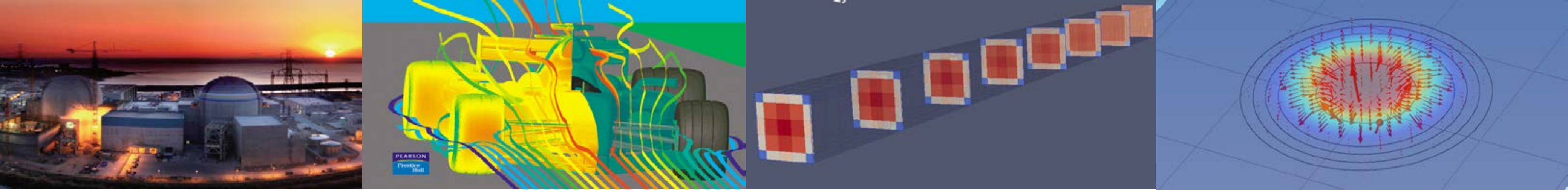


COMPUTATIONAL NUCLEAR THERMAL HYDRAULICS

Cho, Hyoung Kyu

Department of Nuclear Engineering
Seoul National University

NuTHEL
Nuclear Thermal Hydraulic Engineering Lab.

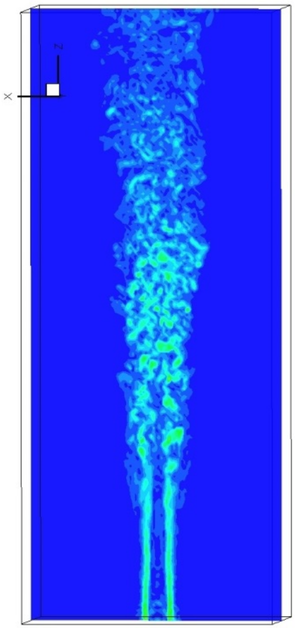
CHAPTER3.
TURBULENCE AND ITS MODELLING

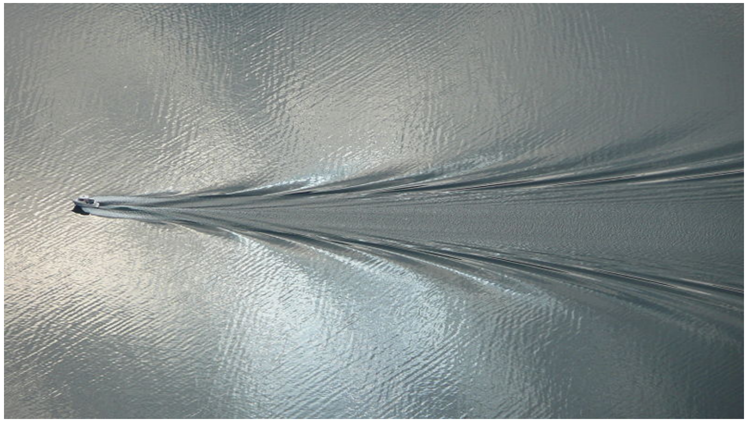
Contents

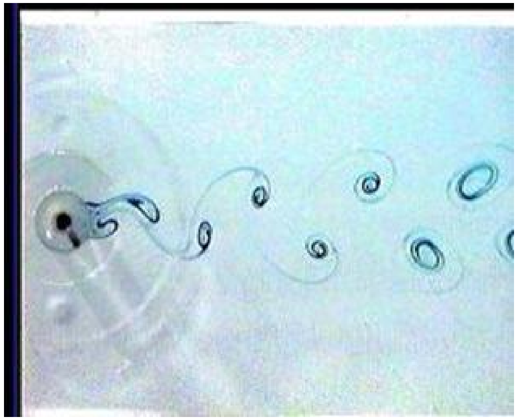
Chap. 3.1: What is turbulence?

Turbulence or **turbulent flow** is a flow regime characterized by chaotic and suspectedly stochastic property changes.

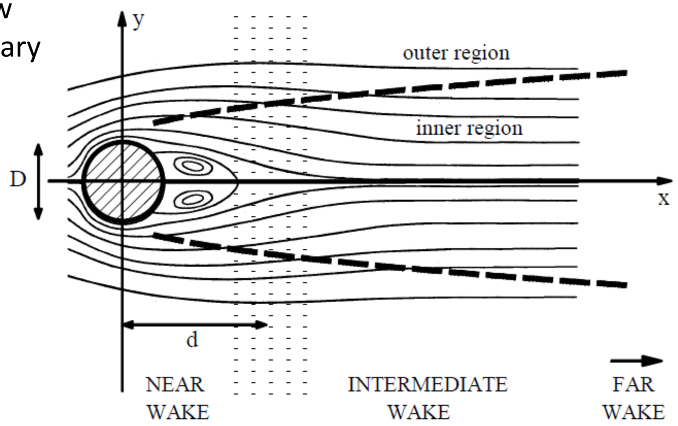
- All flows become unstable above a certain Reynolds number.
  - Simple ones: two-dimensional jets, wakes, pipe flows, flat plate boundary layers
  - More complicated three-dimensional flow







A **wake** is the region of recirculating flow immediately behind a moving or stationary solid body, caused by the flow of surrounding fluid around the body.



A **jet** is an efflux of fluid that is projected into a surrounding medium. Jet fluid has higher momentum compared to the surrounding fluid medium. In the case where the surrounding medium is assumed to be made up of the same fluid as the jet, and this fluid has a viscosity, then the surrounding fluid near the jet is assumed to be carried along with the jet by a process called Entrainment (hydrodynamics)

3.1: What is turbulence?

- Is the flow turbulence?
  - External flows

$$\mathrm{Re}_x \geq 5\times10^5$$

along a surface

$$\mathrm{Re}_D \geq 20{,}000$$

around an obstacle

  - Internal flows

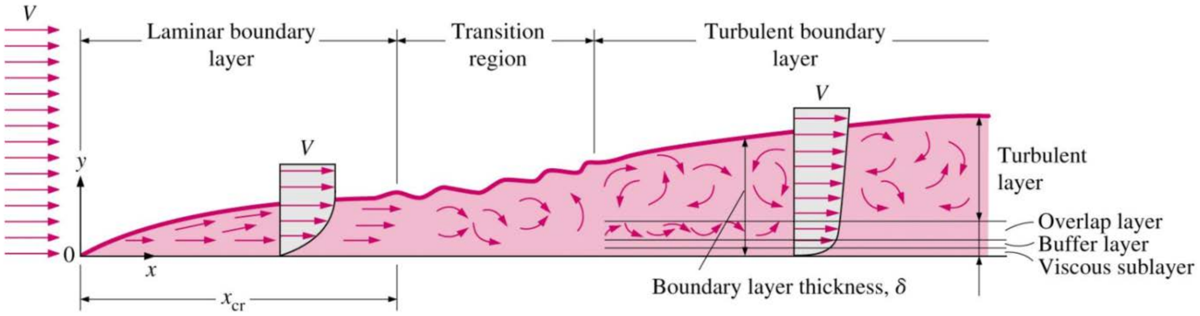
$$\mathrm{Re}_{D_h} \geq 2{,}300$$

  - Natural convection

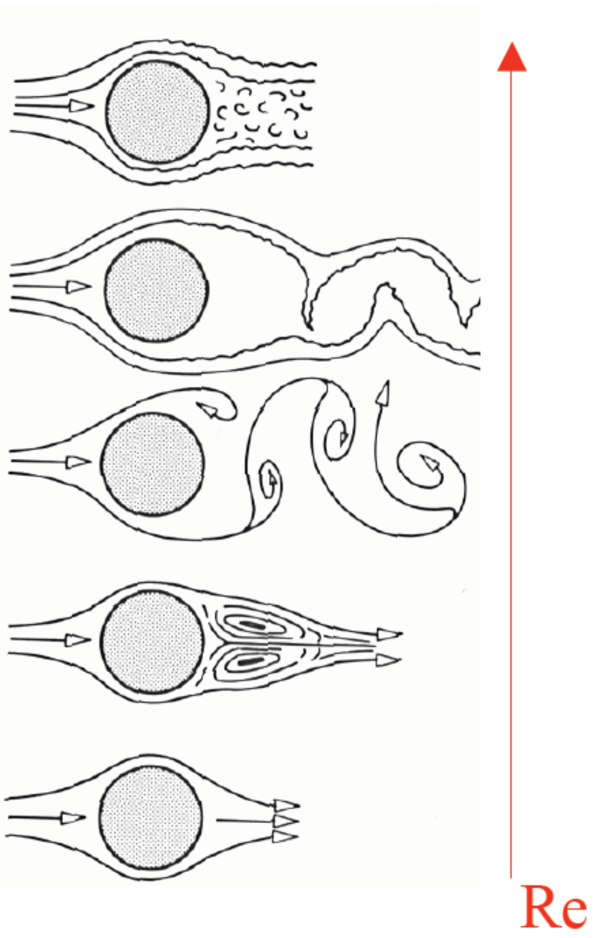
$$Ra \geq 10^9\ \mathrm{Pr} \qquad Ra = \frac{g\beta\Delta T L^3}{\alpha\nu}$$

$$\mathrm{Re}_L \equiv \frac{\rho U L}{\mu}$$

Other factors such as free-stream turbulence, surface conditions, and disturbances may cause earlier transition to turbulent flow.



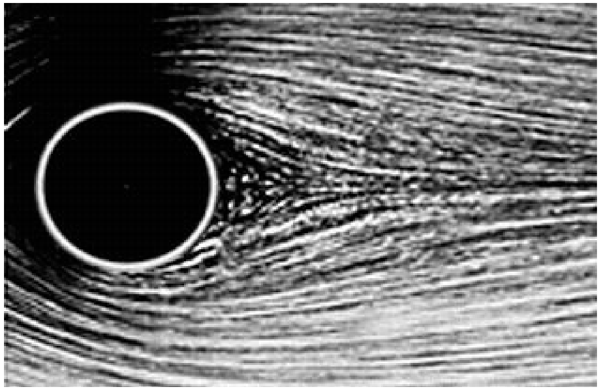
3.1: What is turbulence?



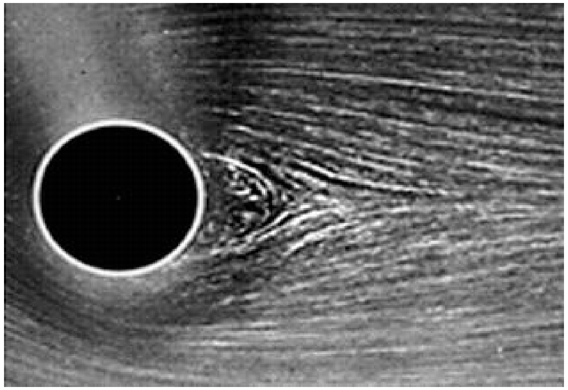
350K < Re	Turbulent Separation Chaotic
200 < Re < 350K	Laminar Separation/Turbulent Wake Periodic
40 < Re < 200	Laminar Separated Periodic
5 < Re < 40	Laminar Separated Steady
Re < 5	Laminar Attached Steady

3.1: What is turbulence?

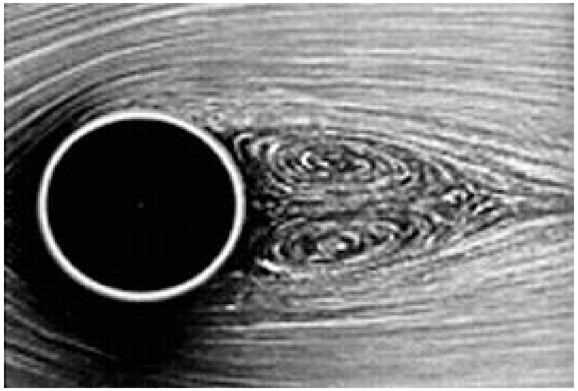
- For flow around a cylinder, the flow starts separating at Re = 5. For Re below 30, the flow is stable. Oscillations appear for higher Re.
- The separation point moves upstream, increasing drag up to Re = 2000.



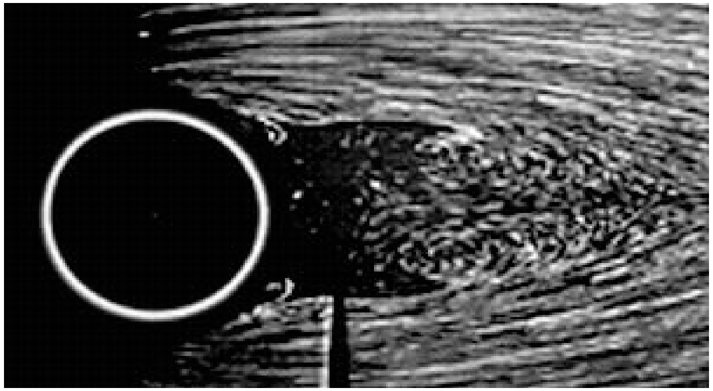
Re = 9.6



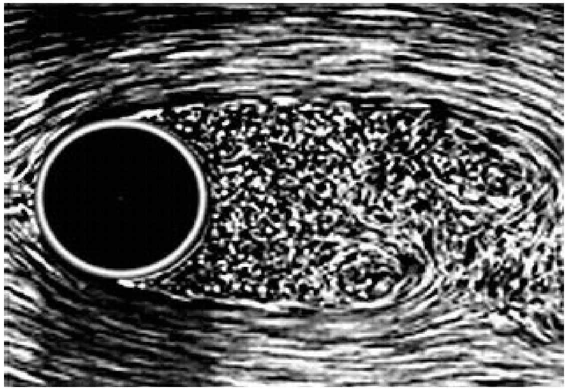
Re = 13.1



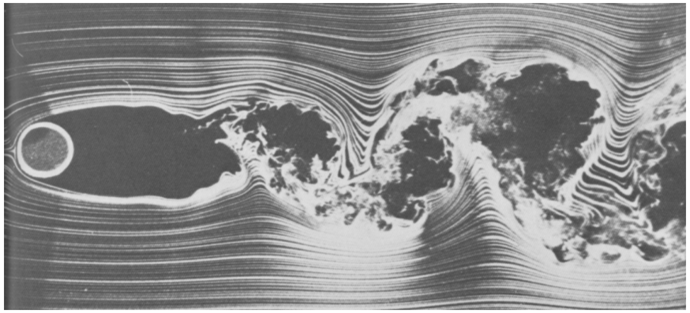
Re = 26



Re = 30.2

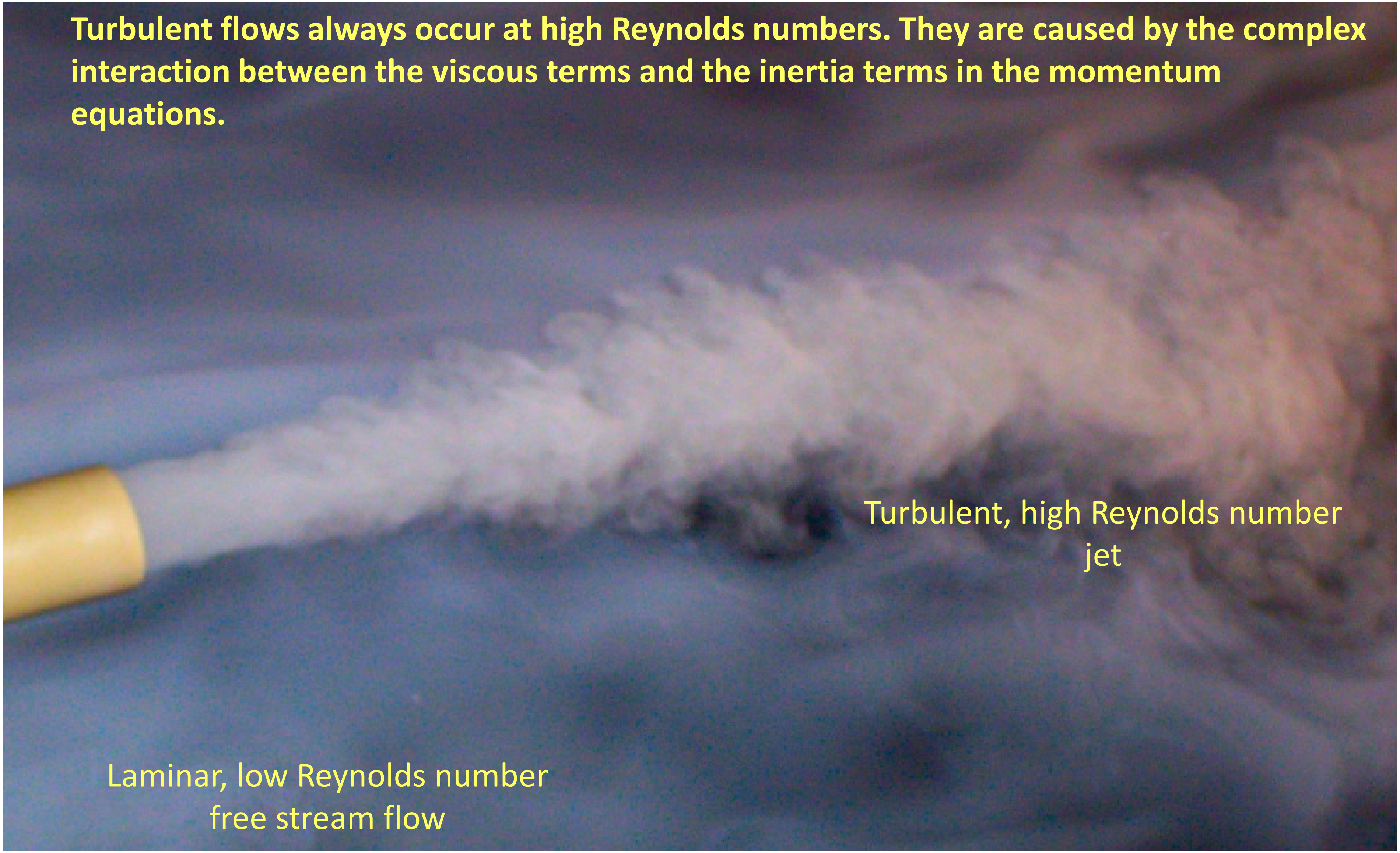


Re = 2000



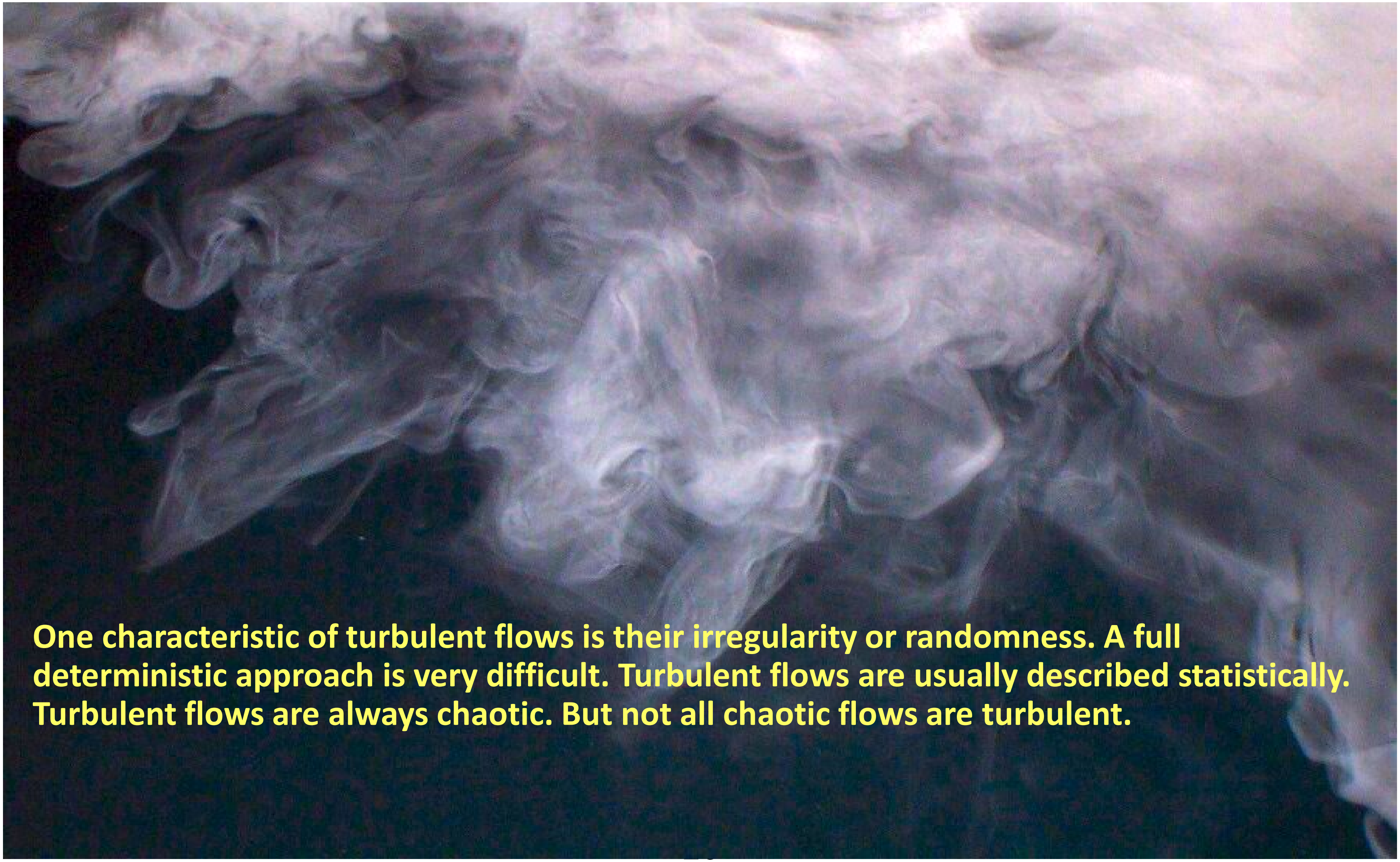
Re = 10,000

Turbulence: high Reynolds numbers



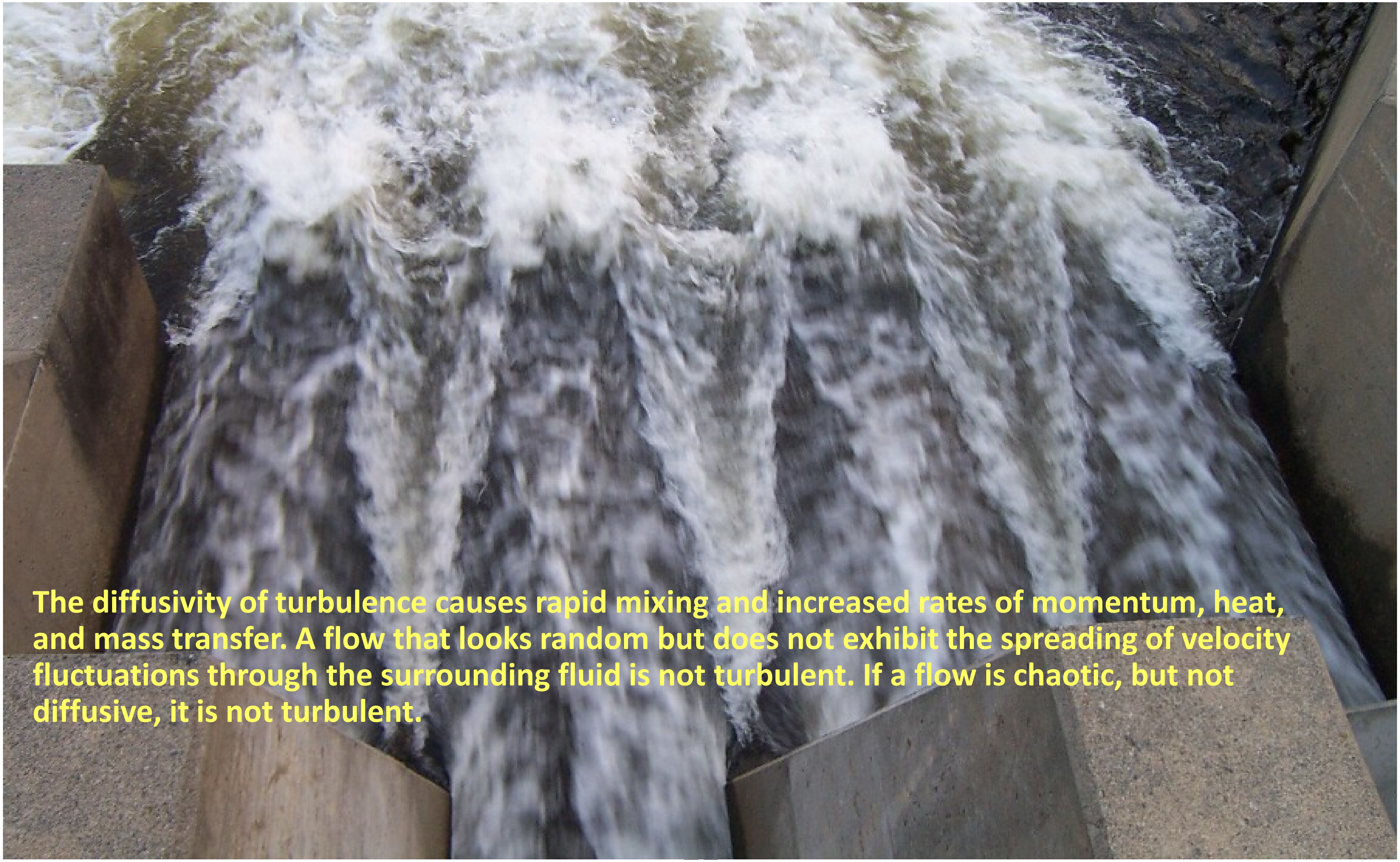
Turbulent flows always occur at high Reynolds numbers. They are caused by the complex interaction between the viscous terms and the inertia terms in the momentum equations.

Turbulent flows are chaotic



One characteristic of turbulent flows is their irregularity or randomness. A full deterministic approach is very difficult. Turbulent flows are usually described statistically. Turbulent flows are always chaotic. But not all chaotic flows are turbulent.

Turbulence: diffusivity

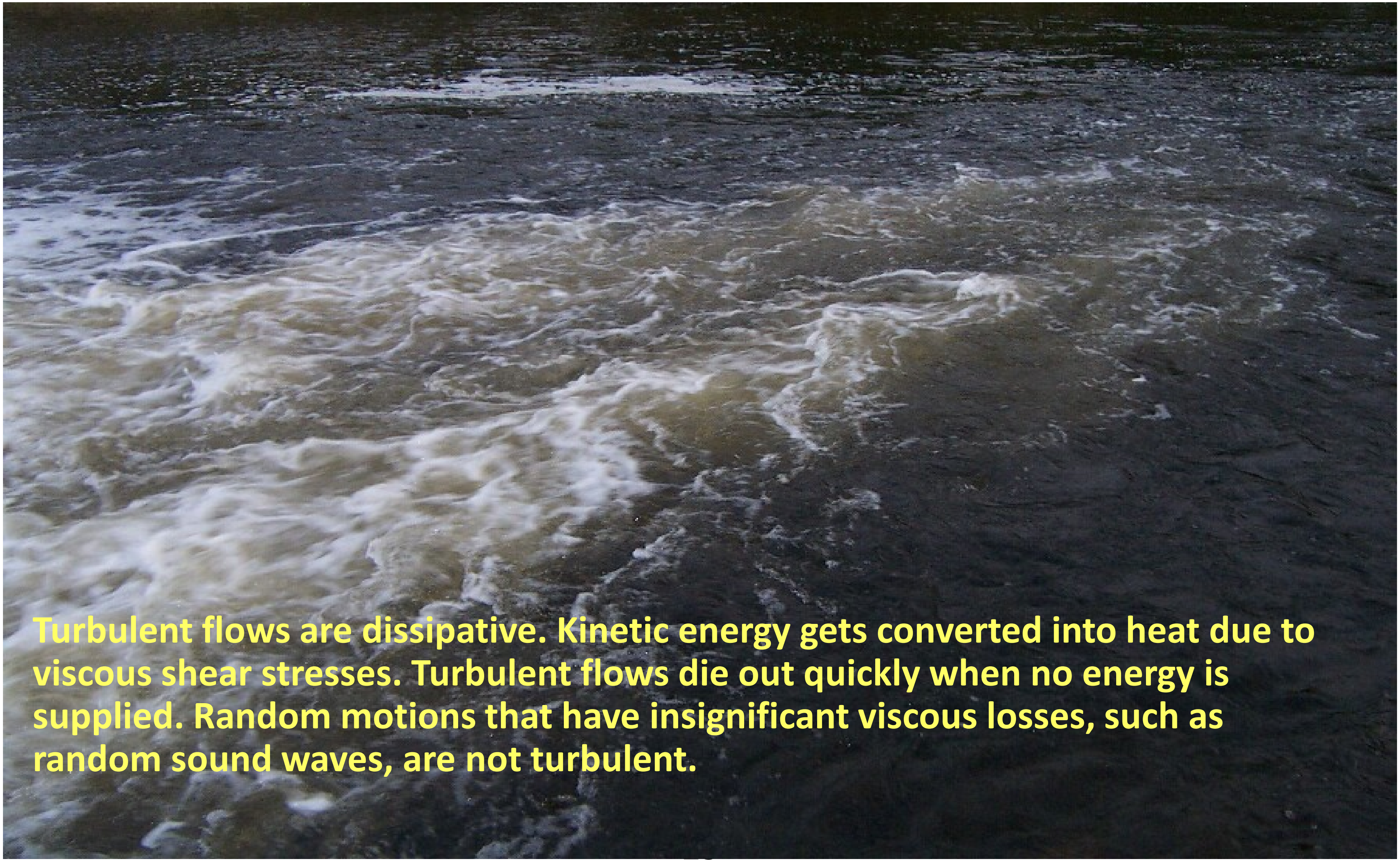


The diffusivity of turbulence causes rapid mixing and increased rates of momentum, heat, and mass transfer. A flow that looks random but does not exhibit the spreading of velocity fluctuations through the surrounding fluid is not turbulent. If a flow is chaotic, but not diffusive, it is not turbulent.

Turbulence: diffusivity

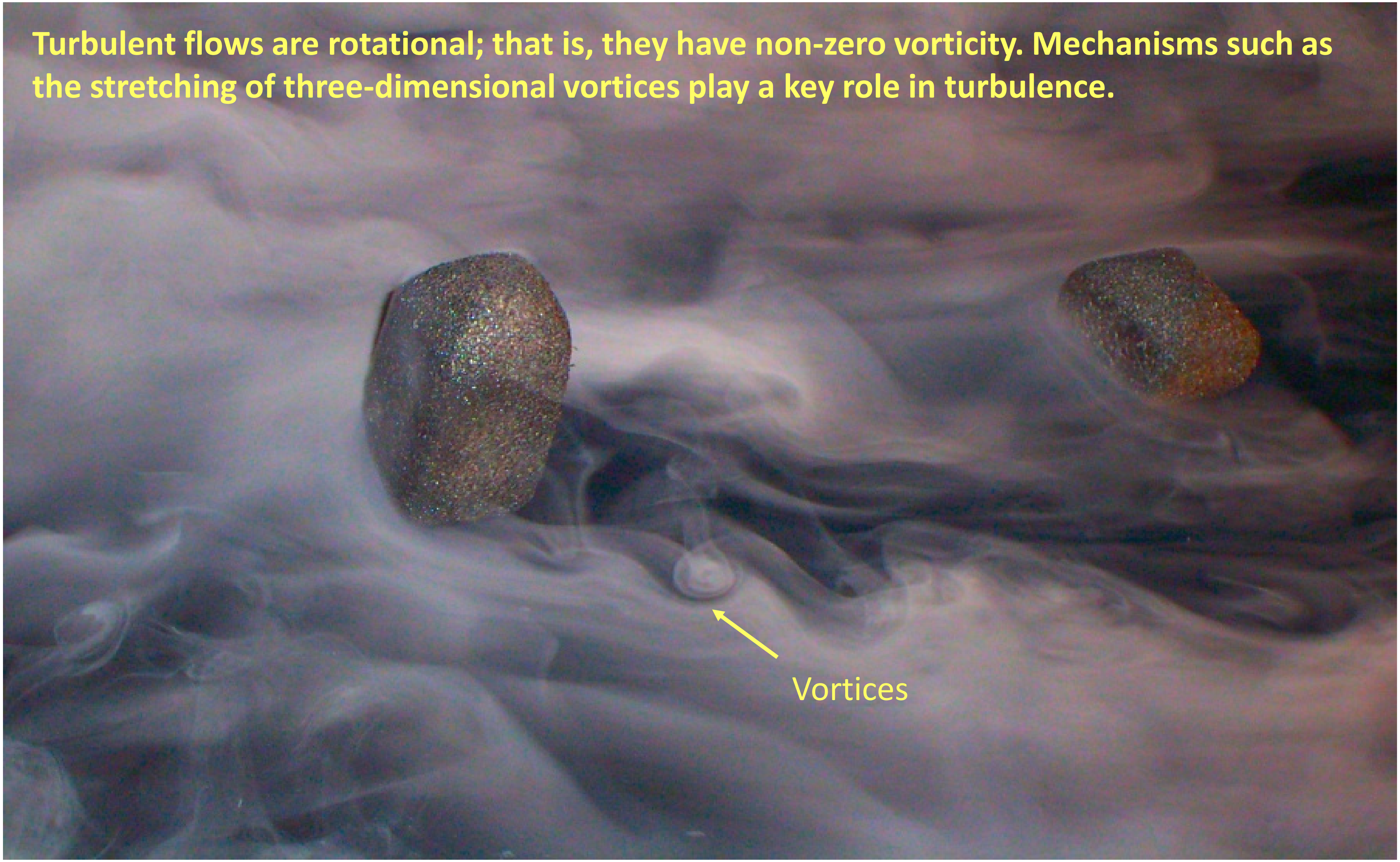
The contrails of a jet aircraft are a case in point: excluding the turbulent region just behind the aircraft, the contrails have a very nearly constant diameter for several miles. Such a flow is not turbulent, even it was turbulent when it was generated.

Turbulence: dissipation



Turbulent flows are dissipative. Kinetic energy gets converted into heat due to viscous shear stresses. Turbulent flows die out quickly when no energy is supplied. Random motions that have insignificant viscous losses, such as random sound waves, are not turbulent.

Turbulence: rotation and vorticity



Turbulent flows are rotational; that is, they have non-zero vorticity. Mechanisms such as the stretching of three-dimensional vortices play a key role in turbulence.

3.1: What is turbulence?

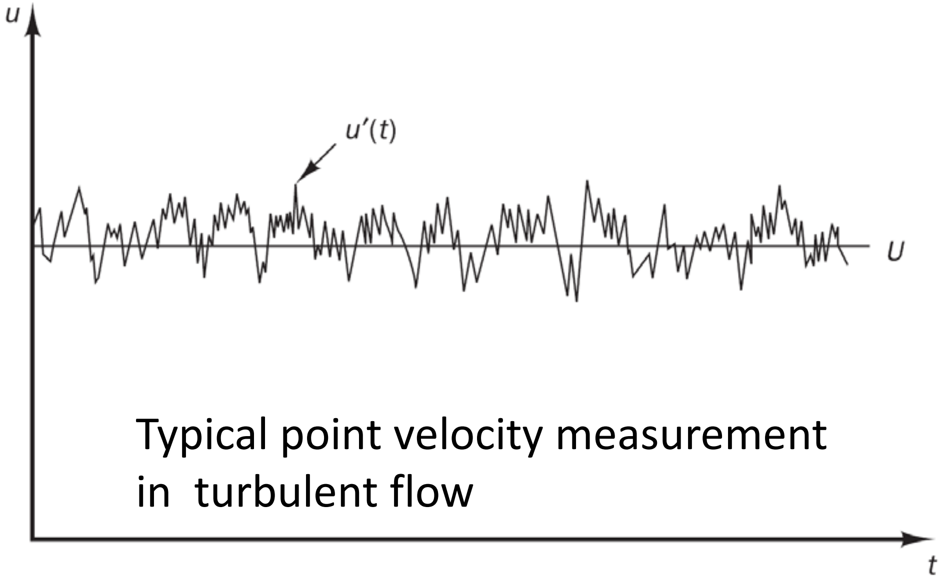
Turbulent flow

- Chaotic and random state of motion develops.
- Velocity and pressure change continuously with time.
  - Intrinsically unsteady even with constant imposed boundary conditions
- The velocity fluctuations give rise to additional stresses on the fluid
  - These are called Reynolds stresses.
  - We will try to model these extra stress terms.
- A streak of dye which is introduced at a point will rapidly break up and dispersed.
  - Effective mixing
- Give rise to high values of diffusion coefficient for mass/momentum and heat

$$u(t) = U + u'(t)$$

Reynolds decomposition

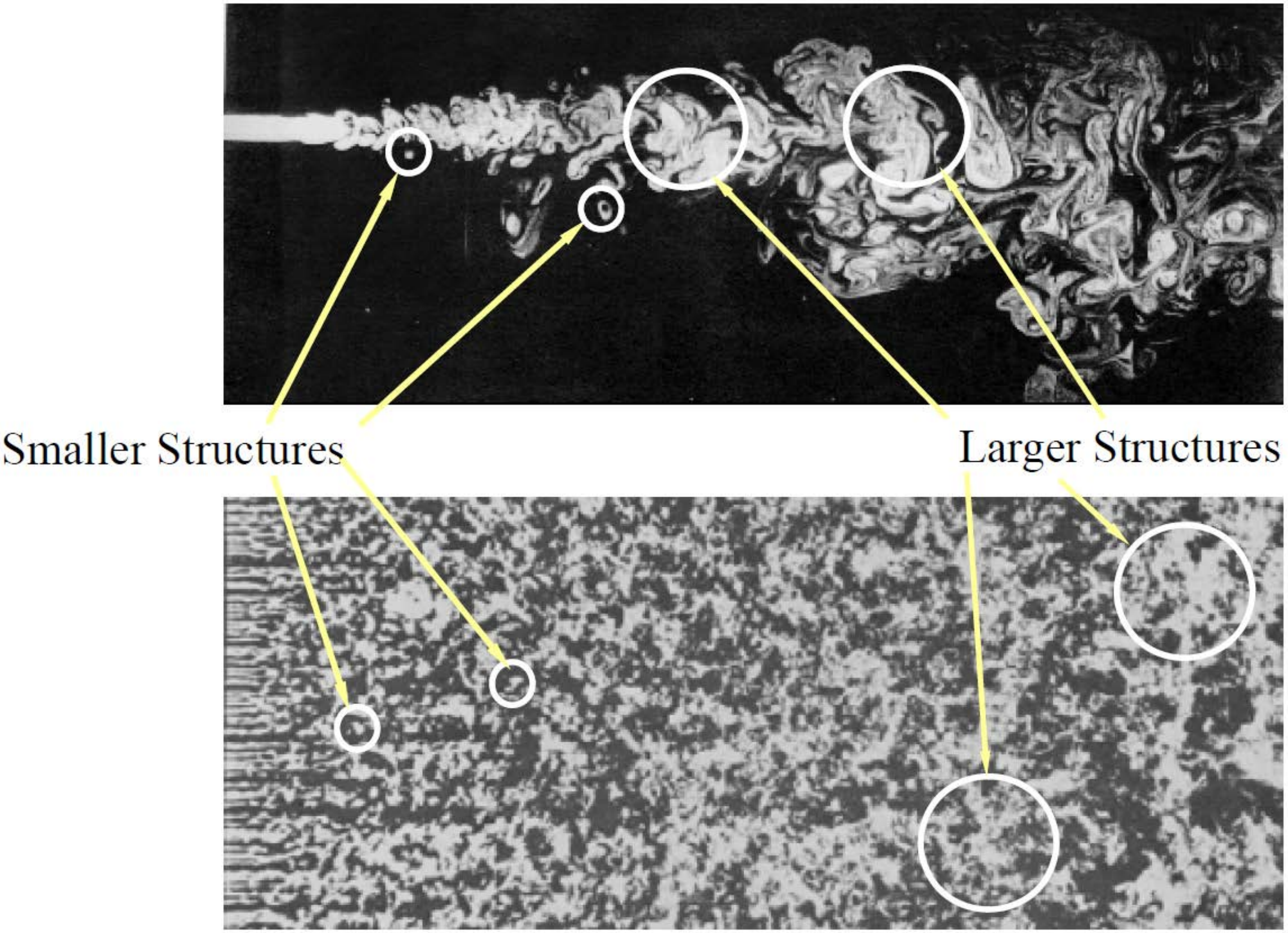
- Mean velocity, 1D, 2D, 3D
- Fluctuation, always 3D

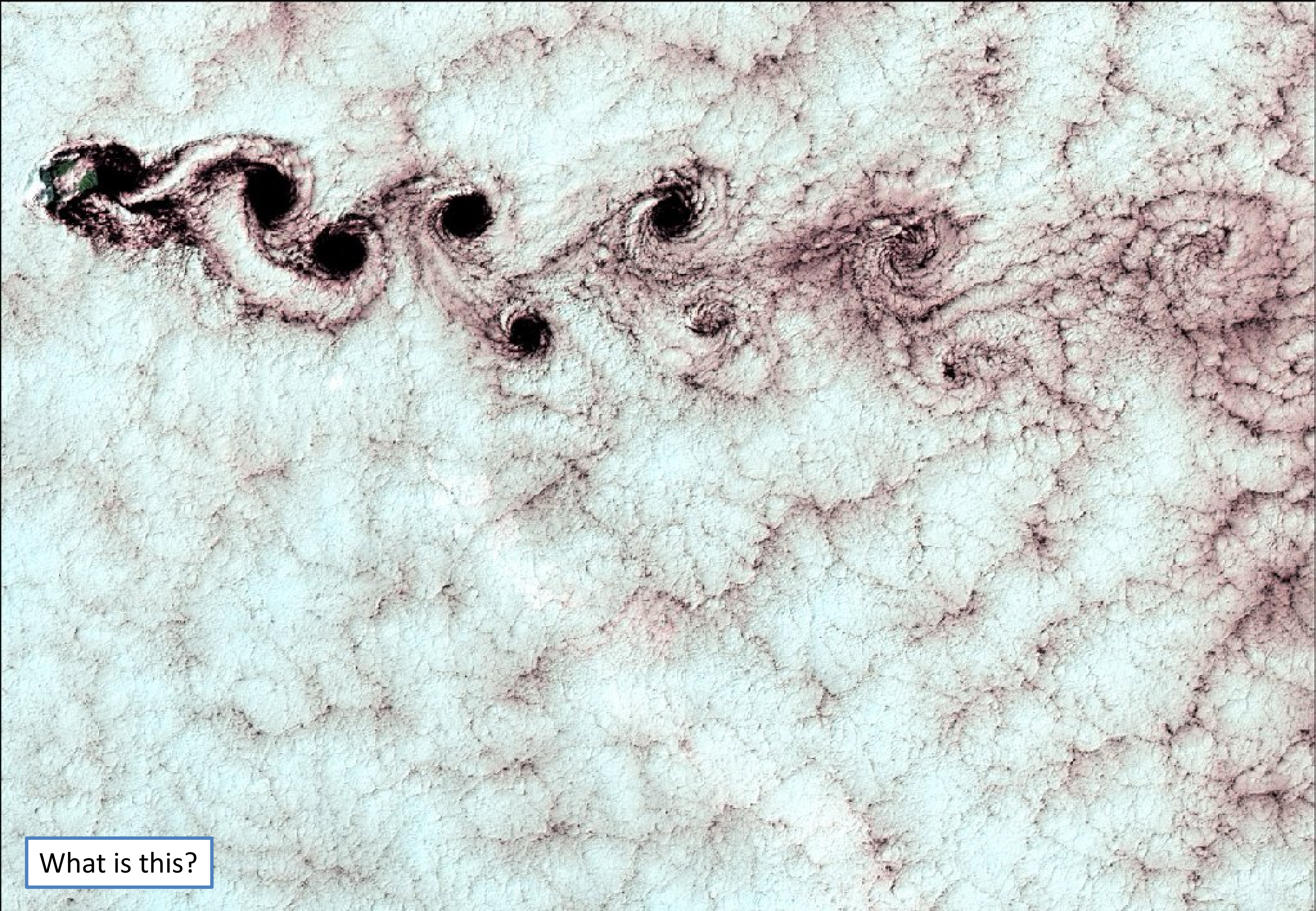


Typical point velocity measurement in turbulent flow

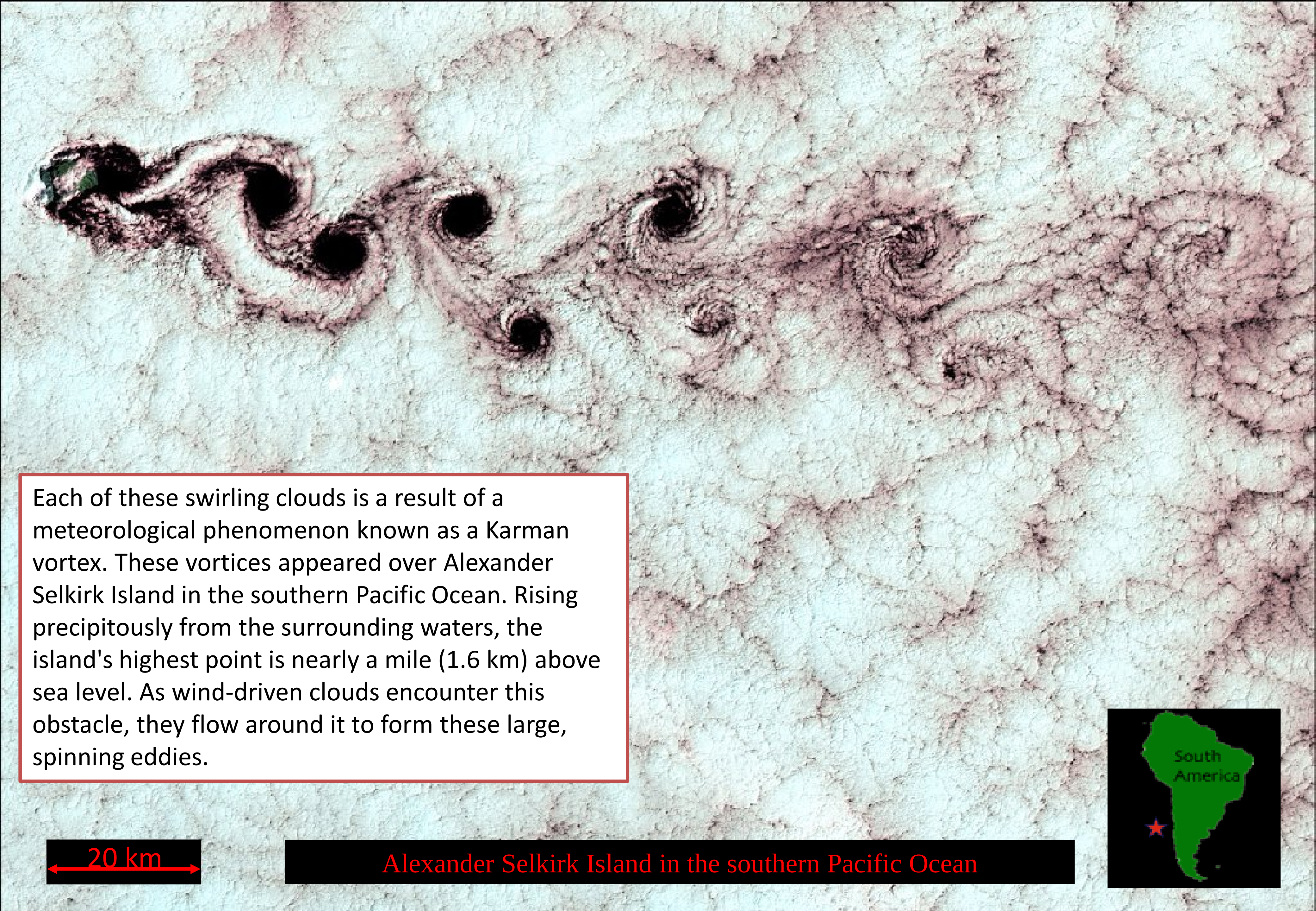
3.1: What is turbulence?

- Turbulent flow
  - In turbulent flows there are rotational flow structures called turbulent eddies, which have a wide range of length scales.



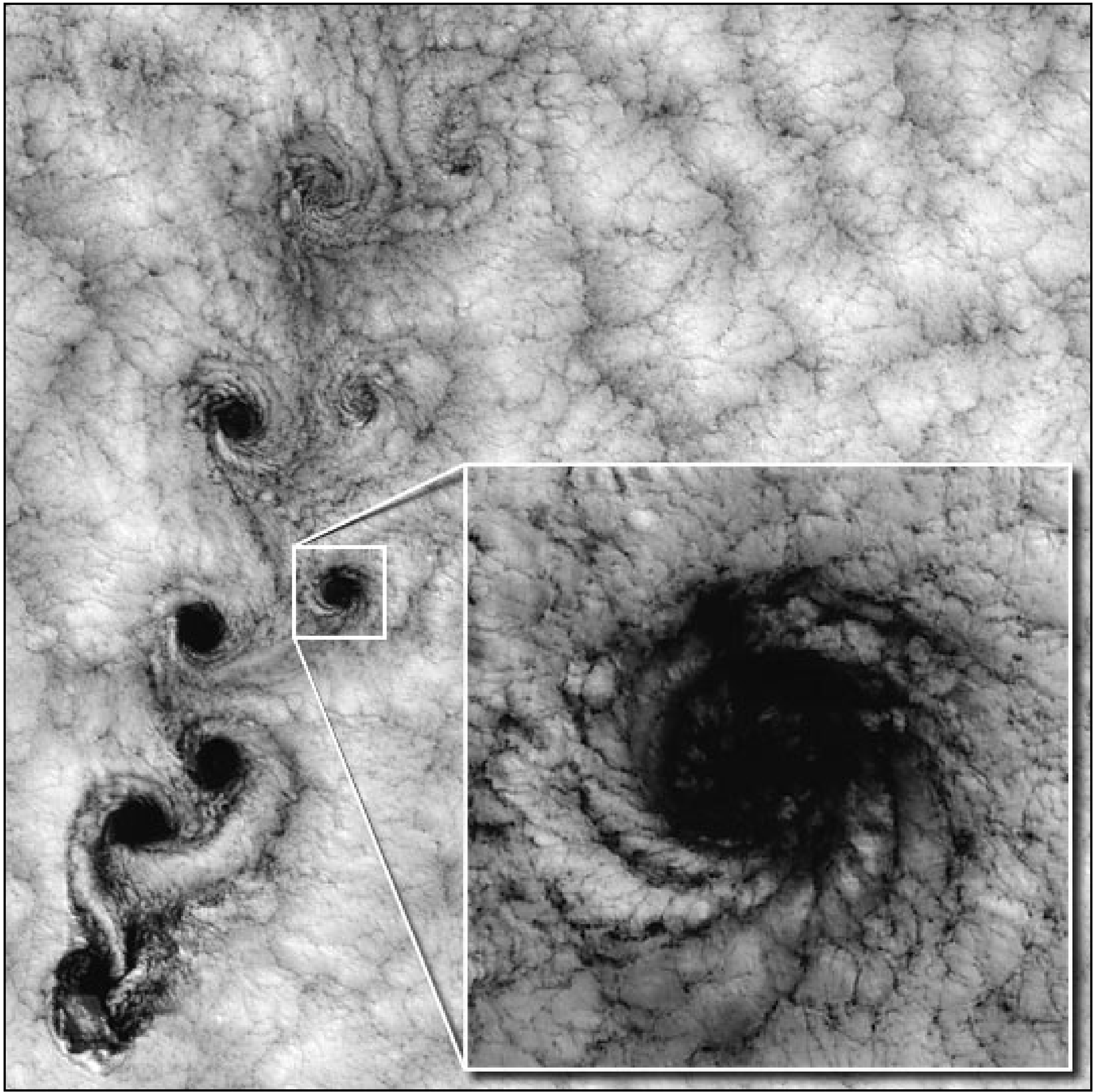


What is this?



Each of these swirling clouds is a result of a meteorological phenomenon known as a Karman vortex. These vortices appeared over Alexander Selkirk Island in the southern Pacific Ocean. Rising precipitously from the surrounding waters, the island's highest point is nearly a mile (1.6 km) above sea level. As wind-driven clouds encounter this obstacle, they flow around it to form these large, spinning eddies.

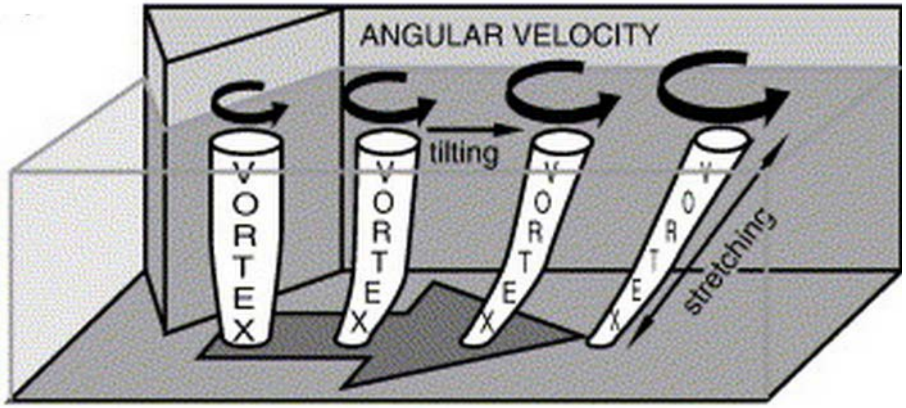
Alexander Selkirk Island in the southern Pacific Ocean



3.1: What is turbulence?

- Turbulent flow
  - Largest turbulent eddies
    - Interact with and extract energy from the mean flow
      - By a process called vortex stretching
    - Vortex stretching
      - The lengthening of vortices in three-dimensional fluid flow, associated with a corresponding increase of the component of vorticity in the stretching direction—due to the conservation of angular momentum
      - Increased rotation rate and decreased radius of their cross-sections
      - Vortex stretching is at the core of the description of the turbulence energy cascade from the large scales to the small scales in turbulence.
      - Creates smaller transverse length scales and smaller time scales
      - Smaller eddies are stretched strongly by somewhat larger eddies
      - In this way the kinetic energy is handed down from large eddies to progressively smaller and smaller eddies in what is termed the energy cascade.

Energy cascade



3.1: What is turbulence?

Turbulent flow

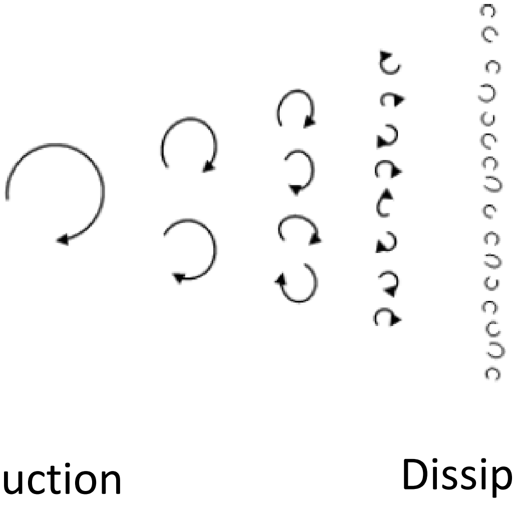
- Smallest turbulent eddies
  - Viscous dissipation in the smallest eddies converts kinetic energy into thermal energy.
  - Kolmogorov microscales

- Large eddies
  - have large eddy Reynolds number,
  - are dominated by inertia effects
  - viscous effects are negligible
  - are effectively inviscid
  - Highly anisotropic, i.e. the fluctuations are different in different directions

$$\mathrm{Re}_l = \frac{\vartheta l}{\nu}$$

- Small Eddies
  - motion is dictated by viscosity
  - Re ≈ 1
  - length scales : 0.1 – 0.01 mm
  - frequencies : ≈ 10 kHz
  - Isotropic

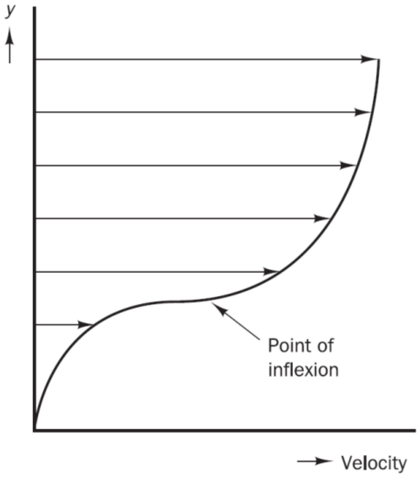
$$\mathrm{Re}_\eta = \frac{\upsilon \eta}{\nu}$$

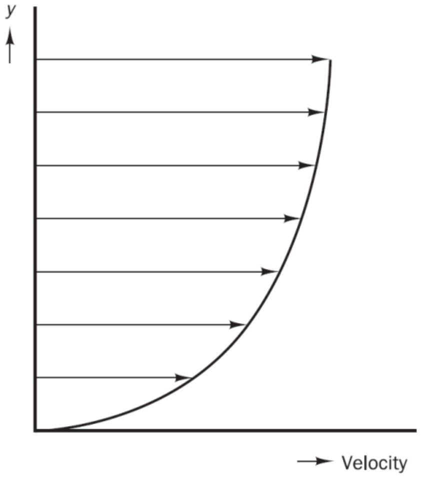


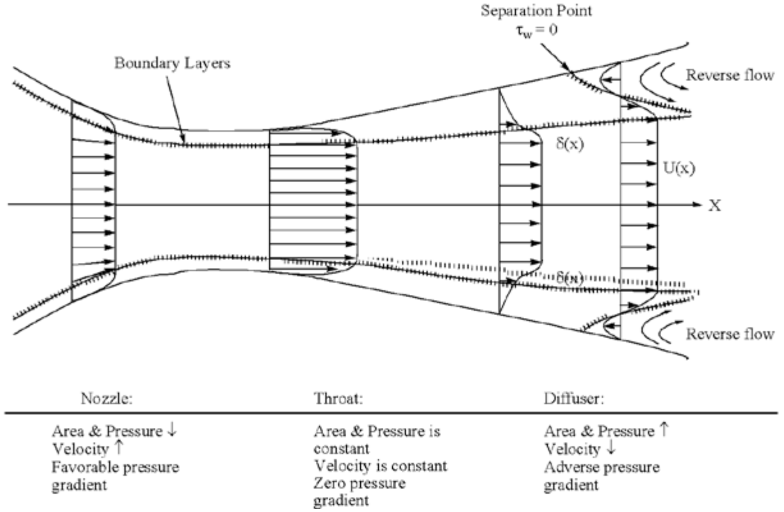
3.2: Transition from laminar to turbulent flow

- Transition
  - Can be explained by considering the stability of laminar flows to small disturbances
    - Hydrodynamic instability

- Hydrodynamic stability of laminar flows
  - Inviscid instability
    - flows with velocity profile having a point of inflection.
    - jet flows
    - mixing layers and wakes
    - boundary layers with adverse pressure gradients
  - Viscous instability:
    - flows with laminar profile having no point of inflection
    - occurs near solid walls



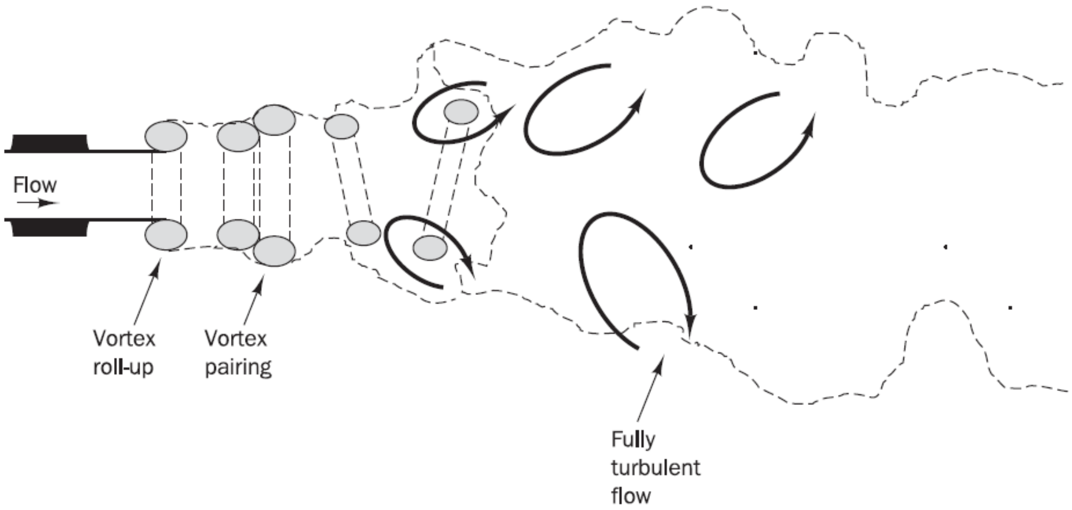


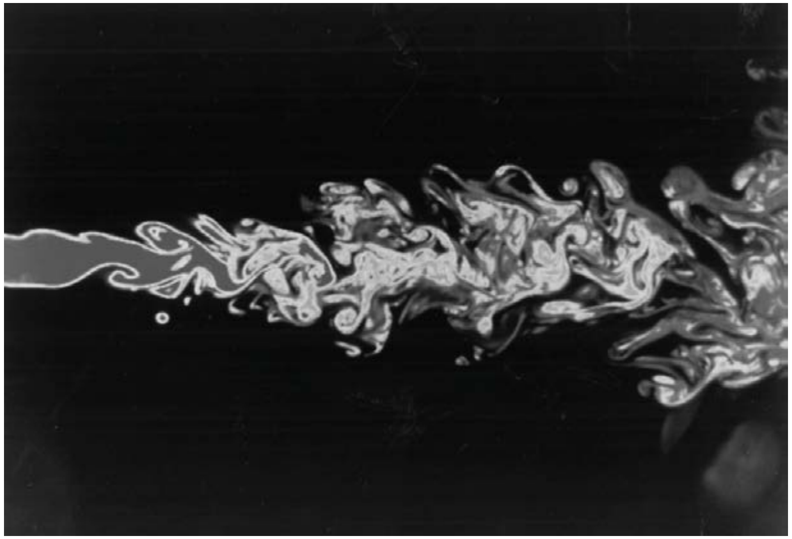


3.2: Transition from laminar to turbulent flow

Inviscid instability

- After the flow emerges from the orifice, the laminar exit flow produces the rolling up of a vortex fairly close to the orifice.
- Subsequent amplification involves the formation of a single vortex of greater strength through the pairing of vortices. A short distance further downstream, three-dimensional disturbances cause the vortices to become heavily distorted and less distinct.
- The flow breaks down, generating a large number of small-scale eddies, and the flow undergoes rapid transition to the fully turbulent regime.
- Mixing layers and wakes behind bluff bodies exhibit a similar sequence of events, leading to transition and turbulent flow.

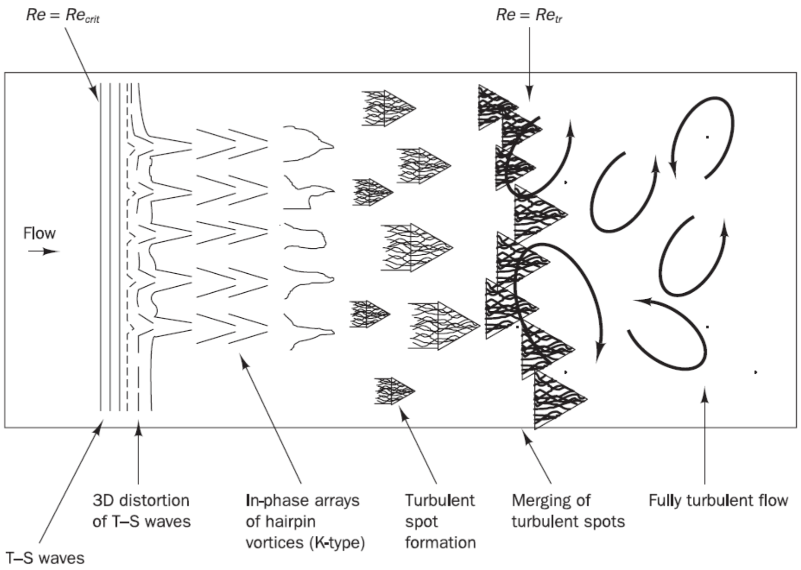


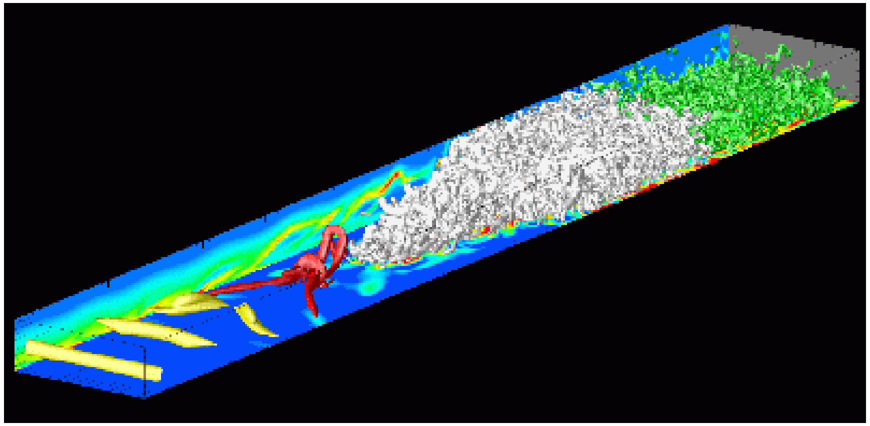


3.2: Transition from laminar to turbulent flow

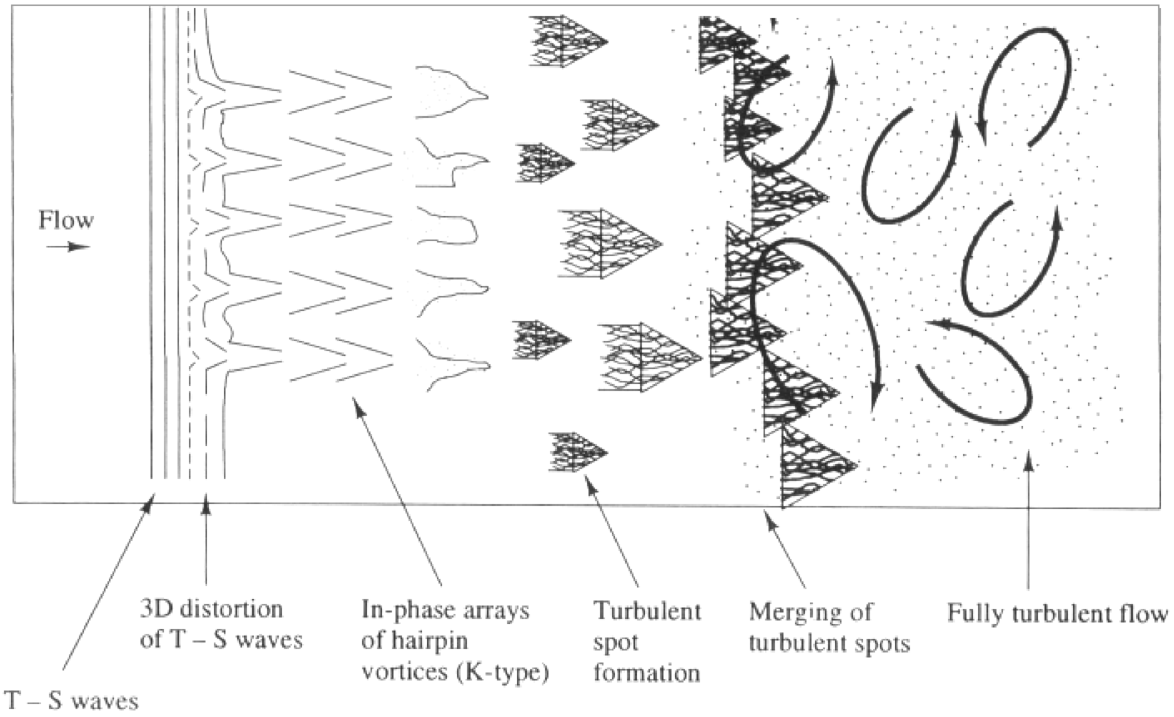
Viscous instability

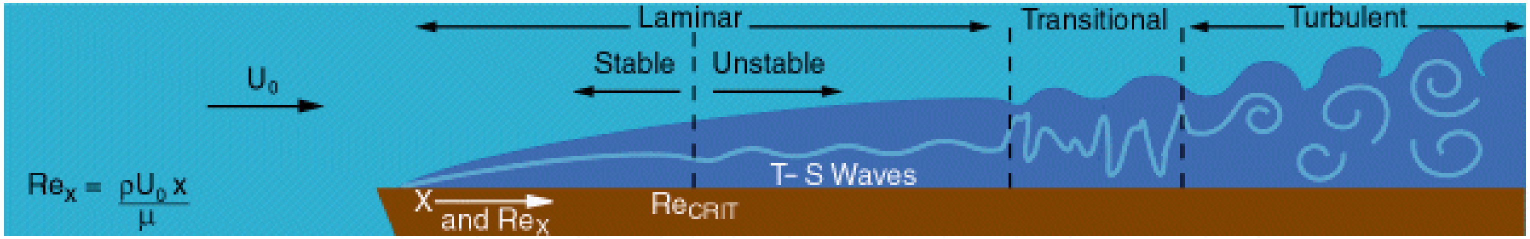
- The unstable two-dimensional disturbances are called Tollmien–Schlichting (T–S) waves.
- These disturbances are amplified in the flow direction.
- If the amplitude is large enough a secondary, non-linear, instability mechanism causes the Tollmien–Schlichting waves to become three-dimensional and finally evolve into hairpin Λ-vortices.
- Above the hairpin vortices a high shear region is induced which subsequently intensifies, elongates and rolls up.





3.2: Transition from laminar to turbulent flow





3.2: Transition from laminar to turbulent flow

- Common features in the transition process
  - The amplification of initially small disturbances
  - The development of areas with concentrated rotational structures
  - The formation of intense small scale motions
  - The growth and merging of these areas of small scale motions into fully turbulent flows

- Transition to turbulence is strongly affected by:
  - Pressure gradient
  - Disturbance levels
  - Wall roughness
  - Heat transfer

- The transition region often comprises only a very small fraction of the flow domain
  - Commercial CFD packages often ignore transition entirely
  - Classify the flow as only laminar or turbulent

3.3: Descriptors of turbulent flow

- Necessity of time-averaged properties
  - In turbulent flow there are eddying motions of a wide range of length scales.
  - A domain of 0.1×0.1m×0.1m contains smallest eddies of 10-100 μm size.
  - We need $10^9 - 10^{12}$ mesh points
  - The frequency of fastest events ≈ 10 kHz $\Rightarrow \Delta t \approx$ 100μs needed.

  - DNS of turbulent pipe flow of Re = 10^5 requires a computer which is 10 million times faster than CRAY supercomputer.

- Engineers need only time-averaged properties of the flow.

3.3: Descriptors of turbulent flow

- Time average or mean

$$\varphi = \Phi + \varphi'$$

$$\Phi = \frac{1}{\Delta t}\int_{0}^{\Delta t} \varphi(t)\, \mathrm{d}t$$

$$\overline{\varphi'} = \frac{1}{\Delta t}\int_{0}^{\Delta t} \varphi'(t)\, \mathrm{d}t \equiv 0$$

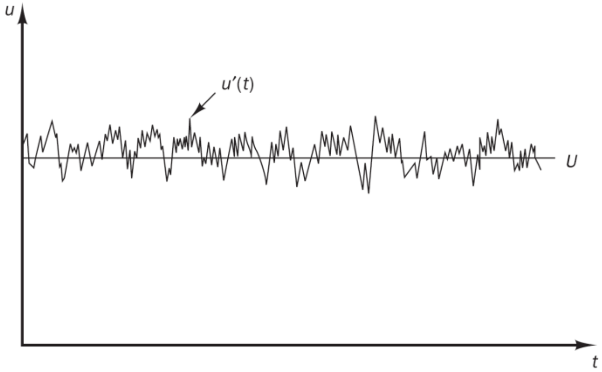
- Variance: the spread of the fluctuations

$$\overline{(\varphi')^2} = \frac{1}{\Delta t}\int_{0}^{\Delta t} (\varphi')^2\, \mathrm{d}t$$

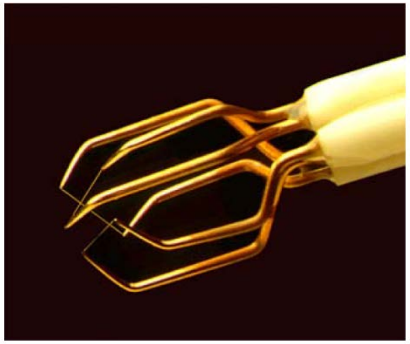
$$\varphi_{rms} = \sqrt{\overline{(\varphi')^2}} = \left[\frac{1}{\Delta t}\int_{0}^{\Delta t} (\varphi')^2\, \mathrm{d}t\right]^{1/2}$$

Reynolds decomposition

$$u(t) = U + u'(t)$$



RMS values of velocity components can be measured (by hot-wire anemometer)



$I^2R_w = hA(T_w - T_a)$

$I^2R_w = Nuk_f/dA(T_w - T_a)$

$Nu = A_1 + B_1 \cdot Re^n = A_2 + B_2 \cdot U^n$

3.3: Descriptors of turbulent flow

- Turbulent kinetic energy, k

$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2} \right)$$

- Turbulence intensity, T_i
  - Average RMS velocity / reference mean flow velocity

$$T_i = \frac{\sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)}}{U_{ref}} = \frac{\left(\frac{2}{3}k\right)^{1/2}}{U_{ref}}$$

3.3: Descriptors of turbulent flow

Moments of different fluctuating variables

- Definition of second moment

$$\varphi = \Phi + \varphi' \qquad \psi = \Psi + \psi'$$

$$\overline{\varphi'} = \overline{\psi'} = 0$$

$$\boxed{\overline{\varphi'\psi'} = \frac{1}{\Delta t}\int_0^{\Delta t} \varphi'\psi' \, \mathrm{d}t}$$

- If velocity fluctuations in different directions were independent random fluctuations, then $u'v'$, $u'w'$ and $v'w'$ would be equal to zero.

- However, although u', v' and w' are chaotic, they are not independent. As a result the second moments $u'v'$, $u'w'$ and $v'w'$ are non-zero.

- In section 3.5, we will come across the second moments in the time average of the NS equations.

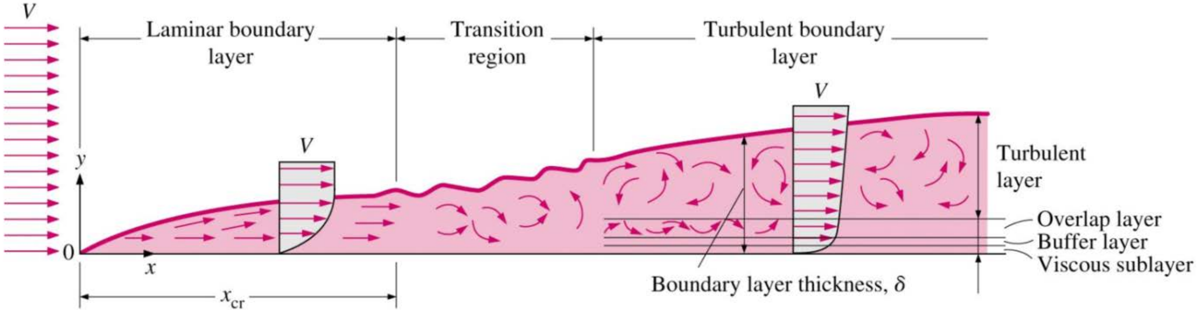
Contents

3.4: Characteristics of simple turbulent flows

- Simple turbulent flows
  - 3.4.2 Boundary layers near solid walls
    - Flat plate boundary layer
    - Pipe flow

- Reynolds number

$$\mathrm{Re} = \frac{\textit{inertia forces}}{\textit{viscous forces}}$$



- Reynolds number based on a distance ***y*** away from the wall

$$\mathrm{Re} = \frac{Uy}{\nu}$$

y: distance away from the wall

  - Near the wall $\Rightarrow y$ is small $\Rightarrow Re_y$ is small $\Rightarrow$ viscous forces dominate
  - Away from the wall $\Rightarrow y$ is large $\Rightarrow Re_y$ is large $\Rightarrow$ inertia forces dominate

3.4: Characteristics of simple turbulent flows

Boundary layers near solid walls

- Near the wall
  - The flow is influenced by viscous effects and does not depend on free stream parameters.

$$U = f(y, \rho, \mu, \tau_w)$$

  - Dimensionless analysis

$$u^+ = \frac{U}{u_\tau} = f\left(\frac{\rho u_\tau y}{\mu}\right) = f(y^+)$$

Law of the wall

$$y^+ \equiv \frac{y}{\delta_v} = \frac{u_\tau y}{\nu} = \frac{\rho u_\tau y}{\mu}$$

    – Friction velocity and viscous length scale

$$u_\tau = \sqrt{\tau_w / \rho} \qquad \delta_v = \frac{\nu}{u_\tau}$$

- Far away from the wall
  - The velocity at a point to be influenced by the retarding effect of the wall through the value of the wall shear stress, but not by the viscosity itself.
  - Length scale: boundary layer thickness (δ)

Velocity defect law

$$U = g(y, \delta, \rho, \tau_w) \qquad u^+ = \frac{U}{u_\tau} = g\left(\frac{y}{\delta}\right) \qquad \frac{U_{\max} - U}{u_\tau} = g\left(\frac{y}{\delta}\right)$$

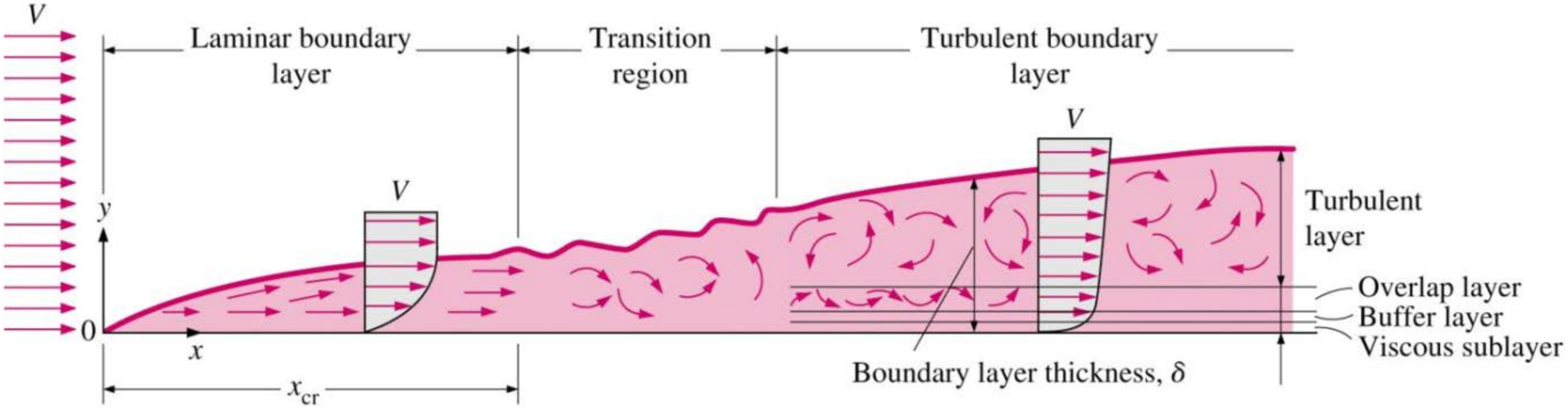
3.4: Characteristics of simple turbulent flows

- Boundary layers near solid walls
  - Very near the wall
    - There is no turbulent shear stress ⇒ flow is dominated by viscous shear
    - $y^+ < 5$ ⇒ shear stress is approximately constant

$$\tau(y) = \mu \frac{\partial U}{\partial y} \cong \tau_w$$

    - $U=0$ at $y=0$,

$$U = \frac{\tau_w y}{\mu}$$

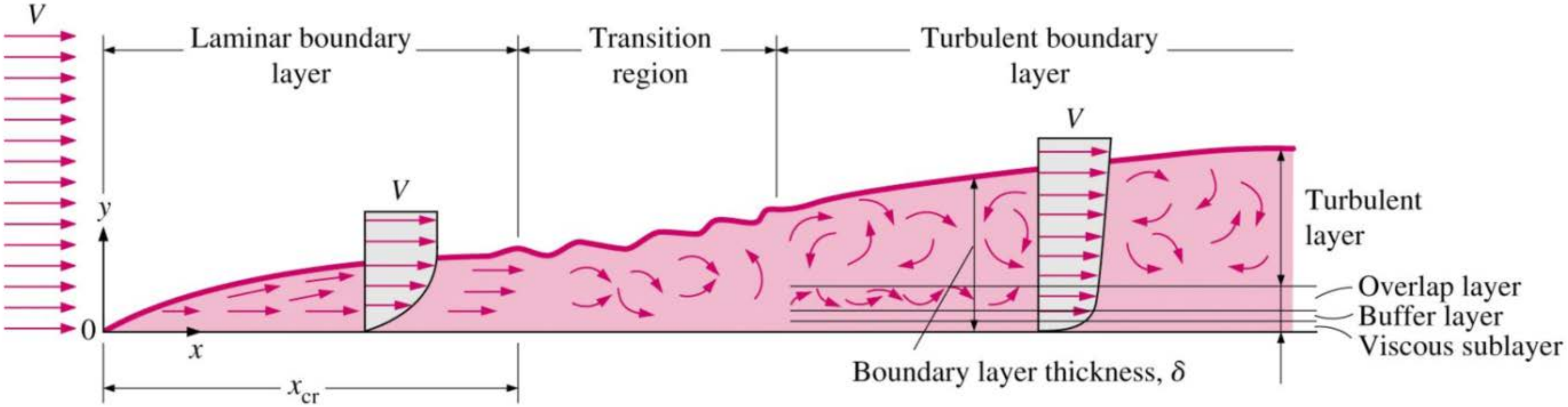


3.4: Characteristics of simple turbulent flows

- Boundary layers near solid walls
  - Very near the wall

$$u^+ = y^+$$

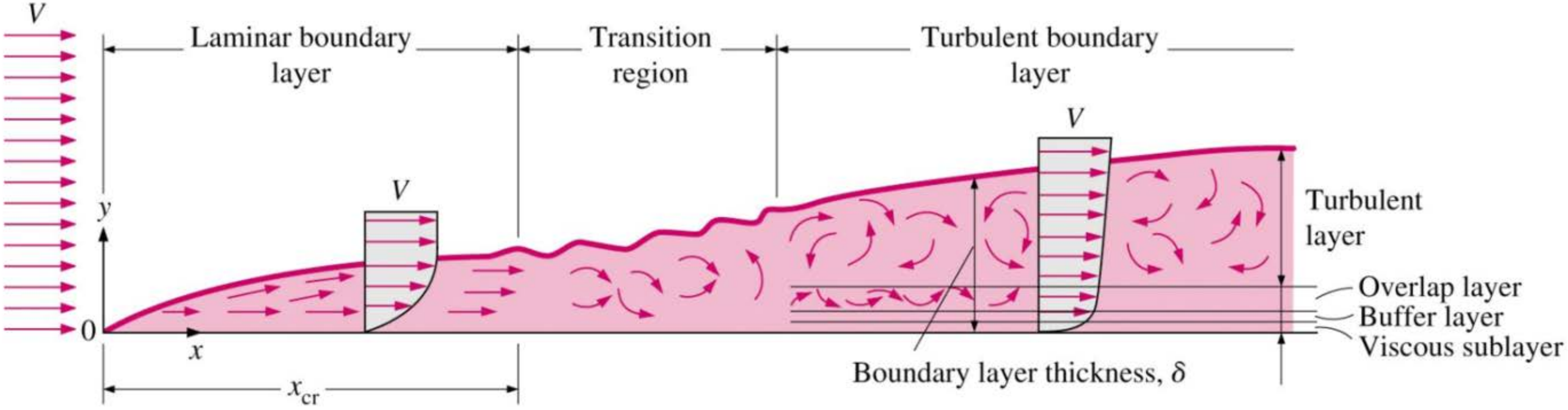
    - Linear relationship between velocity and distance from the wall
    - Linear sub-layer, viscous sublayer, viscous wall layer



3.4: Characteristics of simple turbulent flows

Boundary layers near solid walls

- Outside the viscous sublayer
  - A region where viscous and turbulent effects are both important.
  - $30 < y^{+} < 500$
  - Shear stress varies slowly with distance from the wall.
    - von Karman's constant
      - For smooth walls: $\kappa \approx 0.4$, $B \approx 5.5$



3.4: Characteristics of simple turbulent flows

- Boundary layers near solid walls
  - Outside the viscous sublayer
    - Derivation from Prandtl's mixing length theory

$$l_m = \kappa y \qquad \boxed{\nu_t = \ell_m^2 \left| \frac{\partial U}{\partial y} \right|}$$

$$\tau = \rho \nu_t \left(\frac{dU}{dy} \right) =$$

⬅ $\tau = \tau_w$

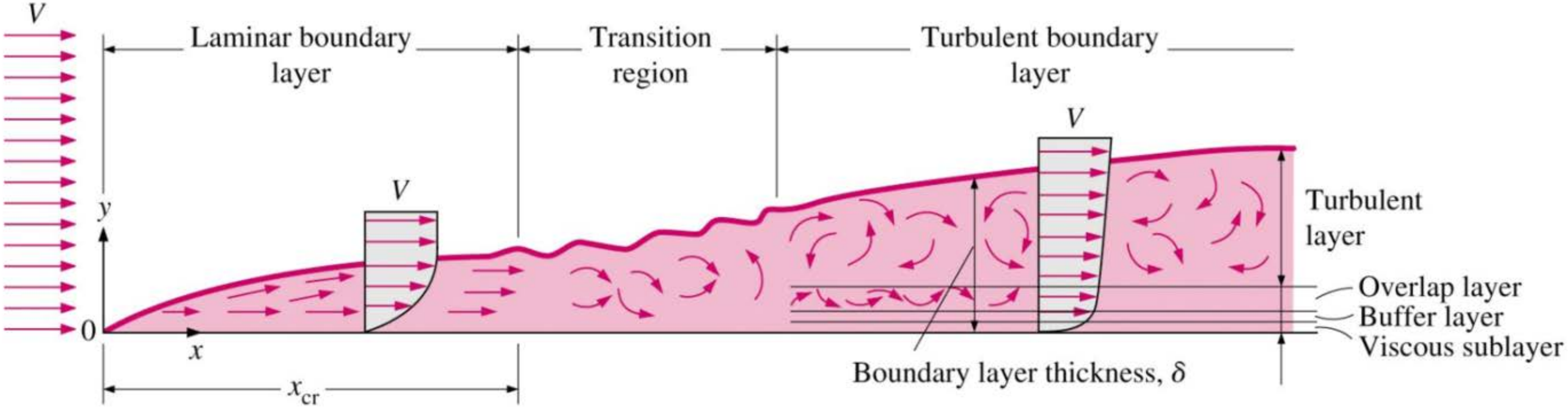
$$1 = \frac{1}{\rho u_\tau^2} \rho \kappa^2 y^2 \left(\frac{dU}{dy} \right)^2 = \kappa^2 y^{+2} \left(\frac{du^+}{dy^+} \right)^2$$

$$\frac{1}{\kappa^2 y^{+2}} = \left(\frac{du^+}{dy^+} \right)^2$$

3.4: Characteristics of simple turbulent flows

- Boundary layers near solid walls
  - Outside the viscous sublayer
    - $30 < y^+ < 500$
    - Logarithmic relationship
      - Log-law
      - Universal
    - Log-law layer, overlap layer

$$u^+ = \frac{1}{\kappa} \ln y^+ + B$$



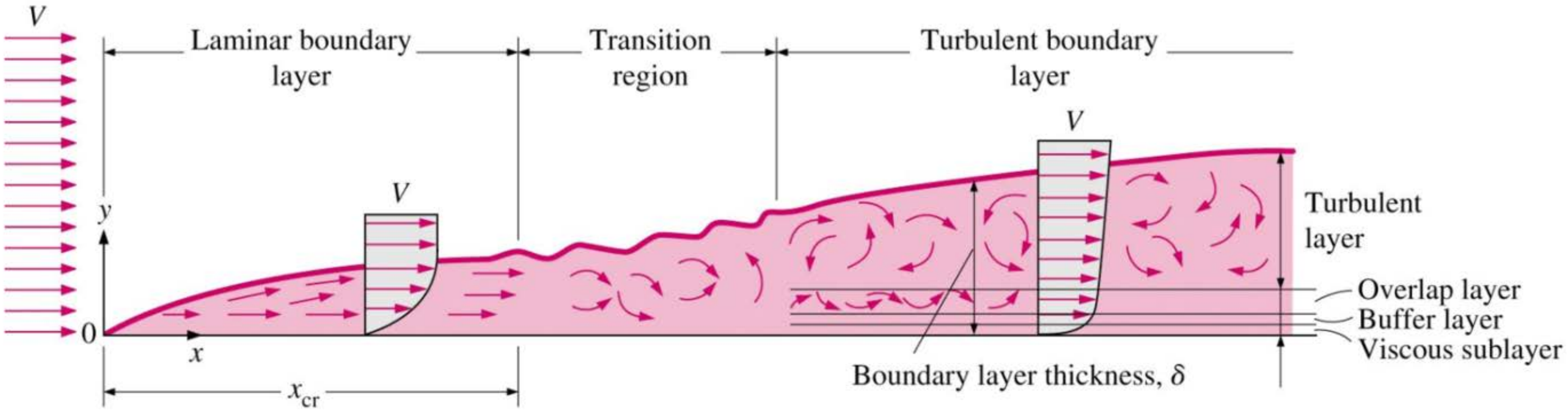
3.4: Characteristics of simple turbulent flows

Boundary layers near solid walls

- Outer layer
  - Inertia-dominated region far from the wall
  - For large values of y, the velocity-defect law provides correct form.
  - In the overlap region, the log-law and the velocity defect law have to be equal

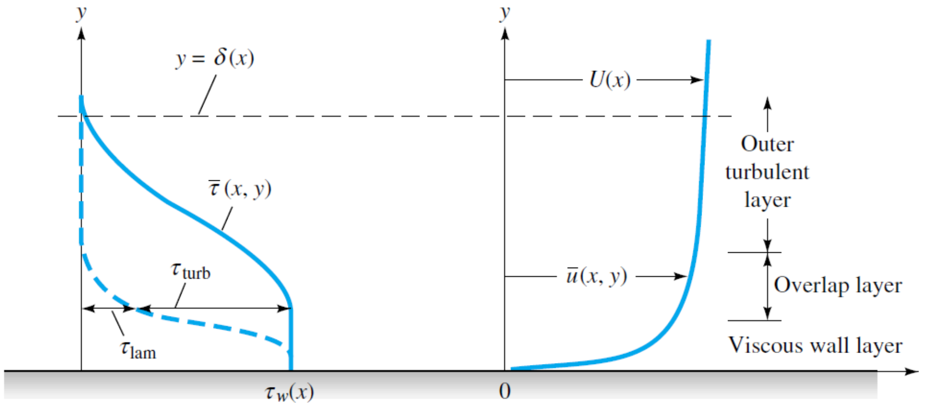
$$\frac{U_{\max} - U}{u_\tau} = g\left(\frac{y}{\delta}\right) \qquad u^+ = \frac{U}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{y}{\delta}\right) + B$$

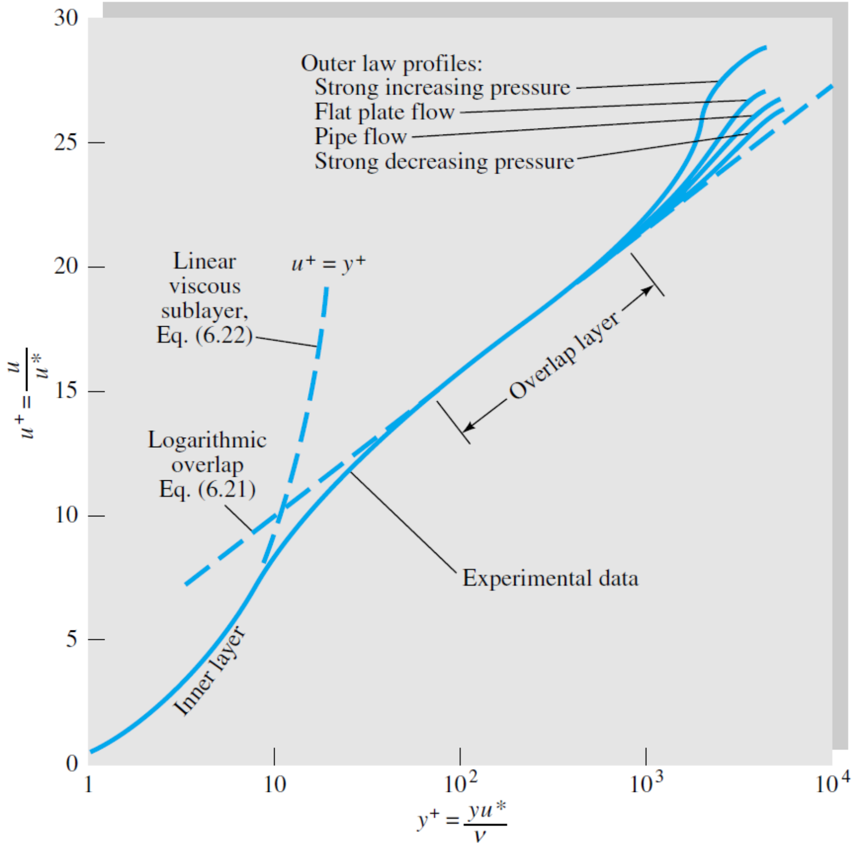
Law of the wake



3.4: Characteristics of simple turbulent flows

- Boundary layers near solid walls
  - Inner region
    - 10–20% of the total thickness of the wall layer
    - Within this region there are three zones.
      - linear sub-layer: viscous stresses dominate the flow adjacent to surface
      - buffer layer: viscous and turbulent stresses are of similar magnitude
      - log-law layer: turbulent (Reynolds) stresses dominate.
  - Outer region
    - Inertia dominated core flow far from wall
    - Free from direct viscous effects





3.4: Characteristics of simple turbulent flows

- Boundary layers near solid walls
  - Mean velocity distribution

$$u^+ = y^+ \qquad \frac{U}{u_\tau} = \frac{y}{\delta}$$

$$u^+ = \frac{1}{\kappa}\ln y^+ + B \qquad \frac{U}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{y}{\delta}\right) + B$$

$$\frac{U_{\max} - U}{u_\tau} = \frac{U_{\max}}{u_\tau} - \frac{1}{\kappa}\ln\left(\frac{y}{\delta}\right) - B = -\frac{1}{\kappa}\ln\left(\frac{y}{\delta}\right) + A$$

- For smooth wall pipe
- Re=10^4

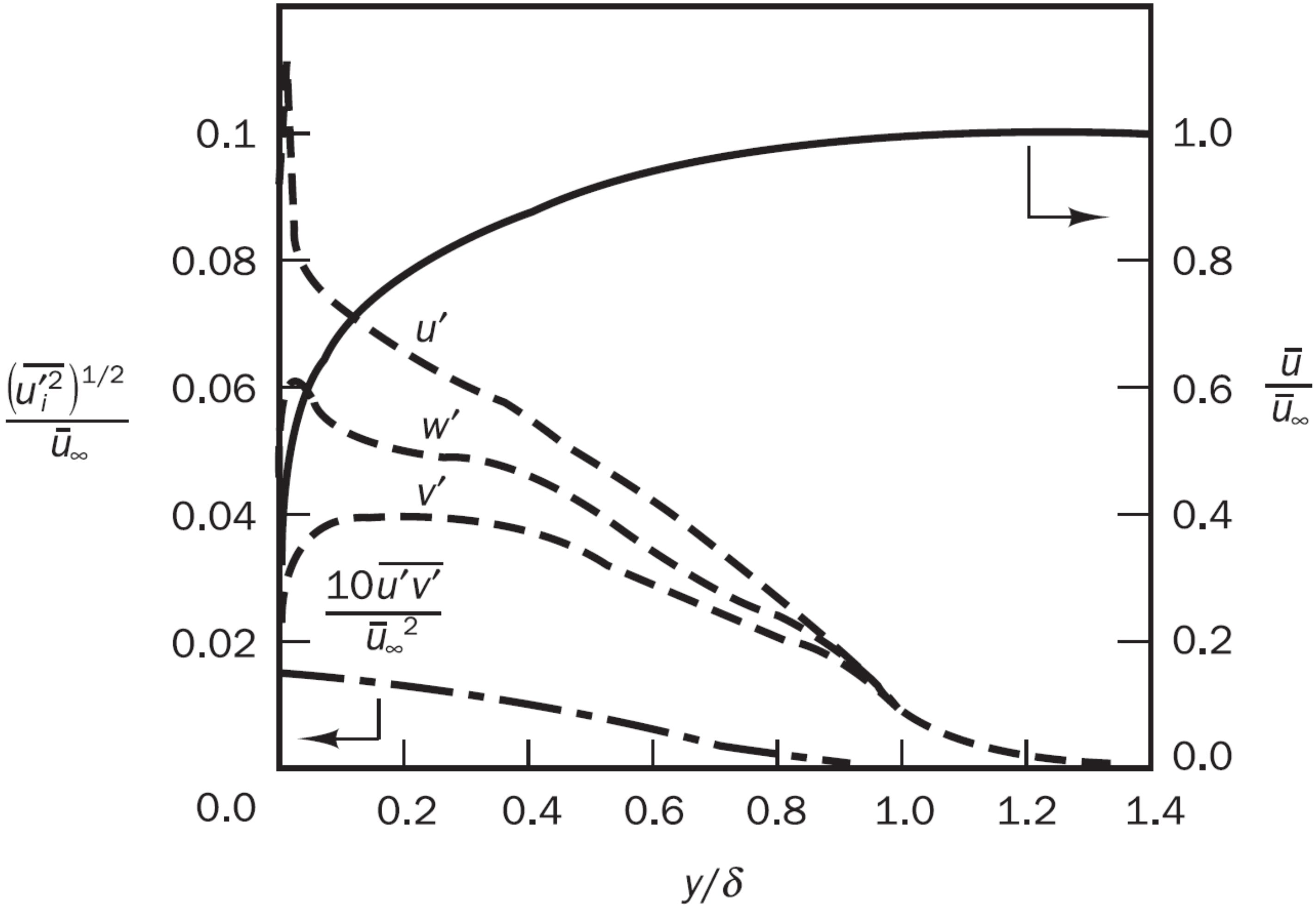
$$f = \frac{0.316}{\mathrm{Re}^{-0.25}}$$

Darcy friction factor

$$\tau_w = \frac{1}{8} f\rho V^2 = \rho u_\tau^2$$

3.4: Characteristics of simple turbulent flows

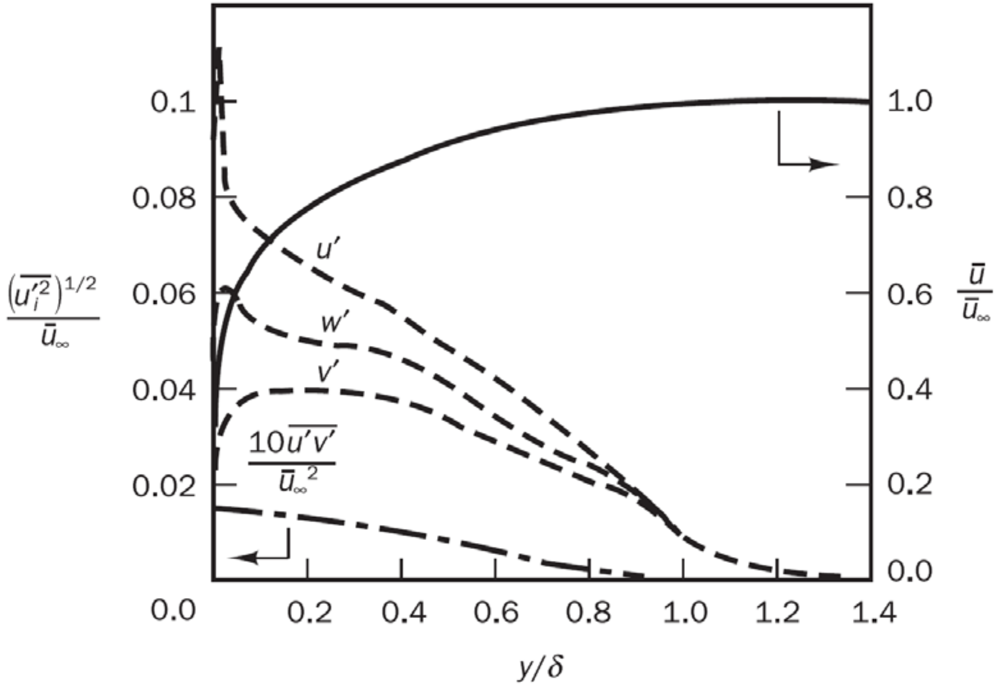
- Distribution of mean velocity and second moments



3.4: Characteristics of simple turbulent flows

- Distribution of mean velocity and second moments
  - For $y/\delta > 0.8$ fluctuating velocities become almost equal
    - Isotropic turbulence structure here. (far away the wall)
  - For $y/\delta < 0.2$ large mean velocity gradients
    - High values of fluctuation $\Rightarrow$ high turbulence production

  - Turbulence is anisotropic near the wall!



3.4: Characteristics of simple turbulent flows

- Summary
  - Turbulence is generated and maintained by shear in the mean flow.
  - Where shear is large the magnitudes of turbulence quantities such as the r.m.s. velocity fluctuations are high and their distribution is anisotropic with higher levels of fluctuations in the mean flow direction.
  - Without shear, or an alternative agency to maintain it, turbulence decays and becomes more isotropic in the process.

Contents

3.5 Effect of Turbulent Fluctuations on Properties of the Mean Flow

- Derivation of the Reynolds-averaged Navier-Stokes equations
  - Fluctuating properties

$$\varphi = \Phi + \varphi' \qquad \psi = \Psi + \psi'$$

$$\overline{\varphi'} = \overline{\psi'} = 0 \quad \overline{\Phi} = \Phi \quad \overline{\frac{\partial \varphi}{\partial s}} = \frac{\partial \Phi}{\partial s} \quad \overline{\int \varphi \mathrm{d}s} = \int \Phi \mathrm{d}s$$

$$\overline{\varphi + \psi} = \Phi + \Psi \quad \overline{\varphi\psi} = \boxed{\phantom{\Phi\Psi + \overline{\varphi'\psi'}}} \quad \overline{\varphi\Psi} = \Phi\Psi \quad \overline{\varphi'\Psi} = 0$$

  - Fluctuating vector quantity

$$\mathbf{a} = \mathbf{A} + \mathbf{a}'$$

$$\overline{\operatorname{div}\mathbf{a}} = \boxed{\phantom{\operatorname{div}\mathbf{A}}}; \ \overline{\operatorname{div}(\varphi\mathbf{a})} = \operatorname{div}(\overline{\varphi\mathbf{a}}) = \boxed{\phantom{\operatorname{div}(\Phi\mathbf{A}) + \operatorname{div}(\overline{\varphi'\mathbf{a}'})}};$$

$$\overline{\operatorname{div}\operatorname{grad}\varphi} = \operatorname{div}\operatorname{grad}\Phi$$

3.5 Effect of Turbulent Fluctuations on Properties of the Mean Flow

❖ Derivation of the Reynolds-averaged Navier-Stokes equations

- For incompressible flow

$$\operatorname{div} \mathbf{u} = 0$$

$$\frac{\partial u}{\partial t} + \operatorname{div}(u\mathbf{u}) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \operatorname{div}(\operatorname{grad}(u))$$

$$\frac{\partial v}{\partial t} + \operatorname{div}(v\mathbf{u}) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \operatorname{div}(\operatorname{grad}(v))$$

$$\frac{\partial w}{\partial t} + \operatorname{div}(w\mathbf{u}) = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \operatorname{div}(\operatorname{grad}(w))$$

- Substitute

$$\mathbf{u} = \mathbf{U} + \mathbf{u}' \quad u = U + u' \quad v = V + v' \quad w = W + w' \quad p = P + p'$$

- Do time average!

3.5 Effect of Turbulent Fluctuations on Properties of the Mean Flow

❖ Derivation of the Reynolds-averaged Navier-Stokes equations

- Continuity eq.

$$\overline{\operatorname{div}\mathbf{u}} = \overline{\operatorname{div}(\mathbf{U}+\mathbf{u}')} = \overline{\operatorname{div}\mathbf{U}} = \operatorname{div}\mathbf{U}$$

  - Continuity equation for the mean flow

- x-momentum equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(u\mathbf{u}) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\,\operatorname{div}(\operatorname{grad}(u))$$

$$\overline{\frac{\partial u}{\partial t}} = \boxed{\quad} \qquad \overline{\operatorname{div}(u\mathbf{u})} = \boxed{\quad} \qquad \overline{-\frac{1}{\rho}\frac{\partial p}{\partial x}} = \boxed{\quad} \qquad \overline{\nu\,\operatorname{div}(\operatorname{grad}(u))} = \boxed{\quad}$$

3.5 Effect of Turbulent Fluctuations on Properties of the Mean Flow

Derivation of the Reynolds-averaged Navier-Stokes equations

- y- and z- momentum equation

$$\frac{\partial V}{\partial t} + \operatorname{div}(V\mathbf{U}) + \boxed{\operatorname{div}(\overline{v'\mathbf{u}'})} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu \operatorname{div}(\operatorname{grad}(V))$$

$$\frac{\partial W}{\partial t} + \operatorname{div}(W\mathbf{U}) + \boxed{\operatorname{div}(\overline{w'\mathbf{u}'})} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu \operatorname{div}(\operatorname{grad}(W))$$

Convective momentum transfer due to turbulent eddies

- Rearrange ⇒ Reynolds-averaged Navier-Stokes equations

$$\frac{\partial U}{\partial t} + \operatorname{div}(U\mathbf{U}) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \operatorname{div}(\operatorname{grad}(U)) + \boxed{\frac{1}{\rho}\left[\frac{\partial(-\rho\overline{u'^2})}{\partial x} + \frac{\partial(-\rho\overline{u'v'})}{\partial y} + \frac{\partial(-\rho\overline{u'w'})}{\partial z}\right]}$$

$$\frac{\partial V}{\partial t} + \operatorname{div}(V\mathbf{U}) = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu \operatorname{div}(\operatorname{grad}(V)) + \boxed{\frac{1}{\rho}\left[\frac{\partial(-\rho\overline{u'v'})}{\partial x} + \frac{\partial(-\rho\overline{v'^2})}{\partial y} + \frac{\partial(-\rho\overline{v'w'})}{\partial z}\right]}$$

$$\frac{\partial W}{\partial t} + \operatorname{div}(W\mathbf{U}) = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu \operatorname{div}(\operatorname{grad}(W)) + \boxed{\frac{1}{\rho}\left[\frac{\partial(-\rho\overline{u'w'})}{\partial x} + \frac{\partial(-\rho\overline{v'w'})}{\partial y} + \frac{\partial(-\rho\overline{w'^2})}{\partial z}\right]}$$

3.5 Effect of Turbulent Fluctuations on Properties of the Mean Flow

- Derivation of the Reynolds-averaged Navier-Stokes equations
  - Extra stress terms ⇒ turbulent shear stress
    - Normal stresses and shear stresses

$$\tau_{xx} = -\rho\overline{u'^2} \quad \tau_{yy} = -\rho\overline{v'^2} \quad \tau_{zz} = -\rho\overline{w'^2}$$

$$\tau_{xy} = \tau_{yx} = -\rho\overline{u'v'} \quad \tau_{xz} = \tau_{zx} = -\rho\overline{u'w'} \quad \tau_{yz} = \tau_{zy} = -\rho\overline{v'w'}$$

    - Called the Reynolds stresses
    - Very large compared with the viscous stresses in a turbulent flow
  - Extra transport terms in scalar equation

$$\frac{\partial\Phi}{\partial t} + \operatorname{div}(\Phi\mathbf{U}) = \frac{1}{\rho}\operatorname{div}(\Gamma_\Phi \operatorname{grad}\Phi) + \left[-\frac{\partial\overline{u'\varphi'}}{\partial x} - \frac{\partial\overline{v'\varphi'}}{\partial y} - \frac{\partial\overline{w'\varphi'}}{\partial z}\right] + S_\Phi$$

3.5 Effect of Turbulent Fluctuations on Properties of the Mean Flow

- Effect of density fluctuation
  - Small density fluctuation: do not affect the flow significantly
    - RMS velocity fluctuations: < 5% of the mean speed
      - Density fluctuations are unimportant up to Mach numbers around 3 to 5.
    - RMS velocity fluctuations: > 20% of the mean speed
      - Density fluctuations affects the turbulence around Mach numbers of 1.

$$\frac{\partial \bar{\rho}}{\partial t} + \operatorname{div}(\bar{\rho}\tilde{\mathbf{U}}) = 0$$

$$\frac{\partial(\bar{\rho}\tilde{U})}{\partial t} + \operatorname{div}(\bar{\rho}\tilde{U}\tilde{\mathbf{U}}) = -\frac{\partial \bar{P}}{\partial x} + \operatorname{div}(\mu \operatorname{grad} \tilde{U}) + \left[-\frac{\partial(\overline{\bar{\rho}u'^2})}{\partial x} - \frac{\partial(\overline{\bar{\rho}u'v'})}{\partial y} - \frac{\partial(\overline{\bar{\rho}u'w'})}{\partial z} \right] + S_{Mx}$$

$$\frac{\partial(\bar{\rho}\tilde{V})}{\partial t} + \operatorname{div}(\bar{\rho}\tilde{V}\tilde{\mathbf{U}}) = -\frac{\partial \bar{P}}{\partial y} + \operatorname{div}(\mu \operatorname{grad} \tilde{V}) + \left[-\frac{\partial(\overline{\bar{\rho}u'v'})}{\partial x} - \frac{\partial(\overline{\bar{\rho}v'^2})}{\partial y} - \frac{\partial(\overline{\bar{\rho}v'w'})}{\partial z} \right] + S_{My}$$

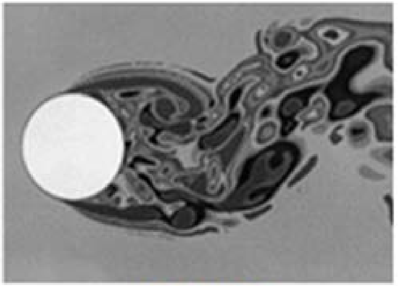
$$\frac{\partial(\bar{\rho}\tilde{W})}{\partial t} + \operatorname{div}(\bar{\rho}\tilde{W}\tilde{\mathbf{U}}) = -\frac{\partial \bar{P}}{\partial z} + \operatorname{div}(\mu \operatorname{grad} \tilde{W}) + \left[-\frac{\partial(\overline{\bar{\rho}u'w'})}{\partial x} - \frac{\partial(\overline{\bar{\rho}v'w'})}{\partial y} - \frac{\partial(\overline{\bar{\rho}w'^2})}{\partial z} \right] + S_{Mz}$$

$$\frac{\partial(\bar{\rho}\tilde{\Phi})}{\partial t} + \operatorname{div}(\bar{\rho}\tilde{\Phi}\tilde{\mathbf{U}}) = \operatorname{div}(\Gamma_{\Phi} \operatorname{grad} \tilde{\Phi}) + \left[-\frac{\partial(\overline{\bar{\rho}u'\varphi'})}{\partial x} - \frac{\partial(\overline{\bar{\rho}v'\varphi'})}{\partial y} - \frac{\partial(\overline{\bar{\rho}w'\varphi'})}{\partial z} \right] + S_{\Phi}$$

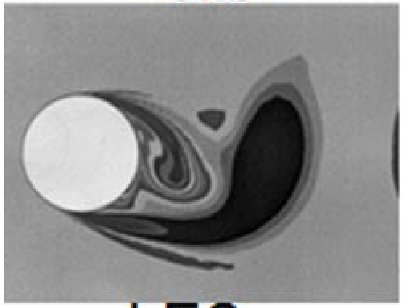
Contents

3.6 Turbulent Flow Calculations

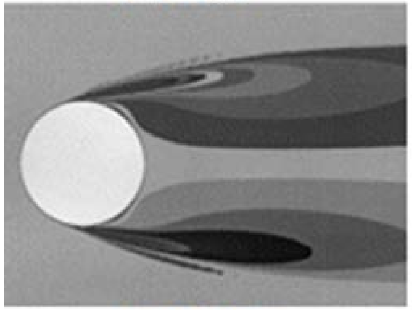
- Three methods to calculate turbulent flow
  - Turbulence models for Reynolds-averaged Navier-Stokes (RANS) equations
    - Attention is focused on the mean flow and the effects of turbulence on mean flow properties.
    - Extra stresses (Reynolds stresses) are modelled .
      - k-ε model
      - Reynolds stress model
  - Large eddy simulation (LES)
    - This is an intermediate form of turbulence calculations which tracks the behaviour of the larger eddies.
    - Space filtering of the unsteady Navier-Stokes equations
      - Passes the larger eddies and rejects the smaller eddies
      - The effects on the resolved flow (mean flow + large eddies) due to the unresolved eddies are included by means of a so-called sub-grid scale model.
  - Direct numerical simulations (DNS)
    - Mean flow and all turbulent velocity fluctuations
    - The unsteady NS equations are solved on spatial grids that are sufficiently fine
    - Resolve the Kolmogorov length scales and the fastest fluctuations



DNS



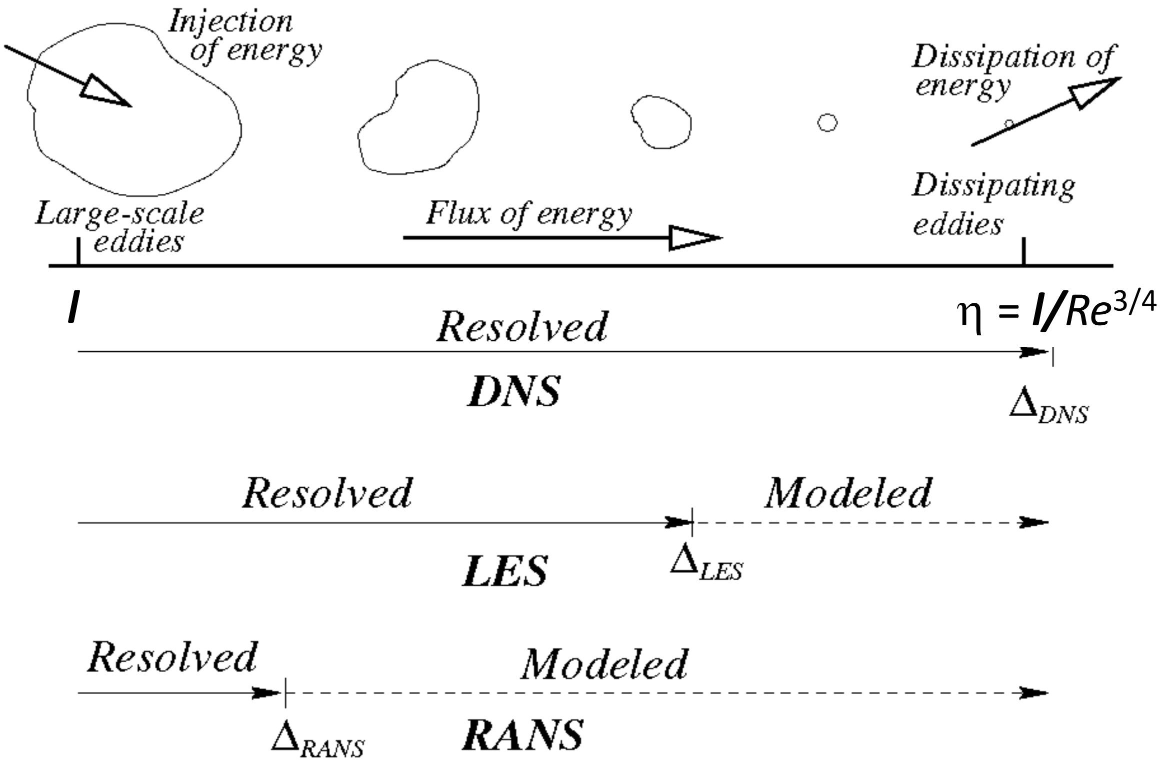
LES



RANS

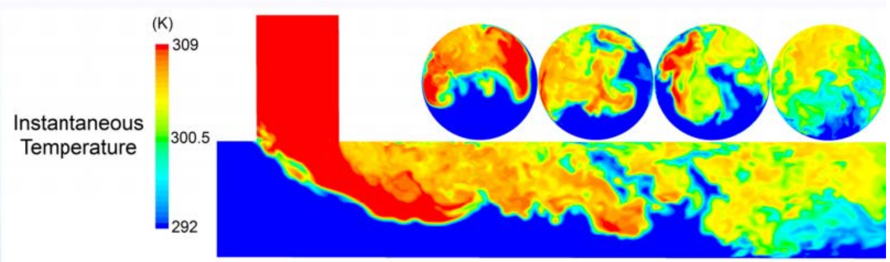
3.6 Turbulent Flow Calculations

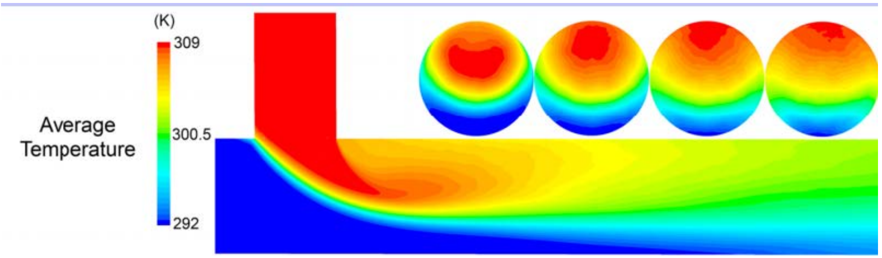
- Three methods to calculate turbulent flow

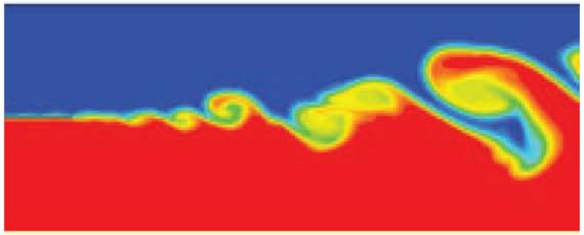


3.6 Turbulent Flow Calculations

- Three methods to calculate turbulent flow

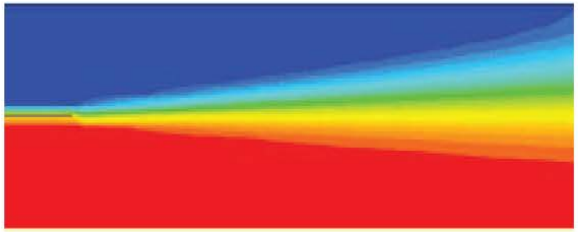






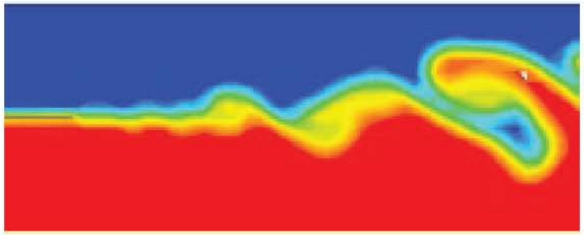
LES

3D, unsteady



RANS

Steady / unsteady



V-LES

3D, unsteady

Contents

3.7 RANS Equations and Classical Turbulence Models

- Reynolds stresses and scalar transport terms

$$\tau_{xx} = -\rho\overline{u'^2} \quad \tau_{yy} = -\rho\overline{v'^2} \quad \tau_{zz} = -\rho\overline{w'^2}$$

$$\tau_{xy} = \tau_{yx} = -\rho\overline{u'v'} \quad \tau_{xz} = \tau_{zx} = -\rho\overline{u'w'} \quad \tau_{yz} = \tau_{zy} = -\rho\overline{v'w'}$$

$$\overline{u'\varphi'} \quad \overline{v'\varphi'} \quad \overline{w'\varphi'}$$

  - It is necessary to develop turbulence models!

No. of extra transport equations	***Name***
Zero	Mixing length model
One	Spalart–Allmaras model
Two	k–ε model
	k–ω model
	Algebraic stress model
Seven	Reynolds stress model

3.7 RANS Equations and Classical Turbulence Models

❖ Eddy viscosity and eddy diffusivity

- Viscous stress ∝ rate of deformation of fluid elements
- For an incompressible fluid

$$\tau_{ij} = \mu s_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\tau_{12} = \tau_{xy} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

- Boussinesq (1877): Reynolds stress ∝ mean rate of deformation of fluid elements

$$\tau_{ij} = -\rho\,\overline{u_i' u_j'} =$$

$$k = \tfrac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

Turbulent kinetic energy per unit mass

  - μ_t : Turbulent or eddy viscosity

$$\delta_{ij} = 1 \text{ if } i = j \text{ and } \delta_{ij} = 0 \text{ if } i \neq j$$

3.7 RANS Equations and Classical Turbulence Models

- Eddy viscosity and eddy diffusivity

$$\tau_{ij} = -\rho\,\overline{u_i' u_j'} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\rho k \delta_{ij}$$

$$\tau_{xx} = -\rho\overline{u'^2} = 2\mu_t\left[\frac{\partial U}{\partial x}\right] - \frac{1}{3}\rho\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$$\tau_{yy} = -\rho\overline{v'^2} = 2\mu_t\left[\frac{\partial V}{\partial y}\right] - \frac{1}{3}\rho\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$$\tau_{zz} = -\rho\overline{w'^2} = 2\mu_t\left[\frac{\partial W}{\partial z}\right] - \frac{1}{3}\rho\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$$-\rho\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) = 2\mu_t\left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right] - \rho\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

3.7 RANS Equations and Classical Turbulence Models

- Eddy viscosity and eddy diffusivity
  - Turbulent transport of a scalar

$$-\rho \overline{u_i'\varphi'} = \Gamma_t \frac{\partial \Phi}{\partial x_i}$$

    - Γ_t: Turbulent or eddy diffusivity
    - Since turbulent transport of momentum and heat or mass is due to the same mechanism – eddy mixing – we expect that the value of the turbulent diffusivity is fairly close to that of the turbulent viscosity.
    - Turbulent Prandtl number

$$\sigma_t = \frac{\mu_t}{\Gamma_t}$$

      - Most CFD procedures assume this to be around unity.

$$\sigma_t \approx 1$$

3.7 RANS Equations and Classical Turbulence Models

- RANS models
  - Mixing length models
  - k-ε model
  - Reynolds stress equation models
  - Algebraic stress models
  - k-ω model

$$\tau_{ij} = -\rho\,\overline{u_i' u_j'} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij}$$

3.7 RANS Equations and Classical Turbulence Models

Mixing length models

- Attempts to describe the turbulent stresses by means of simple algebraic formulae for μ_t as a function of position
- Kinematic turbulent viscosity ν_t
    - Turbulent velocity scale (ϑ)
    - Turbulent length scale (ℓ)

$$\nu_t = \boxed{\quad} \qquad \mu_t = \boxed{\quad}$$

- Most of the kinetic energy of turbulence is contained in the largest eddies.
- Turbulence length scale is therefore characteristic of these eddies which interact with the mean flow.
- When there is significant velocity gradient $\dfrac{\partial U}{\partial y}$, 2D problem

$$\vartheta = \boxed{\quad}$$

$$\boxed{\nu_t = \boxed{\quad}}$$

Prandtl's mixing length model

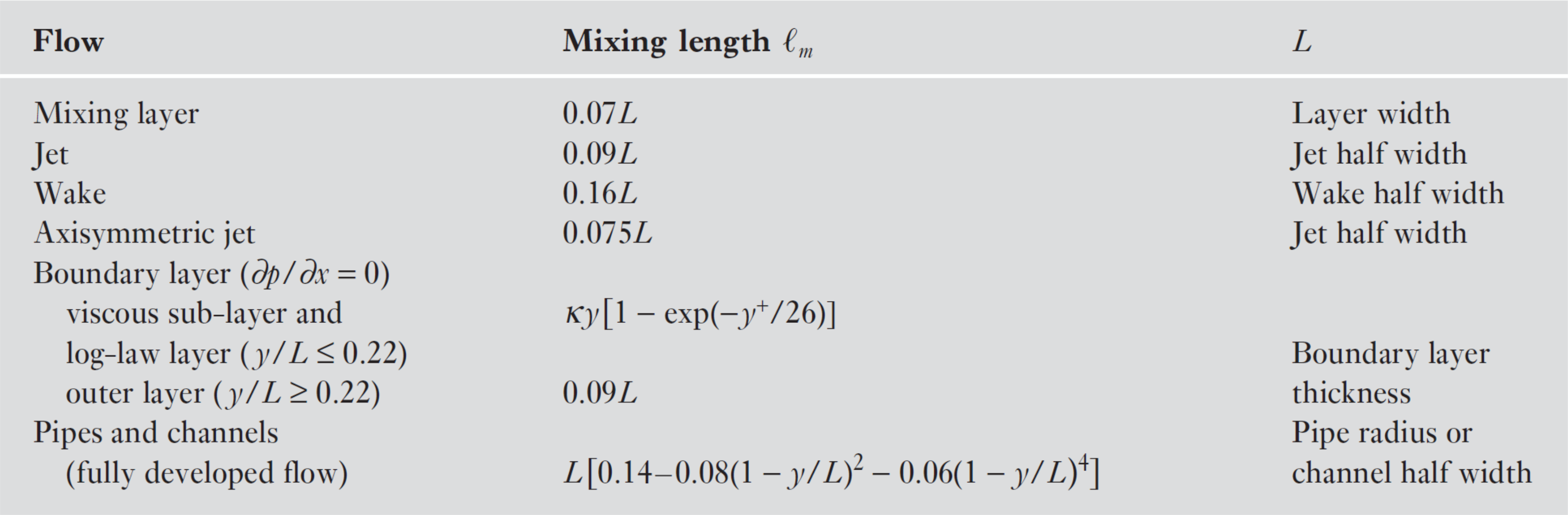
3.7 RANS Equations and Classical Turbulence Models

Mixing length models

- Turbulent Reynolds stress

$$\tau_{xy} = \tau_{yx} = -\rho \overline{u'v'} = \rho \ell_m^2 \left| \frac{\partial U}{\partial y} \right| \frac{\partial U}{\partial y}$$

$$\tau_{ij} = -\rho \overline{u_i'u_j'} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij}$$



Flow	**Mixing length ℓ_m**	L
Mixing layer	$0.07L$	Layer width
Jet	$0.09L$	Jet half width
Wake	$0.16L$	Wake half width
Axisymmetric jet	$0.075L$	Jet half width
Boundary layer ($\partial p/\partial x = 0$)		
viscous sub-layer and	$\kappa y[1 - \exp(-y^+/26)]$	
log-law layer ($y/L \le 0.22$)		Boundary layer
outer layer ($y/L \ge 0.22$)	$0.09L$	thickness
Pipes and channels		Pipe radius or
(fully developed flow)	$L[0.14 – 0.08(1 - y/L)^2 - 0.06(1 - y/L)^4]$	channel half width

3.7 RANS Equations and Classical Turbulence Models

- Mixing length models
  - Scalar transport

$$\frac{\partial \Phi}{\partial t} + \mathrm{div}(\Phi \mathbf{U}) = \frac{1}{\rho}\,\mathrm{div}(\Gamma_{\Phi}\ \mathrm{grad}\ \Phi) + \left[-\frac{\partial \overline{u'\varphi'}}{\partial x} - \frac{\partial \overline{v'\varphi'}}{\partial y} - \frac{\partial \overline{w'\varphi'}}{\partial z} \right] + S_{\Phi}$$

2D problem

$$\boxed{-\rho\,\overline{v'\varphi'} = \Gamma_t \frac{\partial \Phi}{\partial y}}$$

$$\left.\begin{array}{l} \sigma_t = 0.9 \text{ for near wall flows} \\ \sigma_t = 0.5 \text{ for jets and mixing layers} \\ \sigma_t = 0.7 \text{ in axisymmetric jets} \end{array}\right] \rightarrow \text{Rodi(1980)}$$

3.7 RANS Equations and Classical Turbulence Models

- Mixing length models
  - Advantages
    - Easy to implement and cheap in terms of computing resources
    - Good predictions for thin shear layers: jets, mixing layers, wakes and boundary layers
    - Well established

  - Disadvantages
    - Completely incapable of describing flows with separation and recirculation
    - Only calculates mean flow properties and turbulent shear stress

3.7 RANS Equations and Classical Turbulence Models

❖ k-ε model

- If convection and diffusion of turbulence properties are not negligible, and it causes significant differences between production and destruction of turbulence, as in the case of recirculating flows, then the mixing length model is not applicable.
- The k-ε model focuses on the mechanisms that affect the turbulent kinetic energy.

- Some preliminary definitions

$$K =$$

Mean kinetic energy

$$k =$$

Turbulent kinetic energy

$$k(t) = K + k$$

Instantaneous kinetic energy

3.7 RANS Equations and Classical Turbulence Models

❖ k-ε model

- Deformation tensor and stress tensor

$$s_{ij} = \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{bmatrix} \qquad \tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

$$s_{ij}(t) = S_{ij} + s'_{ij}$$

$$s_{xx}(t) = S_{xx} + s'_{xx} = \qquad s_{yy}(t) = S_{yy} + s'_{yy} = \qquad s_{zz}(t) = S_{zz} + s'_{zz} =$$

$$s_{xy}(t) = S_{xy} + s'_{xy} = s_{yx}(t) = S_{yx} + s'_{yx} =$$

$$s_{xz}(t) = S_{xz} + s'_{xz} = s_{zx}(t) = S_{zx} + s'_{zx} = \frac{1}{2}\left[\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right] + \frac{1}{2}\left[\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right]$$

$$s_{yz}(t) = S_{yz} + s'_{yz} = s_{zy}(t) = S_{zy} + s'_{zy} = \frac{1}{2}\left[\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right] + \frac{1}{2}\left[\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right]$$

3.7 RANS Equations and Classical Turbulence Models

❖ k-ε model

- Product of vector and tensor

$$\mathbf{a}b_{ij} \equiv a_i b_{ij} = [a_1\ a_2\ a_3]\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_1b_{11} + a_2b_{21} + a_3b_{31} \\ a_1b_{12} + a_2b_{22} + a_3b_{32} \\ a_1b_{13} + a_2b_{23} + a_3b_{33} \end{bmatrix}^T = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}^T = c_j = \mathbf{c}$$

- Scalar product of two tensors

$$\begin{aligned} a_{ij} \,.\, b_{ij} = {} & a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} \\ & + a_{31}b_{31} + a_{32}b_{32} + a_{33}b_{33} \end{aligned}$$

3.7 RANS Equations and Classical Turbulence Models

❖ k-ε model

- Governing equation for mean flow kinetic energy K
  - (x-directional Reynolds eq.) × U
  - (y-directional Reynolds eq.) × V
  - (w-directional Reynolds eq.) × W
  - Sum

$$\frac{\partial(\rho K)}{\partial t} + \operatorname{div}(\rho K\mathbf{U}) = \operatorname{div}(-P\mathbf{U} + 2\mu\mathbf{U}S_{ij} - \rho\mathbf{U}\,\overline{u_i'u_j'}) - 2\mu S_{ij}\,.\,S_{ij} + \rho\,\overline{u_i'u_j'}\,.\,S_{ij}$$

Rate of change of mean kinetic energy K + Transport of K by convection = Transport of K by pressure + Transport of K by viscous stresses + Transport of K by Reynolds stress − Rate of viscous dissipation of K − Rate of destruction of K due to turbulence production

3.7 RANS Equations and Classical Turbulence Models

k-ε model

- Governing equation for turbulent kinetic energy k
    - (x-directional Reynolds eq.) × u’
    - (y-directional Reynolds eq.) × v’
    - (z-directional Reynolds eq.) × w’
    - Sum

$$\frac{\partial(\rho k)}{\partial t} + \operatorname{div}(\rho k \mathbf{U}) = \operatorname{div}\left(-\overline{p'\mathbf{u}'} + 2\mu\overline{\mathbf{u}'s'_{ij}} - \rho\tfrac{1}{2}\overline{u'_i \,.\, u'_i u'_j}\right) - 2\mu\overline{s'_{ij}\,.\,s'_{ij}} - \rho\overline{u'_i u'_j}\,.\,S_{ij}$$

$$\begin{matrix}\text{Rate of change of} & & \text{Transport} & & \text{Transport} & & \text{Transport of} & & \text{Transport of} & & \text{Rate of} & & \text{Rate of} \\ \text{turbulent kinetic} & + & \text{of } k \text{ by} & = & \text{of } k \text{ by} & + & k \text{ by viscous} & + & k \text{ by Reynolds} & - & \text{dissipation} & + & \text{production} \\ \text{energy } k & & \text{convection} & & \text{pressure} & & \text{stresses} & & \text{stress} & & \text{of } k & & \text{of } k\end{matrix}$$

$$-2\mu\overline{s'_{ij}\,.\,s'_{ij}} = -2\mu\left(\overline{s'^2_{11}} + \overline{s'^2_{22}} + \overline{s'^2_{33}} + 2\overline{s'^2_{12}} + 2\overline{s'^2_{13}} + 2\overline{s'^2_{23}}\right)$$

$$\Phi = \mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right\} + \lambda(\operatorname{div}\mathbf{u})^2$$

$$\boxed{\varepsilon = 2\nu\overline{s'_{ij}\,.\,s'_{ij}}}$$

Rate of dissipation of turbulent kinetic energy per unit mass

3.7 RANS Equations and Classical Turbulence Models

k-ε model

- k-ε model equations (standard k-ε model)
  - Two model equations
    - k
    - ε

$$\vartheta = k^{1/2} \quad \ell = \frac{k^{3/2}}{\varepsilon}$$

ε: dissipation, [m^2/s^3]

  - Small eddy variable to define the large eddy scale!
  - Because at high Reynolds numbers the rate at which large eddies extract energy from the mean flow is broadly matched to the rate of transfer of energy across the energy spectrum to small, dissipating, eddies if the flow does not change too rapidly.

$$\mu_t =$$

3.7 RANS Equations and Classical Turbulence Models

k-ε model

- k-ε model equations (standard k-ε model)

$$\frac{\partial(\rho k)}{\partial t} + \operatorname{div}(\rho k \mathbf{U}) = \operatorname{div}\left[\frac{\mu_t}{\sigma_k} \operatorname{grad} k\right] + 2\mu_t S_{ij} \,.\, S_{ij} - \rho\varepsilon$$

$$\frac{\partial(\rho \varepsilon)}{\partial t} + \operatorname{div}(\rho \varepsilon \mathbf{U}) = \operatorname{div}\left[\frac{\mu_t}{\sigma_\varepsilon} \operatorname{grad} \varepsilon\right] + C_{1\varepsilon}\frac{\varepsilon}{k} 2\mu_t S_{ij} \,.\, S_{ij} - C_{2\varepsilon}\rho\frac{\varepsilon^2}{k}$$

$$C_\mu = 0.09 \quad \sigma_k = 1.00 \quad \sigma_\varepsilon = 1.30 \quad C_{1\varepsilon} = 1.44 \quad C_{2\varepsilon} = 1.92$$

3.7 RANS Equations and Classical Turbulence Models

k-ε model

- k-ε model equations (standard k-ε model)

$$\frac{\partial(\rho k)}{\partial t} + \mathrm{div}(\rho k \mathbf{U}) = \mathrm{div}\left(-\overline{p'\mathbf{u}'} + 2\mu\overline{\mathbf{u}'s'_{ij}} - \rho\tfrac{1}{2}\overline{u'_i \,.\, u'_i u'_j}\right) - 2\mu\overline{s'_{ij} \,.\, s'_{ij}} - \rho\overline{u'_i u'_j} \,.\, S_{ij}$$

$$\varepsilon =$$

$$\tau_{ij} = -\rho\,\overline{u'_i u'_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\rho k \delta_{ij}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$$\tau_{ij} = -\rho\overline{u'_i u'_j} =$$

$$-\rho\overline{u'_i u'_j} \cdot S_{ij} =$$

$$\frac{\partial(\rho k)}{\partial t} + \mathrm{div}(\rho k \mathbf{U}) = \mathrm{div}\left[\frac{\mu_t}{\sigma_k}\,\mathrm{grad}\ k\right] +$$

3.7 RANS Equations and Classical Turbulence Models

- *k*-*ε* model
  - *k*-*ε* model equations (standard *k*-*ε* model)

$$\frac{\partial(\rho k)}{\partial t} + \mathrm{div}(\rho k\mathbf{U}) = \boxed{\mathrm{div}\left(-\overline{p'\mathbf{u}'} + 2\mu\overline{\mathbf{u}'s'_{ij}} - \rho\tfrac{1}{2}\overline{u'_i\,.\,u'_iu'_j}\right)} - 2\mu\overline{s'_{ij}\,.\,s'_{ij}} - \rho\overline{u'_iu'_j}\,.\,S_{ij}$$

$$\Big\downarrow$$

Transport terms

$$\frac{\partial(\rho k)}{\partial t} + \mathrm{div}(\rho k\mathbf{U}) = \boxed{\qquad\qquad\qquad} + 2\mu_t S_{ij}\,.\,S_{ij} - \rho\varepsilon$$

Modeled using the gradient of k

3.7 RANS Equations and Classical Turbulence Models

k-ε model

- k-ε model equations (standard k-ε model)

$$\frac{\partial(\rho k)}{\partial t} + \operatorname{div}(\rho k \mathbf{U}) = \operatorname{div}\left[\frac{\mu_t}{\sigma_k}\operatorname{grad} k\right] + 2\mu_t S_{ij} \,.\, S_{ij} - \rho\varepsilon$$

$$\frac{\partial(\rho \varepsilon)}{\partial t} + \operatorname{div}(\rho \varepsilon \mathbf{U}) = \operatorname{div}\left[\frac{\mu_t}{\sigma_\varepsilon}\operatorname{grad} \varepsilon\right] + C_{1\varepsilon}\frac{\varepsilon}{k}2\mu_t S_{ij} \,.\, S_{ij} - C_{2\varepsilon}\rho\frac{\varepsilon^2}{k}$$

  - Production and destruction of turbulent kinetic energy are always closely linked.
  - Dissipation rate is larger where production of k is large.

$$\frac{\varepsilon}{k}\left(C_{1\varepsilon} 2\mu_t S_{ij} \cdot S_{ij} - C_{2\varepsilon}\rho\varepsilon\right)$$

$\dfrac{\varepsilon}{k}$: to make the terms dimensionally correct

$$\boxed{\mu_t = C\rho\vartheta\ell = \rho C_\mu \frac{k^2}{\varepsilon}}$$

$$\boxed{\tau_{ij} = -\rho\,\overline{u_i' u_j'} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\rho k \delta_{ij}}$$

3.7 RANS Equations and Classical Turbulence Models

k-ε model: Boundary conditions

- k-ε model equations: elliptic equation by virtue of the gradient diffusion term
- Inlet: distribution of k and ε must be given
- Outlet, symmetry axis

  $$\frac{\partial k}{\partial n} = 0 \ , \ \frac{\partial \varepsilon}{\partial n} = 0$$

- Free stream: k and ε must be given or $\frac{\partial k}{\partial n} = 0 \ , \ \frac{\partial \varepsilon}{\partial n} = 0$
- Solid walls: approach depends on Reynolds number
- In real simulations,
  - At the inlet

    $$k = \frac{2}{3}\left(U_{ref} T_i\right)^2 \qquad \varepsilon = C_\mu^{3/4} \frac{k^{3/2}}{l} \qquad l = 0.07L$$

  - High Re case

    $$u^+ = \frac{1}{\kappa} \ln E y^+ \quad \text{for} \qquad 30 < y^+ < 500$$

the rate of turbulence production
= rate of dissipation

$$k = \frac{u_\tau^{\ 2}}{\sqrt{C_\mu}} \qquad \varepsilon = \frac{u_\tau^{\ 3}}{\kappa y}$$

3.7 RANS Equations and Classical Turbulence Models

- k-ε model

the rate of turbulence production
= rate of dissipation

$$P = \frac{\mu_t}{\rho} S^2 = \frac{\mu_t}{\rho}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)^2 \approx \frac{\mu_t}{\rho}\left(\frac{\partial U}{\partial y}\right)^2 = \varepsilon$$

wall function

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$$

turbulent viscosity model

$$\mu_t = C\rho\vartheta\ell = \rho C_\mu \frac{k^2}{\varepsilon}$$

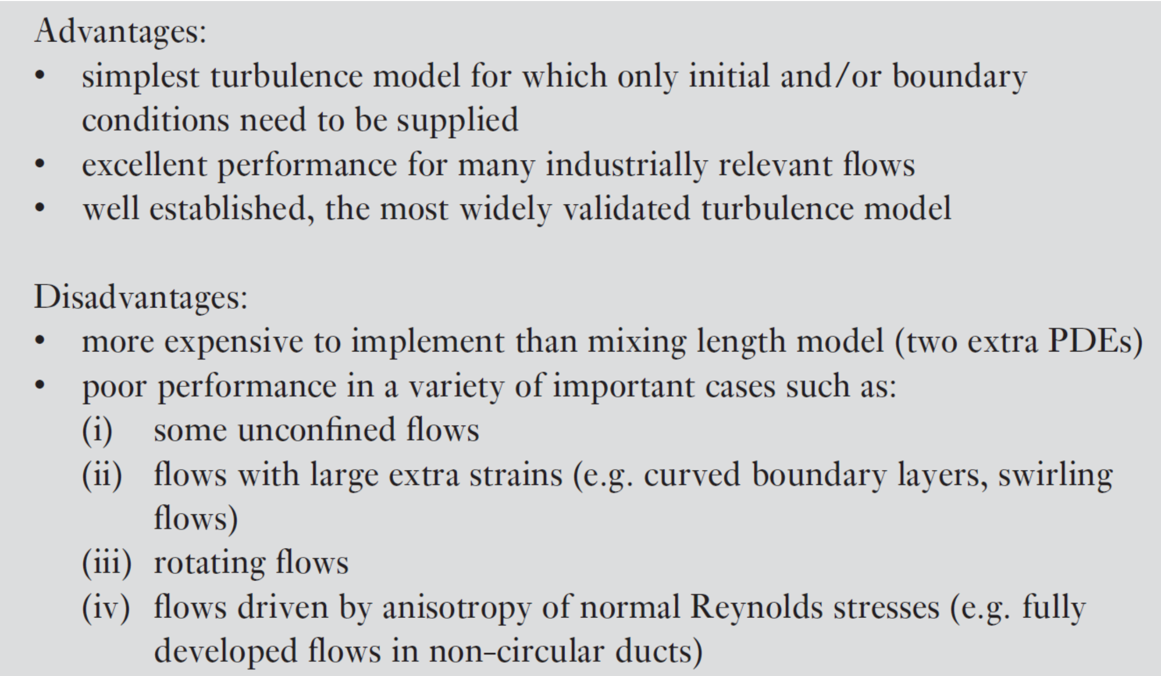
Wall shear stress

$$\tau_w = \rho u_\tau^2 = \mu_t \frac{\partial U}{\partial y}$$

3.7 RANS Equations and Classical Turbulence Models

- k-ε model
  - ε-equation: main sources of accuracy limitations
  - Boussinesq isotropic assumption
    - Causes problems in swirling flows
    - Flows with large rapid extra strains
    - Secondary flows in long non-circular ducts
      - Driven by anisotropic normal Reynolds stresses
      - Cannot be predicted

$$\tau_{ij} = -\rho \overline{u_i' u_j'} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij}$$



Advantages:

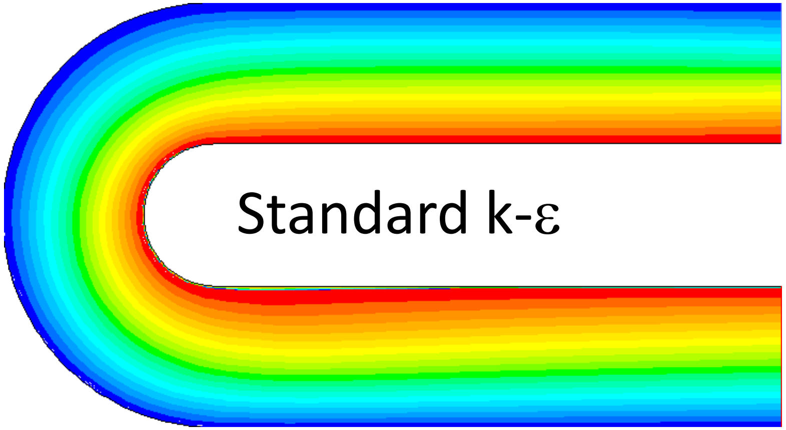
- simplest turbulence model for which only initial and/or boundary conditions need to be supplied
- excellent performance for many industrially relevant flows
- well established, the most widely validated turbulence model

Disadvantages:

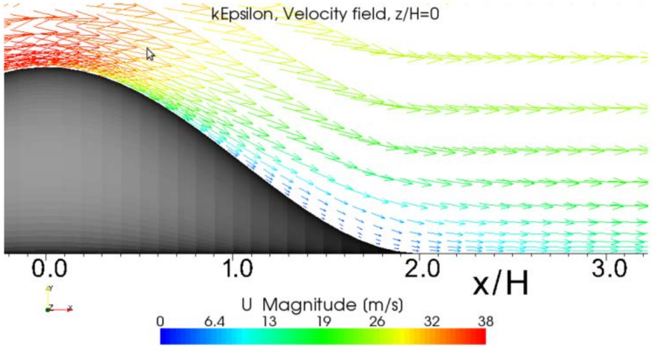
- more expensive to implement than mixing length model (two extra PDEs)
- poor performance in a variety of important cases such as:
  - (i) some unconfined flows
  - (ii) flows with large extra strains (e.g. curved boundary layers, swirling flows)
  - (iii) rotating flows
  - (iv) flows driven by anisotropy of normal Reynolds stresses (e.g. fully developed flows in non-circular ducts)

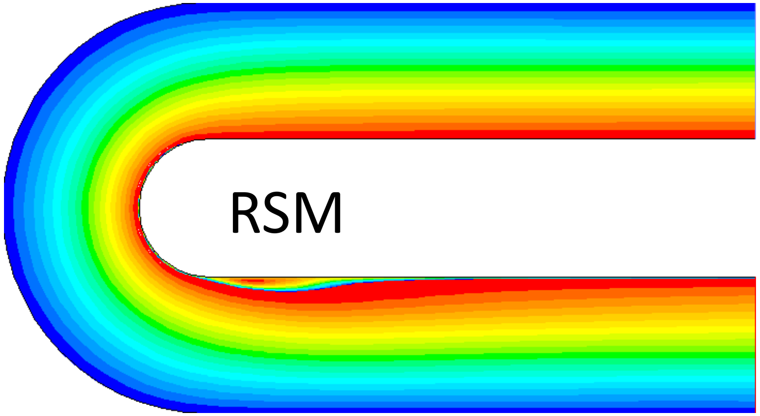
3.7 RANS Equations and Classical Turbulence Models

- k-ε model

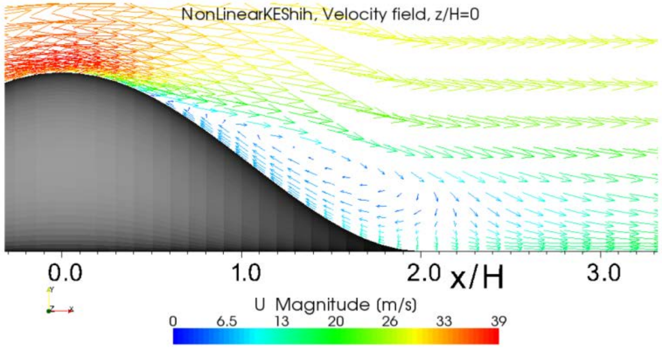


Standard k-ε





RSM



3.7 RANS Equations and Classical Turbulence Models

- RANS models
  - Mixing length models
  - k-ε model
  - Reynolds stress equation models
  - Algebraic stress models
  - k-ω model

- LES and DNS