

457.212 Statistics for Civil & Environmental Engineers
In-Class Material: Class 13
Multiple Random Variables – Part I (A&T: 3.3, Supp #2)

<< Complete description of random variable(s) >>

- Single random variable, X
 - PMF (discrete) $p_X(x) =$
 - PDF (continuous) $f_X(x) =$
 - CDF (discrete/continuous) $F_X(x) =$

- $P(X \leq 20)?$

- Random variables, X (weight) and Y (height):
 - $P(X \leq x_{0.1} \cap Y \leq y_{0.1})?$
 - $P(X \leq x_{0.1} \cup Y > y_{0.9})?$

- Complete description of multiple random variables: “joint” PMF, PDF, and CDF

1. **Joint** cumulative distribution function (**CDF**) ~ continuous/discrete r.v.

$$F_{XY}(x, y) = P(\quad \cap \quad) = P(\quad, \quad)$$

(a) $F_{XY}(-\infty, -\infty) =$, $F_{XY}(\infty, \infty) =$

(b) $F_{XY}(-\infty, y) =$, $F_{XY}(\infty, y) =$
 $F_{XY}(x, -\infty) =$, $F_{XY}(x, \infty) =$

(c) $F_{XY}(x, y)$ is non-negative, non-decreasing function of x and y .

2. **Joint** probability mass function (**PMF**) ~ discrete r.v.

$$p_{XY}(x, y) = P(\quad, \quad)$$

(a) $F_{XY}(x, y) = \sum_{\{x_i \leq x, y_i \leq y\}} p_{XY}(x_i, y_i)$

(b) **Conditional Probability Mass Function**

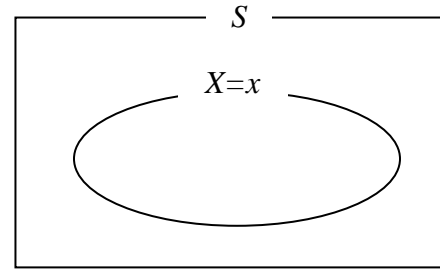
$$p_{X|Y}(x | y) = P(\quad | \quad) = \frac{P(\quad)}{P(\quad)} = \text{---}$$

$$p_{Y|X}(y | x) = \text{---}$$

(c) $p_X(x)$ from $p_{XY}(x, y)$ (“marginal” from “joint”)

$$\begin{aligned}
 p_X(x) &= \sum_{\text{all } y_i} P(X = x | \quad) P(\quad) \\
 &= \sum_{\text{all } y_i} P(\quad) \\
 &= \sum
 \end{aligned}$$

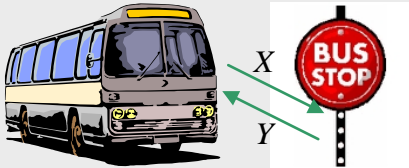
$$p_Y(y) = \sum \quad \text{(marginalization rule)}$$



(d) If X and Y are statistically independent, $p_{X|Y}(x | y) =$

Thus, $p_{XY}(x, y) =$

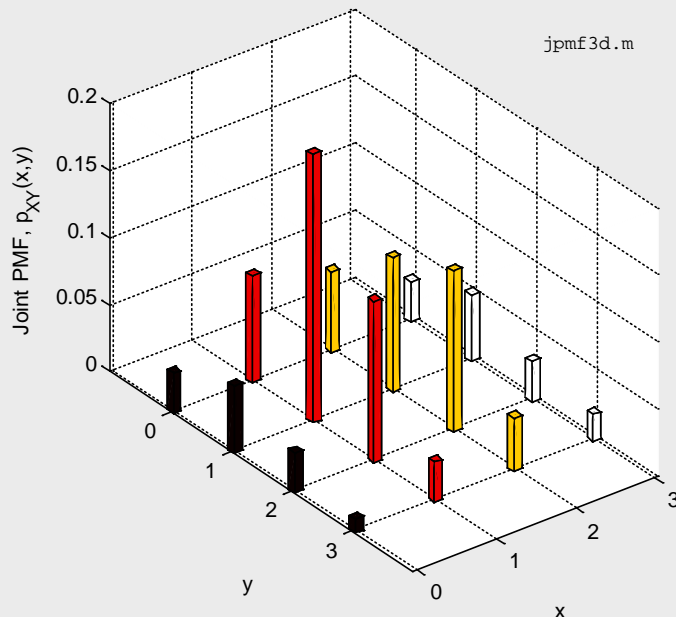
Example 1: Uncertain number of passengers getting off/on at bust stop A



X : number of passengers getting off at bus stop A
 Y : number of passengers getting on at bus stop A

Joint PMF of X and Y , $P_{XY}(x, y) = P(X = x \cap Y = y)$ is known as

$y \backslash x$	0	1	2	3	$p_Y(y)$
0	0.03	0.08	0.06	0.03	
1	0.05	0.20	0.10	0.05	
2	0.03	0.12	0.12	0.03	
3	0.01	0.03	0.04		
$p_X(x)$					



- (1) Obtain the marginal PMF's of X and Y from the joint PMF
- (2) Find the value of $p_{XY}(3,3)$
- (3) Suppose there are two passengers waiting for the bus to arrive. What's the probability that one passenger will get off the bus?
- (4) Are $X = x$ and $Y = y$ statistically independent?

For your information, the following table shows $p_X(x) \times p_Y(y)$.

$y \backslash x$	0	1	2	3
0	0.024	0.086	0.064	0.026
1	0.048	0.172	0.128	0.052
2	0.036	0.129	0.096	0.039
3	0.012	0.043	0.032	0.013

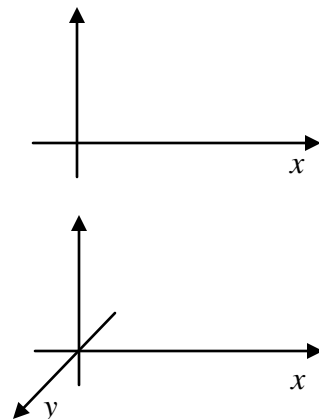
Another way to check is to compare the marginal PMF's with the conditional PMF's. For example, compare the following conditional PMF's $p_{Y|X}(1|x)$ with the marginal PMF $p_Y(1) = 0.40$.

$y \backslash x$	0	1	2	3
1	0.417	0.465	0.313	0.385

- (5) Are $X = 3$ and $Y = 2$ mutually exclusive?

3. **Joint probability density (PDF)** ~ continuous r.v.

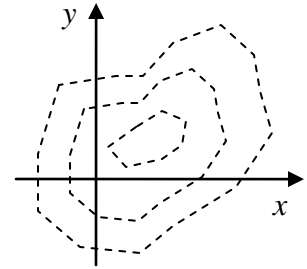
- Marginal PDF: $f_X(x) = \lim_{\Delta x \rightarrow 0} \frac{P(x < X \leq x + \Delta x)}{\Delta x}$
 ~ Probability over unit ()
- Joint PDF:
 $f_{XY}(x, y) = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{P(x < X \leq x + \Delta x \cap y < Y \leq y + \Delta y)}{\Delta x \Delta y}$
 ~ Probability over unit ()



(a) Relationship with joint CDF

$$F_{XY}(x, y) =$$

$$f_{XY}(x, y) = \text{—————}$$

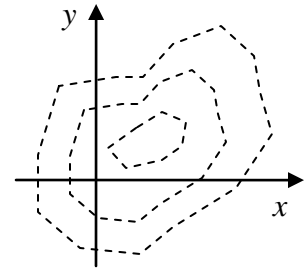


(b) $P(a < X \leq b, c < Y \leq d) =$

(c) **Conditional Probability Density Function**

~ Density of a random variable when the outcome of the other r.v. is known

$$f_{X|Y}(x | y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$



Multiplication rule:

$$f_{XY}(x, y) =$$

$$=$$

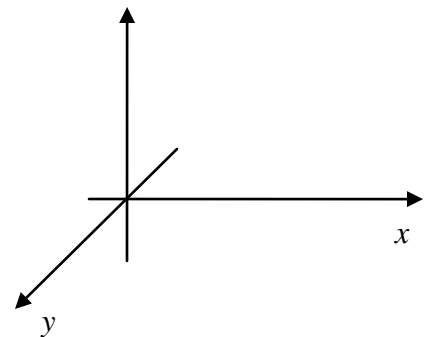
If, X and Y are statistically independent,

(d) Joint PDF \rightarrow marginal PDF

(Recall ~ Discrete case: marginalization rule – sum over the other random variable)

$$f_X(x) =$$

$$f_Y(y) =$$

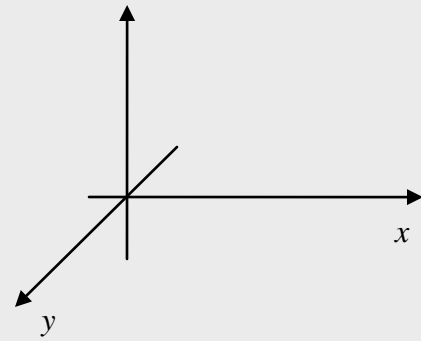


Example 2: Consider non-negative random variables X and Y . The likelihood of the pair (x, y) is proportional to $\exp[-3(x + y)]$.

(a) Joint PDF?

(a-2) Sketch of joint PDF?

(b) Joint CDF?



(c) Check the result in (b)? How?

(d) Marginal PDF's?

(e) Check

$$F_{XY}(-\infty, -\infty)$$

$$F_{XY}(\infty, \infty)$$

$$F_{XY}(-\infty, y)$$

$$F_{XY}(\infty, y)$$

(f) Conditional PDF's?

(g) $P(1 \leq X \leq 2, 1 \leq Y \leq 2) =$