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457.212 Statistics for Civil & Environmental Engineers

In-Class Material: Class 18

Mathematical Expectations of Nonlinear Functions (A&T: 4.3; Supplement #7)

Nonlinear Function: $Y_k = g_k(X_1, ..., X_n)$, $k = 1, ..., m$

Given: (　　), (　　) and (　　)

Want to know: (　　), (　　), (　　) and (　　)

e.g. Natural period of a pendulum: $T = g(M, K) = 2\pi\sqrt{\frac{M}{K}}$

Strain energy of a prismatic bar: $U = g(L, A, E, F) = \frac{L}{2AE}F^2$

Unlike linear functions, we do not have formulas for exact mathematical expectations that work for any nonlinear functions.

Usually, we obtain the mathematical expectations approximately by the following steps:

(1) Linearize the nonlinear function(s) using a truncated Taylor series expansion
(2) Use the formulas originally given for linear functions for the linearized one from (1)

1. **Nonlinear Function of a Single Random Variable**

$$Y = g(X)$$

(a) First-order approximation of $g(X)$ by a Taylor series expansion at the mean value of X:

$$Y = g(\mu_X) + \left.\frac{dg}{dx}\right|_{X=\mu_X}(X - \mu_X) + \frac{1}{2}\left.\frac{d^2g}{dx^2}\right|_{X=\mu_X}(X - \mu_X)^2 + \cdots$$

$$\cong g(\mu_X) + \left.\frac{dg}{dx}\right|_{X=\mu_X}(X - \mu_X)$$

Higher-order terms truncated

(b) Recall, for $Y = a_0 + a_1 X$,

$$\mu_Y = a_0 + a_1\mu_X \text{ and } \sigma_Y^2 = a_1^2\sigma_X^2$$

(c) Therefore,

$$\mu_Y \cong g(\mu_X) + \left.\frac{dg}{dx}\right|_{X=\mu_X}(\mu_X - \mu_X) = g(\mu_X)$$ ~ First-order approximation of μ_Y

$$\sigma_Y^2 \cong \left(\left.\frac{dg}{dx}\right|_{X=\mu_X}\right)^2 \sigma_X^2$$ ~ First-order approximation of σ_Y^2

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Example 1: Consider the error of a measurement device, X. Its mean is 3 cm and standard deviation is 0.3 cm. What is the first-order approximation of the mean and standard deviation of the squared error X^2?

2. **Nonlinear Functions of Multiple Random Variables**

$$Y_k = g_k(X_1, \ldots, X_n), \; k = 1, \ldots, m$$

Linearize them by truncated Taylor series expansions around the mean values and use the algebraic formulas or matrix formulations for linear functions.

(a) Taylor series expansion

$$Y_k \cong g_k(\mathbf{M_X}) + \sum_{i=1}^{n} \overbrace{\left(\left. \frac{\partial g_k}{\partial x_i} \right|_{\mathbf{X}=\mathbf{M_X}} \right)}^{g_{k,i}} (X_i - \mu_{X_i}) \quad \text{Higher-order terms truncated}$$

$$= a_{k,0} + \sum_{i=1}^{n} a_{k,i} X_i$$

Note $a_{k,0} = g_k(\mathbf{M_X}) - \sum_{i=1}^{n} g_{k,i}\mu_{X_i}$ and $a_{k,i} = g_{k,j}$

(b) Algebraic formula ($n \le 2$)

- Mean: $\mu_{Y_k} \cong a_{k,0} + \sum_{i=1}^{n} a_{k,i}\mu_{X_i} = g_k(\mathbf{M_X}) - \sum_{i=1}^{n} g_{k,i}\mu_{X_i} + \sum_{i=1}^{n} g_{k,i}\mu_{X_i} = g_k(\mathbf{M_X})$
- Variance:

$$\sigma_{Y_k}^2 \cong \sum_{i=1}^{n} g_{k,i}^2 \sigma_{X_i}^2 + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} g_{k,i} g_{k,j} \sigma_{X_i} \sigma_{X_j} \rho_{X_i X_j}$$

- Covariance:

$$\text{Cov}[Y_k, Y_l] \cong \sum_{i=1}^{n} g_{k,i} g_{l,i} \sigma_{X_i}^2 + \sum_{\substack{i=1 \\ i \ne j}}^{n} \sum_{j=1}^{n} g_{k,i} g_{l,j} \sigma_{X_i} \sigma_{X_j} \rho_{X_i X_j}$$

(c) Matrix formula ($n \ge 3$)

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- Mean: $\mathbf{M_Y} = \mathbf{g}(\mathbf{M_X})$
- Covariance matrix: $\mathbf{\Sigma_{YY}} = \mathbf{A\Sigma_{XX}A}^{\mathrm{T}}$
  (Use $g_{k,i}$ instead of $a_{k,i}$ in constructing $\mathbf{A}$ matrix)

Example 2: Consider the natural period of a single-degree-of-freedom oscillator,

$$T = 2\pi\sqrt{\frac{M}{K}}$$

The means, standard deviations and correlation coefficient of the random variables M and K are given as

	μ	σ	ρ
M (kg)	120	30	−0.40
K (N)	2,000	400	

(a) The first-order approximation of the mean of T :

(b) The first-order approximation of the standard deviation of T :

(c) The covariance between T and K :

(d) The correlation coefficient between T and K :

(e) The importance measure of M and K in contributing the uncertainty in T , i.e. $\left|\frac{\partial T}{\partial M}\right|\sigma_M$ and $\left|\frac{\partial T}{\partial K}\right|\sigma_K$

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Example 3: The roof displacement of a structure during an earthquake event is often approximated in the form

$$D = C\frac{S_a}{\omega^2}$$

where C is a factor accounting for various uncertain effects, S_a is the spectral acceleration (g) and ω is the natural frequency (rad/s) of a structure. Suppose C and S_a are random variables with means $\mu_C = 1.0$, $\mu_{S_a} = 2.0$, and coefficient of variations $\delta_C = 0.1$, $\delta_{S_a} = 0.5$. The two random variables are assumed to be statistically independent. Let us consider a structure which has the natural frequency $\omega = 10$ rad/s.

a) Obtain the first-order approximation on the mean of D.

b) Obtain the first-order approximation on the standard deviation of D.

Let us now assume C and S_a are statistically independent lognormal random variables with the means and coefficients of variation given above. Based on these assumptions, answer the following questions. (Hint: The logarithm of a lognormal random variable is a normal random variable).

c) Identify the distribution of D (give its name) and provide your reasoning.

d) Determine the exact mean and coefficient of variation of D.

e) Determine the probability density function of D using the type given in (c) and parameter values found in (d).