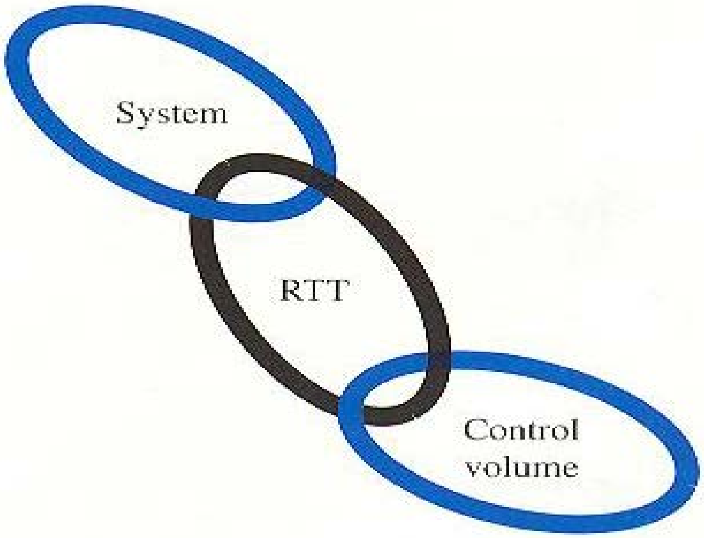
Chapter 4

Continuity Equation and Reynolds Transport Theorem



Contents

Objectives

- Apply the concept of the control volume to derive equations for the conservation of mass for steady one- and two-dimensional flows
- Derive the Reynolds transport theorem for three-dimensional flow
- Show that continuity equation can recovered by simplification of the Reynolds transport theorem

4.1 Control Volume

Physical system

~ is defined as a particular collection of matter or a region of space chosen for study

~ is identified as being separated from everything external to the system by closed boundary

- The boundary of a system: fixed vs. movable (moving) boundary
- Two types of system:

- closed system (control mass) ~ consists of a fixed mass, no mass can cross its boundary

- open system (control volume) ~ mass and energy can cross the boundary of a control volume

4.1 Control Volume

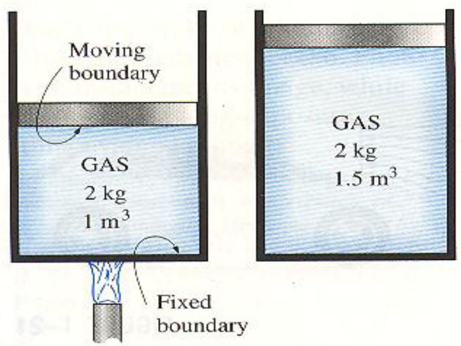
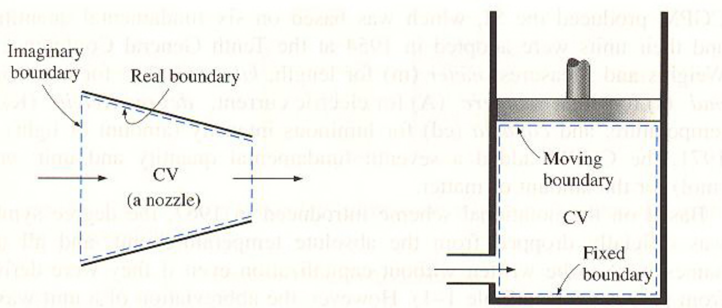


FIGURE 1–24
A closed system with a moving boundary.



(*a*) A control volume (CV) with real and imaginary boundaries

(*b*) A control volume (CV) with fixed and moving boundaries

FIGURE 1–25
A control volume may involve fixed, moving, real, and imaginary boundaries.

4.1 Control Volume

A system-based analysis of fluid flow leads to the Lagrangian equations of motion in which particles of fluid are tracked.

However, a fluid system is mobile and very deformable.

A large number of engineering problems involve mass flow in and out of a system.

→ This suggests the need to define a convenient object for analysis.

→ control volume

4.1 Control Volume

• **Control volume**

~ a volume which is fixed in space and through whose boundary matter, mass, momentum, energy can flow

~ The boundary of control volume is a control surface.

~ The control volume can be any size (finite or infinitesimal), any space.

~ The control volume can be fixed in size and shape.

→ This approach is consistent with the Eulerian view of fluid motion, in which attention is focused on particular points in the space filled by the fluid rather than on the fluid particles.

4.2 Continuity Equation for One-Dimensional Steady Flow

- Principle of conservation of mass

The application of principle of conservation of mass to a steady flow in a streamtube results in the continuity equation.

- Continuity equation

~ describes the continuity of flow from section to section of the streamtube

4.2 Continuity Equation for One-Dimensional Steady Flow

- One-dimensional steady flow

Consider the element of a finite streamtube

- no net velocity normal to a streamline

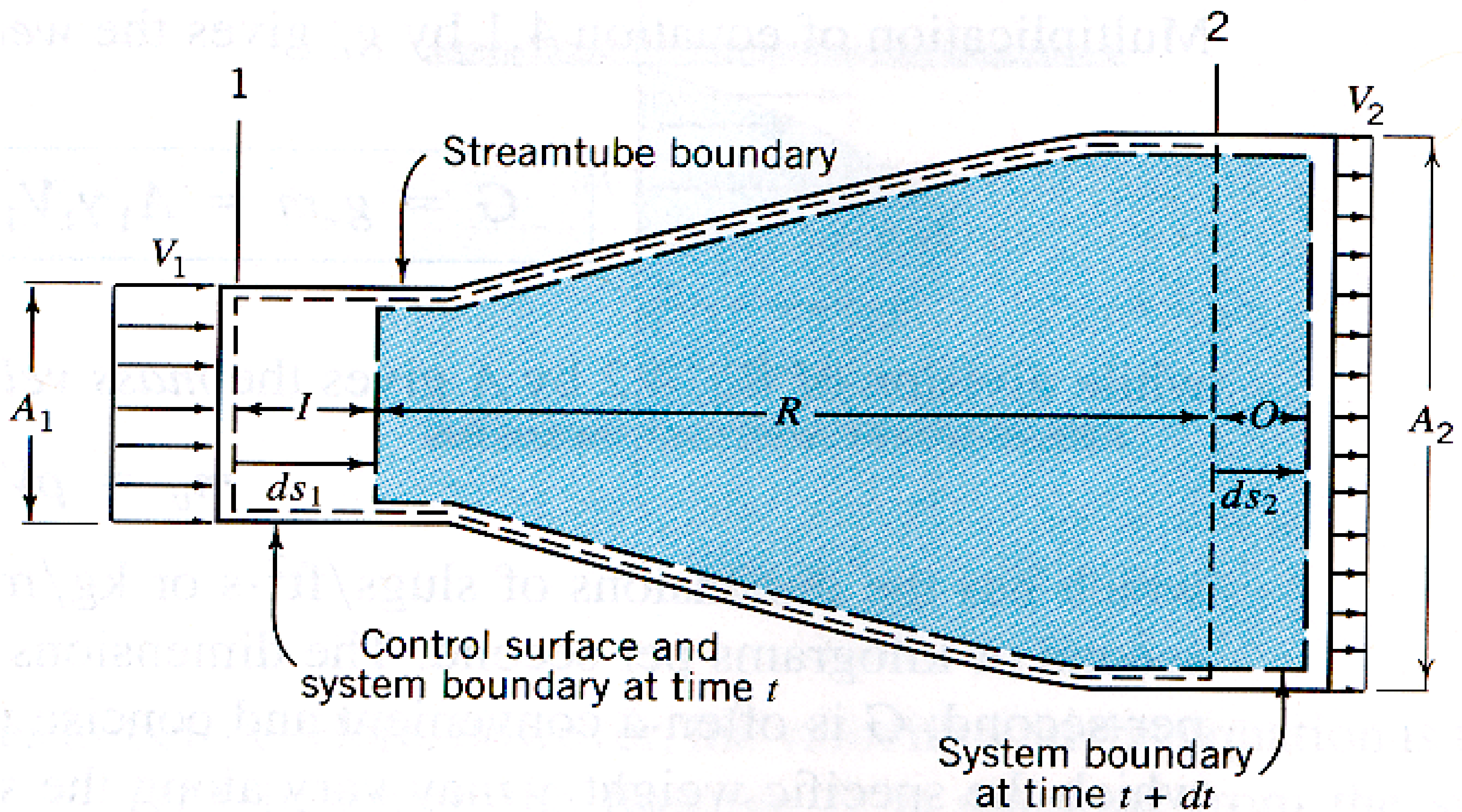
- no fluid can leave or enter the stream tube except at the ends
- Now, define the control volume as marked by the control surface that bounds the region between sections 1 and 2.

→ To be consistent with the assumption of one-dimensional steady flow, the velocities at sections 1 and 2 are assumed to be uniform.

→ The control volume comprises volumes *I* and *R*.

→ The control volume is fixed in space, but in dt the system moves downstream.

4.2 Continuity Equation for One-Dimensional Steady Flow



4.2 Continuity Equation for One-Dimensional Steady Flow

From the conservation of system mass

$$(m_I + m_R)_t = (m_R + m_O)_{t+\Delta t} \tag{1}$$

For steady flow, the fluid properties at points in space are not functions of time, $\frac{\partial m}{\partial t} = 0$

$$\rightarrow (m_R)_t = (m_R)_{t+\Delta t} \tag{2}$$

Substituting (2) into (1) yields

$$(m_I)_t = (m_O)_{t+\Delta t} \tag{3}$$

Inflow

Outflow

4.2 Continuity Equation for One-Dimensional Steady Flow

Express inflow and outflow in terms of the mass of fluid moving across the control surface in time dt

$$(m_I)_t = \rho_1 A_1 ds_1$$

$$(m_O)_{t+\Delta t} = \rho_2 A_2 ds_2 \qquad (4)$$

Substituting (4) into (3) yields

$$\rho_1 A_1 ds_1 = \rho_2 A_2 ds_2$$

Dividing by dt gives

$$\boxed{\frac{ds_1}{dt} = V_1}$$

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad (4.1)$$

→ Continuity equation

4.2 Continuity Equation for One-Dimensional Steady Flow

In steady flow, the mass flow rate, $\dot{m}$

passing all sections of a stream tube is constant.

$\dot{m} = \rho AV$ = constant (kg/sec)

$$d(\rho AV) = 0 \qquad (4.2a)$$

$$\rightarrow \; d\rho(AV) + dA(\rho V) + dV(\rho A) = 0 \qquad (5)$$

Dividing (a) by ρAV results in

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \qquad (4.2b)$$

→ 1-D steady <u>compressible</u> fluid flow

4.2 Continuity Equation for One-Dimensional Steady Flow

For incompressible fluid flow; constant density

$$\rightarrow \quad d\rho = 0, \frac{\partial \rho}{\partial t} = 0 \tag{4.3}$$

From Eq. (4.2a)

$$\rho d(AV) = 0 \tag{4.4}$$

$$d(AV) = 0 \tag{6}$$

Set Q = volume flowrate (m³/s, cms)

Then (6) becomes

$$Q = AV = \text{const.} = A_1 V_1 = A_2 V_2 \tag{4.5}$$

4.2 Continuity Equation for One-Dimensional Steady Flow

For 2-D flow, flowrate is usually quoted per unit distance normal to the plane of the flow, b

$\rightarrow$ $q =$ flowrate per unit distance normal to the plane of flow $\left(\mathrm{m}^3/\mathrm{s}\cdot\mathrm{m}\right)$

$$q = \frac{Q}{b} = \frac{AV}{b} = hV \qquad (4.6)$$

$$h_1V_1 = h_2V_2 \qquad (4.7)$$

[Re] For unsteady flow

$$mass_{t+\Delta t} = mass_t + \text{inflow} - \text{outflow}$$

$$(m_R)_{t+\Delta t} - (m_R)_t = (m_I)_t - (m_O)_{t+\Delta t}$$

4.2 Continuity Equation for One-Dimensional Steady Flow

Divide by dt

$$\frac{(m_R)_{t+\Delta t} - (m_R)_t}{dt} = (m_I)_t - (m_O)_{t+\Delta t}$$

Define

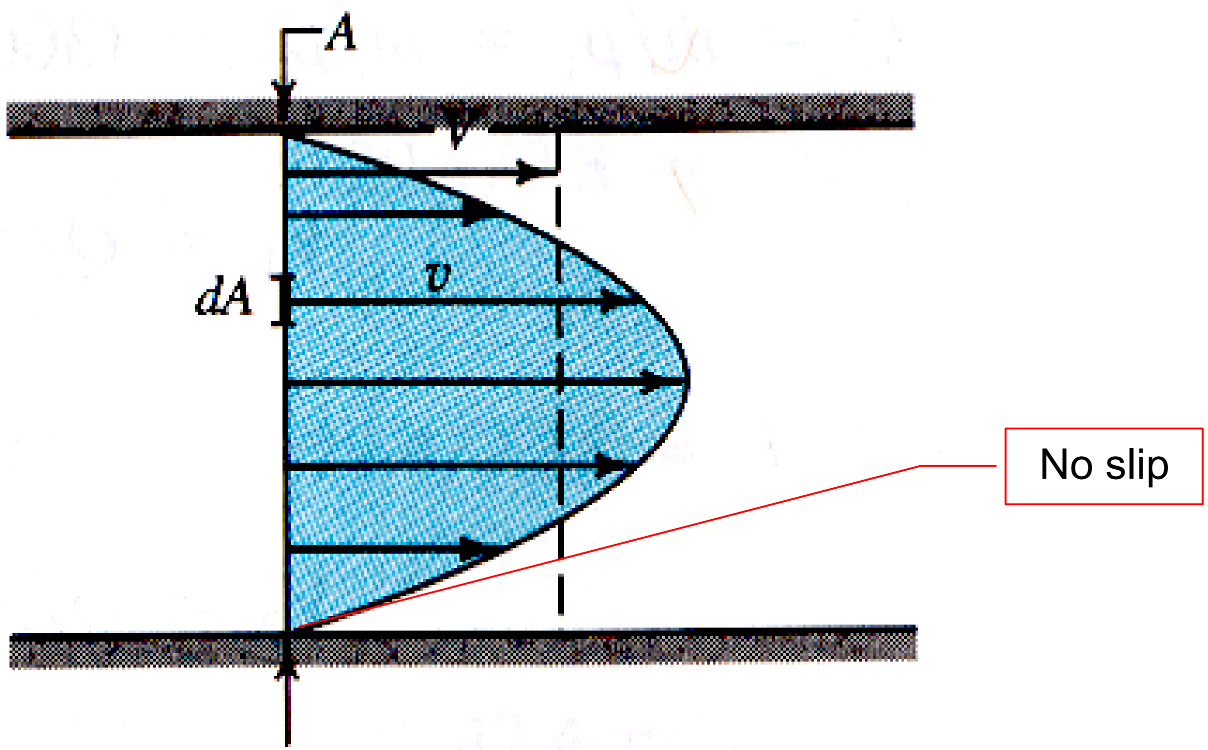
$$\frac{\partial m}{\partial t} = \frac{(m_R)_{t+\Delta t} - (m_R)_t}{dt} = \frac{\partial(\rho vol)}{\partial t}$$

Then

$$\frac{\partial(\rho vol)}{\partial t} = (m_I)_t - (m_O)_{t+\Delta t}$$

4.2 Continuity Equation for One-Dimensional Steady Flow

- Non-uniform velocity distribution through flow cross section



4.2 Continuity Equation for One-Dimensional Steady Flow

Eq. (4.5) can be applied. However, velocity in Eq. (4.5) should be the mean velocity.

$$V = \frac{Q}{A} \qquad Q = \int_A dQ = \int_A v dA$$

$$\therefore \quad V = \frac{1}{A} \int_A v dA$$

•The product AV remains constant along a streamline in a fluid of constant density.

→ As the cross-sectional area of stream tube increases, the velocity must decrease.

→ Streamlines <u>widely spaced indicate regions of low velocity</u>, streamlines <u>closely spaced indicate regions of high velocity</u>.

$$A_1 V_1 = A_2 V_2 : \; A_1 > A_2 \rightarrow V_1 < V_2$$

4.2 Continuity Equation for One-Dimensional Steady Flow

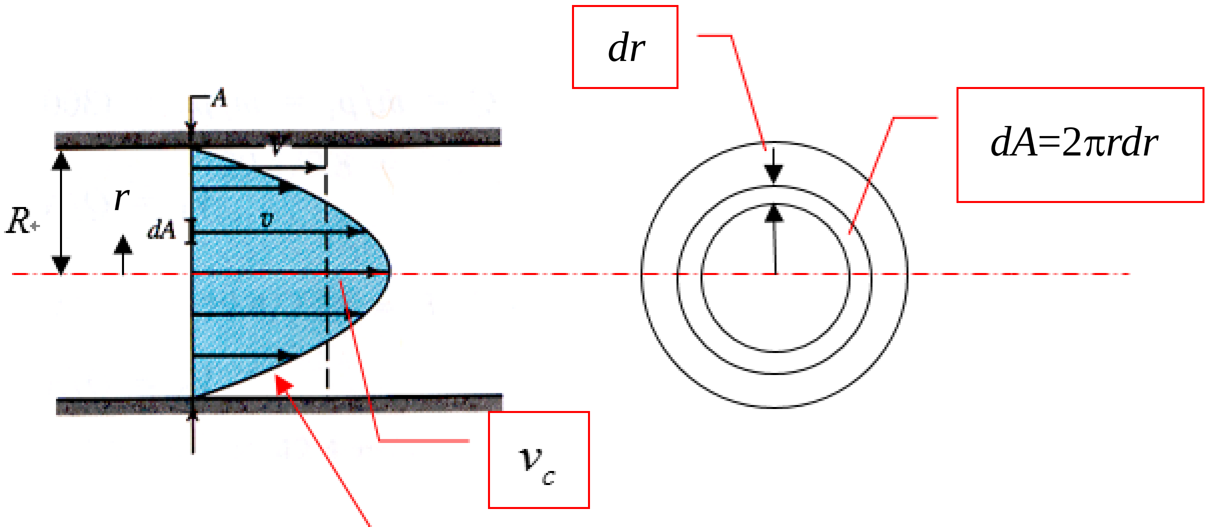
[IP 4.3] p. 113

The velocity in a cylindrical pipe of radius R

is represented by an axisymmetric parabolic distribution (laminar flow).

What is V in terms of maximum velocity, v_c ?

4.2 Continuity Equation for One-Dimensional Steady Flow



$$v = v_c\left(1 - \frac{r^2}{R^2}\right)$$ ← equation of paraboloid

4.2 Continuity Equation for One-Dimensional Steady Flow

$$V = \frac{Q}{A} = \frac{1}{A}\int_A v\,dA = \frac{1}{\pi R^2}\int_0^R v_c\left(1 - \frac{r^2}{R^2}\right)2\pi r\,dr$$

$$= \frac{2v_c}{R^2}\int_0^R\left(r - \frac{r^3}{R^2}\right)dr = \frac{2v_c}{R^2}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = \frac{2v_c}{R^2}\left[\frac{R^2}{2} - \frac{R^2}{4}\right] = \frac{v_c}{2}$$ → Laminar flow

[Cf] Turbulent flow

→ logarithmic velocity distribution

4.2 Continuity Equation for One-Dimensional Steady Flow

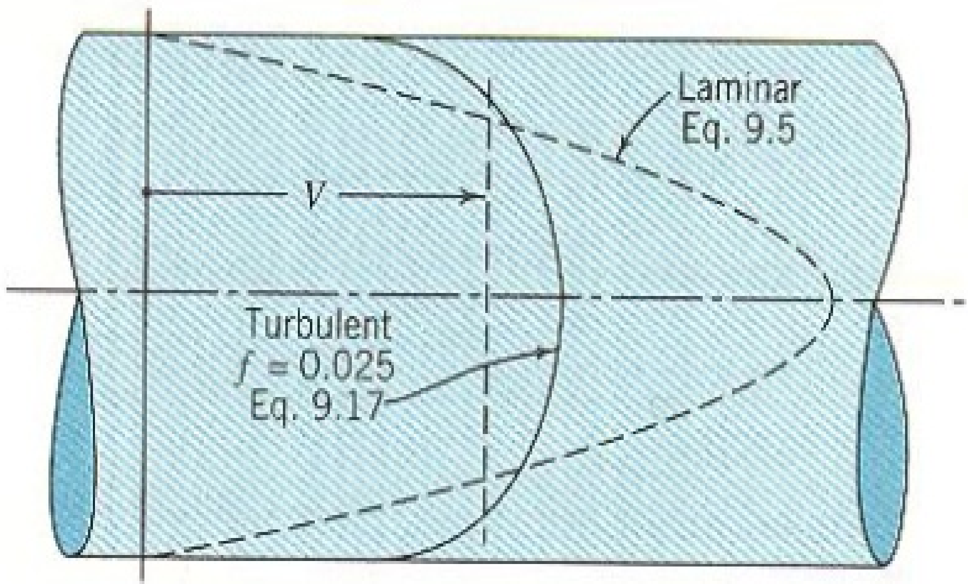
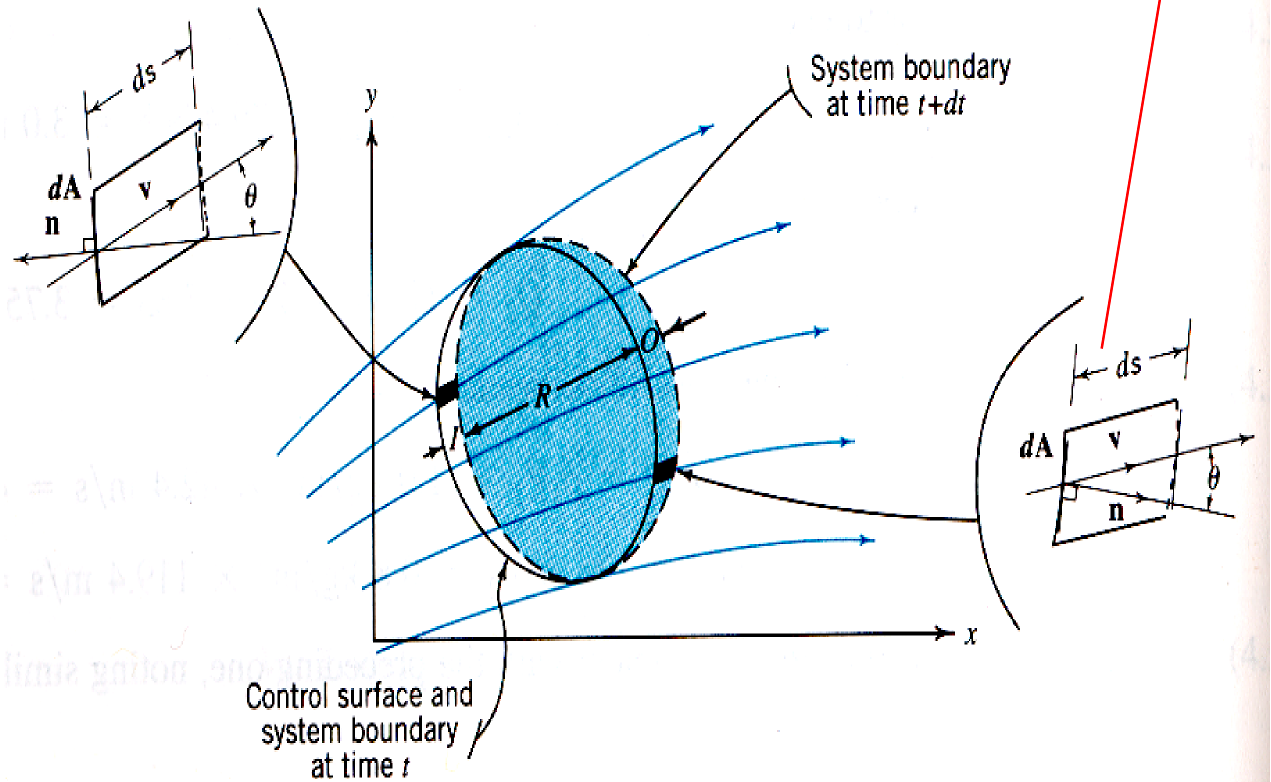


Fig. 9.5

4.3 Continuity Equation for Two-Dimensional Steady Flow

(1) Finite control volume

Area of parallelogram =
$(ds\cos\theta)dA$



4.3 Continuity Equation for Two-Dimensional Steady Flow

Consider a general control volume, and apply conservation of mass

$$(m_I + m_R)_t = (m_R + m_O)_{t+\Delta t} \qquad \text{(a)}$$

For steady flow: $(m_R)_t + (m_R)_{t+\Delta t}$

Then (a) becomes

$$(m_I)_t = (m_O)_{t+\Delta t} \qquad \text{(b)}$$

i) Mass in *O* moving out through control surface

$$(m_O)_{t+\Delta t} = \int_{C.S.out} \rho (ds\cos\theta) dA$$

$$\boxed{\text{mass} = \rho\ vol = \rho \times \text{area} \times 1 = \rho\, ds\, dA \cos\theta}$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

- Displacement along a streamline is

$$ds = vdt \qquad \text{(c)}$$

Substituting (c) into (b) gives

$$(m_O)_{t+\Delta t} = \int_{C.S.out} \rho(v\cos\theta)\,dA\,dt \qquad \text{(d)}$$

By the way, $v\cos\theta$ = normal velocity component normal to C.S. at dA

Set $\vec{n}$ = outward unit normal vector at dA $\left(|\vec{n}|=1\right)$

$$\therefore v_n = \vec{v}\cdot\vec{n} = v\cos\theta \leftarrow \text{scalar or dot product} \qquad \text{(e)}$$

Substitute (e) into (d)

$$(m_O)_{t+\Delta t} = dt\int_{C.S.out} \rho\vec{v}\cdot\vec{n}\,dA = dt\int_{C.S.out} \rho\vec{v}\cdot\overrightarrow{dA} \qquad \text{(4.8)}$$

where $\overrightarrow{dA} = \vec{n}\,dA$ =directed area element

4.3 Continuity Equation for Two-Dimensional Steady Flow

[Cf] Tangential component of velocity does not contribute to flow through the C.S.

→ It contributes to circulation

ii) Mass flow into CV

$$(m_I)_t = \int_{C.S.in} \rho(ds\cos\theta)dA$$

$$\boxed{\theta > 90^\circ \rightarrow \cos\theta < 0}$$

$$\int_{C.S.in} \rho(v\cos\theta)dAdt = dt\int_{C.S.in} \rho\vec{v}\cdot(-\vec{n})dA$$

$$= dt\left\{-\int_{C.S.in} \rho\vec{v}\cdot\vec{n}dA\right\} = dt\left\{-\int_{C.S.in} \rho\vec{v}\cdot\overrightarrow{dA}\right\}$$

For steady flow, *mass in = mass out*

$$dt\int_{C.S.out} \rho\vec{v}\cdot\overrightarrow{dA} = dt\left\{-\int_{C.S.in} \rho\vec{v}\cdot\overrightarrow{dA}\right\}$$

Divide by dt

$$-\int_{C.S.in} \rho\vec{v}\cdot\overrightarrow{dA} = \int_{C.S.out} \rho\vec{v}\cdot\overrightarrow{dA}$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

$$\int_{C.S.out} \rho \vec{v} \cdot \overrightarrow{dA} + \int_{C.S.in} \rho \vec{v} \cdot \overrightarrow{dA} = 0 \qquad \text{(f)}$$

Combine C.S. in and C.S. out

Integral form

$$\boxed{\oint_{C.S.} \rho \vec{v} \cdot \overrightarrow{dA} = \oint_{C.S.} \rho \vec{v} \cdot \vec{n} dA = 0}$$

where $\oint_{C.S.} =$ integral around the control surface in the counterclockwise direction

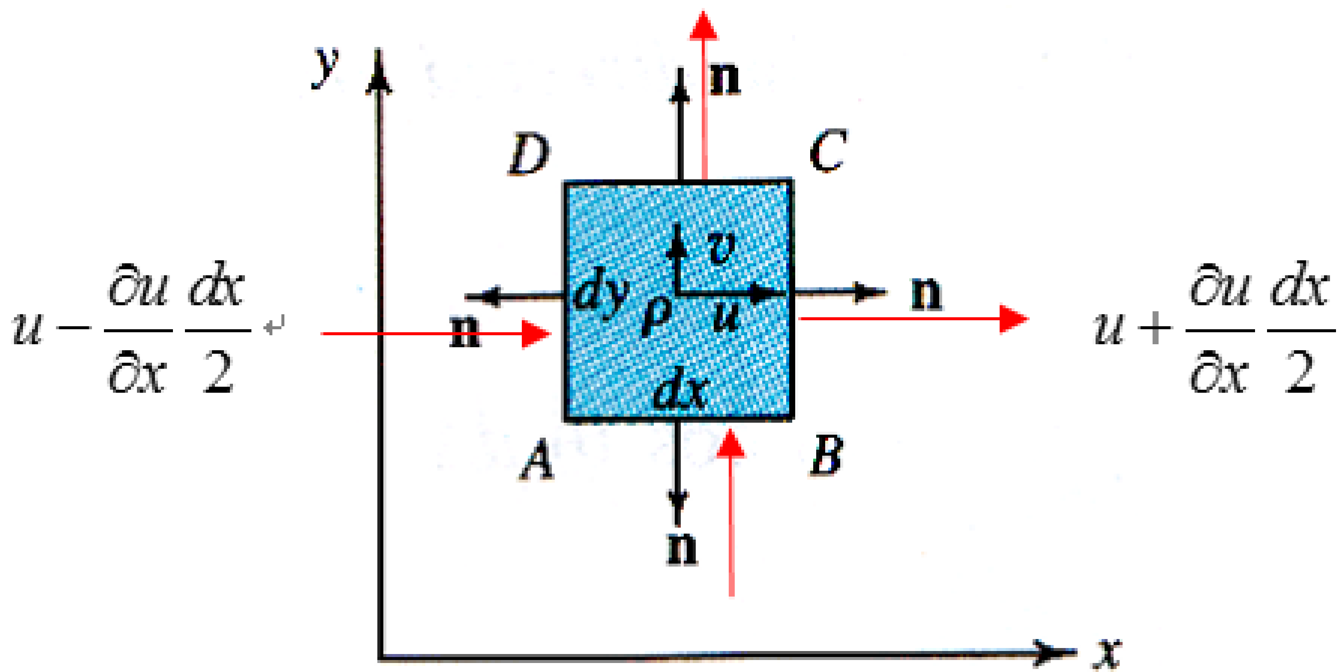
→ Continuity equation for 2-D steady flow of compressible fluid

[Cf] For unsteady flow

$$\frac{\partial}{\partial t}(mass\ inside\ c.v.) = \textit{mass flowrate in} - \textit{mass flowrate out}$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

(2) Infinitesimal control volume



4.3 Continuity Equation for Two-Dimensional Steady Flow

Apply (4.9) to control volume ABCD

$$\int_{AB} \rho \vec{v}\cdot\vec{n}dA + \int_{BC} \rho \vec{v}\cdot\vec{n}dA + \int_{CD} \rho \vec{v}\cdot\vec{n}dA + \int_{DA} \rho \vec{v}\cdot\vec{n}dA = 0 \qquad \text{(f)}$$

Expand to first-order accuracy

$$\rho_{AB} \approx \rho - \frac{\partial \rho}{\partial x}\frac{dx}{2}$$

$$\int_{AB} \rho \vec{v}\cdot\vec{n}dA \cong -\left(\rho - \frac{\partial \rho}{\partial y}\frac{dy}{2}\right)\left(v - \frac{\partial v}{\partial y}\frac{dy}{2}\right)dx$$

$$\vec{v}\cdot\vec{n} = -\left(v - \frac{\partial v}{\partial y}\frac{dy}{2}\right)$$

$$\int_{BC} \rho \vec{v}\cdot\vec{n}dA \cong \left(\rho + \frac{\partial \rho}{\partial x}\frac{dx}{2}\right)\left(u + \frac{\partial u}{\partial x}\frac{dx}{2}\right)dy$$

$$\int_{CD} \rho \vec{v}\cdot\vec{n}dA \cong \left(\rho + \frac{\partial \rho}{\partial y}\frac{dy}{2}\right)\left(v + \frac{\partial v}{\partial y}\frac{dy}{2}\right)dx \qquad \text{(g)}$$

$$\int_{DA} \rho \vec{v}\cdot\vec{n}dA \cong -\left(\rho - \frac{\partial \rho}{\partial x}\frac{dx}{2}\right)\left(u - \frac{\partial u}{\partial x}\frac{dx}{2}\right)dy$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

Substitute (g) to (f), and expand products, and then retain only terms of lowest order (largest order of magnitude)

$$-\rho v dx + \rho \frac{\partial v}{\partial y}\frac{dy}{2}dx + v\frac{\partial \rho}{\partial y}\frac{dy}{2}dx - \frac{\partial \rho}{\partial y}\frac{\partial v}{\partial y}\frac{(dy)^2}{4}dx$$

$$+\rho v dx + \rho \frac{\partial v}{\partial y}\frac{dy}{2}dx + v\frac{\partial \rho}{\partial y}\frac{dy}{2}dx + \frac{\partial \rho}{\partial y}\frac{\partial v}{\partial y}\frac{(dy)^2}{4}dx$$

$$+\rho u dy + \rho \frac{\partial u}{\partial x}\frac{dx}{2}dy + u\frac{\partial \rho}{\partial x}\frac{dx}{2}dy + \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x}\frac{(dx)^2}{4}dy$$

$$-\rho u dy + \rho \frac{\partial u}{\partial x}\frac{dx}{2}dy + u\frac{\partial \rho}{\partial x}\frac{dx}{2}dy - \frac{\partial \rho}{\partial x}\frac{\partial v}{\partial x}\frac{(dx)^2}{4}dy = 0$$

$$\therefore \quad \rho\frac{\partial v}{\partial y}dxdy + v\frac{\partial \rho}{\partial y}dxdy + \rho\frac{\partial u}{\partial x}dxdy + u\frac{\partial \rho}{\partial x}dxdy = 0$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

$$\rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0$$

Point form

$$\boxed{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0} \qquad (4.10)$$

→ Continuity equation for 2-D steady flow of compressible fluid

• Continuity equation of incompressible flow for both steady and unsteady flow ($\rho =$ const.)

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \qquad (4.11)$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

Continuity equation for unsteady 3-D flow of compressible fluid

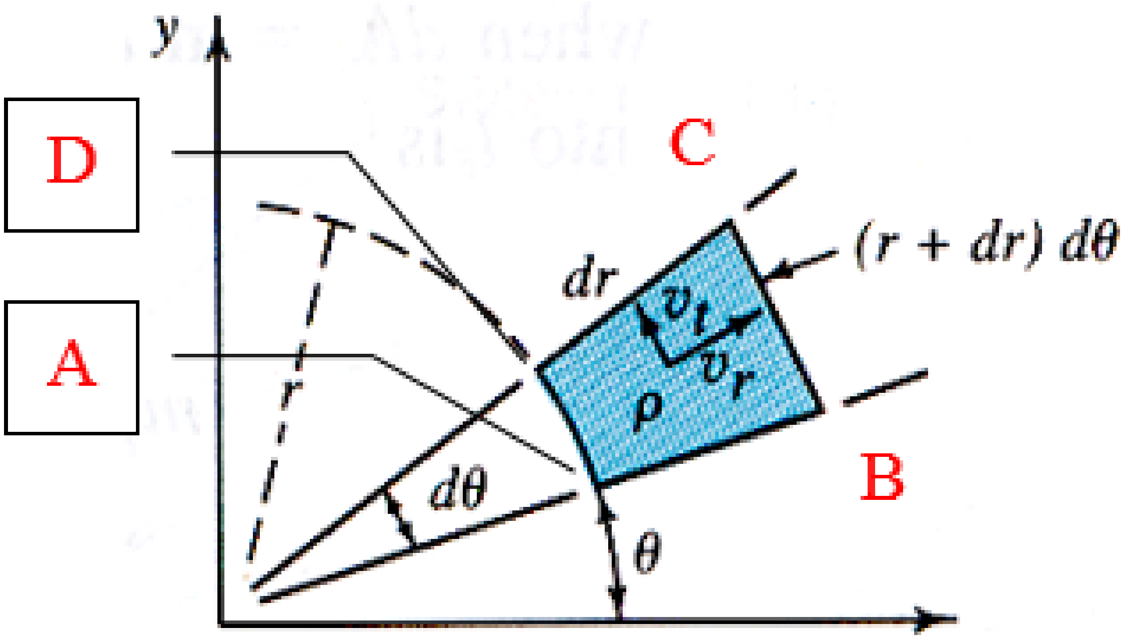
$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

For steady 3-D flow of incompressible fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

- Continuity equation for polar coordinates



4.3 Continuity Equation for Two-Dimensional Steady Flow

Apply (4.9) to control volume ABCD

$$\int_{AB} \rho \vec{v} \cdot \vec{n} dA + \int_{BC} \rho \vec{V} \cdot \vec{n} dA + \int_{CD} \rho \vec{V} \cdot \vec{n} dA + \int_{DA} \rho \vec{V} \cdot \vec{n} dA = 0$$

$$\int_{AB} \rho \vec{V} \cdot \vec{n} dA \cong -\left(\rho - \frac{\partial \rho}{\partial \theta} \frac{d\theta}{2} \right) \left(v_t - \frac{\partial v_t}{\partial \theta} \frac{d\theta}{2} \right) dr$$

$$\int_{BC} \rho \vec{V} \cdot \vec{n} dA \cong \left(\rho + \frac{\partial \rho}{\partial r} \frac{dr}{2} \right) \left(v_r + \frac{\partial v_r}{\partial r} \frac{dr}{2} \right) (r + dr) d\theta$$

$$\int_{CD} \rho \vec{V} \cdot \vec{n} dA \cong \left(\rho + \frac{\partial \rho}{\partial \theta} \frac{d\theta}{2} \right) \left(v_t + \frac{v_t}{\partial \theta} \frac{d\theta}{2} \right) dr$$

$$\int_{DA} \rho \vec{V} \cdot \vec{n} dA \cong -\left(\rho - \frac{\partial \rho}{\partial r} \frac{dr}{2} \right) \left(v_r - \frac{\partial v_r}{\partial r} \frac{dr}{2} \right) r d\theta$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

$$-\cancel{\rho v_t dr} + \rho \frac{\partial v_t}{\partial \theta}\frac{d\theta}{2}dr + v_t \frac{\partial \rho}{\partial \theta}\frac{d\theta}{2}dr - \frac{\partial \rho}{\partial \theta}\frac{\partial v_t}{\partial \theta}\cancel{\frac{(d\theta)^2}{4}}dr$$

$$+\cancel{\rho v_t dr} + \rho \frac{\partial v_t}{\partial \theta}\frac{d\theta}{2}dr + v_t \frac{\partial \rho}{\partial \theta}\frac{d\theta}{2}dr + \frac{\partial \rho}{\partial \theta}\frac{\partial v_t}{\partial \theta}\cancel{\frac{(d\theta)^2}{4}}dr$$

$$+\cancel{\rho v_r r d\theta} + \rho v_r dr d\theta + \rho \frac{\partial v_r}{\partial r}\frac{dr}{2}r d\theta + \rho \frac{\partial v_r}{\partial r}\frac{dr}{2}dr d\theta$$

$$+v_r \frac{\partial \rho}{\partial r}\frac{dr}{2}r d\theta + v_r \frac{\partial \rho}{\partial r}\frac{dr}{2}dr d\theta + \frac{\partial \rho}{\partial r}\cancel{\left(\frac{dr}{2}\right)^2}\frac{\partial v_r}{\partial r}r d\theta + \frac{\partial \rho}{\partial r}\left(\frac{dr}{2}\right)^2\frac{\partial v_r}{\partial r}dr d\theta$$

$$-\cancel{\rho v_r r d\theta} + \rho \frac{\partial v_r}{\partial r}\frac{dr}{2}r d\theta + v_r \frac{\partial \rho}{\partial r}\frac{dr}{2}r d\theta - \frac{\partial \rho}{\partial r}\frac{\partial v_r}{\partial r}\cancel{\left(\frac{dr}{2}\right)^2}r d\theta = 0$$

$$\rho \frac{\partial v_t}{\partial \theta}d\theta dr + v_t \frac{\partial \rho}{\partial \theta}d\theta dr + \rho \frac{\partial v_r}{\partial r}r dr d\theta + v_r \frac{\partial \rho}{\partial r}r dr d\theta$$

$$+\rho v_r dr d\theta + \rho \frac{\partial v_r}{\partial r}\frac{1}{2}(dr)^2 d\theta + v_r \frac{\partial \rho}{\partial r}\frac{1}{2}(dr)^2 d\theta + \frac{\partial \rho}{\partial r}\frac{\partial v_r}{\partial r}\frac{1}{2}(dr)^3 d\theta = 0$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

Divide by $dr d\theta$

$$\rho \frac{\partial v_t}{\partial \theta} + v_t \frac{\partial \rho}{\partial \theta} + \rho \frac{\partial v_r}{\partial r} + v_r \frac{\partial \rho}{\partial r} r + \rho v_r + \rho \frac{\partial v_r}{\partial r} \frac{1}{2} dr + v_r \frac{\partial \rho}{\partial r} \frac{1}{2} dr + \frac{\partial \rho}{\partial r} \frac{\partial v_r}{\partial r} \frac{1}{2} dr = 0$$

$$\therefore \quad \rho \frac{\partial v_r}{\partial r} r + v_r \frac{\partial \rho}{\partial r} r + \rho v_r + \rho \frac{\partial v_t}{\partial \theta} + v_t \frac{\partial \rho}{\partial \theta} = 0$$

Divide by r

$$\rho \frac{\partial v_r}{\partial r} + v_r \frac{\partial \rho}{\partial r} + \rho \frac{v_r}{r} + \rho \frac{\partial v_t}{r \partial \theta} + v_t \frac{\partial \rho}{r \partial \theta} = 0$$

$$\therefore \quad \frac{\partial (\rho v_r)}{\partial r} + \frac{\rho v_r}{r} + \frac{\partial (\rho v_t)}{r \partial \theta} = 0 \qquad (4.12)$$

For incompressible fluid

$$\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_t}{r \partial \theta} = 0 \qquad (4.13)$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

[IP 4.4] p. 117

A mixture of ethanol and gasoline, called "gasohol," is created by pumping the two liquids into the "wye" pipe junction. Find Q_{eth} and V_{eth}

$$\rho_{mix} = 691.1 \text{ kg/m}^3$$

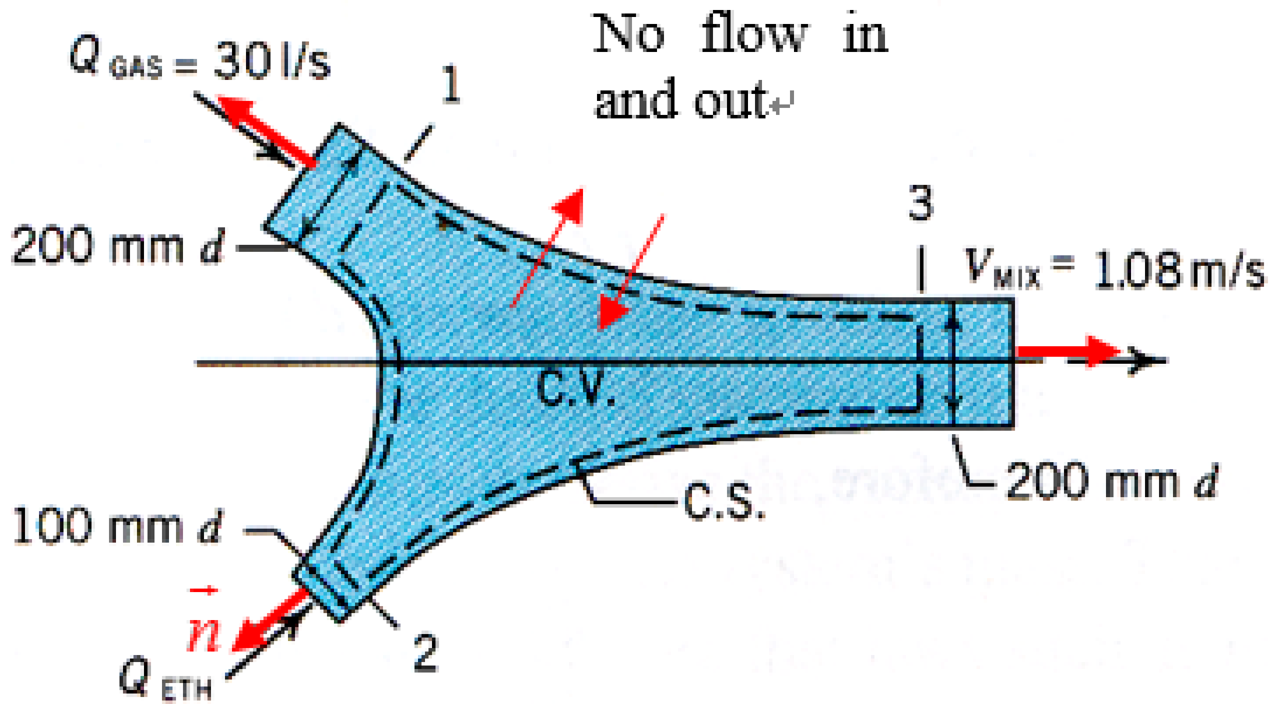
$$V_{mix} = 1.08 \text{ m/s}$$

$$Q_{gas} = 30\ l\ /\ s = 30 \times 10^{-3} \text{ m}^3\text{/s}$$

$$\rho_{gas} = 680.3 \text{ kg/m}^3$$

$$\rho_{eth} = 788.6 \text{ kg/m}^3$$

4.3 Continuity Equation for Two-Dimensional Steady Flow



4.3 Continuity Equation for Two-Dimensional Steady Flow

[Sol]

$$A_1 = \frac{\pi}{4}(0.2)^2 = 0.031\text{m}^2 \;\; A_2 = 0.0079\,\text{m}^2 \;\; A_3 = 0.031\text{m}^2$$

$$V_1 = 30 \times 10^{-3} / 0.031 = 0.97 \text{ m/s}$$

$$\boxed{\int_1 \rho \vec{v} \cdot \vec{n}\, dA + \int_2 \rho \vec{v} \cdot \vec{n}\, dA + \int_3 \rho \vec{v} \cdot \vec{n}\, dA = 0}$$

$$\int_1 \rho \vec{v} \cdot \vec{n}\, dA = -680.3 \times 0.97 \times 0.031 = -20.4 \text{ kg/s}$$

$$\int_2 \rho \vec{v} \cdot \vec{n}\, dA = -788.6 \times V_2 \times 0.0079 = -6.23\, V_2$$

$$\int_3 \rho \vec{v} \cdot \vec{n}\, dA = 691.1 \times 1.08 \times 0.031 = 23.1 \text{ kg/s}$$

$$\therefore \quad \oint_{c.s} \rho \vec{v} \cdot \vec{n} dA = -20.4 - 6.23\, V_2 + 23.1 = 0$$

$$V_2 = 0.43 \text{ m/s}$$

$$\rightarrow \quad Q_{eth} = V_2\, A_2 = (0.43)(0.0079) = 3.4 \times 10^{-3} \text{ m}^3\text{/s} = 3.4\ l\text{/s}$$

4.4 The Reynolds Transport Theorem

• **Reynolds Transport Theorem (RTT)**

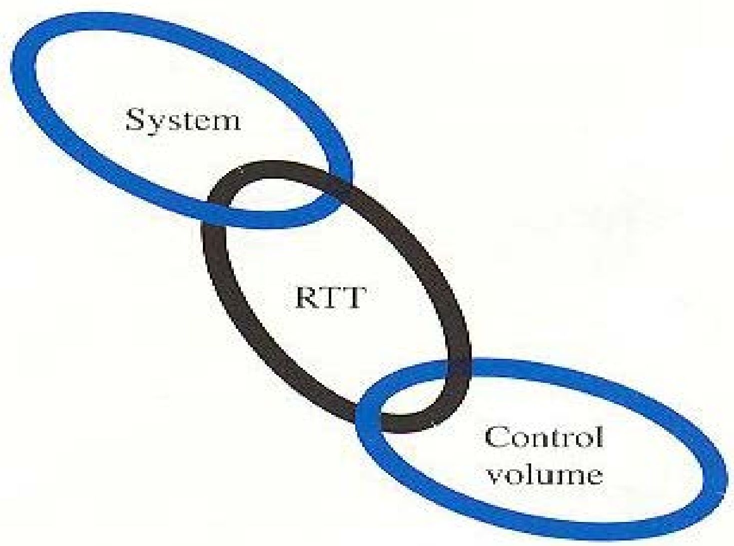
Osborne Reynolds (1842-1912); English engineer

~ A general relationship that converts the laws such as mass conservation and Newton's 2nd law <u>from the system to the control volume</u>

Most principles of fluid mechanics are adopted from solid mechanics, where the physical laws dealing with the time rates of change of extensive properties are expressed for systems.

→ There is a need to relate the changes in a control volume to the changes in a system.

4.4 The Reynolds Transport Theorem



- Two types of properties

Extensive properties (E): total system mass, momentum, energy

Intensive properties (i): mass, momentum, energy per unit mass

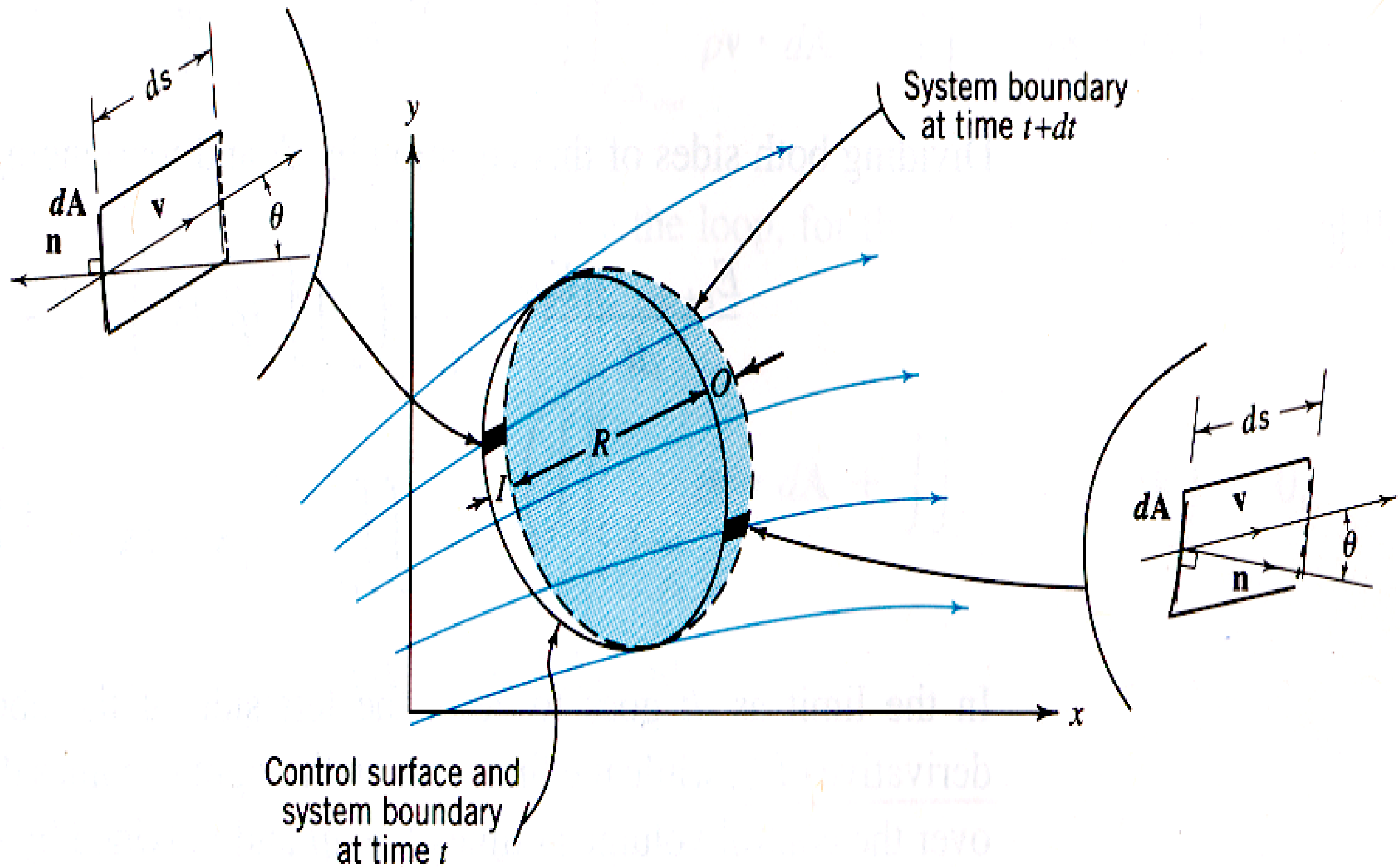
4.4 The Reynolds Transport Theorem

$\underline{E}$	$\underline{\boldsymbol{i}}$
system mass, m	1
system momentum, $m\vec{v}$	$\vec{v}$
system energy, $m(\vec{v})^2$	$(\vec{v})^2$

$$E = \iiint_{system} i \, dm = \iiint_{system} i\rho \, dvol \qquad (4.14)$$

4.4 The Reynolds Transport Theorem

- Derivation of RTT



4.4 The Reynolds Transport Theorem

Consider time rate of change of a system property

$$E_{t+dt} - E_t = (E_R + E_0)_{t+dt} - (E_R + E_I)_t \qquad \text{(a)}$$

$$(E_0)_{t+dt} = dt \iint_{c.s.out} i\rho \vec{v} \cdot \overrightarrow{dA} \qquad \text{(b.1)}$$

$$(E_I)_t = dt\left(-\iint_{c.s.in} i\rho \vec{v} \cdot \overrightarrow{dA}\right) \qquad \text{(b.2)}$$

$$(E_R)_{t+dt} = \left(\iiint_R i\rho\, dvol\right)_{t+dt} \qquad \text{(b.3)}$$

$$(E_R)_t = \left(\iiint_R i\rho\, dvol\right)_t \qquad \text{(b.4)}$$

4.4 The Reynolds Transport Theorem

Substitute (b) into (a) and divide by dt

$$\therefore \frac{E_{t+dt} - E_t}{dt} = \frac{1}{dt}\left\{\left(\iiint_R i\rho\, dvol\right)_{t+dt} - \left(\iiint_R i\rho\, dvol\right)_t\right\}$$

$$+\iint_{c.s.out} i\rho\, \vec{v} \cdot \overrightarrow{dA} + \iint_{c.s.in} i\rho\, \vec{v} \cdot \overrightarrow{dA}$$

$$\boxed{\frac{dE}{dt} = \frac{d}{dt}\left(\iiint_{c.v.} i\rho\, dvol\right) = \frac{\partial}{\partial t}\left(\iiint_{c.v.} i\rho\, dvol\right) + \oint\oint_{c.s.} i\rho\, \vec{v} \cdot \overrightarrow{dA}} \qquad (4.15)$$

4.4 The Reynolds Transport Theorem

① $\dfrac{dE}{dt}$ = time rate of change of E in the system

② $\dfrac{\partial}{\partial t}\left(\iiint_{c.v.} i\rho \, dvol\right)$ = time rate change within the control volume → unsteady term

③ $\oint\oint_{c.s.} i\rho \vec{v}\cdot\vec{dA}$ = fluxes of E across the control surface

- Application of RTT to conservation of mass

For application of RTT to the conservation of mass,

in Eq. (4.15), $E = m,\ i = 1$ and $\dfrac{dm}{dt} = 0$ because mass is conserved.

$$\therefore \frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho \, dvol\right) = -\oint\oint_{c.s.} \rho\vec{v}\cdot\vec{dA} = -\left(\iint_{c.s.out} \rho\vec{v}\cdot\vec{dA} + \iint_{c.s.in} \rho\vec{v}\cdot\vec{dA}\right) \quad (4.16)$$

Unsteady flow: mass within the control volume may change if the density changes

4.4 The Reynolds Transport Theorem

For flow of uniform density or steady flow, (4.16) becomes

$$\iint_{c.s.out} \rho \vec{v} \cdot \overrightarrow{dA} + \iint_{c.s.in} \rho \vec{v} \cdot \overrightarrow{dA} = 0$$ ~ same as Eq. (4.9)

For one-dimensional flow

$$\iint_{c.s.out} \rho \vec{v} \cdot \overrightarrow{dA} = \rho_2 V_2 A_2$$

$$\iint_{c.s.in} \rho \vec{v} \cdot \overrightarrow{dA} = -\rho_1 V_1 A_1$$

$$\therefore \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad (4.1)$$

- In Ch. 5 & 6, RTT is also used to derive the work-energy, impulse-momentum, and moment of momentum principles.

4.4 The Reynolds Transport Theorem

Homework Assignment # 4

Due: 1 week from today

Prob. 4.9

Prob. 4.12

Prob. 4.14

Prob. 4.20

Prob. 4.31