion
    - $u = v = 2$ m/s everywhere
    - Velocity field is uniform and parallel to the diagonal (solid line) across the grid.
    - The boundary conditions for the scalar
      - $\phi = 0$ along the south and east boundaries
      - $\phi = 100$ on the west and north boundaries
      - $\phi = 50$ at the first and the last
    - Pure convection with Γ=0

UDS is not entirely suitable for accurate flow calculations and considerable research has been directed towards finding improved discretization schemes.

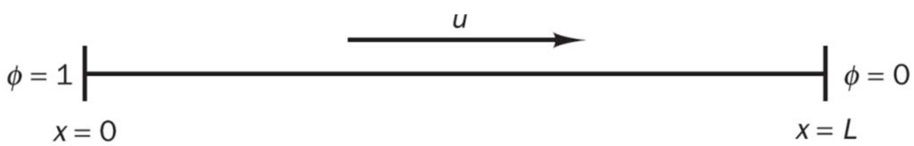




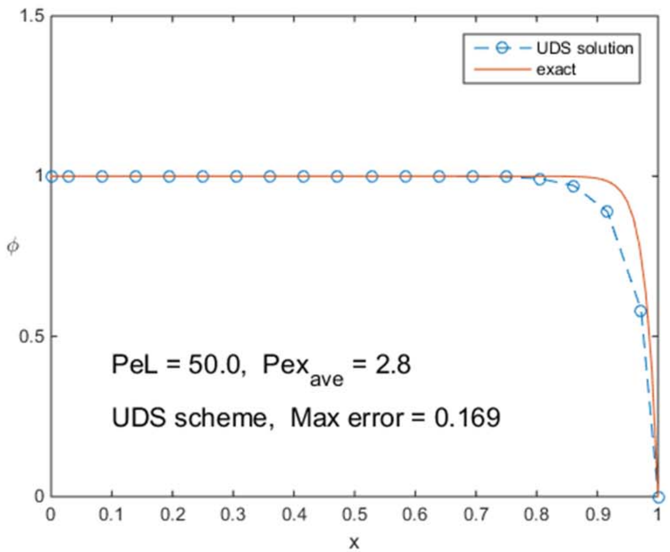
Homework #2-1

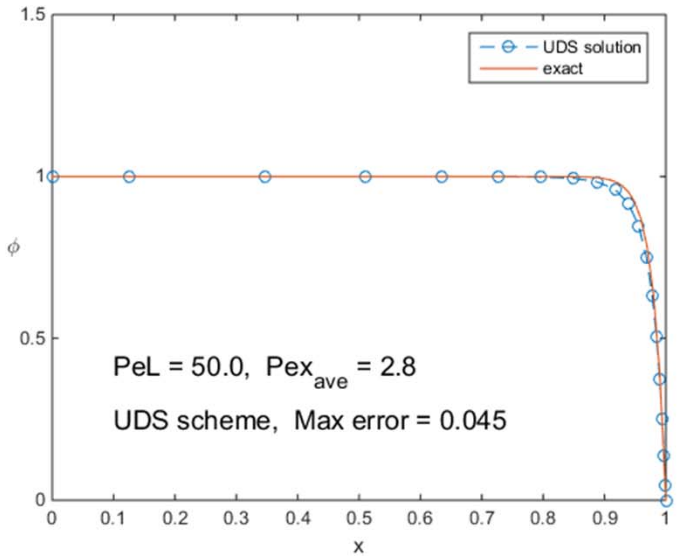
- Steady 1D C-D problem
  - With CDS and UDS

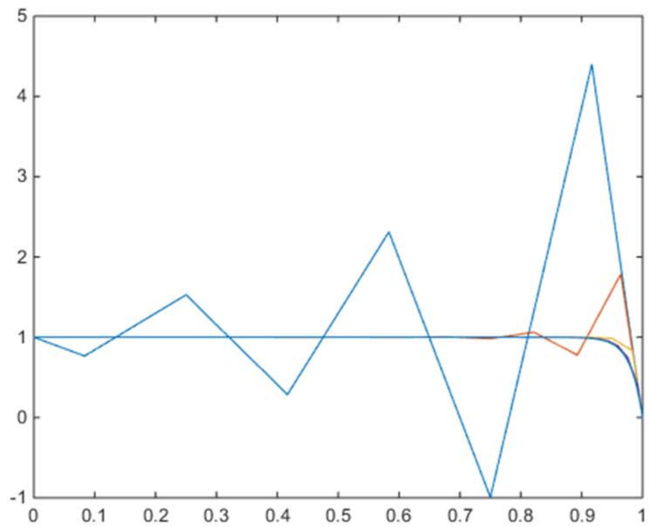
$$\mathrm{Pe} = 50 \; (L = 1.0,\ \rho = 1.0,\ u = 1.0,\ \Gamma = 0.02,\ \phi_0 = 0 \text{ and } \phi_L = 1.0)$$



Sample code is available on ETL!







Contents

The hybrid differencing scheme

- Hybrid differencing scheme (Spalding)
  - Combination of CDS and UDS
  - CDS: second order accurate $\Rightarrow$ not transportive
  - UDS: first order accurate $\Rightarrow$ transportive

Spalding

Patankar

- HDS uses
  - CDS for Pe < 2
  - UDS for Pe > 2

$$Pe_w = \frac{F_w}{D_w} = \frac{(\rho u)_w}{\Gamma_w / \delta x_{WP}}$$

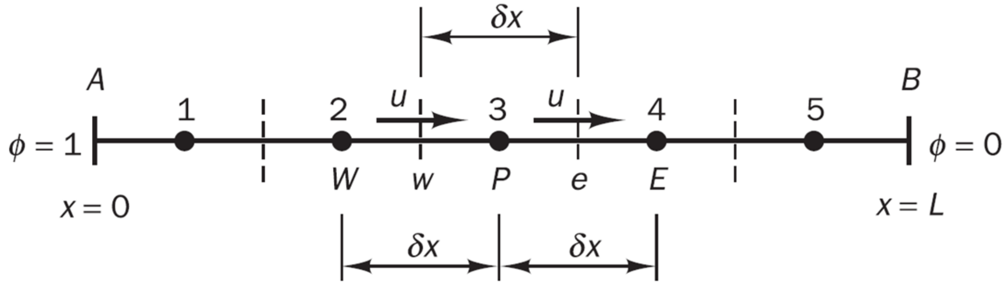
$$F_e\phi_e - F_w\boxed{\phi_w} = D_e(\phi_E - \phi_P) - \boxed{D_w(\phi_P - \phi_W)}$$

$$q_w = \boxed{}$$

The net flux per unit area through the west face

The hybrid differencing scheme

- HDS uses
  - UDS for |Pe| > 2



$$q_w = F_w \phi_w + D_w(\phi_W - \phi_P)$$

$$Pe_w = \frac{F_w}{D_w} = \frac{(\rho u)_w}{\Gamma_w / \delta x_{WP}}$$

$$\text{for } Pe_w \geq 2 \qquad q_w = F_w \phi_W$$

$$\text{for } Pe_w \leq -2 \qquad q_w = F_w \phi_P$$

Diffusion term is ignored!

  - CDS for |Pe| <2

Diffusion term is included!

$$q_w = F_w \left[\frac{1}{2}\left(1 + \frac{2}{Pe_w}\right)\phi_W + \frac{1}{2}\left(1 - \frac{2}{Pe_w}\right)\phi_P \right]$$

$$= \left[\frac{1}{2} F_w(\phi_W + \phi_P) + F_w\left(\frac{\phi_W}{Pe_w} - \frac{\phi_P}{Pe_w}\right)\right] = \frac{1}{2} F_w(\phi_W + \phi_P) + D_w(\phi_W - \phi_P)$$

The hybrid differencing scheme

- General form

$$a_P\phi_P = a_W\phi_W + a_E\phi_E \qquad a_P = a_W + a_E + (F_e - F_w)$$

CDS

a_W	a_E
$D_w + \frac{F_w}{2}$	$D_e - \frac{F_e}{2}$

UDS

a_W	a_E
$D_w + \max(F_w, 0)$	$D_e + \max(0, -F_e)$

$$q_w = F_w\phi_w + D_w(\phi_W - \phi_P) \qquad \text{for } Pe_w \geq 2 \quad q_w = F_w\phi_W \qquad \text{for } Pe_w \leq -2 \quad q_w = F_w\phi_P$$

$$\text{for } -2 < Pe_w < 2 \qquad q_w = \frac{1}{2}F_w(\phi_W + \phi_P) + D_w(\phi_W - \phi_P)$$

a_W	a_E
$\max\left[F_w, \left(D_w + \frac{F_w}{2}\right), 0\right]$	$\max\left[-F_e, \left(D_e - \frac{F_e}{2}\right), 0\right]$

The hybrid differencing scheme

- Example 5.3
  - Solve the problem considered in Case 2 of Example 5.1 using the hybrid scheme for u = 2.5 m/s. Compare a 5-node solution with a 25-node solution.

$$F = F_e = F_w = \rho u = 2.5 \qquad D = D_e = D_w = \Gamma / \delta x = 0.5 \qquad Pe_w = Pe_e = \rho u \delta x / \Gamma = 5$$

$$a_P \phi_P = a_W \phi_W + a_E \phi_E \qquad a_P = a_W + a_E + (F_e - F_w)$$

a_W	a_E
$\max\left[F_w, \left(D_w + \frac{F_w}{2}\right), 0\right]$	$\max\left[-F_e, \left(D_e - \frac{F_e}{2}\right), 0\right]$

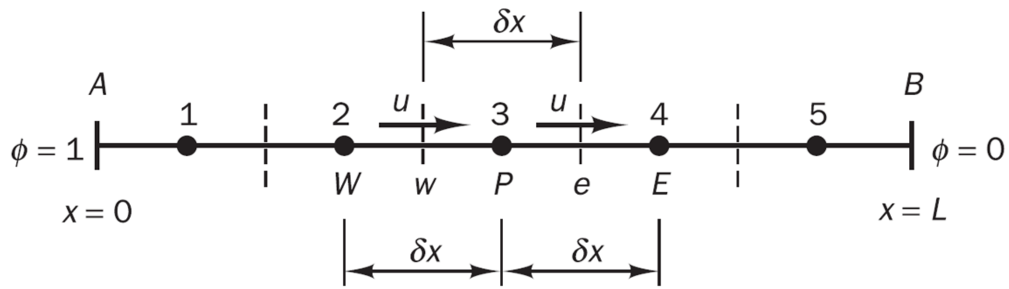
a_W	a_E
F_w	0

  - For boundary cells,

Diffusion term is included for boundaries !

$$F_e \phi_P - F_A \phi_A = 0 - D_A(\phi_P - \phi_A)$$

$$F_B \phi_P - F_w \phi_W = D_B(\phi_B - \phi_P) - 0$$



The hybrid differencing scheme

- Example 5.3
  - General form

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + S_u \qquad a_P = a_W + a_E + (F_e - F_w) - S_P$$

Node	a_W	a_E	S_P	S_u
1	0	0	$-(2D + F)$	$(2D + F)\phi_A$
2,3,4	F	0	0	0
5	F	0	$-2D$	$2D\phi_B$

For internal cells

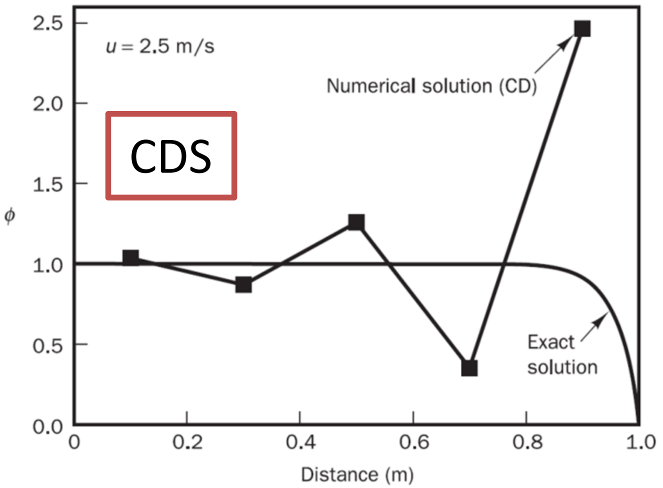
a_W	a_E
F_w	0

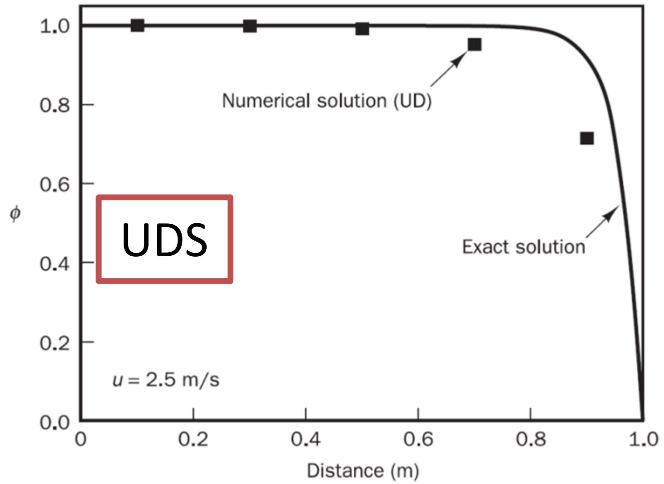
$$F_e \phi_P - F_A \phi_A = 0 - D_A(\phi_P - \phi_A)$$

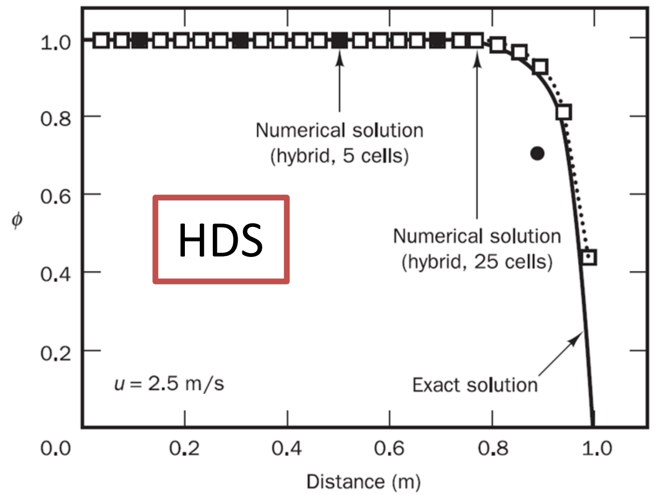
$$F_B \phi_P - F_w \phi_W = D_B(\phi_B - \phi_P) - 0$$

The hybrid differencing scheme

- Example 5.3
  - Comparison with the analytical solutions
  - Assessment of HDS
    - fully conservative
    - Unconditionally bounded (since the coefficients are always positive)
    - Satisfies the transportiveness property
    - Produces physical realistic solutions
    - Highly stable compared with higher order scheme
    - Widely used in various CFD procedures and has proved to be very useful for predicting practical flows.
    - Only first order accurate







The hybrid differencing scheme

- HDS for multi-D C-D problem
  - Simple extension from 1D discretized equation

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + a_B\phi_B + a_T\phi_T$$

$$a_P = a_W + a_E + a_S + a_N + a_B + a_T + \Delta F$$

	One-dimensional flow	*Two-dimensional flow*	*Three-dimensional flow*
a_W	$\max\left[F_w, \left(D_w + \frac{F_w}{2}\right), 0\right]$	$\max\left[F_w, \left(D_w + \frac{F_w}{2}\right), 0\right]$	$\max\left[F_w, \left(D_w + \frac{F_w}{2}\right), 0\right]$
a_E	$\max\left[-F_e, \left(D_e - \frac{F_e}{2}\right), 0\right]$	$\max\left[-F_e, \left(D_e - \frac{F_e}{2}\right), 0\right]$	$\max\left[-F_e, \left(D_e - \frac{F_e}{2}\right), 0\right]$
a_S	–	$\max\left[F_s, \left(D_s + \frac{F_s}{2}\right), 0\right]$	$\max\left[F_s, \left(D_s + \frac{F_s}{2}\right), 0\right]$
a_N	–	$\max\left[-F_n, \left(D_n - \frac{F_n}{2}\right), 0\right]$	$\max\left[-F_n, \left(D_n - \frac{F_n}{2}\right), 0\right]$
a_B	–	–	$\max\left[F_b, \left(D_b + \frac{F_b}{2}\right), 0\right]$
a_T	–	–	$\max\left[-F_t, \left(D_t - \frac{F_t}{2}\right), 0\right]$
ΔF	$F_e - F_w$	$F_e - F_w + F_n - F_s$	$F_e - F_w + F_n - F_s + F_t - F_b$

The hybrid differencing scheme

- HDS for multi-D C-D problem
  - Simple extension from 1D discretized equation

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + a_B\phi_B + a_T\phi_T$$

$$a_P = a_W + a_E + a_S + a_N + a_B + a_T + \Delta F$$

Face	***w***	***e***	***s***	***n***	***b***	***t***
F	$(\rho u)_w A_w$	$(\rho u)_e A_e$	$(\rho v)_s A_s$	$(\rho v)_n A_n$	$(\rho w)_b A_b$	$(\rho w)_t A_t$
D	$\frac{\Gamma_w}{\delta x_{WP}} A_w$	$\frac{\Gamma_e}{\delta x_{PE}} A_e$	$\frac{\Gamma_s}{\delta y_{SP}} A_s$	$\frac{\Gamma_n}{\delta y_{PN}} A_n$	$\frac{\Gamma_b}{\delta x_{BP}} A_b$	$\frac{\Gamma_t}{\delta x_{PT}} A_t$

Contents

The power-law scheme

- Power-law scheme (Patankar)
  - More accurate and produces better results than the HDS
  - For Pe >10 $\Rightarrow$ diffusion is set to zero.
  - For 0 <Pe<10 $\Rightarrow$ the flux is evaluated by a polynomial expression

$$F_e\phi_e - F_w\boxed{\phi_w} = D_e(\phi_E - \phi_P) - \boxed{D_w(\phi_P - \phi_W)}$$

$$q_w = F_w\phi_w + D_w\left(\phi_W - \phi_P\right)$$

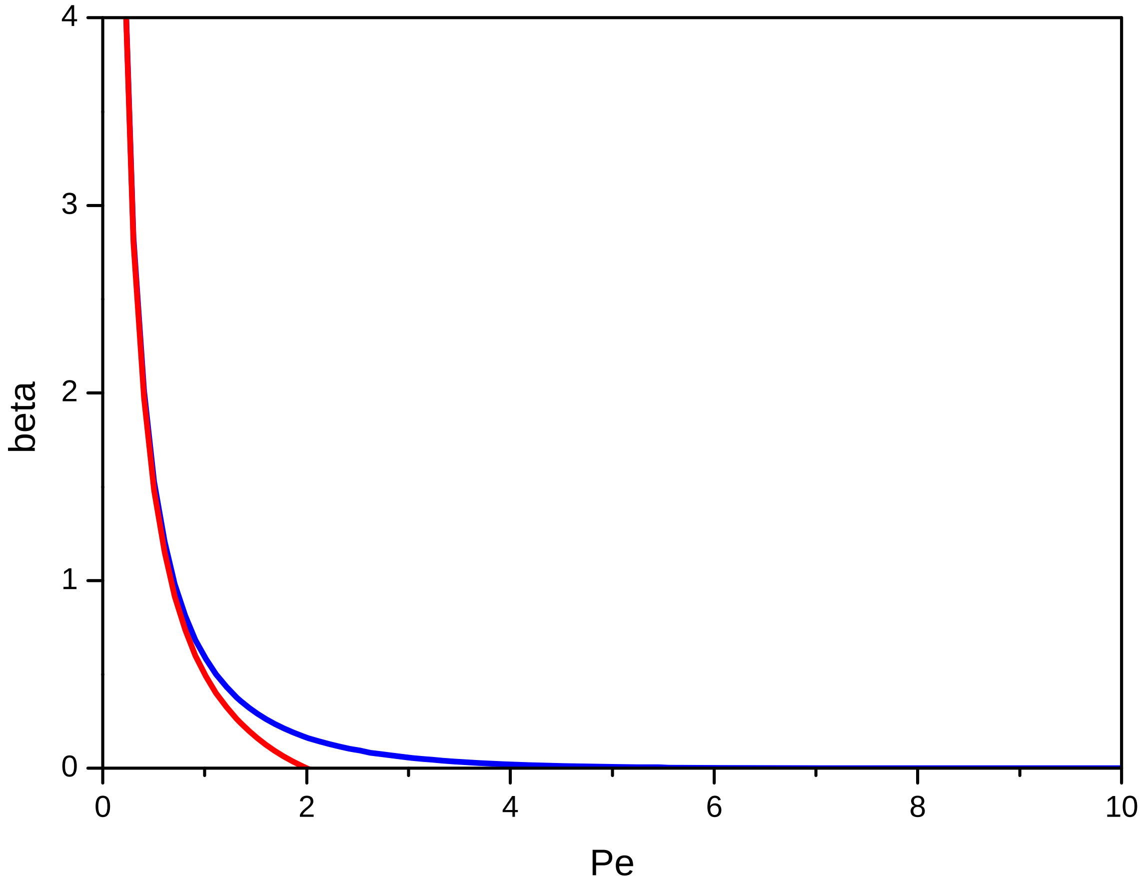
$$q_w = F_w[\phi_W - \beta_w(\phi_P - \phi_W)] \quad \text{for } 0 < Pe < 10 \quad \text{where } \beta_w = (1 - 0.1Pe_w)^5/Pe_w$$

$$q_w = F_w\phi_W \text{ for } Pe > 10$$

$$q_w = F_w\left[\frac{1}{2}\left(1 + \frac{2}{Pe_w}\right)\phi_W + \frac{1}{2}\left(1 - \frac{2}{Pe_w}\right)\phi_P\right] \qquad q_w = F_w\phi_W$$

The power-law scheme

- Power-law scheme (Patankar)



The power-law scheme

- Power-law scheme (Patankar)
  - General form

$$a_P\phi_P = a_W\phi_W + a_E\phi_E \qquad a_P = a_W + a_E + (F_e - F_w)$$

a_W	a_E
$D_w \max[0, (1 - 0.1\|Pe_w\|)^5] + \max[F_w, 0]$	$D_e \max[0, (1 - 0.1\|Pe_e\|)^5] + \max[-F_e, 0]$

$$q_w = F_w[\phi_W - \beta_w(\phi_P - \phi_W)] \qquad \text{where } \beta_w = (1 - 0.1Pe_w)^5/Pe_w$$

$$q_w = F_w\phi_W \text{ for } Pe > 10$$

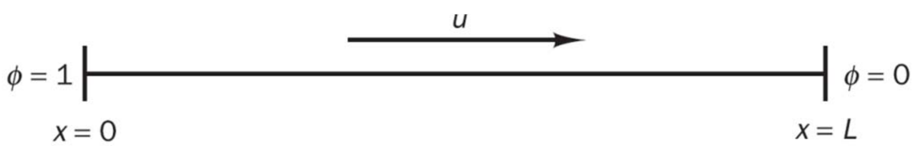
The power-law scheme

- Power-law scheme (Patankar)
  - Assessment
    - Similar characteristics with HDS
    - More accurate for 1D problems
    - Useful in practical flow calculations
    - Employed in commercial CFD codes

Homework #2-2

- Steady 1D C-D problem
  - With CDS and UDS

$$\text{Pe} = 50 \ (L = 1.0,\ \rho = 1.0,\ u = 1.0,\ \Gamma = 0.02,\ \phi_0 = 0 \text{ and } \phi_L = 1.0)$$



  - Repeat this problem with hybrid scheme and power-law scheme
  - Test with different Pe number
  - Compare with CDS and UDS

Contents

Higher-order differencing schemes for C-D problems

- Hybrid and Upwind schemes are
  - Stable
  - Obey the transportiveness requirement
  - But have first order accuracy $\Rightarrow$ are prone to numerical diffusion errors
  - Such errors can be minimized by employing higher order discretizations.
- Central difference scheme is second order accurate but is not stable.
- Formulations that do not take into account the flow direction are unstable.
- For more accuracy:
  - use higher order schemes, which preserve upwinding for stability

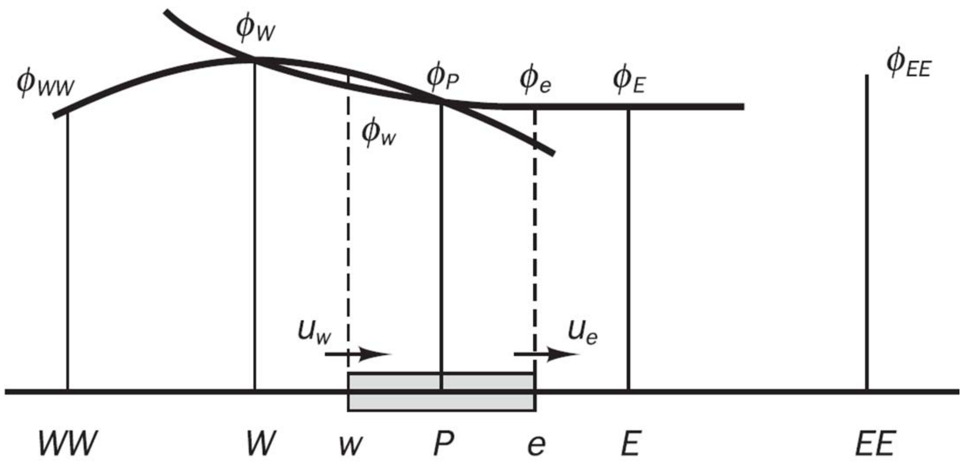
Higher-order differencing schemes for C-D problems

- QUICK scheme (Leonard)
  - Quadratic upwind differencing scheme
  - Quadratic Upstream Interpolation for Convective Kinetic (QUICK)
  - Three-point upstream-weighted quadratic interpolation for cell face values

$$F_e\boxed{\phi_e} - F_w\boxed{\phi_w} = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

$$\phi_w = f(\phi_P, \phi_W, \phi_{WW}) \quad \text{when } u_w > 0 \qquad \phi_w = f(\phi_W, \phi_P, \phi_E) \quad u_w < 0$$

$$\phi_e = f(\phi_E, \phi_P, \phi_W) \quad u_e > 0 \qquad \phi_e = f(\phi_{EE}, \phi_E, \phi_P) \quad u_e < 0$$



Two upstream nodes and one downstream node

Higher-order differencing schemes for C-D problems

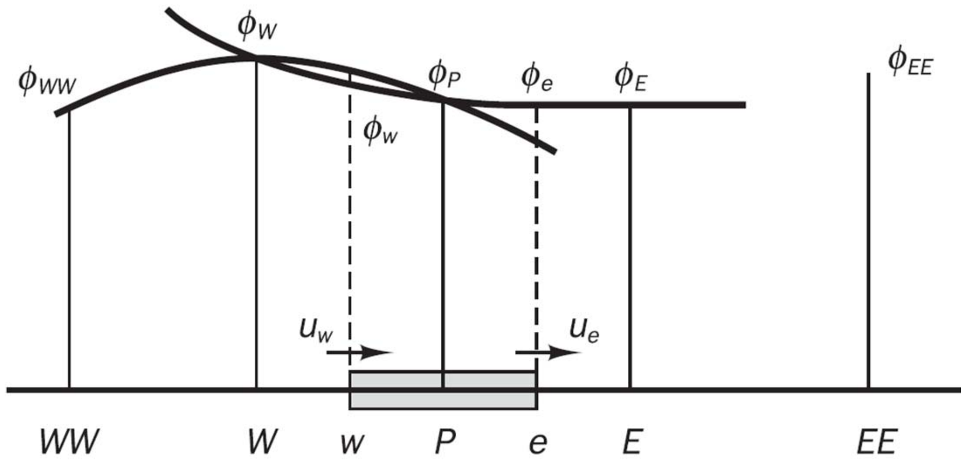
❖ Quadratic function for uniform grid

$$\phi_{face} = \frac{6}{8}\phi_{i-1} + \frac{3}{8}\phi_i - \frac{1}{8}\phi_{i-2}$$

How?

When $u_w > 0$ $$\phi_w = \frac{6}{8}\phi_W + \frac{3}{8}\phi_P - \frac{1}{8}\phi_{WW}$$

When $u_e > 0$ $$\phi_e = \frac{6}{8}\phi_P + \frac{3}{8}\phi_E - \frac{1}{8}\phi_W$$



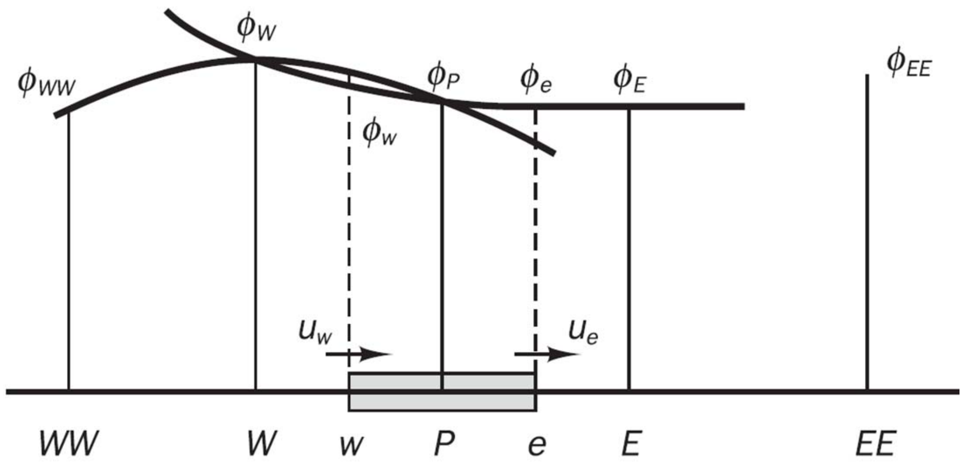
Higher-order differencing schemes for C-D problems

- Quadratic function for uniform grid

$$\phi_{face} = \frac{6}{8}\phi_{i-1} + \frac{3}{8}\phi_i - \frac{1}{8}\phi_{i-2}$$

How?

$$\phi(x) = ax^2 + bx + c$$



$$\phi(0) = \phi_P = c$$

$$\phi(-\delta x) = \phi_W = a(-x)^2 + b(-x) + c$$

$$\phi(-2\delta x) = \phi_{WW} = a(-2\delta x)^2 + b(-2\delta x) + c$$

Evaluate a,b,c

$$\phi(-0.5\delta x) = \phi_w = a(-0.5\delta x)^2 + b(-0.5\delta x) + c$$

$$\phi_{face} = \frac{6}{8}\phi_{i-1} + \frac{3}{8}\phi_i - \frac{1}{8}\phi_{i-2}$$

Higher-order differencing schemes for C-D problems

- Discretized equation by QUICK scheme

$$F_e\phi_e - F_w\phi_w = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

Quadratic function for convection
Central differencing for diffusion

If $F_w > 0$ and $F_e > 0$

$$\left[F_e\left(\frac{6}{8}\phi_P + \frac{3}{8}\phi_E - \frac{1}{8}\phi_W\right) - F_w\left(\frac{6}{8}\phi_W + \frac{3}{8}\phi_P - \frac{1}{8}\phi_{WW}\right)\right] = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

$$\left[D_w - \frac{3}{8}F_w + D_e + \frac{6}{8}F_e\right]\phi_P = \left[D_w + \frac{6}{8}F_w + \frac{1}{8}F_e\right]\phi_W + \left[D_e - \frac{3}{8}F_e\right]\phi_E - \frac{1}{8}F_w\phi_{WW}$$

- General form

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_{WW}\phi_{WW}$$

a_W	a_E	a_{WW}	a_P
$D_w + \frac{6}{8}F_w + \frac{1}{8}F_e$	$D_e - \frac{3}{8}F_e$	$-\frac{1}{8}F_w$	$a_W + a_E + a_{WW} + (F_e - F_w)$

Higher-order differencing schemes for C-D problems

- Discretized equation by QUICK scheme

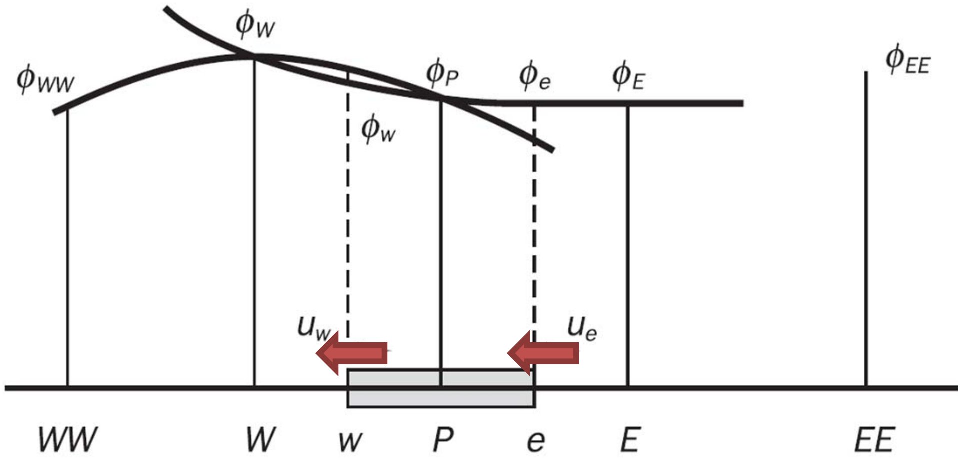
$$F_e\phi_e - F_w\phi_w = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

Quadratic function for convection
Central differencing for diffusion

For $F_w < 0$ and $F_e < 0$

$$\phi_w = \frac{6}{8}\phi_P + \frac{3}{8}\phi_W - \frac{1}{8}\phi_E$$

$$\phi_e = \frac{6}{8}\phi_E + \frac{3}{8}\phi_P - \frac{1}{8}\phi_{EE}$$



- General form

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_{EE}\phi_{EE}$$

a_W	a_E	a_{EE}	a_P
$D_w + \frac{3}{8}F_w$	$D_e - \frac{6}{8}F_e - \frac{1}{8}F_w$	$\frac{1}{8}F_e$	$a_W + a_E + a_{EE} + (F_e - F_w)$

Higher-order differencing schemes for C-D problems

- Discretized equation by QUICK scheme
  - General form

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_{WW}\phi_{WW} + a_{EE}\phi_{EE}$$

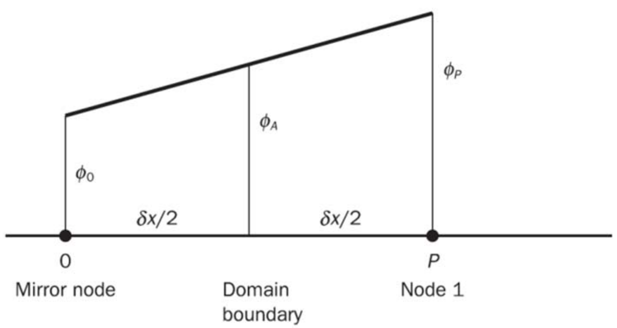
$$a_P = a_W + a_E + a_{WW} + a_{EE} + (F_e - F_w)$$

a_W	a_{WW}	a_E	a_{EE}
$D_w + \frac{6}{8}\alpha_w F_w + \frac{1}{8}\alpha_e F_e + \frac{3}{8}(1-\alpha_w)F_w$	$-\frac{1}{8}\alpha_w F_w$	$D_e - \frac{3}{8}\alpha_e F_e - \frac{6}{8}(1-\alpha_e)F_e - \frac{1}{8}(1-\alpha_w)F_w$	$\frac{1}{8}(1-\alpha_e)F_e$

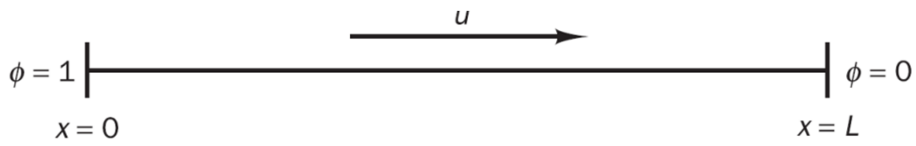
$$\alpha_w = 1 \text{ for } F_w > 0 \text{ and } \alpha_e = 1 \text{ for } F_e > 0$$
$$\alpha_w = 0 \text{ for } F_w < 0 \text{ and } \alpha_e = 0 \text{ for } F_e < 0$$

Higher-order differencing schemes for C-D problems

- Example 5.4
  - Will not be covered in detail
  - Boundary treatment
    - Linear extrapolation



$$\phi_0 = 2\phi_A - \phi_P$$



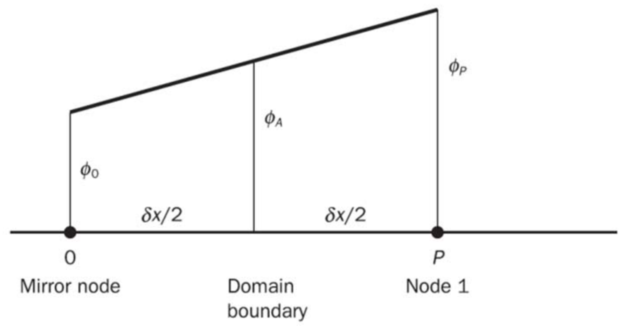
$$\phi_e = \frac{6}{8}\phi_P + \frac{3}{8}\phi_E - \frac{1}{8}\phi_W$$

$$\phi_e = \frac{6}{8}\phi_P + \frac{3}{8}\phi_E - \frac{1}{8}(2\phi_A - \phi_P)$$

$$= \frac{7}{8}\phi_P + \frac{3}{8}\phi_E - \frac{2}{8}\phi_A$$

Higher-order differencing schemes for C-D problems

- Example 5.4
  - Will not be covered in detail
  - Boundary treatment



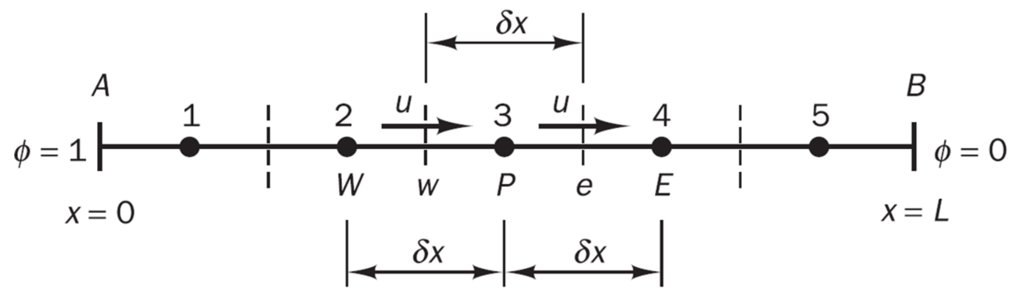
$$\Gamma \frac{\partial \phi}{\partial x}\bigg|_A = \frac{D_A^*}{3}(9\phi_P - 8\phi_A - \phi_E)$$

$$\phi = f(\phi_A, \phi_P, \phi_E) = ax^2 + bx + c$$

$$\phi(0) = \phi_A$$

$$\phi\left(\frac{\delta x}{2}\right) = \phi_P$$

$$\phi\left(\frac{3\delta x}{2}\right) = \phi_E$$



Higher-order differencing schemes for C-D problems

- Assessment of QUICK
  - Uses consistent quadratic profiles $\Rightarrow$ conservative
  - Is based on a quadratic function $\Rightarrow$ has 3rd order truncation error
  - Is based on 2 upstream and 1 downstream nodes $\Rightarrow$ has transportiveness
  - Boundedness is not guaranteed.
  - Gives rise to stability problems and unbounded solutions.
  - QUICK scheme is conditionally stable.
  - Involves ϕ_{WW} and ϕ_{EE} which are not immediate-neighbor nodes
    - Complicated matrix structure

a_W	a_{WW}	a_E	a_{EE}
$D_w + \frac{6}{8}\alpha_w F_w + \frac{1}{8}\alpha_e F_e + \frac{3}{8}(1-\alpha_w)F_w$	$-\frac{1}{8}\alpha_w F_w$	$D_e - \frac{3}{8}\alpha_e F_e - \frac{6}{8}(1-\alpha_e)F_e - \frac{1}{8}(1-\alpha_w)F_w$	$\frac{1}{8}(1-\alpha_e)F_e$

Higher-order differencing schemes for C-D problems

- Stability problems of the QUICK scheme and remedies
  - Iterative method (Hayase et al.)
  - Deferred correction

$$\phi_w = \frac{6}{8}\phi_W + \frac{3}{8}\phi_P - \frac{1}{8}\phi_{WW}$$

$$\phi_w = \phi_W + \frac{1}{8}[3\phi_P - 2\phi_W - \phi_{WW}]$$

$$\phi_w^{n+1} = \phi_W^{n+1} + \frac{1}{8}[3\phi_P^n - 2\phi_W^n - \phi_{WW}^n]$$

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + \bar{S}$$

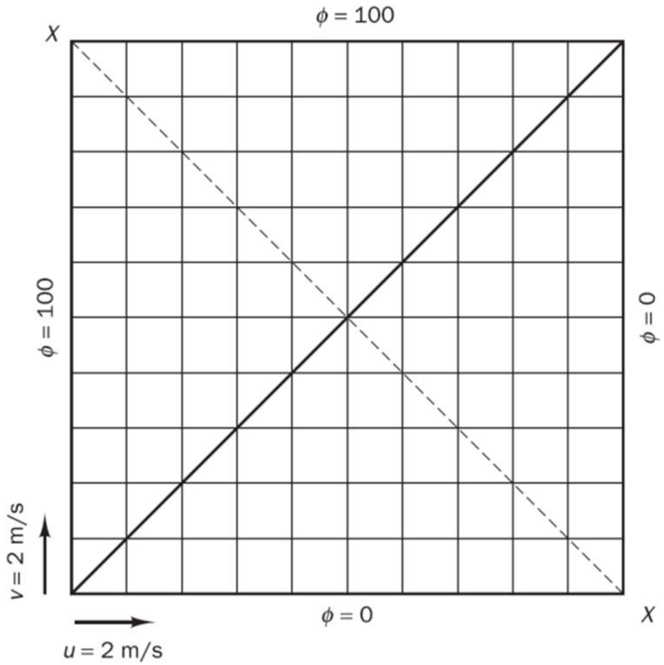
It should be noted that in some articles the function (or virtue) of the deferred correction technique seems to be exaggerated.
For example, in [21], after adopting the defer correction technique to the coefficient rearrangement of QUICK, it is claimed that ``the coefficients are always positive and now satisfy the requirements for conservativeness, boundedness and transportiveness.''

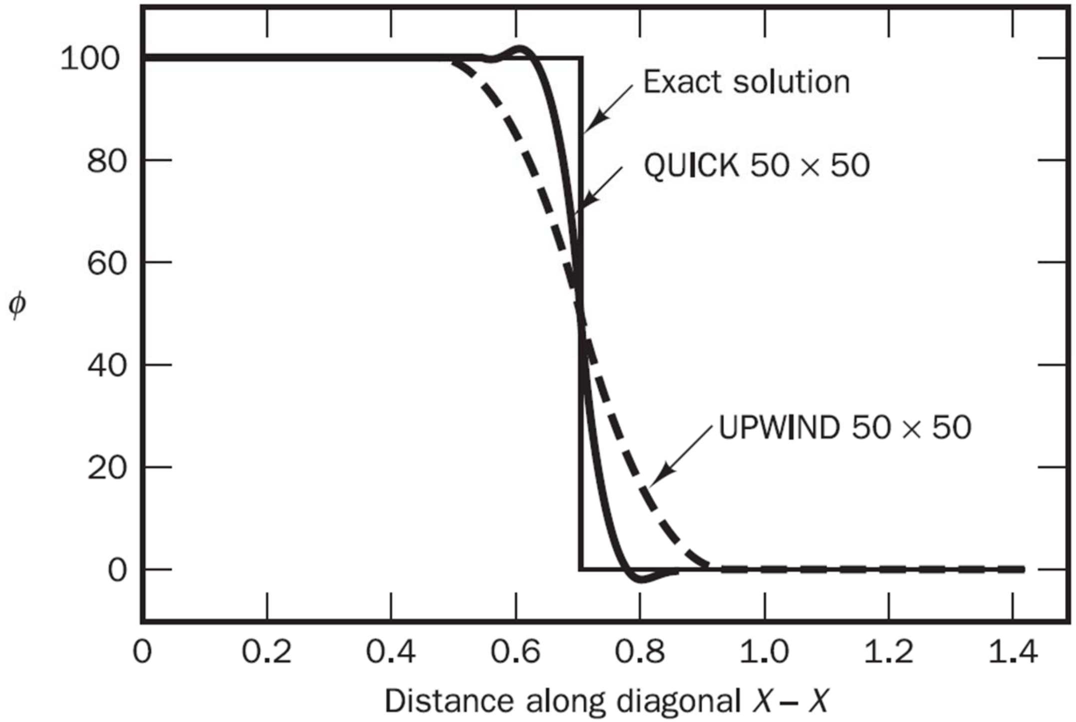
From this statement it seems that once the coefficients of the discretization equation are reorganized by the deferred correction technique, any scheme could be absolutely stable and bounded.

Our numerical practice does not support such viewpoint. As illustrated in [22], deferred correction is an efficient way to ensure the solution stability of discretization equation, but it cannot change the inherent convective stability and boundedness of a scheme.

Higher-order differencing schemes for C-D problems

- General comments on QUICK
  - Undershoot and overshoot

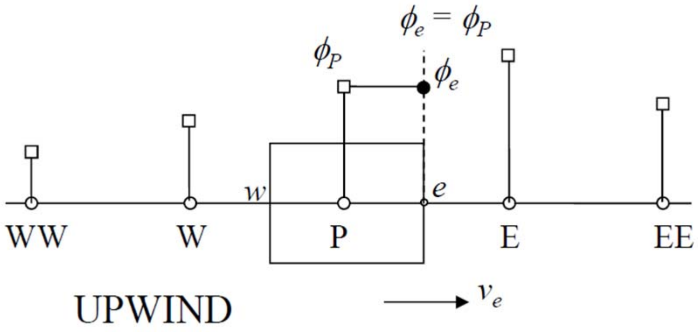




Contents

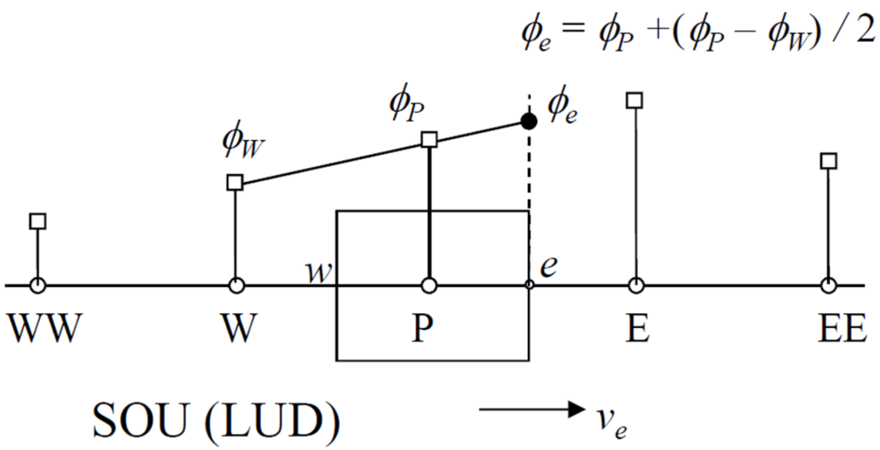
TVD schemes

- Generalization of upwind-biased schemes
  - Standard Upwind Differencing Scheme (UD)



$$\phi_e = \phi_P$$

  - Linear Upwind Differencing Scheme (LUD)

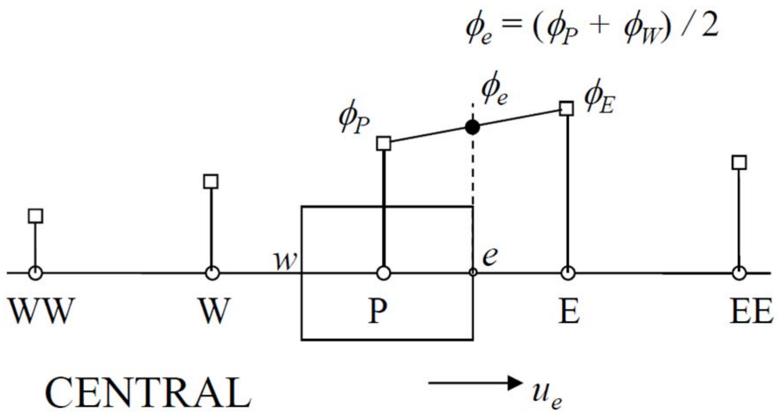


$$\phi_e = \phi_P + \frac{(\phi_P - \phi_W)}{\delta x}\frac{\delta x}{2}$$

$$= \phi_P + \frac{1}{2}(\phi_P - \phi_W)$$

TVD schemes

- Generalization of upwind-biased schemes
  - Central Differencing Scheme (CD)

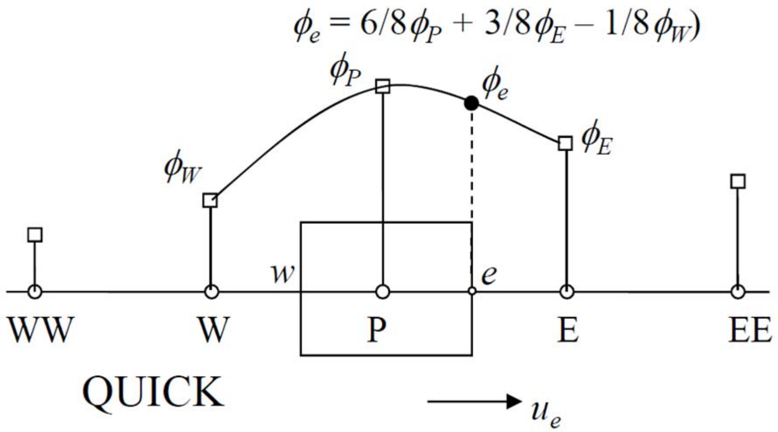


$$\phi_e = \frac{(\phi_P + \phi_E)}{2}$$

or

$$\phi_e = \phi_P + \frac{1}{2}(\phi_E - \phi_P)$$

  - QUICK Scheme



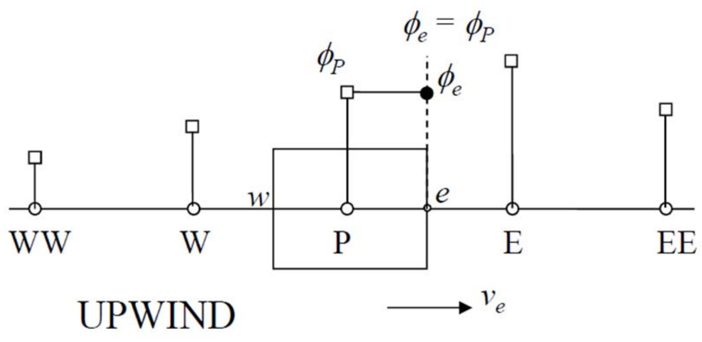
$$\phi_e = \frac{6}{8}\phi_P + \frac{3}{8}\phi_E - \frac{1}{8}\phi_W$$

$$\phi_e = \phi_P + \frac{1}{8}(3\phi_E - 2\phi_P - \phi_W)$$

TVD schemes

- Generalization of upwind-biased schemes
  - General form

$$\phi_e = \phi_P + \frac{1}{2}\psi(\phi_E - \phi_P)$$



$$\psi = 0 \text{ for UD scheme} \qquad \phi_e = \phi_P$$

$$\psi = 1 \text{ for CD scheme} \qquad \phi_P + \frac{1}{2}(\phi_P - \phi_W)$$

$$\psi = \left(\frac{\phi_P - \phi_W}{\phi_E - \phi_P}\right) \text{ for LUD scheme} \qquad \phi_P + \frac{1}{2}(\phi_E - \phi_P)$$

$$\psi = \left(3 + \frac{\phi_P - \phi_W}{\phi_E - \phi_P}\right)\frac{1}{4} \text{ for QUICK scheme} \qquad \phi_P + \frac{1}{8}(3\phi_E - 2\phi_P - \phi_W)$$

TVD schemes

- Generalization of upwind-biased schemes
  - General form

$$\phi_e = \phi_P + \frac{1}{2}\psi(\phi_E - \phi_P) \quad \Rightarrow \quad \phi_e = \phi_P + \frac{1}{2}\psi(r)(\phi_E - \phi_P)$$

$$r = \frac{\phi_P - \phi_W}{\phi_E - \phi_P}$$

$\psi = 0$ for UD scheme $\quad\Rightarrow\quad \psi(r) = 0$

$\psi = 1$ for CD scheme $\quad\Rightarrow\quad \psi(r) = 1$

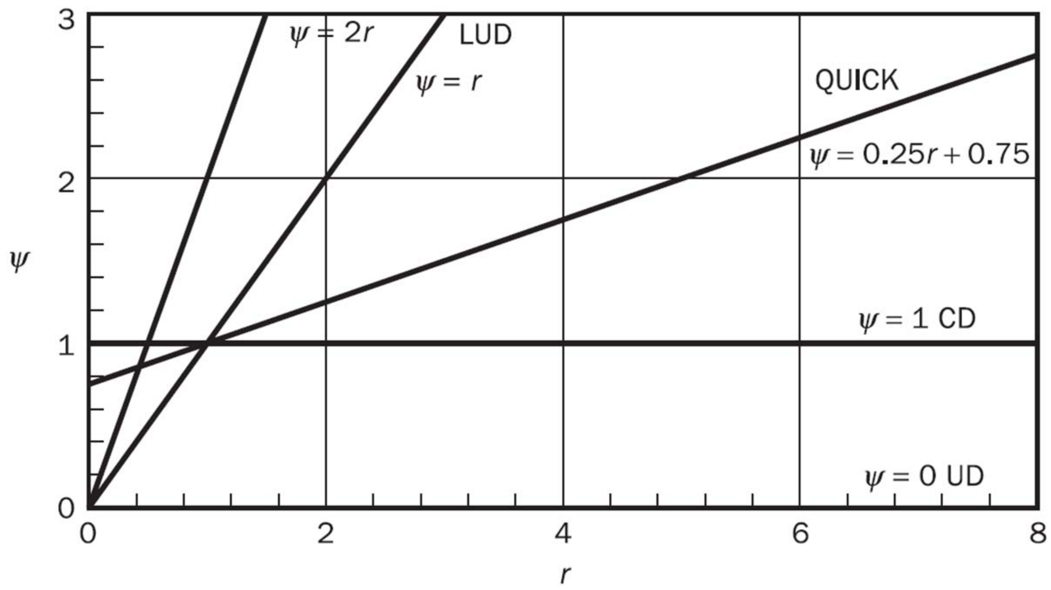
$\psi = \left(\dfrac{\phi_P - \phi_W}{\phi_E - \phi_P}\right)$ for LUD scheme $\quad\Rightarrow\quad \psi(r) = r$

$\psi = \left(3 + \dfrac{\phi_P - \phi_W}{\phi_E - \phi_P}\right)\dfrac{1}{4}$ for QUICK scheme $\quad\Rightarrow\quad \psi(r) = (3 + r)/4$

TVD schemes

- Generalization of upwind-biased schemes
  - General form

$$\phi_e = \phi_P + \frac{1}{2}\psi(r)(\phi_E - \phi_P)$$



$$\psi(r) = 0$$

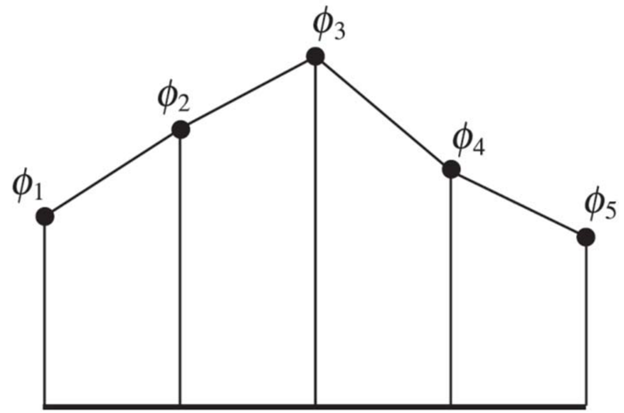
$$\psi(r) = 1$$

$$\psi(r) = r$$

$$\psi(r) = (3 + r)/4$$

- Total variation and TVD schemes

$$TV(\phi) = |\phi_2 - \phi_1| + |\phi_3 - \phi_2| + |\phi_4 - \phi_3| + |\phi_5 - \phi_4|$$
$$= |\phi_3 - \phi_1| + |\phi_5 - \phi_3|$$



- Monotonicity preserving
  - It must not create local extrema.
  - The value of an existing local minimum must be non-decreasing and that of a local maximum must be non-increasing.
- For monotonicity, this *TV* must not increase with time.
- Monotonicity preserving schemes do not create new undershoots and overshoots.

We measure the amount of oscillation in a solution by defining the total variation of a function:

The more the variation between cells, the larger the *total variation* ➔ the more oscillatory the function.

TVD schemes

- Total variation and TVD schemes

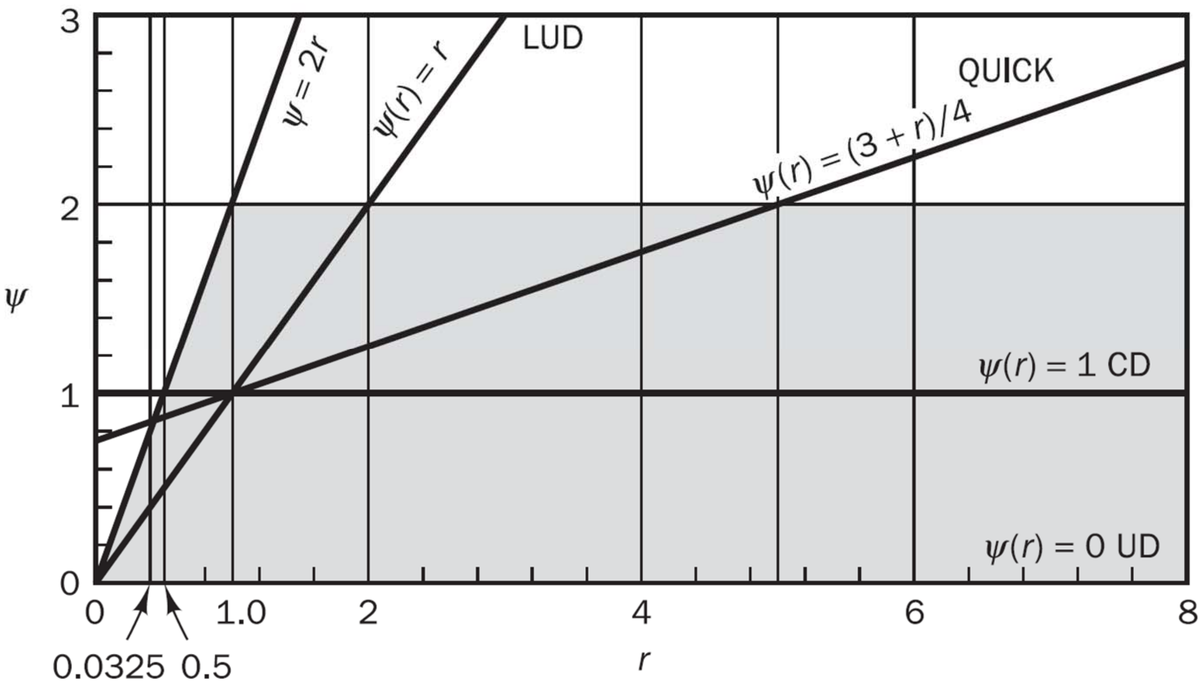
$$TV(\phi) = |\phi_2 - \phi_1| + |\phi_3 - \phi_2| + |\phi_4 - \phi_3| + |\phi_5 - \phi_4|$$
$$= |\phi_3 - \phi_1| + |\phi_5 - \phi_3|$$

- In other words *TV* must diminish with time.
- Hence, the term total variation diminishing or TVD.

- Originally TVD was developed for time-dependent flows.

$$TV(\phi^{n+1}) \leq TV(\phi^n)$$

❖ Criteria for TVD Schemes



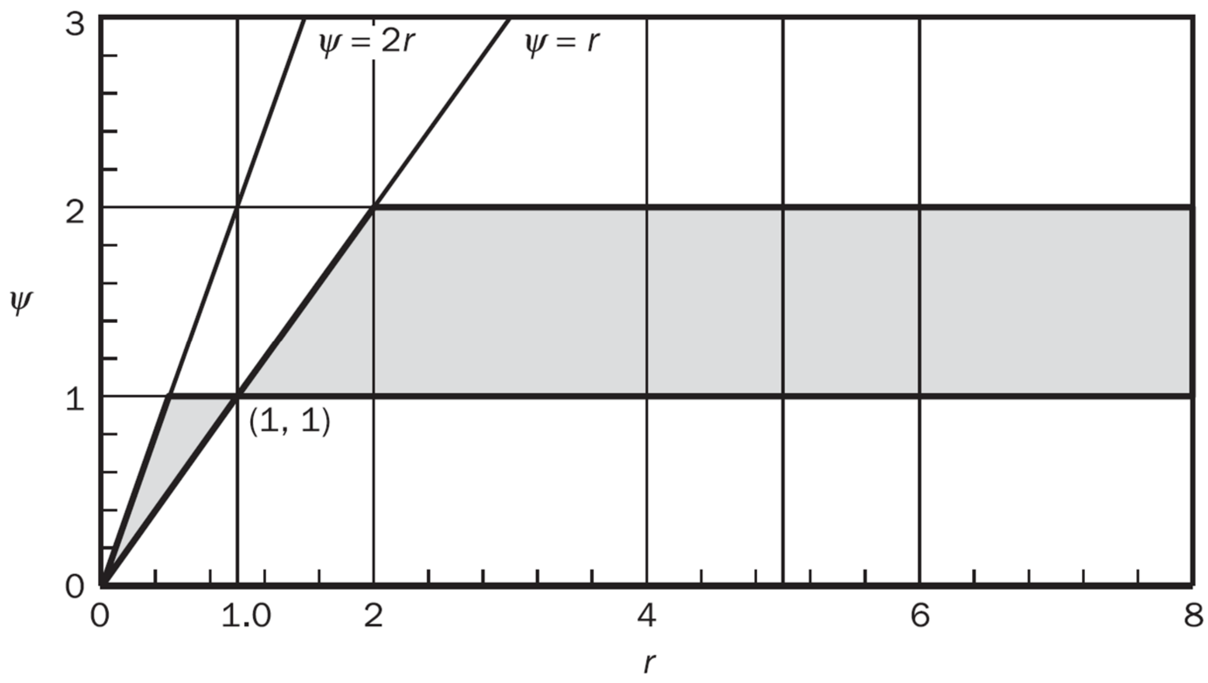
- Sweby (1984) has given necessary and sufficient ***conditions for a scheme to be TVD*** in terms of the *r* − ψ relationship:

$$\text{For } 0 < r < 1 \rightarrow \psi(r) \leq 2r$$
$$\text{For } r \geq 1 \rightarrow \psi(r) \leq 2$$

$$r = \frac{\phi_P - \phi_W}{\phi_E - \phi_P}$$

TVD schemes

❖ Criteria for 2nd order accuracy



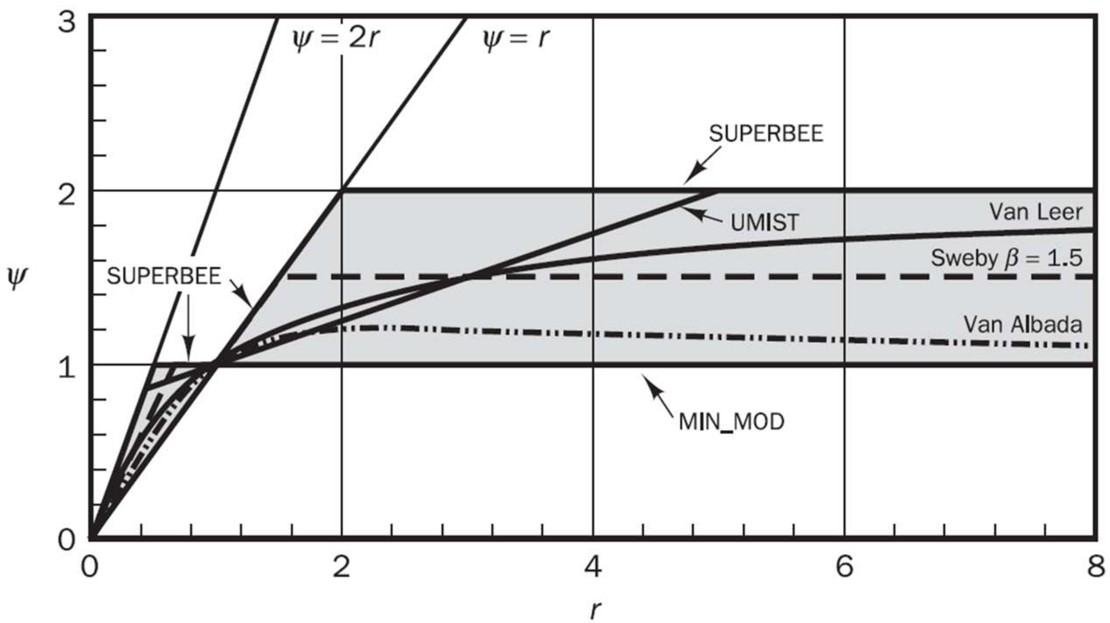
- Sweby also showed that the ***range of possible second-order schemes*** is bounded by the central difference and linear upwind schemes:

$$\text{For } 0 < r < 1 \text{ for TVD to be second order } \; r \le \psi(r) \le 1$$
$$\text{For } \; r \ge 1 \text{ for TVD to be second order } \; 1 \le \psi(r) \le r$$

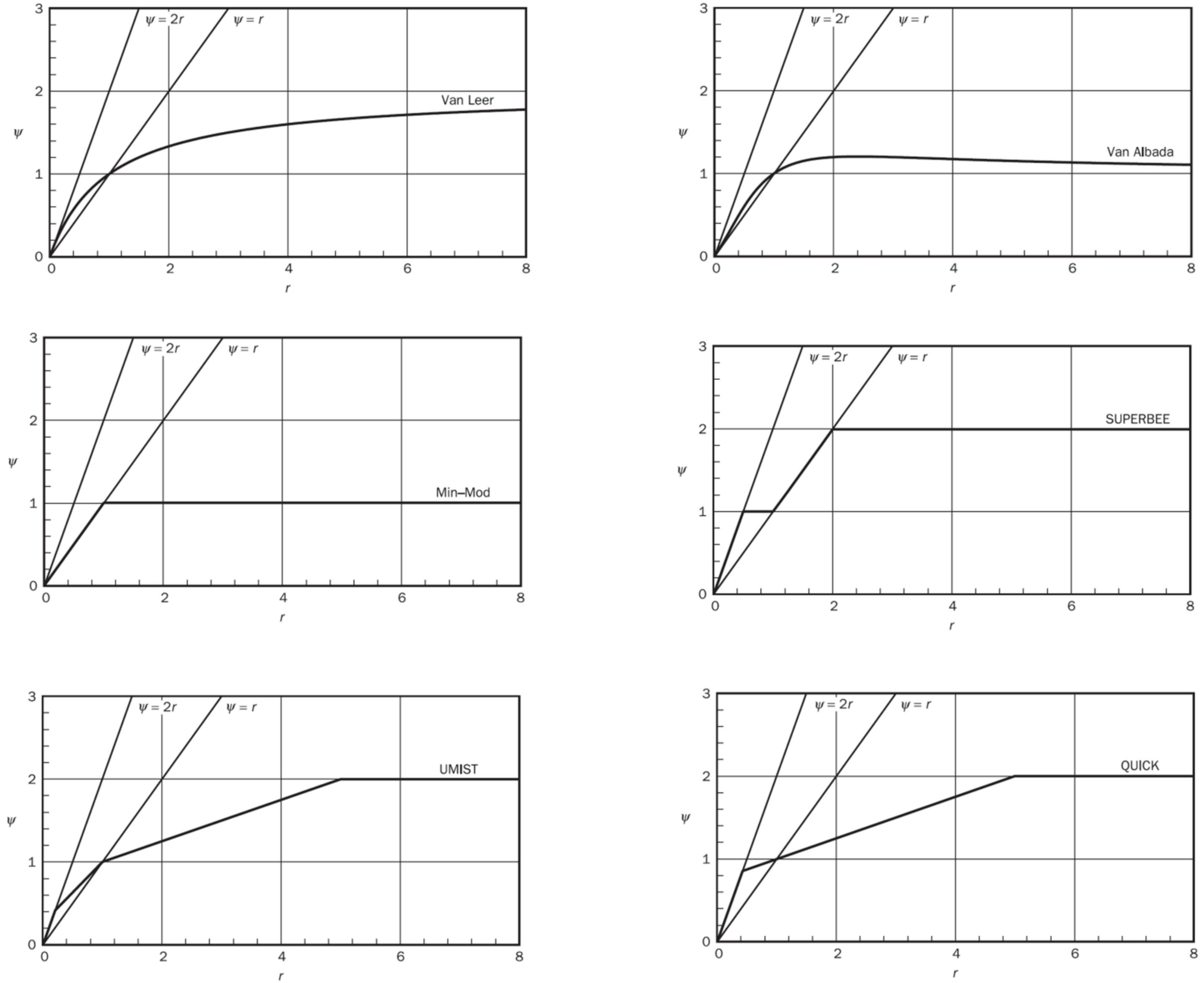
TVD schemes

- Limiter functions

Name	*Limiter function ψ(r)*	*Source*
Van Leer	$\frac{r+\|r\|}{1+r}$	Van Leer (1974)
Van Albada	$\frac{r+r^2}{1+r^2}$	Van Albada *et al.* (1982)
Min-Mod	$\psi(r)=\begin{cases}\min(r,1) & \text{if } r>0\\ 0 & \text{if } r\le 0\end{cases}$	Roe (1985)
SUPERBEE	$\max[0, \min(2r, 1), \min(r, 2)]$	Roe (1985)
Sweby	$\max[0, \min(\beta r, 1), \min(r, \beta)]$	Sweby (1984)
QUICK	$\max[0, \min(2r, (3+r)/4, 2)]$	Leonard (1988)
UMIST	$\max[0, \min(2r, (1+3r)/4, (3+r)/4, 2)]$	Lien and Leschziner (1993)



- Limiter functions



- Implementation of TVD scheme
  - With deferred correction

$$\phi_e = \phi_P + \frac{1}{2}\psi(r)(\phi_E - \phi_P)$$

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + S_u^{DC}$$

where $a_W = D_w + F_w$

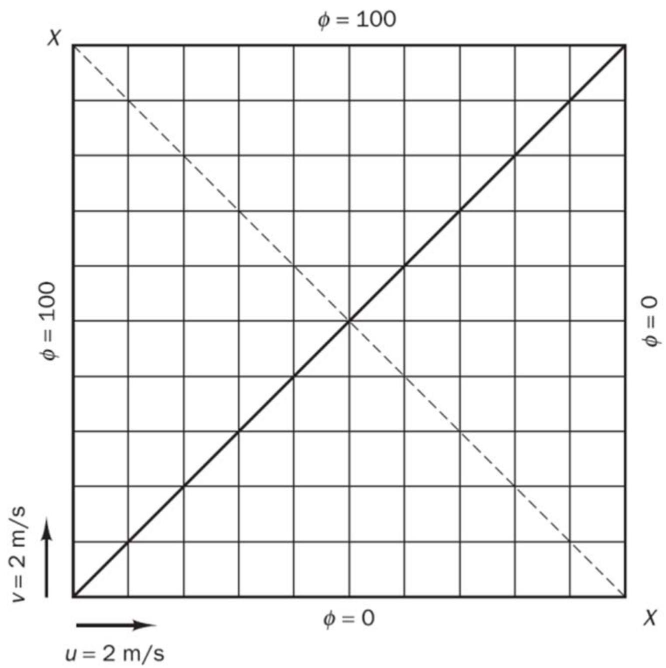
$$a_E = D_e$$

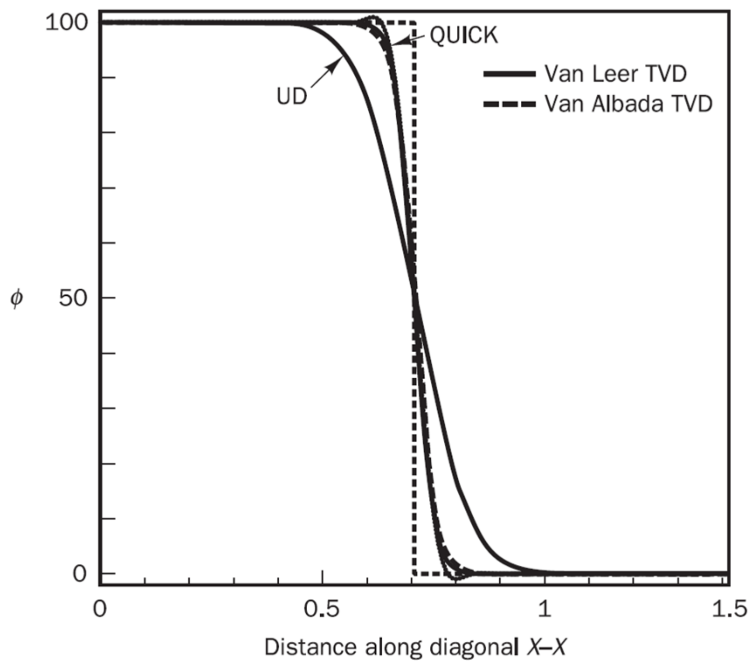
$$a_P = a_W + a_E + (F_e - F_w)$$

$$S_u^{DC} = -F_e\left[\frac{1}{2}\psi(r_e)(\phi_E - \phi_P)\right] + F_w\left[\frac{1}{2}\psi(r_w)(\phi_P - \phi_W)\right]$$

TVD schemes

- Evaluation of TVD scheme
  - Second order accurate without unphysical oscillation
  - TVD schemes require about 15% more CPU time.





Summary

- Introduction
- Steady one-dimensional convection and diffusion
- The central differencing scheme
- Properties of discretization schemes
- Assessment of the central differencing scheme for convection–diffusion problems
- The upwind differencing scheme
- The hybrid differencing scheme
- The power-law scheme
- Higher-order differencing schemes for convection–diffusion problems
- TVD schemes
- Summary

Homework #2-3

- Temperature profile in fully developed flow
  - Consider a fluid at a uniform temperature *Ti* entering a channel whose surface is maintained at a different temperature *Ts*. A *Thermal boundary layer* along the tube develops, after which the form of the temperature profile does not change. Assume that the flow profile is constant in the channel where the velocities are given by

$$\frac{u}{u_{\max}} = 1 - \left(\frac{2y}{H} - 1\right)^2 \quad \text{and} \quad v = 0$$

$$u_{\max} = 1.5 u_{\text{mean}}$$

$$\frac{\partial(\rho u T)}{\partial x} + \frac{\partial(\rho v T)}{\partial y} = \frac{\partial}{\partial x}\left(\frac{k}{c_p}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{k}{c_p}\frac{\partial T}{\partial y}\right)$$

$$\text{Re} = \rho u_{mean} H/\mu = 10 \qquad \text{Pr} = \mu c_p/k = 1 \qquad L_x/H = 5 \qquad k/c_p = \mu/\text{Pr}$$

$$T_{in} = 0,\ T_{walls} = 100,\ u_{mean} = 1\ \text{m/s} \qquad \rho = 1$$

  - Use upwind method