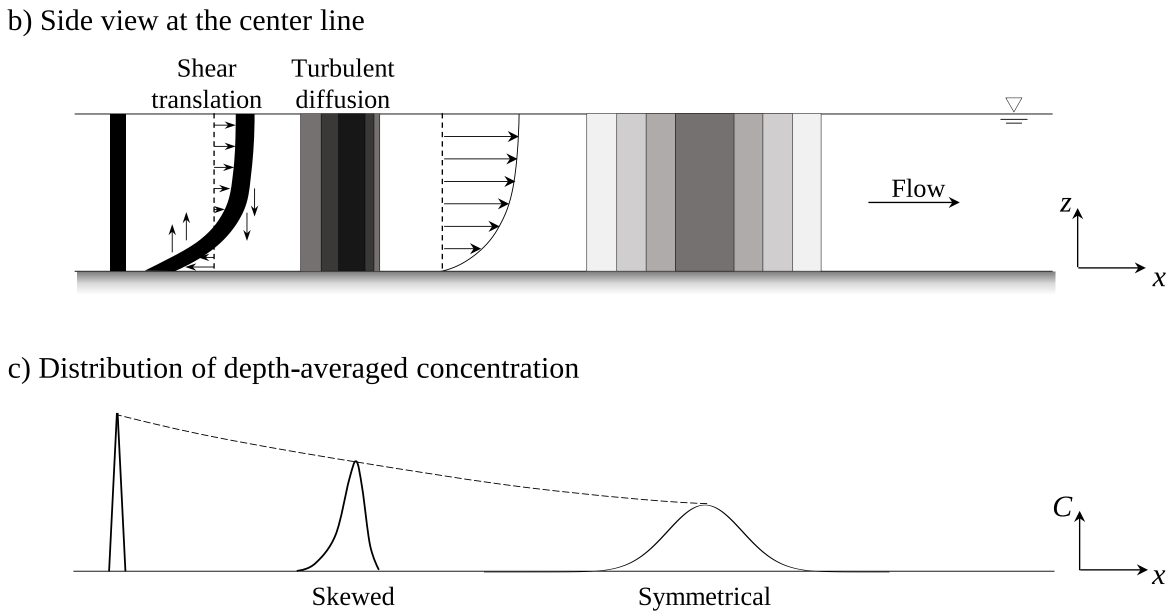
Chapter 4

Shear Flow Dispersion



Chapter 4 Shear Flow Dispersion

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Taylor, Geoffrey –
English fluid mechanician

Objectives

- Describe the spreading of particles in shear flows
- Derive shear flow dispersion equation using Taylor' analysis (1953, 1954) for laminar flow in pipe and turbulent flow
- Extend dispersion analysis to unsteady flow and two-dimensional flow

4.1 Description of Dispersion in Shear Flow

4.1.1 Introductory Remarks

• Dispersion - the spreading of particles in the direction of flow cause primarily by the velocity profile in the cross section

Flows with velocity gradients are often referred to as “shear flows.”

→ shear effect

This process can be described with the analysis of diffusion by continuous movements in turbulent flows (1921).

However, Taylor developed <u>a completely new method</u> in analyzing the spread of dissolved contaminants both in laminar flow in pipe and in turbulent flow (1953, 1954). In this analysis, he derived a solution for mass flux in the flow direction, and relate it with Fick’s law.

4.1 Description of Dispersion in Shear Flow

Thus, we can classify analyses in two categories:

(i) Non-Fickian model: use random walk theory

(ii) Fickian model: use Fick's law

4.1.2 Non-Fickian Description of spreading of particles in shear flow

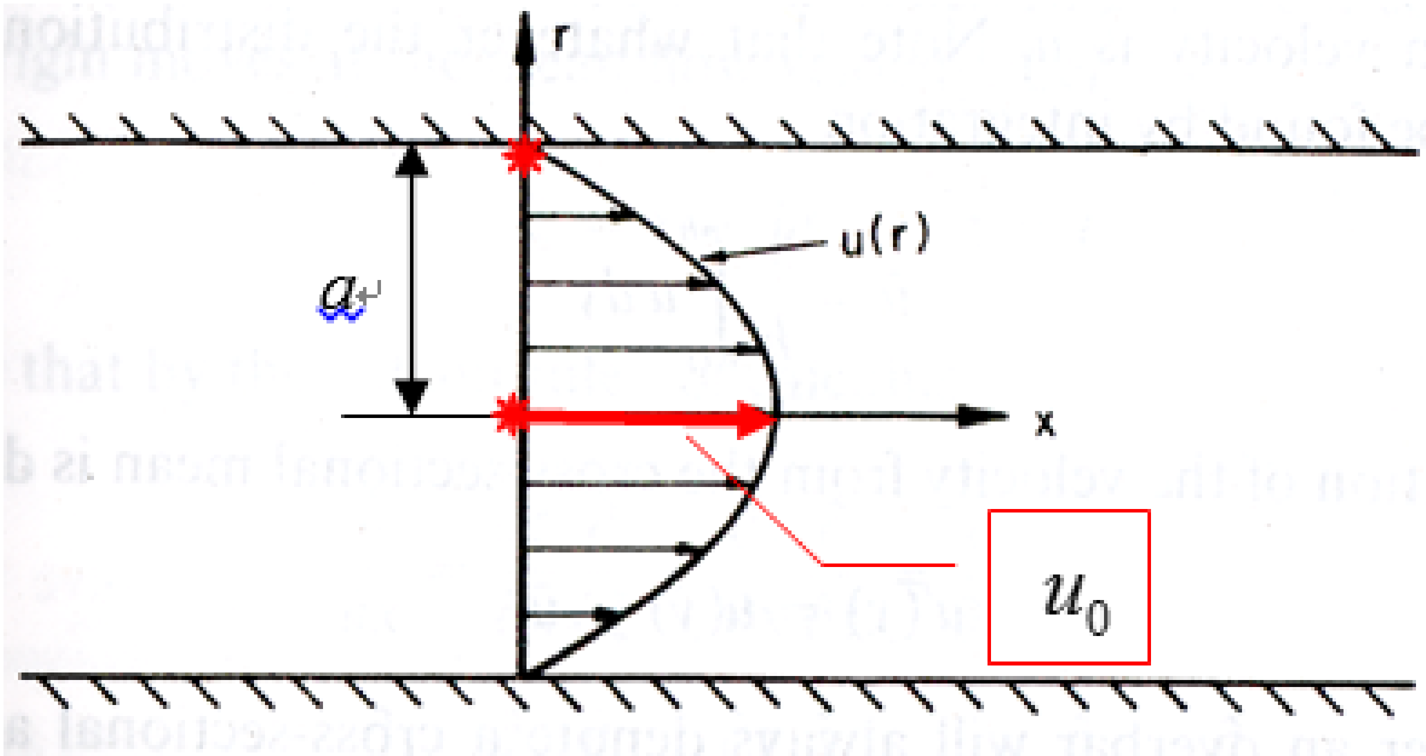
Consider laminar flow in pipe with velocity profile shown below.

1) Assume two molecules are being carried in the flow; one in the center and one near the wall.

Rate of separation caused by the difference in advective velocity

≫ separation by molecular motion in $x-$direction

4.1 Description of Dispersion in Shear Flow



4.1 Description of Dispersion in Shear Flow

1) Assume two molecules are being carried in the flow; one in the center and one near the wall.

Rate of separation caused by the difference in advective velocity

≫ separation by molecular motion in x-direction

2) Because of molecular diffusion in r-direction, given enough time, any single molecule would wander randomly throughout the cross section, and would sample at random all the advective velocities.

→ Therefore, if a long enough averaging time was available, a single molecule's time-averaged velocity would be equal to the instantaneous cross-sectional average of all molecules' velocities.

4.1 Description of Dispersion in Shear Flow

3) After some long enough "forgetting time" its location is independent of the initial location, and therefore its velocity is independent of its initial velocity.

→ Thus, we can imagine that the motion of a single molecule is the sum of a series of independent steps of random length.

4) If we adopt a coordinate system moving at the mean velocity, the random steps are likely to be back and forward with respect to the moving coordinate system.

→ This motion is similar to the random walk, if the flow continues unchanged for a time much longer than the "forgetting time."

4.1 Description of Dispersion in Shear Flow

→ Fickian diffusion equation, Eq. (2.4) can describe the spread of particles along the axis of the pipes, except that since the step length and time increment are much different from those of molecular diffusion we expect to find a different value of diffusion coefficient.

→ dispersion coefficient

Now, find the rate of spreading for laminar shear flow in pipe

For turbulent flow, the rate of spreading is described by a turbulent diffusion coefficient as

$$\varepsilon = \langle U^2 \rangle T_L$$

where U = velocity deviation; T_L = Lagrangian time scale.

4.1 Description of Dispersion in Shear Flow

The motion of a single molecule in laminar pipe flow is similar to the motion of a fluid particle in turbulent flow in that the velocity of the molecule is a stationary random function of time.

For laminar flow in pipe, the Lagrangian time scale is will be proportional to the time required to sample whole field of velocities, which is proportional to the time scale for cross sectional mixing as

$$T_L \propto \frac{a^2}{D}$$

where D is molecular diffusion coefficient.

4.1 Description of Dispersion in Shear Flow

The mean square velocity deviation of the molecule, $<U^2>$ results primarily wandering of the molecule across the cross section, during which it samples velocities ranging from zero at the wall to the peak velocity u_0 at the centerline.

$$<U^2> \propto u_0^{\,2}$$

where u_0 = maximum velocity at the centerline of pipe

Thus, longitudinal dispersion coefficient due to combined action of shear advection and molecular diffusion is described, in the limit $t >> T_L$, by the relation of the form

$$K = <U^2> T_L \propto u_0^{\,2} \frac{a^2}{D} \qquad (4.1)$$

→ K is inversely proportional to molecular diffusion.

4.1 Description of Dispersion in Shear Flow

▸Consider the x-position of a single molecule in the shear flow.

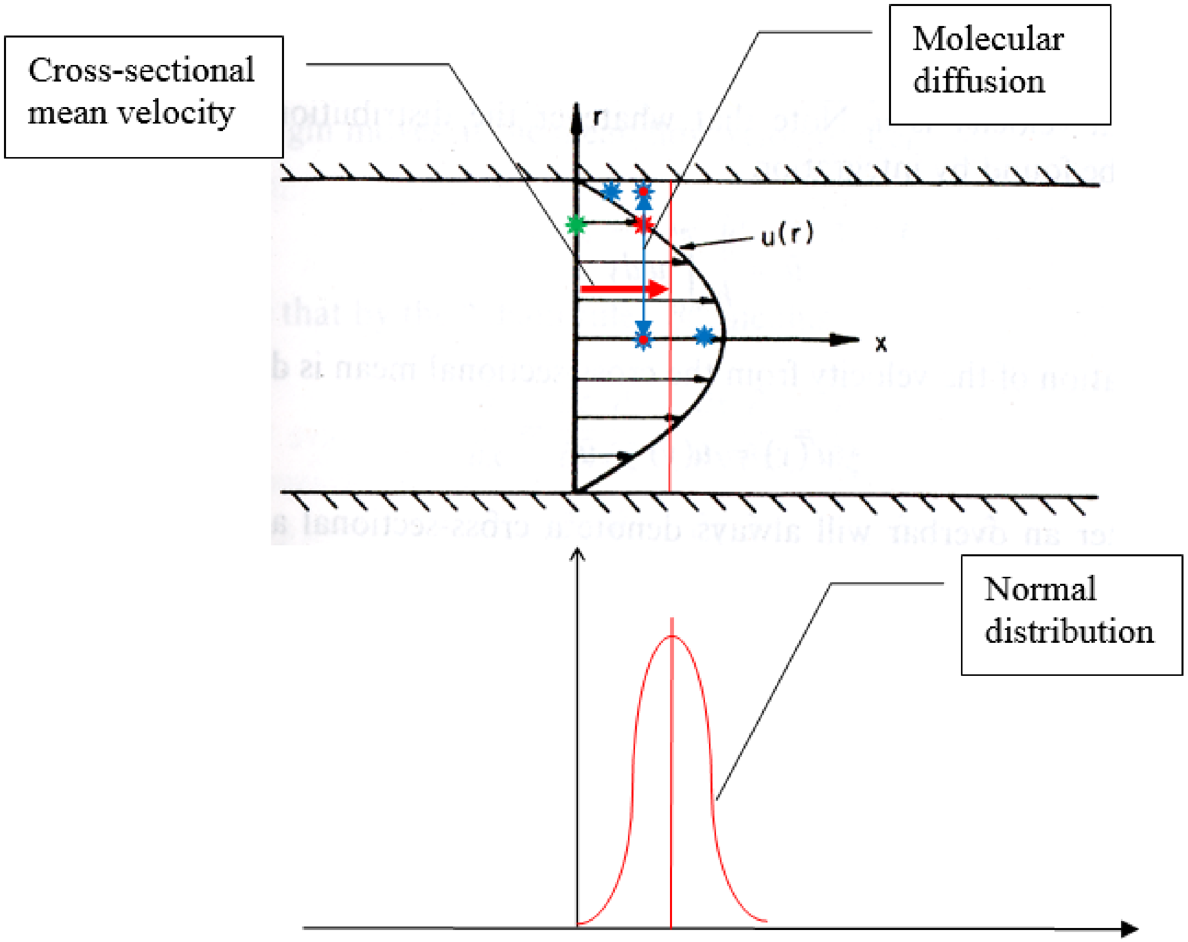
After the shear advection, its location in the x-direction is $u_1 \triangle t$.

Then, after the molecular diffusion across the cross section, its location in the x-direction would be $u_i \triangle t$, because the molecular diffusion causes the molecule moving at <u>random back and forth across the cross section</u>.

→ This motion is similar to the <u>random walk</u>, if the flow continues unchanged for a time much longer than the “forgetting time.”

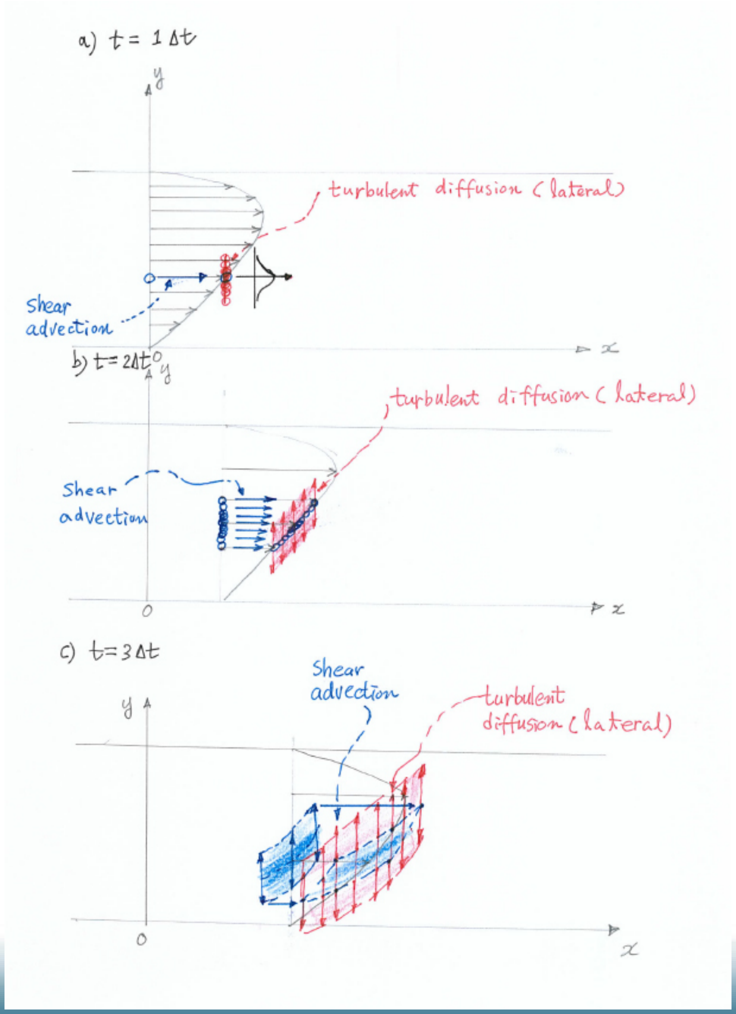
→ Thus, in the limit, the probability of the molecule being between x and $x + \triangle x$ approaches the <u>normal distribution</u> with mean μ and a variance σ^2.

4.1 Description of Dispersion in Shear Flow

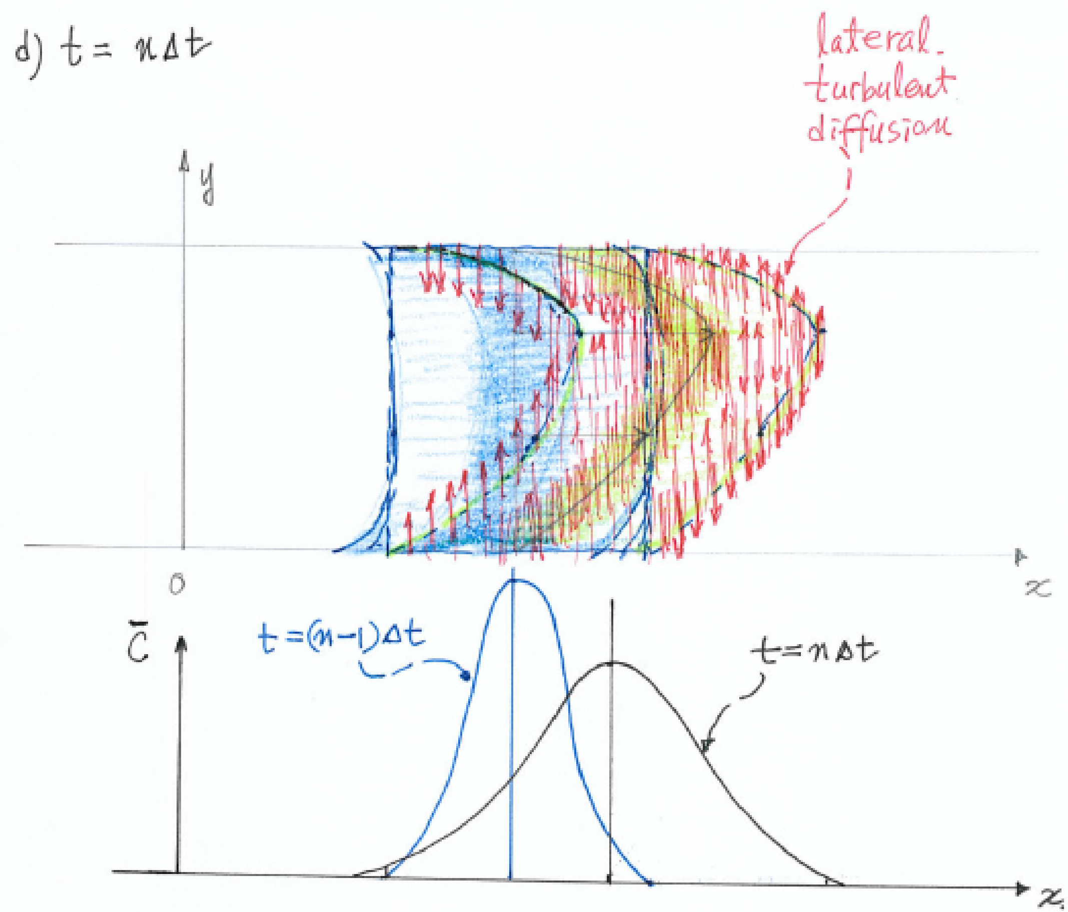


4.1 Description of Dispersion in Shear Flow

I. Spreading of a single particle

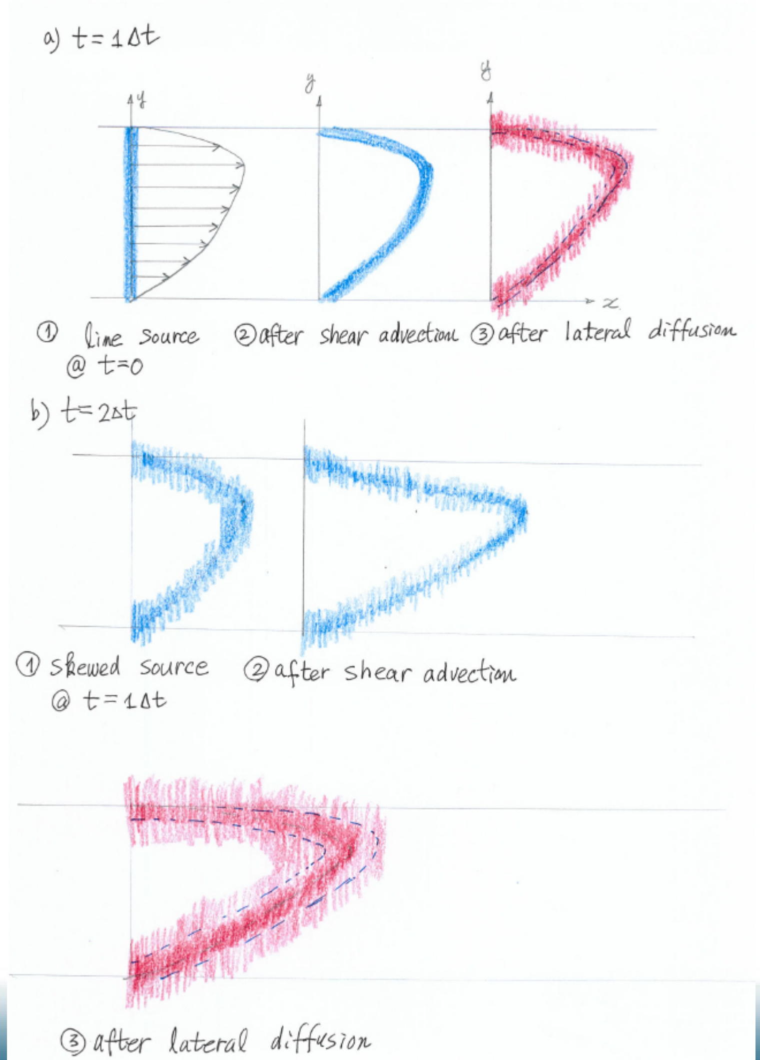


4.1 Description of Dispersion in Shear Flow

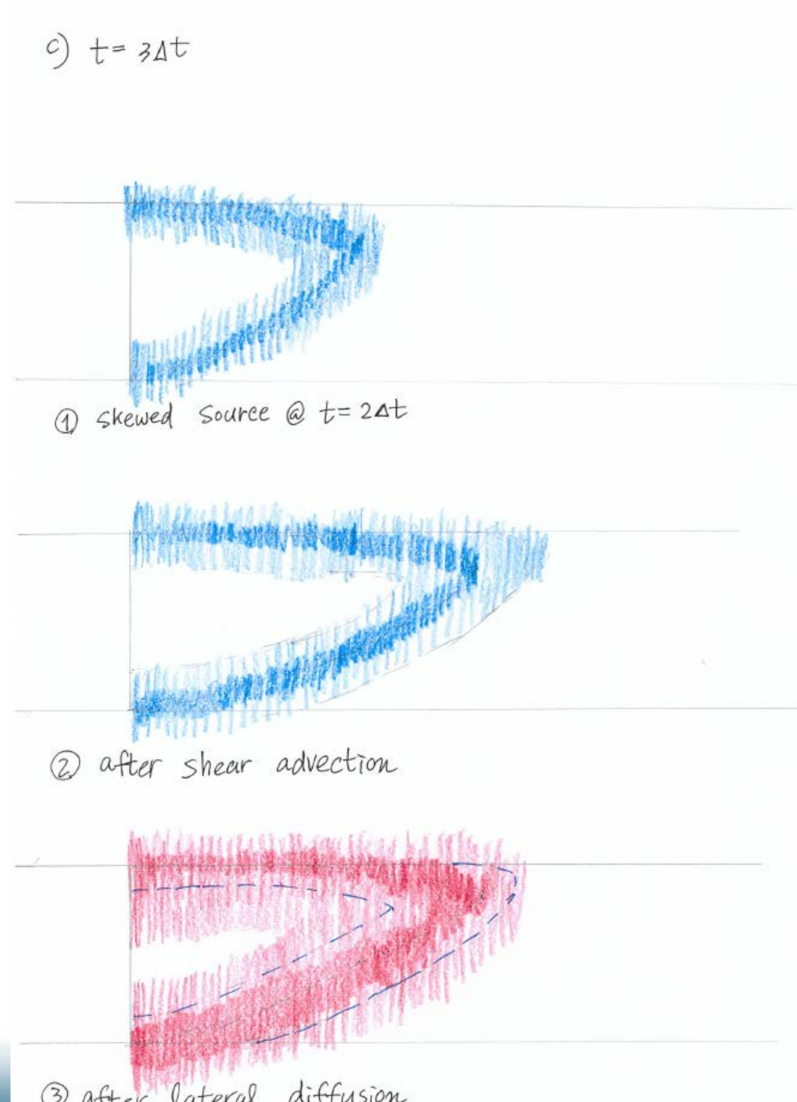


4.1 Description of Dispersion in Shear Flow

II. Spreading of a line source



4.1 Description of Dispersion in Shear Flow



4.1 Description of Dispersion in Shear Flow

4.1.3 Non-Fickian models of shear flow dispersion

Non-Fickian modeling of dispersion mechanism

Consider mixing in a hypothetical river

Assumption:

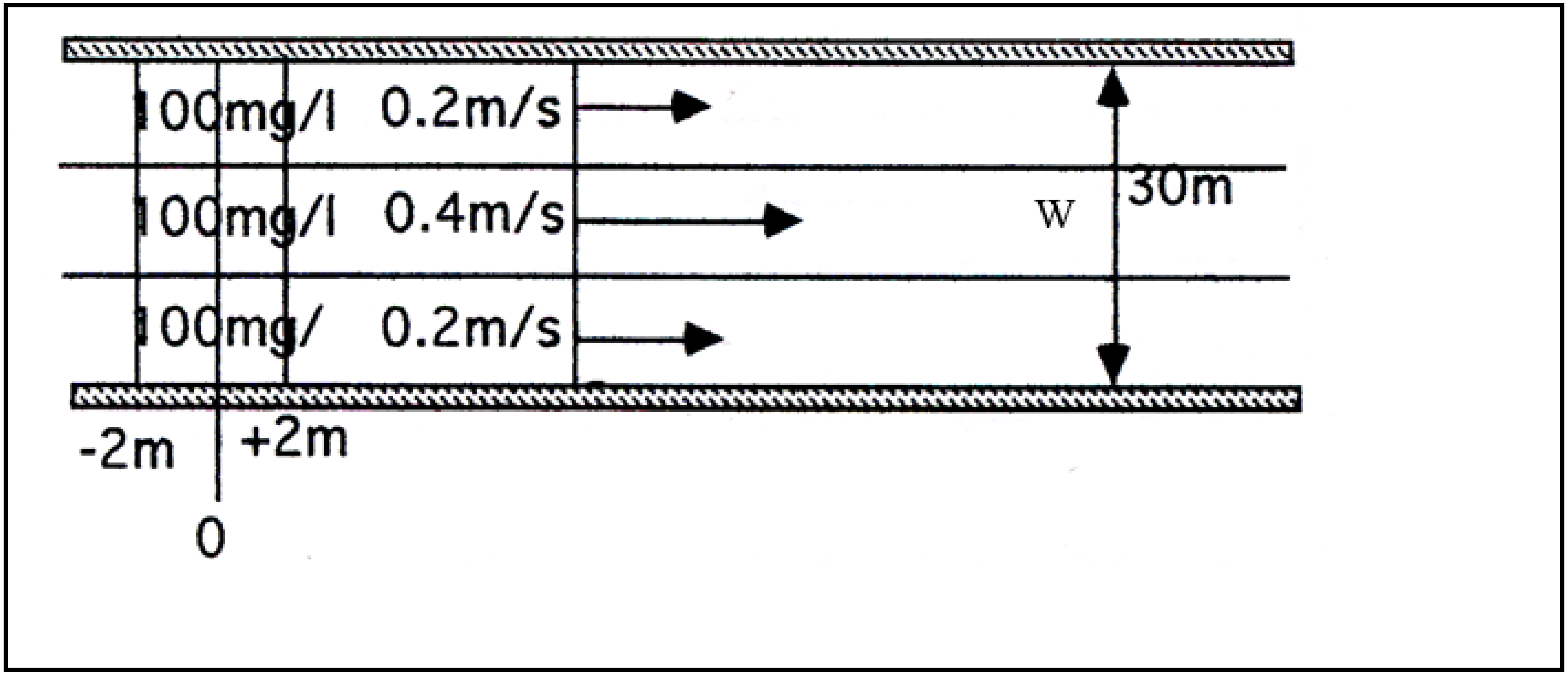
1) A hypothetical river with 3 lanes of different velocities

2) Every t_m seconds complete mixing occurs across the cross section of the river (but not longitudinally) occurs, after shear advection is completed.

→ sequential mixing process

$$\frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial C}{\partial x}\right) \to 0$$

4.1 Description of Dispersion in Shear Flow



4.1 Description of Dispersion in Shear Flow

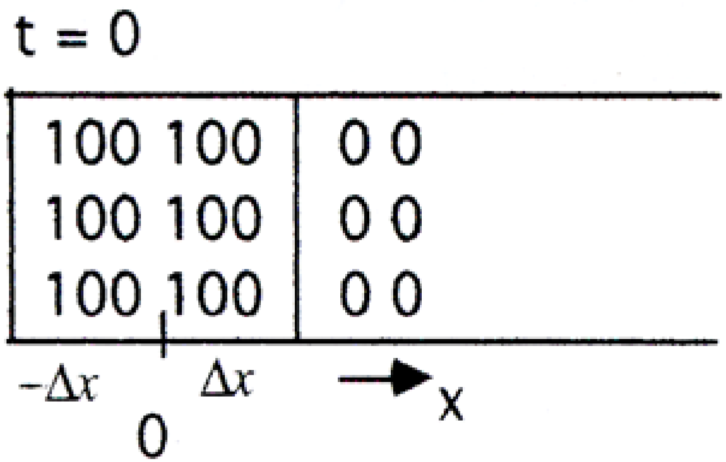
Actually, time needed for complete cross-sectional mixing is very large which is given as

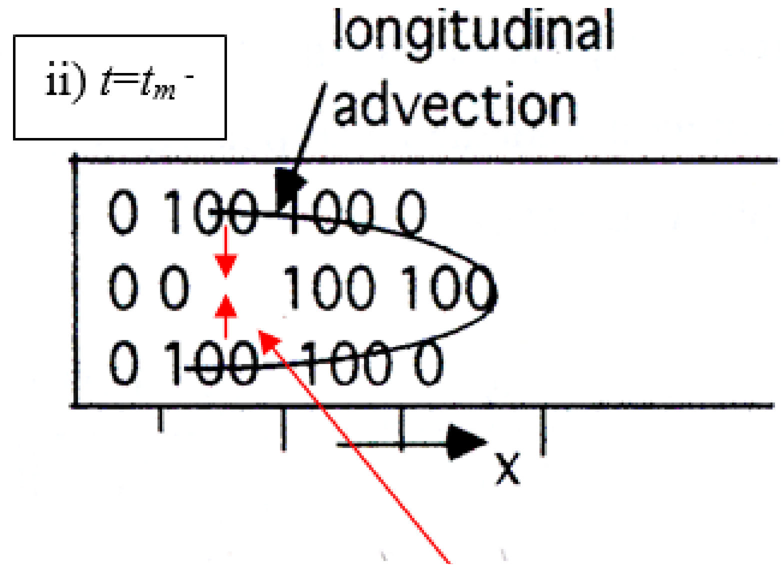
$$t_c \cong \frac{W^2}{\varepsilon_y}$$

Now solve for an instantaneous injection of a line source at x =0

$$t_m = 10\ s;\ \ u_a = 0.2\ m/s;\ \ \Delta x = 2\ m$$

4.1 Description of Dispersion in Shear Flow





$t=t_m^+$: After lateral mixing

0	67	100	33	0
0	67	100	33	0
0	67	100	33	0

4.1 Description of Dispersion in Shear Flow

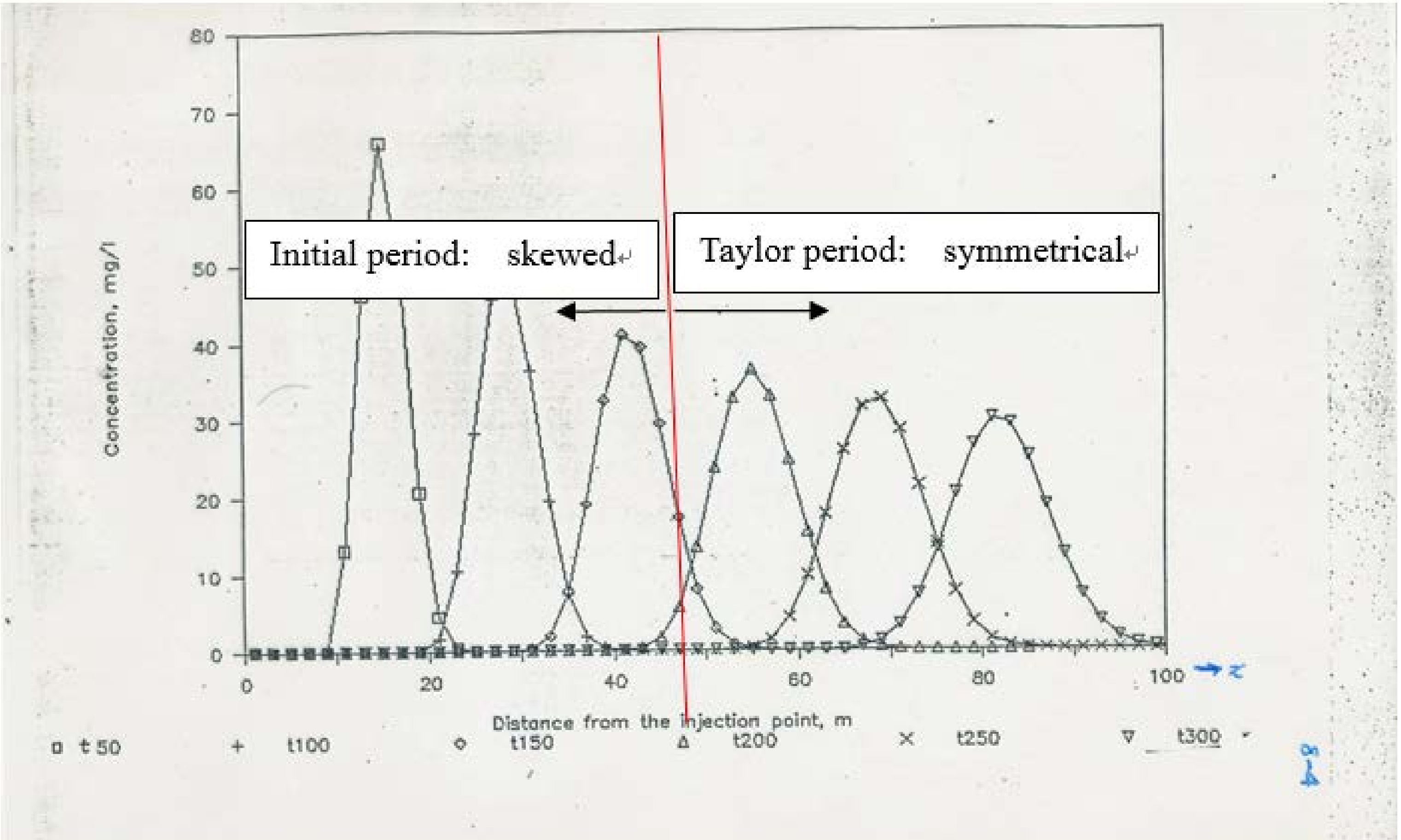
(iii) $t = 2\,t_m^{\,-}$: After shear advection

0	0	67	100	33	0
0	0	0	67	100	33
0	0	67	100	33	0

$t = 2\,t_m^{\,+}$: After lateral mixing

0	0	45	89	55	11	0
0	0	45	89	55	11	0
0	0	45	89	55	11	0

4.1 Description of Dispersion in Shear Flow



4.1 Description of Dispersion in Shear Flow

· Longitudinal dispersion in two-lane river

$\alpha =$ area fraction of river occupied by slow lane, $0 \le \alpha \le 1$

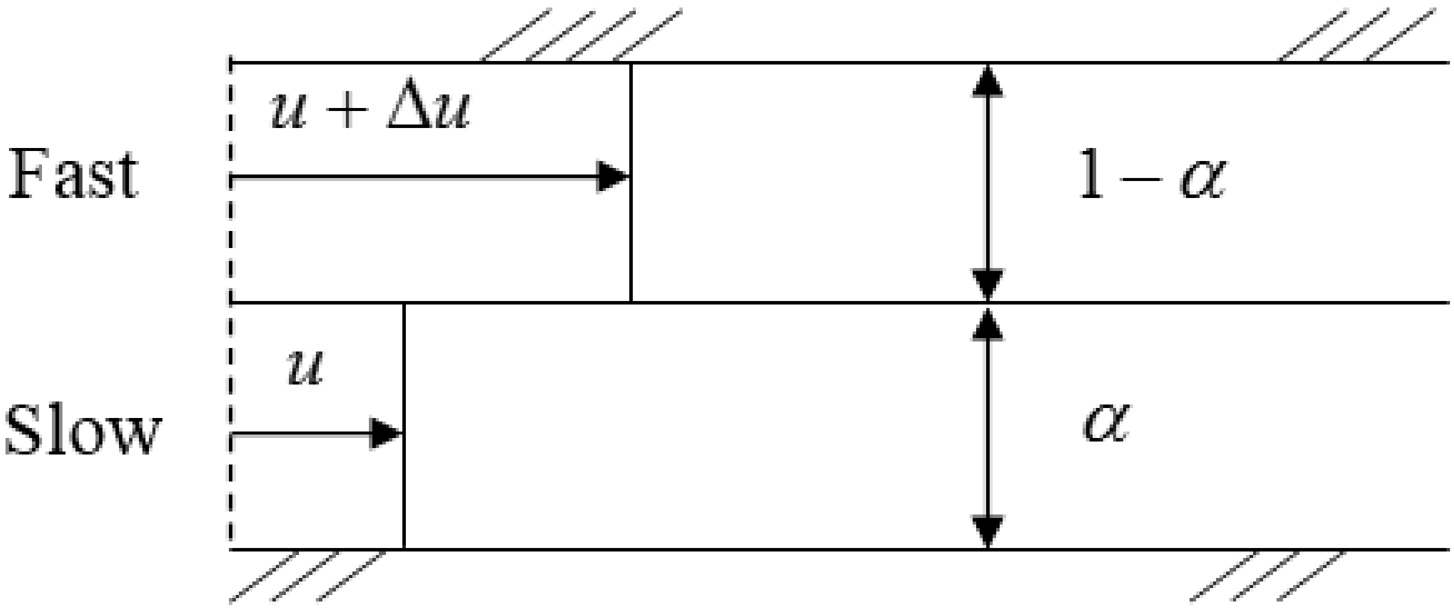
$$u_S = u$$

$$u_F = u + \Delta u$$

$\bar{u} =$ cross-sectional mean velocity

$$= \alpha u + (1-\alpha)(u + \Delta u) \quad \text{(a)}$$

4.1 Description of Dispersion in Shear Flow



4.1 Description of Dispersion in Shear Flow

Consider velocity deviations:

$$u_S' = u_S - \bar{u} = u - \alpha u - (1-\alpha)(u + \Delta u)$$

$$= u - \alpha u - u - \Delta u + \alpha u + \alpha \Delta u = \underline{-(1-\alpha)\Delta u}$$

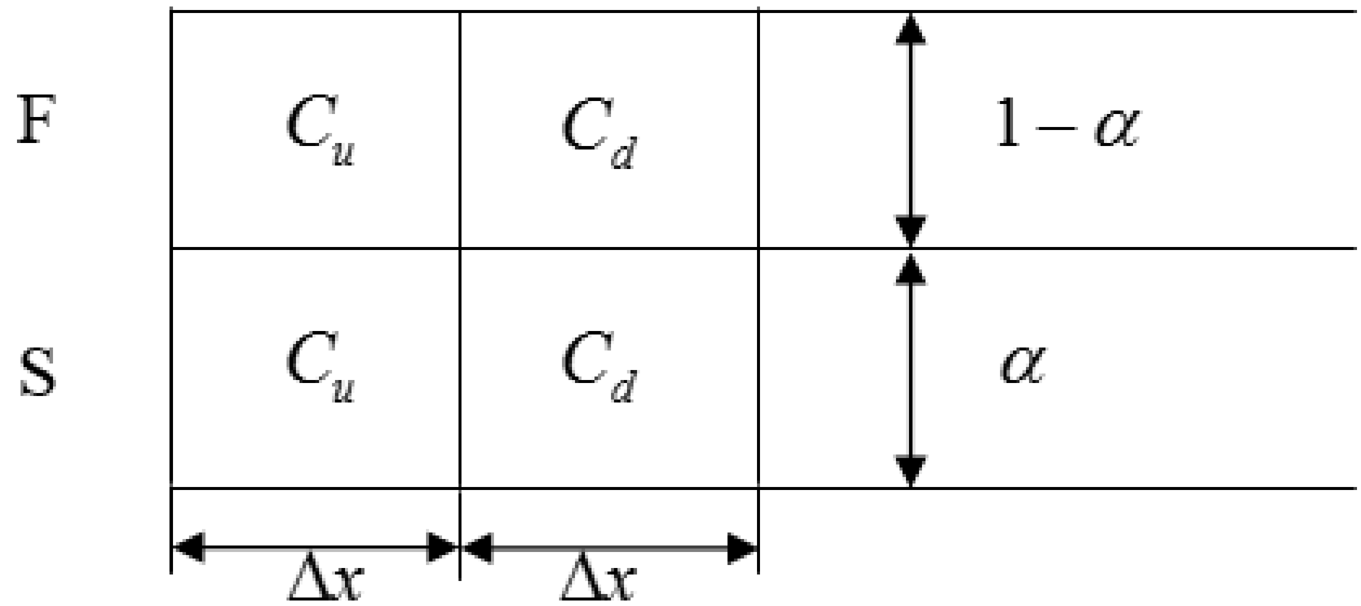
$$u_F' = u_F - \bar{u} = u + \Delta u - \bar{u} = u + \Delta u - \alpha u - (1-\alpha)(u + \Delta u)$$

$$= \underline{\alpha \Delta u}$$

$$\Delta x = \Delta u \cdot t_m$$

4.1 Description of Dispersion in Shear Flow

(i) Before any processes



4.1 Description of Dispersion in Shear Flow

(ii) Just before mixing (JBM) …. after advection only

F	0	C_u	C_d		$1-\alpha$
S	C_u	C_d	0		α

$$\overline{C} = \alpha C_d + (1-\alpha) C_u$$

4.1 Description of Dispersion in Shear Flow

Now, concentration deviations are

$$C'_S = C_d - \overline{C} = C_d - \alpha C_d - (1-\alpha) C_u$$
$$= (1-\alpha)(C_d - C_u)$$

$$C'_F = C_u - \overline{C} = C_u - \alpha C_d - (1-\alpha) C_u$$
$$= -\alpha (C_d - C_u)$$

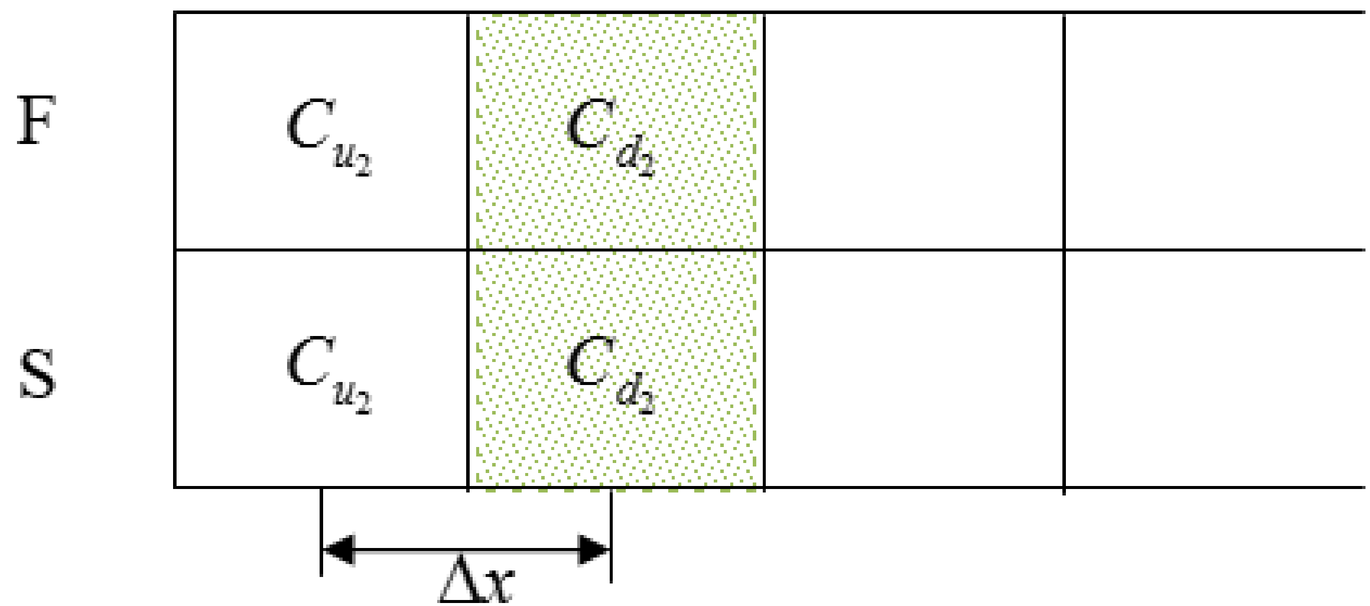
(iii) Just after mixing (JAM)

$$\overline{C} = C_{d_2}$$

$$C'_S = 0$$

$$C'_F = 0$$

4.1 Description of Dispersion in Shear Flow



4.1 Description of Dispersion in Shear Flow

$$\overline{u'C'} = \frac{1}{A}\int_A u'C'\,dA$$

$$\begin{aligned}
\overline{u'C'} &\cong \frac{1}{2}\left\{\left(\overline{u'C'}\right)_{\text{JBM}} + \cancel{\left(\overline{u'C'}\right)_{\text{JAM}}}\right\} \\
&= \frac{1}{2}\left\{\alpha\left(u'C'\right)_S + \left(1-\alpha\right)\left(u'C'\right)_F\right\} \\
&= \frac{1}{2}\left\{\alpha\left[-\left(1-\alpha\right)\Delta u\right]\left[\left(1-\alpha\right)\left(C_d - C_u\right)\right] + \left(1-\alpha\right)\left[\alpha\Delta u\right]\left[\left(-\alpha\right)\left(C_d - C_u\right)\right]\right\} \\
&= \frac{1}{2}\left(\alpha^2 - \alpha\right)\Delta u\left(C_d - C_u\right)
\end{aligned}$$

4.1 Description of Dispersion in Shear Flow

Now, introduce the gradient model

$$\overline{u'C'} = K \frac{\partial \overline{C}}{\partial x}$$

Then, the concentration gradient is

$$\frac{\partial \overline{C}}{\partial x} \approx \frac{C_d - C_u}{\Delta u \, t_m}$$

$$K = -\frac{\overline{u'C'}}{\frac{\partial \overline{C}}{\partial x}} = \frac{\frac{1}{2}\left(\alpha - \alpha^2\right)\Delta u \left(C_d - C_u\right)}{\frac{\left(C_d - C_u\right)}{\Delta u \, t_m}}$$

$$\boxed{= \frac{1}{2}\left(\alpha - \alpha^2\right)\left(\Delta u\right)^2 t_m} \qquad \text{(b)}$$

4.1 Description of Dispersion in Shear Flow

[Example] A three-lane river

$$\alpha = \frac{2}{3}; \quad \Delta u = 0.2; \quad t_m = 10\,\mathrm{sec}$$

$$K = \frac{1}{2}\left[\frac{2}{3} - \left(\frac{2}{3}\right)^2\right](0.2)^2\, t_m = 0.0044\, t_m$$

$t_m = 5$	10	20	30
$K = 0.0222$	0.0444	0.0889	0.1333

4.1 Description of Dispersion in Shear Flow

[Re] Taylor Model vs. Non-Fickian Model for Couette flow

$$K = \frac{U^2 h^2}{120D} \qquad (1)$$

$$K = \frac{1}{2}\left(\alpha - \alpha^2\right)\left(\Delta u\right)^2 t_m \qquad (2)$$

Compare (1) and (2)

$$\alpha = 0.5; \Delta u = U$$

$$\frac{1}{8} U^2 t_m = \frac{U^2 h^2}{120D}$$

$$t_m = \frac{h^2}{15D}$$

4.1 Description of Dispersion in Shear Flow

Homework Assignment No. 4-1

Due: Two weeks from today

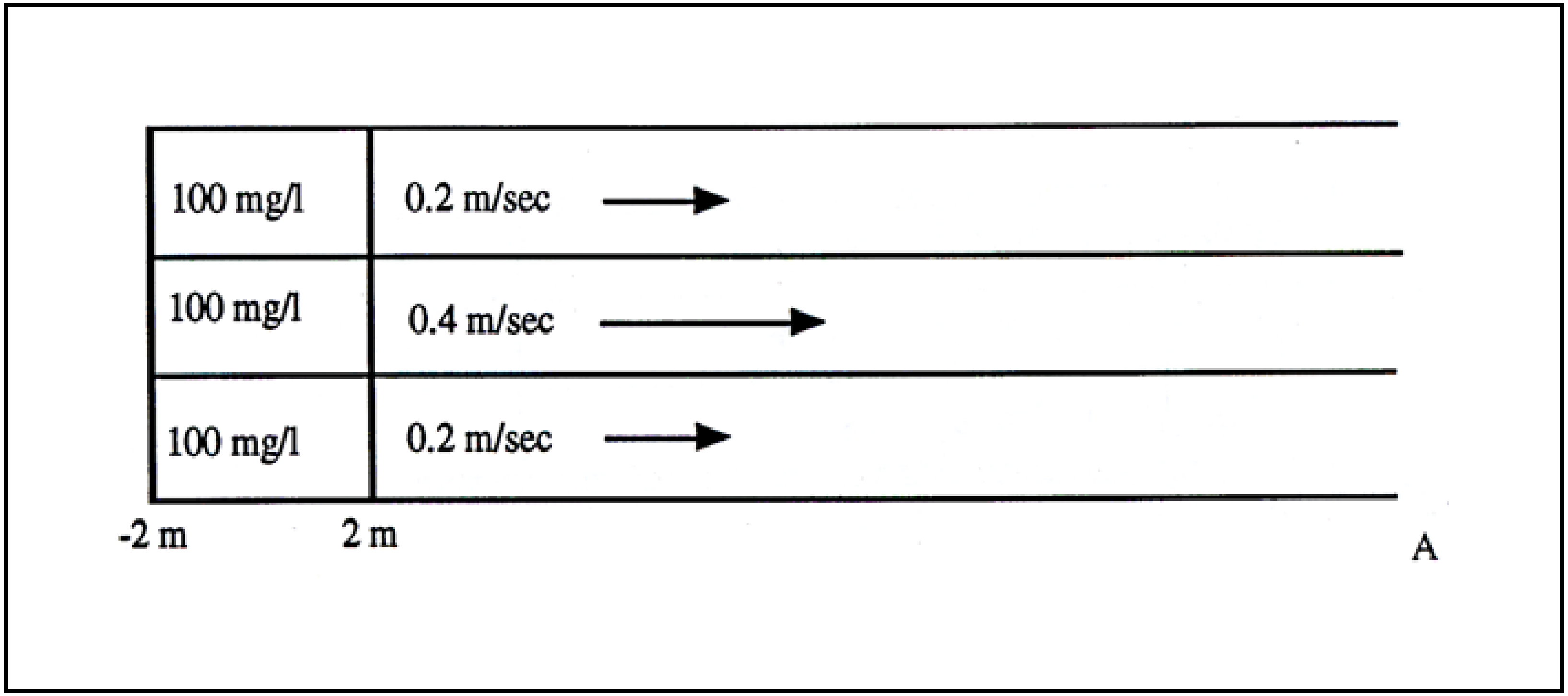
A hypothetical river is 30 m wide and consists of three "lanes", each 10m in width. The two outside lanes move at 0.2 m/sec and the middle lane at 0.4m/sec. Every t_m seconds complete mixing across the cross section of the river (but not longitudinally) occurs, after the shear advection is completed. An instantaneous line injection of a conservative tracer results in a uniform of 100mg/ℓ in the water 2 m upstream and downstream of the injection point. The concentration is initially zero elsewhere.

4.1 Description of Dispersion in Shear Flow

As the tracer is carried downstream and is mixed across the cross-section of the stream, it also becomes mixed longitudinally, due to the <u>velocity difference between lanes</u>, even though there is no longitudinal diffusion within lanes. We call this type of mixing "dispersion".

1) Mathematically simulate the tracer concentration profile (concentration vs. longitudinal distance) as a function of time for several (at least four) values of t_m including 10 sec.
2) Compare the profiles and decide whether you think the effective longitudinal mixing increases or decrease as t_m increases.

4.1 Description of Dispersion in Shear Flow



4.1 Description of Dispersion in Shear Flow

This "scenario" represents the one-dimensional unsteady-state advection and longitudinal dispersion of an instantaneous impulse of tracer for which the concentration profile follow the Gaussian plume equation

$$C(x,t)=\frac{M}{\sqrt{4\pi Kt}}exp\left\{-\frac{\left(x-Ut\right)^{2}}{4Kt}\right\}$$

in which x = distance downstream of the injection point, M = mass injected width of the stream, K = longitudinal dispersion coefficient, U = bulk velocity of the stream (flowrate/cross-sectional area), t = elapsed time since injection.

4.1 Description of Dispersion in Shear Flow

3) Using your best guess of a value for U, find a best-fit value for K for each and for which you calculated a concentration profile. Tabulate of plot the effective K as a function t_m of and make a guess of what you think the functional form is.

4.2 Fickian Dispersion Model

4.2.1 A generalized model

Consider the 2-D laminar flow with velocity variation $u(y)$ between walls

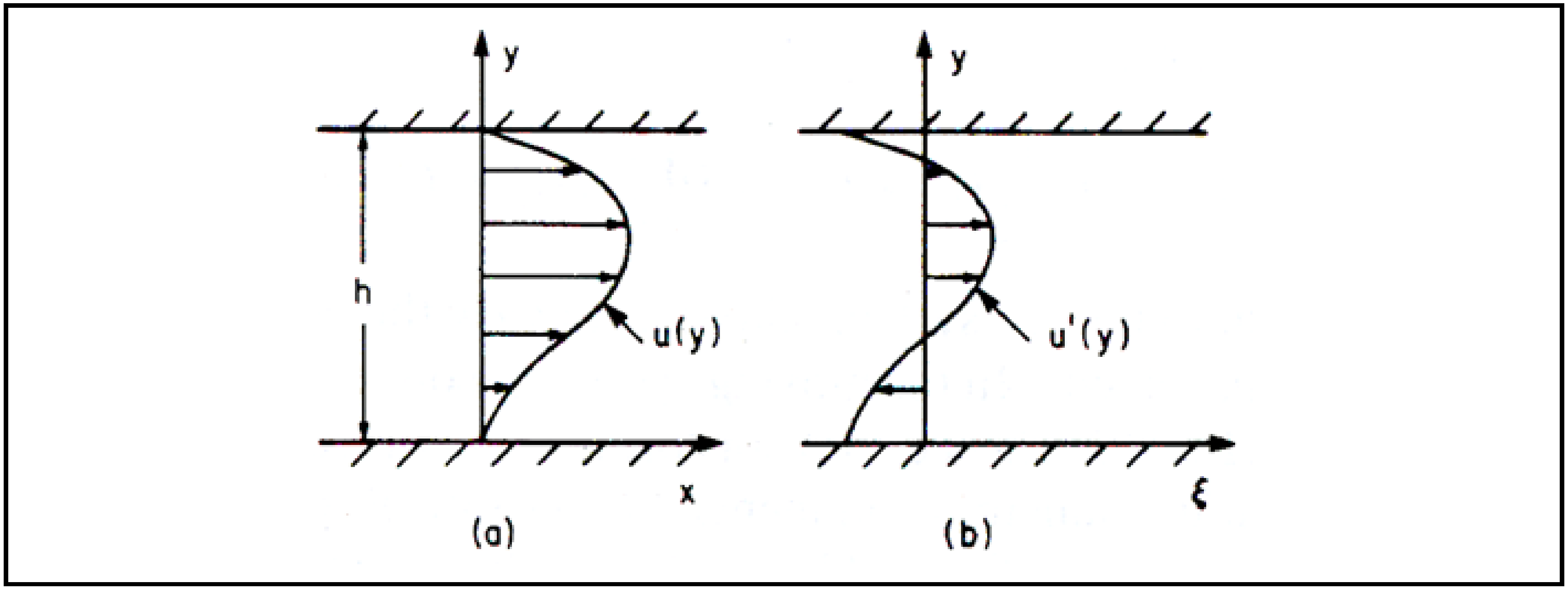
Define the cross-sectional mean velocity as

$$\overline{u} = \frac{1}{h}\int_0^h u\,dy \tag{4.2}$$

Then, velocity deviation is

$$u' = u(y) - \overline{u} \tag{4.3}$$

4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model

Let flow carry a solute with concentration $C(x, y)$ and molecular diffusion coefficient D.

Define the mean concentration at any cross section as

$$\bar{C} = \frac{1}{h}\int_0^h C dy, \qquad \bar{C} = f(x) \neq f(y) \tag{4.4}$$

Then, concentration deviation is

$$C' = C(y) - \bar{C}, \quad C' = C'(x, y) \tag{4.5}$$

4.2 Fickian Dispersion Model

Now, use 2-D diffusion equation with only flow in x-direction (v =0)

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+\cancel{v\frac{\partial C}{\partial y}}=D\frac{\partial^2 C}{\partial x^2}+D\frac{\partial^2 C}{\partial y^2} \quad (1)$$

Substitute (4.2)~(4.4) into (1)

$$\boxed{\frac{\partial \bar{C}}{\partial y}=0}$$

$$\frac{\partial}{\partial t}(\bar{C}+C^{'})+\left(\bar{u}+u^{'}\right)\frac{\partial}{\partial x}(\bar{C}+C^{'})=D\left[\frac{\partial^2}{\partial x^2}(\bar{C}+C^{'})+\frac{\partial^2}{\partial y^2}(\cancel{\bar{C}}+C^{'})\right] \quad (4.5)$$

4.2 Fickian Dispersion Model

Now, simplify (4.5) by a transformation of coordinate system whose origin moves at the mean flow velocity

$$\xi = x - \bar{u}t \quad \rightarrow \frac{\partial \xi}{\partial x} = 1 \quad \frac{\partial \xi}{\partial t} = -\bar{u} \tag{4.6}$$

$$\tau = t \quad \rightarrow \frac{\partial \tau}{\partial x} = 0 \quad \frac{\partial \tau}{\partial t} = 1 \tag{4.7}$$

Chain rule

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \xi} \tag{b}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -\bar{u}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau} \tag{c}$$

4.2 Fickian Dispersion Model

Substitute Eq. (b)-(c) into Eq. (4.5)

$$-\cancel{\bar{u}}\frac{\partial}{\partial \xi}(\bar{C}+C^{'})+\frac{\partial}{\partial \tau}(\bar{C}+C^{'})+\left(\cancel{\bar{u}}+u^{'}\right)\frac{\partial}{\partial \xi}(\bar{C}+C^{'})=D\left[\frac{\partial^2}{\partial \xi^2}(\bar{C}+C^{'})+\frac{\partial^2 C^{'}}{\partial y^2}\right]$$

$$\frac{\partial}{\partial \tau}(\bar{C}+C^{'})+u^{'}\frac{\partial}{\partial \xi}(\bar{C}+C^{'})=D\left[\frac{\partial^2}{\partial \xi^2}(\bar{C}+C^{'})+\frac{\partial^2 C^{'}}{\partial y^2}\right] \qquad (4.8)$$

→ view the flow as an observer moving at the mean velocity

→ $u^{'}$ is the only observable velocity as shown in Gig. 4.2 (b).

4.2 Fickian Dispersion Model

Now, neglect longitudinal diffusion because rate of spreading along the flow direction due to velocity difference greatly exceed that due to molecular diffusion.

$$u' \frac{\partial}{\partial \xi}(\bar{C} + C') \gg D \frac{\partial^2}{\partial \xi^2}(\bar{C} + C')$$

$$\frac{\partial \bar{C}}{\partial \tau} + \frac{\partial C'}{\partial \tau} + u' \frac{\partial \bar{C}}{\partial \xi} + u' \frac{\partial C'}{\partial \xi} = D \frac{\partial^2 C'}{\partial y^2} \qquad (4.9)$$

→ This equation is still intractable because u' varies with y.

→ General solution cannot be found because a general procedure for dealing with differential equations with variable coefficients is not available.

4.2 Fickian Dispersion Model

Now introduce Taylor's assumption

→ discard three terms to leave the easily solvable equation for $C^{'}(y)$

$$\cancel{\frac{\partial \overline{C}}{\partial \tau}} + \cancel{\frac{\partial C^{'}}{\partial \tau}} + u^{'}\frac{\partial \overline{C}}{\partial \xi} + u^{'}\cancel{\frac{\partial C^{'}}{\partial \xi}} = D\frac{\partial^2 C^{'}}{\partial y^2}$$

$$u^{'}\frac{\partial \overline{C}}{\partial \xi} = D\frac{\partial^2 C^{'}}{\partial y^2} \qquad (4.10)$$

[Re] Derivation of Eq. (4.10) using order of magnitude analysis

Take average over the cross section of Eq. (4.9)

→ apply the operator $\frac{1}{h}\int_0^h (\ )dy$

4.2 Fickian Dispersion Model

$$\overline{\frac{\partial \bar{C}}{\partial \tau}} + \cancel{\overline{\frac{\partial C'}{\partial \tau}}} + \overline{u' \cancel{\frac{\partial \bar{C}}{\partial \xi}}} + \overline{u' \frac{\partial C'}{\partial \xi}} = \overline{D \cancel{\frac{\partial^2 C'}{\partial y^2}}}$$

Apply Reynolds rule of average, then we have

$$\frac{\partial \bar{C}}{\partial \tau} + \overline{u' \frac{\partial C'}{\partial \xi}} = 0 \qquad (4.11)$$

Subtract Eq.(4.11) from Eq.(4.9)

$$\frac{\partial C'}{\partial \tau} + u' \frac{\partial \bar{C}}{\partial \xi} + u' \frac{\partial C'}{\partial \xi} - \overline{u' \frac{\partial C'}{\partial \xi}} = D \frac{\partial^2 C'}{\partial y^2} \qquad (4.12)$$

4.2 Fickian Dispersion Model

Assume $\bar{C}, C'$ are well behaved, slowly varying functions and $\bar{C} >> C'$

Then $u' \dfrac{\partial \bar{C}}{\partial \xi} >> u' \dfrac{\partial C'}{\partial \xi}, \overline{u' \dfrac{\partial C'}{\partial \xi}}$

Thus we can drop $u' \dfrac{\partial C'}{\partial \xi}$ and $\overline{u' \dfrac{\partial C'}{\partial \xi}}$

$$\frac{\partial C'}{\partial \tau} = D \frac{\partial^2 C'}{\partial y^2} - u' \frac{\partial \bar{C}}{\partial \xi} \qquad \text{(d)}$$

$-u' \dfrac{\partial \bar{C}}{\partial \xi}$ = source term of variable strength

4.2 Fickian Dispersion Model

→ Net addition by source term is zero because the average of $u^{'}$ is zero.

Assume that $\dfrac{\partial \bar{C}}{\partial \xi}$ remains constant for a long time, so that the source is constant.

Then, Eq. (d) can be assumed as steady state.

$$\rightarrow \frac{\partial C^{'}}{\partial \tau} = 0$$

Then (a) becomes

longitudinal advective transport

$$\underbrace{u^{'} \frac{\partial \bar{C}}{\partial \xi}}_{(a)} = \underbrace{D \frac{\partial^2 C^{'}}{\partial y^2}}_{(b)}$$

cross-sectional diffusive transport

4.2 Fickian Dispersion Model

→ same as Eq. (4.10)

→ The cross sectional concentration profile $C^{'}(y)$ is established by a balance between longitudinal advective transport and cross sectional diffusive transport as shown in Fig. 4.3.

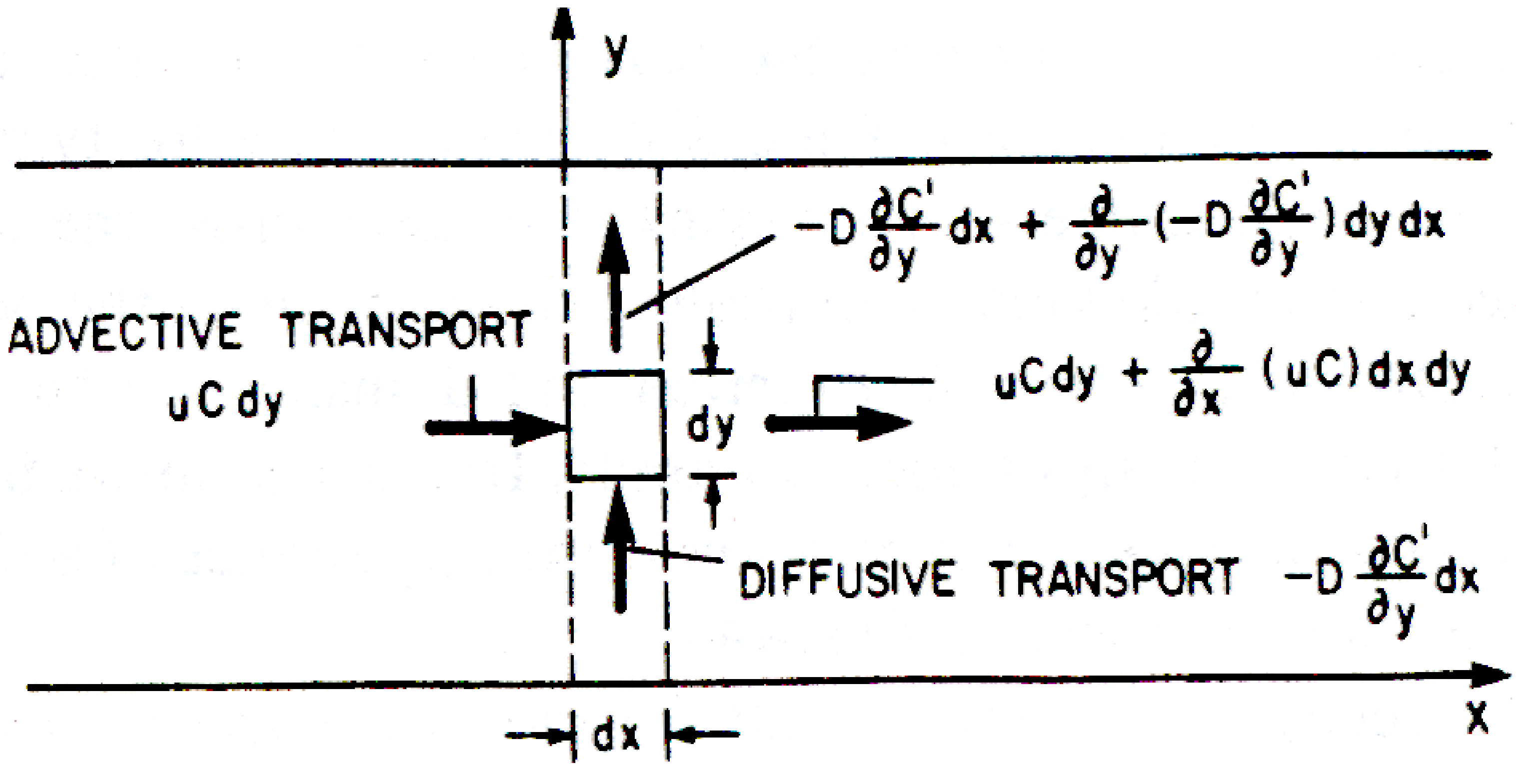
In balance, net transport = 0

$$u^{'}\overline{C}dy-\left\{u^{'}\overline{C}dy+\frac{\partial}{\partial x}\left(u^{'}\overline{C}\right)dxdy\right\}+\left\{-D\frac{\partial C^{'}}{\partial y}dx-\left[-D\frac{\partial C^{'}}{\partial y}dx+\frac{\partial}{\partial y}\left(-D\frac{\partial C^{'}}{\partial y}\right)dydx\right]\right\}=0$$

$$-\frac{\partial}{\partial x}\left(u^{'}\overline{C}\right)dxdy+\frac{\partial}{\partial y}\left(D\frac{\partial C^{'}}{\partial y}\right)dydx=0$$

$$\frac{\partial}{\partial x}\left(u^{'}\overline{C}\right)=\frac{\partial}{\partial y}\left(D\frac{\partial C^{'}}{\partial y}\right) \tag{4.13}$$

4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model

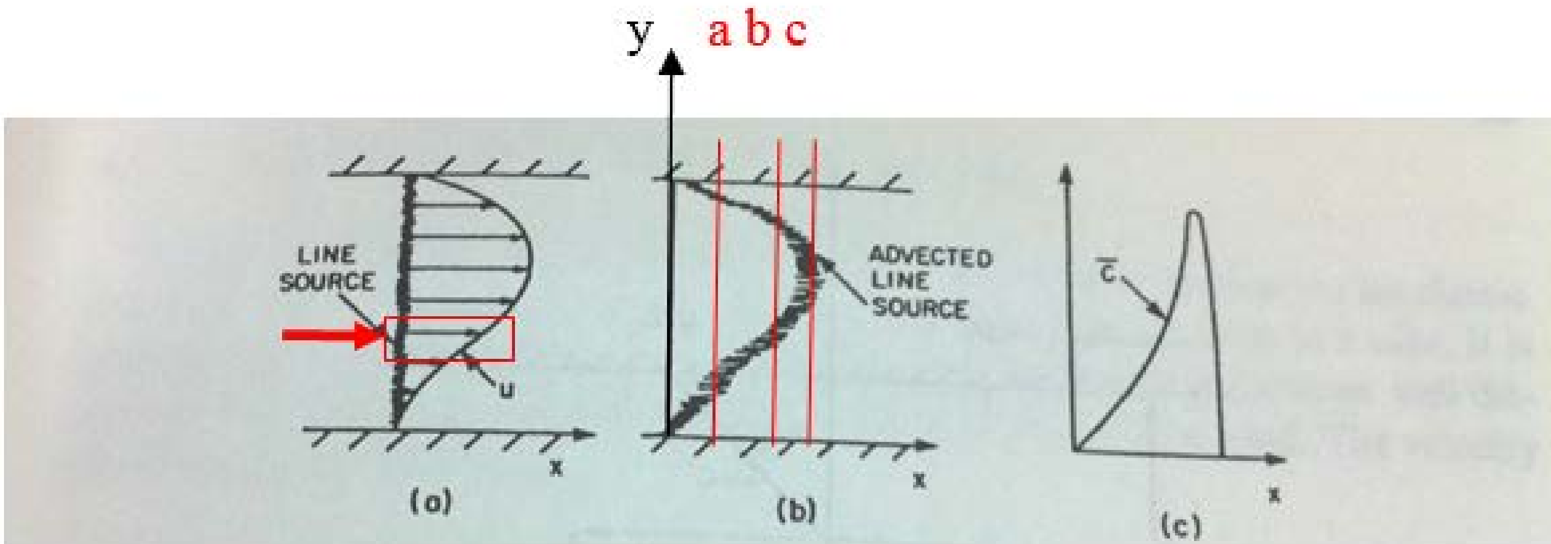
Now, let's find a solution of Eq. (4.10)

$$\frac{\partial^2 C^{'}(y)}{\partial y^2} = \frac{1}{D}\frac{\partial \overline{C}}{\partial \xi}u^{'} = \frac{1}{D}\frac{\partial \overline{C}}{\partial x}u^{'}(y) \qquad \text{(e)}$$

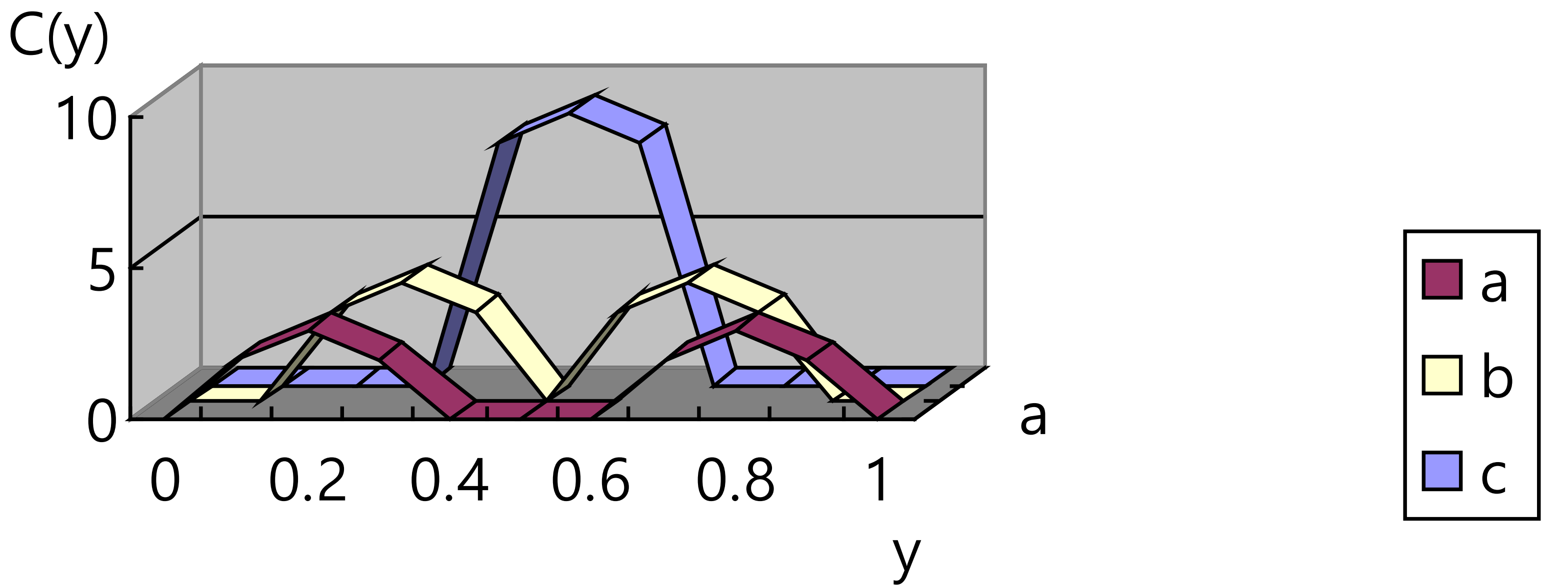
Integrate (e) twice w.r.t. y

$$\boxed{C^{'}(y) = \frac{1}{D}\frac{\partial \overline{C}}{\partial x}\int_0^y \int_0^y u^{'} dy dy + C^{'}(0)} \qquad (4.14)$$

4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model

Now, consider the rate of mass transport in the streamwise direction.

The mass transport, relative to the moving coordinate axis, is given by

$$\dot{M} = \int_0^h q_x dy = \int_0^h \left[u^{'} C^{'} + \cancel{\left(-D \frac{\partial C^{'}}{\partial x} \right)} \right] dy \qquad \text{(f)}$$

Substitute (4.14) in (f)

$$\boxed{\dot{M} = \int_0^h u^{'} C^{'} dy = \frac{1}{D} \frac{\partial \overline{C}}{\partial x} \int_0^h u^{'} \int_0^y \int_0^y u^{'} dy dy dy} \qquad (4.15)$$

$$\int_0^h u^{'} \left\{ C^{'}(0) \right\} dy = 0 \quad \text{since} \quad \int_0^h u^{'} dy = 0$$

4.2 Fickian Dispersion Model

→ Eq. (4.15) means that total mass transport in the streamwise direction is proportional to the concentration gradient in that direction.

$$\dot{M} \propto \frac{\partial \overline{C}}{\partial x} \qquad \text{(g)}$$

→This is exactly the same result that we found for molecular diffusion.

$$q \propto \frac{\partial C}{\partial x} = -D\frac{\partial C}{\partial x}$$

But Eq. (g) is the integral sense for diffusion due to whole field of flow.

Now define a bulk transport coefficient, or “dispersion” coefficient,

in analogy to the molecular diffusion coefficient, by the equation

4.2 Fickian Dispersion Model

$$q = \frac{\dot{M}}{h \times 1} = -K \frac{\partial \overline{C}}{\partial x} \qquad \text{(h)}$$

where q = rate of mass transport per unit area per unit time; h = depth = area per unit width of flow

K = longitudinal dispersion coefficient (= bulk transport coefficient)

→ express as the diffusive property of the velocity distribution (shear flow)

Then, (h) becomes

$$\dot{M} = -hK \frac{\partial \overline{C}}{\partial x} \qquad (4.16)$$

4.2 Fickian Dispersion Model

Comparing Eq. (4.15) and Eq. (4.16) we see that

$$\boxed{K = -\frac{1}{hD}\int_0^h u' \int_0^y \int_0^y u' \, dy dy dy} \tag{4.17}$$

$$K \propto \frac{1}{D}$$

Now, we can express this transport process due to velocity distribution as a one-dimensional Fickian-type diffusion equation, following Eq. (2.4), in moving coordinate system.

$$\frac{\partial \overline{C}}{\partial \tau} = K \frac{\partial^2 \overline{C}}{\partial \xi^2} \tag{4.18}$$

4.2 Fickian Dispersion Model

Return to fixed coordinate system

$$\boxed{\frac{\partial \bar{C}}{\partial t} + \bar{u}\frac{\partial \bar{C}}{\partial x} = K\frac{\partial^2 \bar{C}}{\partial x^2}} \tag{4.19}$$

→ 1-D advection-dispersion equation

$\bar{C}$, $\bar{u}$ = cross-sectional average values

- Balance of advection and diffusion in Eq. (4.10)

Suppose that at some initial time t = 0 a line source of tracer is deposited in the flow.

→ Initially, the line source is advected and distorted by the velocity profile (Fig. 4.4).

4.2 Fickian Dispersion Model

At the same time the distorted source begins to diffuse across the cross section.

→ Shortly we see a smeared cloud with trailing stringers along the boundaries (Fig. 4.4b).

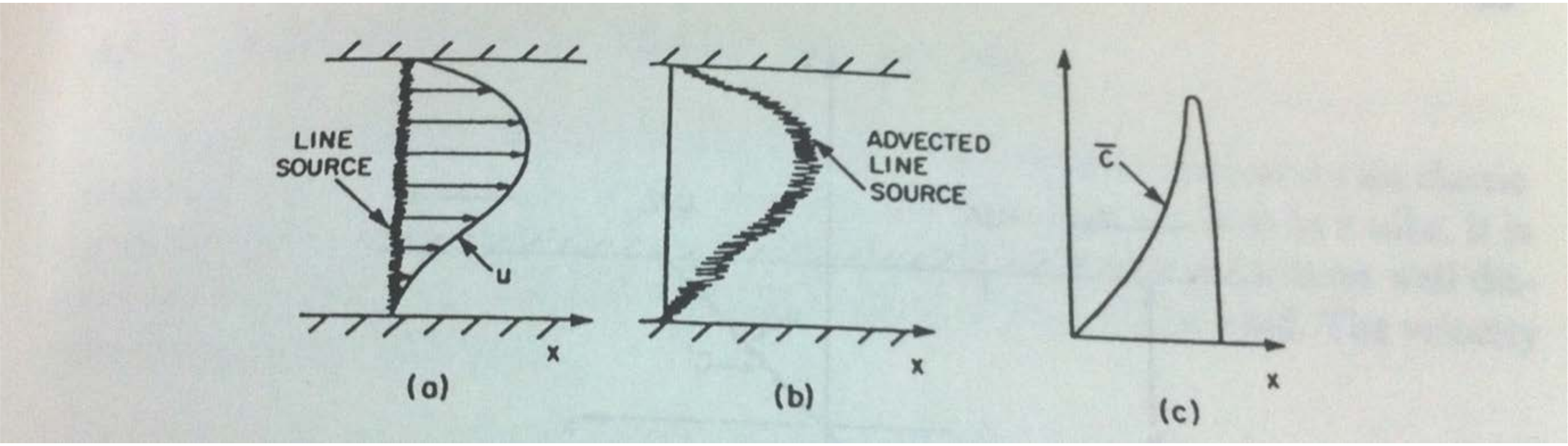
During this period, advection and diffusion are by no means in balance.

→ Cross-sectional average concentration is skewed distribution (Fig. 4.4c).

→ Taylor's assumption does not apply.

If we wait a much longer time, the cloud of tracer extends over a long distance in the x direction.

4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model

→ $\bar{C}$ varies slowly along the channel, and $\dfrac{\partial \bar{C}}{\partial x}$ is essentially constant over a long period of time.

→ C' becomes small because cross-sectional diffusion evens out cross-sectional concentration gradient.

Once the balance is established further longitudinal spreading follows Eq. (4.19), whose solution is normally distributed cloud moving at the mean speed $\bar{u}$, and continuing to spread with $\dfrac{d\sigma^2}{dt} = 2K$

- Chatwin (1970) suggested

i) Initial period: $t < 0.4\dfrac{h^2}{D}$

→ advection > diffusion

4.2 Fickian Dispersion Model

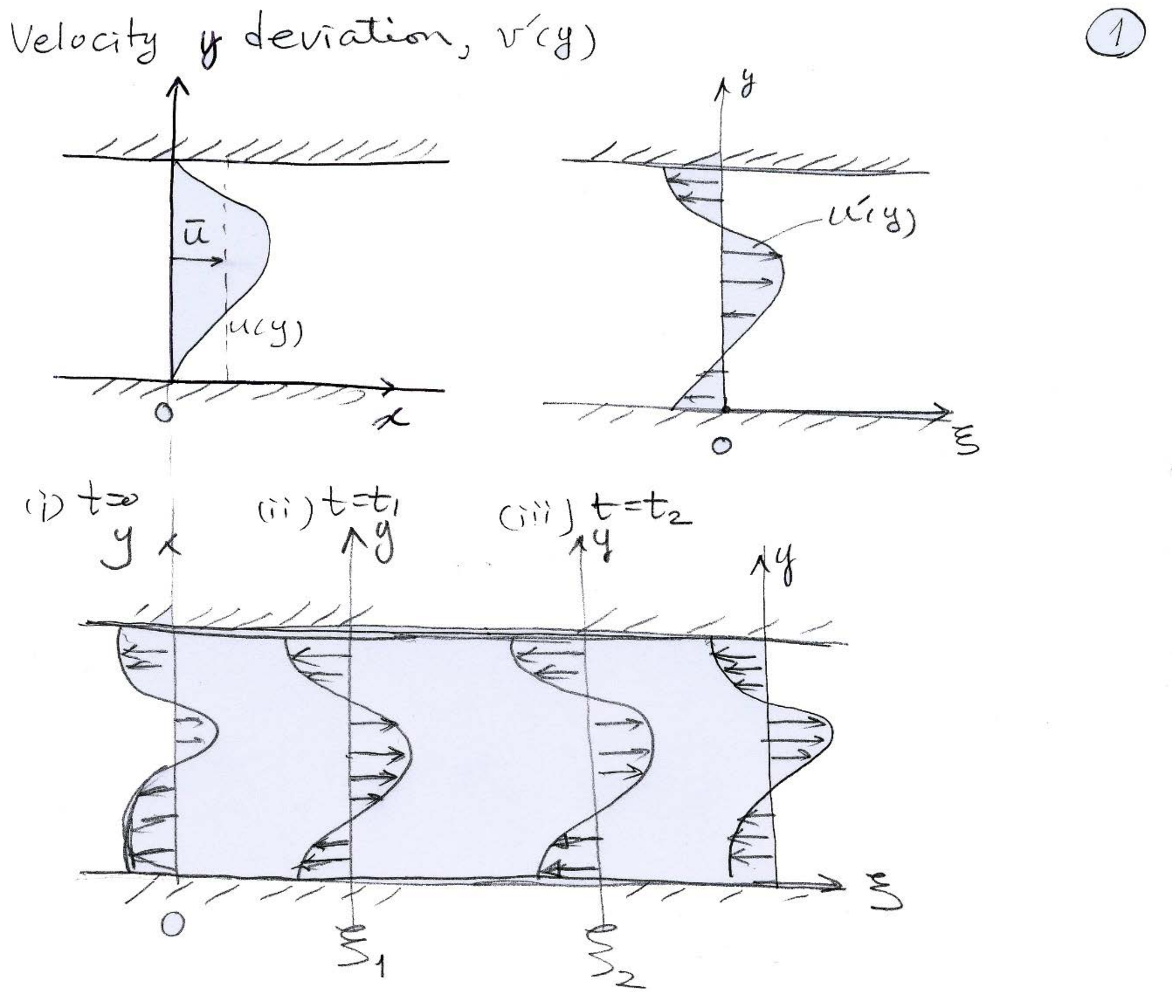
ii) Taylor period: $t > 0.4\dfrac{h^2}{D}$

→ advection ≈ diffusion

→ can use Eq. (4.19)

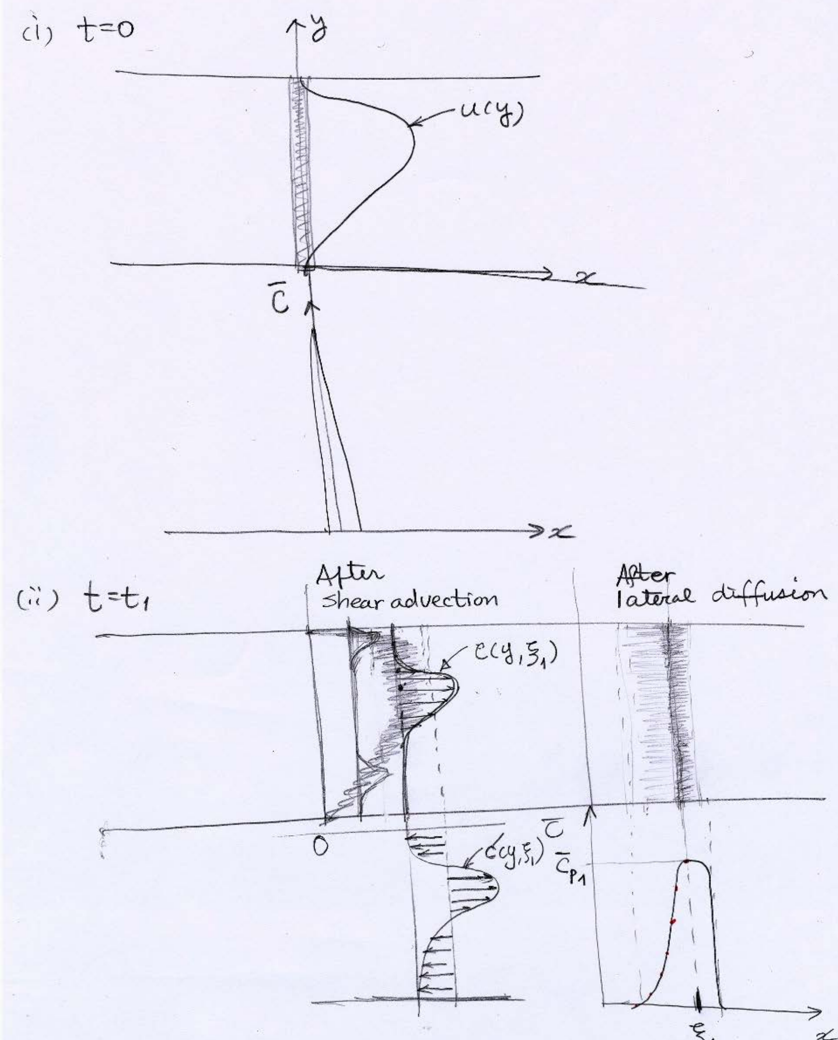
→ The initial skew degenerates into the normal distribution.

4.2 Fickian Dispersion Model

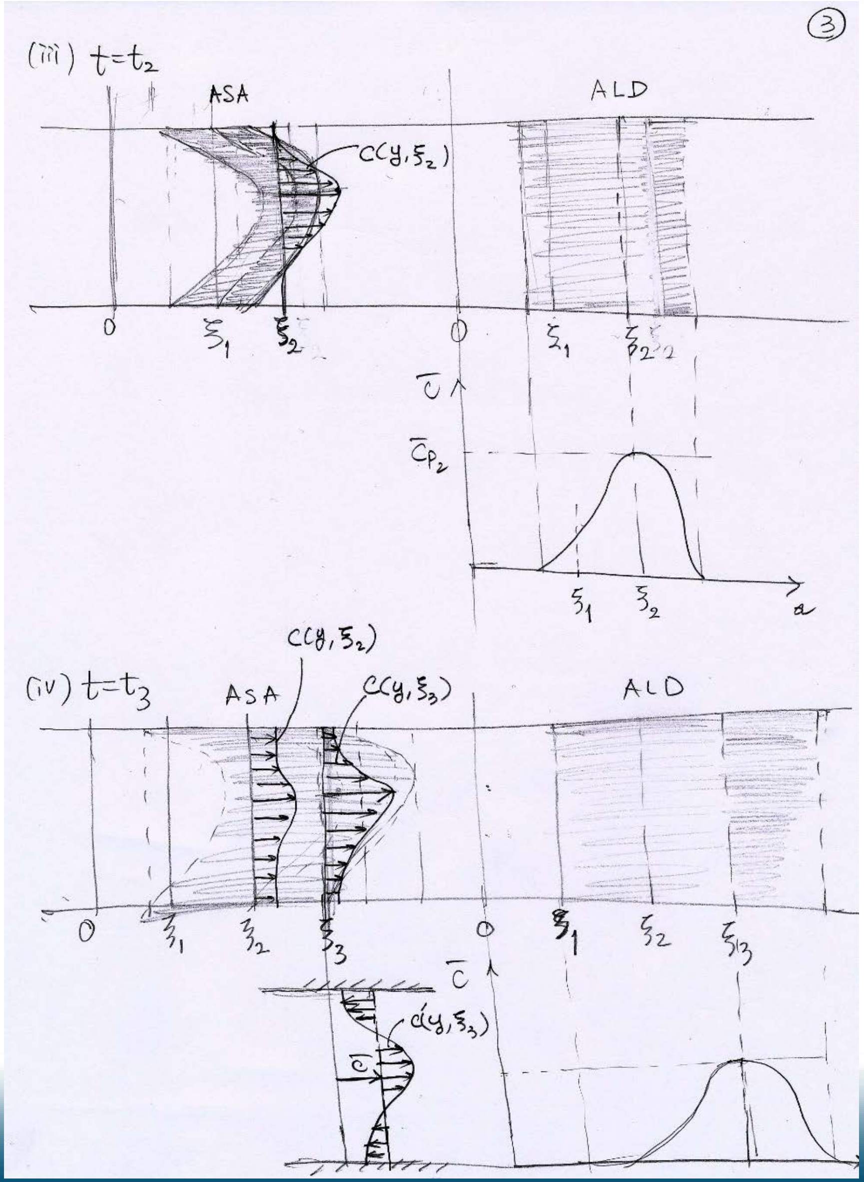


4.2 Fickian Dispersion Model

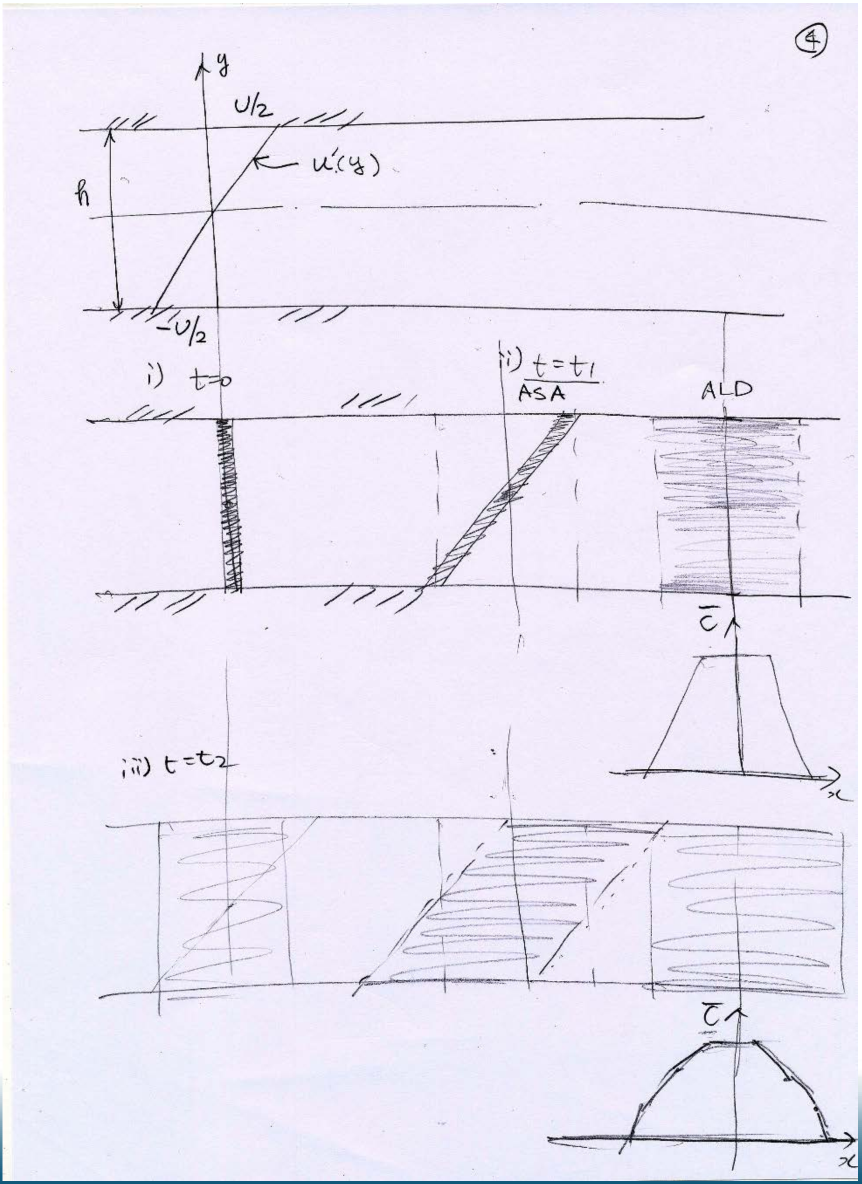
B. Concentration deviation, $c'(y)$ and cross-sectional average concentration, $\bar{c}(x)$



4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model

	region	criteria
Chatwin (1970)	Taylor period	$t > 0.4\frac{h^2}{D}$
Fischer et al. (1979)	Complete transverse mixing	$x > 0.1U\frac{W^2}{\varepsilon_t}$, centerline injection $x > 0.4U\frac{W^2}{\varepsilon_t}$, side injection

4.2 Fickian Dispersion Model

4.2.2 Laminar flows

(1) **Laminar flow between two plates**

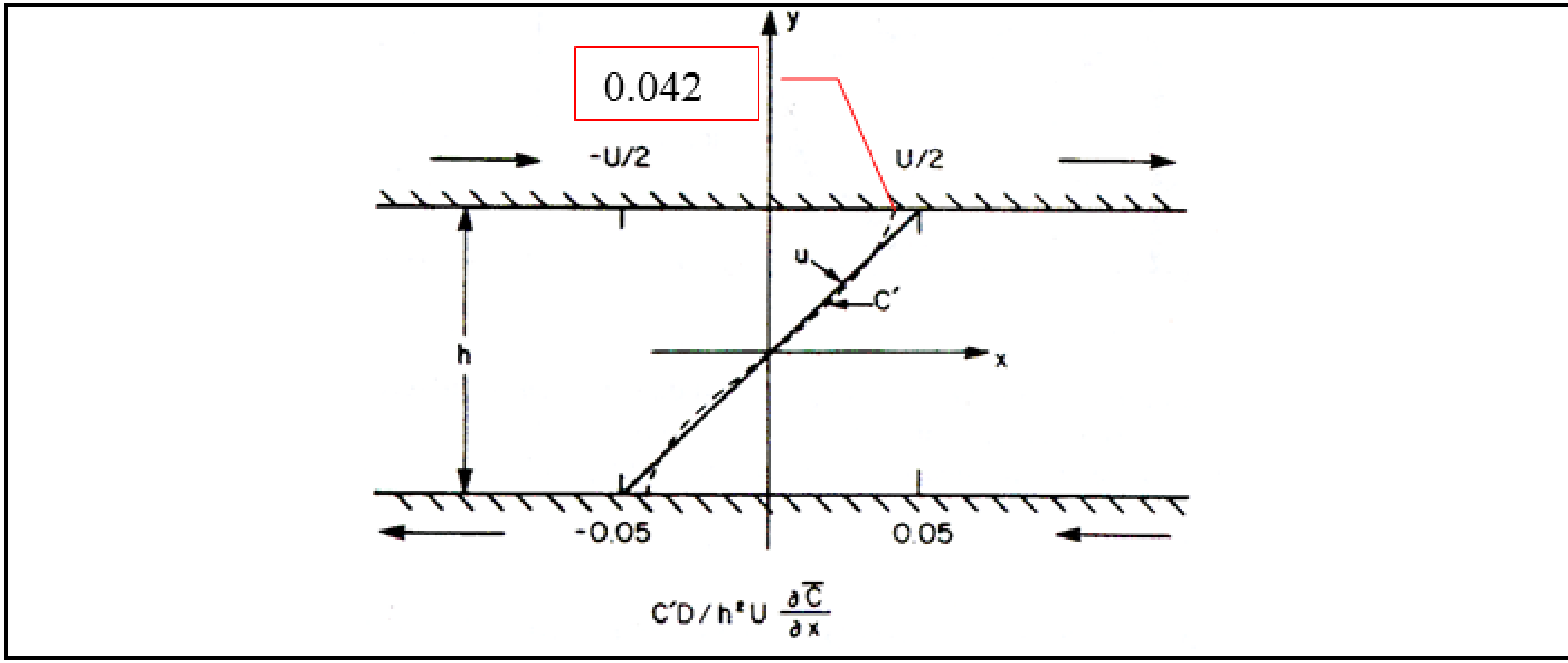
Consider laminar flow between two plates → Couette flow

$$u(y) = U\frac{y}{h}$$

$$\bar{u} = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} U\frac{y}{h}\,dy = 0$$

$$\therefore u' = u$$

4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model

Suppose $t > \frac{h^2}{D}$ → tracer is well distributed → Taylor's analysis can be applied

From Eq. (4.14)

$$C^{'}(y) = \frac{1}{D}\frac{\partial \overline{C}}{\partial x}\int_0^y \int_0^y u^{'} dy dy + C^{'}(0) \qquad (4.20)$$

$$= \frac{1}{D}\frac{\partial \overline{C}}{\partial x}\int_{-\frac{h}{2}}^y \int_{-\frac{h}{2}}^y \frac{Uy}{h} dy dy + C^{'}(-\frac{h}{2}) \qquad (a)$$

$$= \frac{1}{D}\frac{\partial \overline{C}}{\partial x}\int_{-\frac{h}{2}}^y \left[\frac{U}{2h}y^2\right]_{-\frac{h}{2}}^y dy + C^{'}(-\frac{h}{2})$$

4.2 Fickian Dispersion Model

$$= \frac{1}{D}\frac{\partial \bar{C}}{\partial x}\int_{-\frac{h}{2}}^{y}\left[\frac{Uy^2}{2h}-\frac{Uh}{8}\right]dy + C'(-\frac{h}{2})$$

$$= \frac{1}{D}\frac{\partial \bar{C}}{\partial x}\left[\frac{Uy^3}{6h}-\frac{Uh}{8}y\right]_{-\frac{h}{2}}^{y} + C'(-\frac{h}{2})$$

$$= \frac{1}{D}\frac{\partial \bar{C}}{\partial x}\left[\frac{Uy^3}{6h}-\frac{Uh}{8}y+\frac{Uh^2}{48}-\frac{Uh^2}{16}\right] + C'(-\frac{h}{2})$$

$$= \frac{1}{D}\frac{\partial \bar{C}}{\partial x}\frac{U}{2h}\left[\frac{y^3}{3}-\frac{h^2}{4}y-\frac{h^3}{12}\right] + C'\left(-\frac{h}{2}\right) \qquad (4.21)$$

4.2 Fickian Dispersion Model

By symmetry $C^{'}=0 \quad @ \quad y=0$

$$0=\frac{1}{D}\frac{\partial \overline{C}}{\partial x}\frac{U}{2h}\left[-\frac{h^3}{12}\right]+C^{'}\left(-\frac{h}{2}\right)$$

$$C^{'}\left(-\frac{h}{2}\right)=\frac{1}{D}\frac{\partial \overline{C}}{\partial x}\frac{Uh^2}{24}$$

$$\therefore C^{'}(y)=\frac{1}{D}\frac{\partial \overline{C}}{\partial x}\frac{U}{2h}\left[\frac{y^3}{3}-\frac{h^2}{4}y\right]$$

$$\boxed{\rightarrow @\, y=\frac{h}{2};\ C^{'}=\frac{1}{D}\frac{\partial \overline{C}}{\partial x}U\left[-\frac{h^2}{24}\right]} \qquad (4.22)$$

4.2 Fickian Dispersion Model

$$\rightarrow \frac{C^{'} D}{\frac{\partial \bar{C}}{\partial x} U h^{2}} = -\frac{1}{24} = -0.042$$

Dispersion coefficient K

$$K = -\frac{1}{hD} \int_{-\frac{h}{2}}^{\frac{h}{2}} u^{'} \underbrace{\int_{-\frac{h}{2}}^{y} \int_{-\frac{h}{2}}^{y} u^{'} dy dy}_{(A)} dy$$

From (a):

$$(A) = \frac{D C^{'}(y)}{\frac{\partial \bar{C}}{\partial x}} \left[C^{'}(y) - C^{'}\left(-\frac{h}{2}\right) \right]$$

$$= -\frac{1}{hD} \int_{-\frac{h}{2}}^{\frac{h}{2}} u^{'} \frac{D}{\frac{\partial \bar{C}}{\partial x}} \left[C^{'}(y) - C^{'}\left(-\frac{h}{2}\right) \right] dy$$

4.2 Fickian Dispersion Model

$$= -\frac{1}{h\frac{\partial \bar{C}}{\partial x}}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} u' C' dy + C'\left(-\frac{h}{2}\right)\cancel{\int_{-\frac{h}{2}}^{\frac{h}{2}} u' dy}\right]$$

$$= -\frac{1}{h\frac{\partial \bar{C}}{\partial x}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{Uy}{h}\right)\left\{\frac{1}{D}\frac{\partial \bar{C}}{\partial x}\frac{U}{2h}\left(\frac{y^3}{3} - \frac{h^2}{4}y\right)\right\} dy$$

$$= -\frac{U^2}{2h^3 D}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\frac{y^4}{3} - \frac{h^2 y^2}{4}\right] dy$$

$$= -\frac{U^2}{2h^3 D}\left[\frac{y^5}{15} - \frac{h^2 y^3}{12}\right]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$= \frac{U^2 h^2}{120 D} \qquad (4.23)$$

4.2 Fickian Dispersion Model

Note that $K \propto \frac{1}{D}$

→ Larger lateral mixing coefficient makes C' to be decreased.

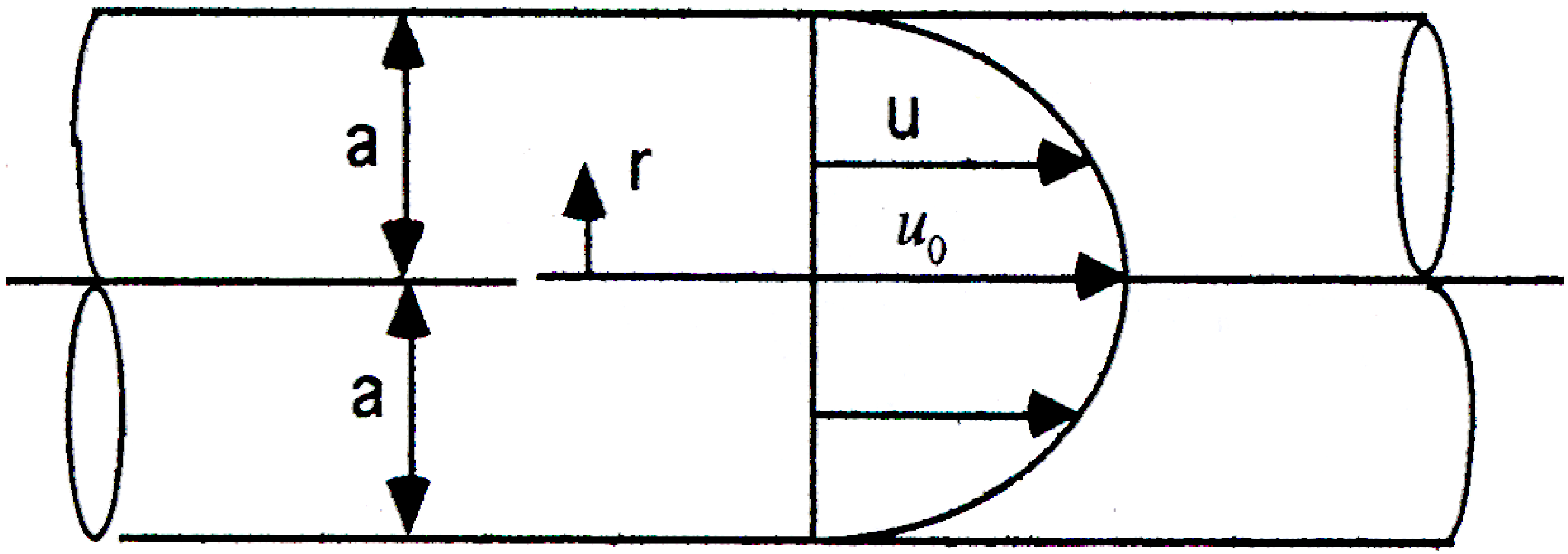
(2) **Laminar Flow in a Tube**

Consider axial symmetrical flow in a tube → Poiseuille flow

Tracer is well distributed over the cross section.

$$u(r) = u_0\left(1 - \frac{r^2}{a^2}\right) \quad \text{→ paraboloid} \qquad \text{(a)}$$

4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model

Integrate u to obtain mean velocity

$$dQ \cong u\,2\pi r dr$$

$$\therefore Q = \int_0^a 2\pi r \left\{ u_0 \left(1 - \frac{r^2}{a^2} \right) \right\} dr$$

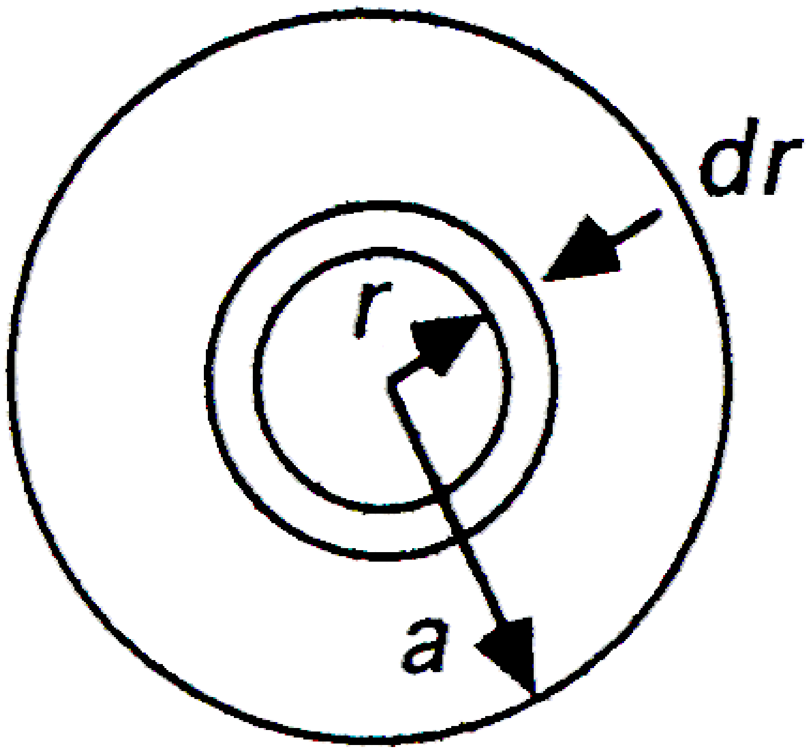
$$= 2\pi u_0 a^2 \int_0^1 \frac{r}{a} \left(1 - \frac{r^2}{a^2} \right) d\left(\frac{r}{a} \right)$$

$$= 2\pi u_0 a^2 \int_0^1 z(1 - z^2) dz$$

$$= 2\pi u_0 a^2 \left[\frac{z^2}{2} - \frac{z^2}{4} \right]_0^1$$

$$= \frac{\pi}{2} a^2 u_0 \tag{4.24}$$

4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model

By the way, $Q = \bar{u} \cdot \pi a^2$

$$\therefore \bar{u} = \frac{u_0}{2} \tag{4.25}$$

2-D advection-dispersion equation in cylindrical coordinate is

$$\frac{\partial C}{\partial t} + u_0\left(1 - \frac{r^2}{a^2}\right)\frac{\partial C}{\partial x} = D\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2}\right) \tag{b}$$

Shift to a coordinate system moving at velocity

Neglect $\dfrac{\partial C}{\partial t}$ and $\dfrac{\partial^2 C}{\partial x^2}$ as before

4.2 Fickian Dispersion Model

Let $z = \dfrac{r}{a}, \xi = x - \bar{u}t, \tau = t$

Decompose C, then (b) becomes

$$\frac{u_0 a^2}{D}(\frac{1}{2} - z^2)\frac{\partial \bar{C}}{\partial \xi} = \frac{\partial^2 C'}{\partial z^2} + \frac{1}{z}\frac{\partial C'}{\partial z}$$

$$\frac{\partial C'}{\partial z} = 0 \quad \text{at} \quad z = 1 \qquad (4.26)$$

Integrate twice w.r.t. z

$$\boxed{C' = \frac{u_0 a^2}{8D}\left(z^2 - \frac{1}{2}z^4\right)\frac{\partial \bar{C}}{\partial x} + const} \qquad \text{(c)}$$

4.2 Fickian Dispersion Model

$$K = -\frac{\dot{M}}{A\dfrac{\partial \overline{C}}{\partial x}} = -\frac{1}{A\dfrac{\partial \overline{C}}{\partial x}}\int_A u^{'}C^{'}dA \qquad \text{(d)}$$

Substitute (a), (c) into (d), and then perform integration

$$\boxed{K = \frac{a^2 u_0^{\ 2}}{192D}} \qquad (4.27)$$

[Example] Salt in water flowing in a tube

$$D = 10^{-5}\,cm^2/\sec$$

$$u_0 = 1\,cm/\sec$$

4.2 Fickian Dispersion Model

$$a = 2mm$$

$$R_e = \frac{ud}{v} = \frac{(0.01)(0.004)}{1\times 10^{-6}} = 40 << 2000$$ → laminar flow

$$K = \frac{a^2 u_0{}^2}{192D} = \frac{(0.2)^2 (1)^2}{192(10^{-5})} = 21cm^2/\sec \approx 10^6 D$$

☞ Initial period

$$t_0 = 0.4\frac{a^2}{D} = \frac{0.4(0.2)^2}{(10^{-5})} = 1600\sec = 27\min$$

$$x_0 = \bar{u}t_0 = \frac{u_0}{2}t_0$$

4.2 Fickian Dispersion Model

$$=(0.5)(1600)=800cm$$

$$=\frac{800}{0.2}=4,000a$$

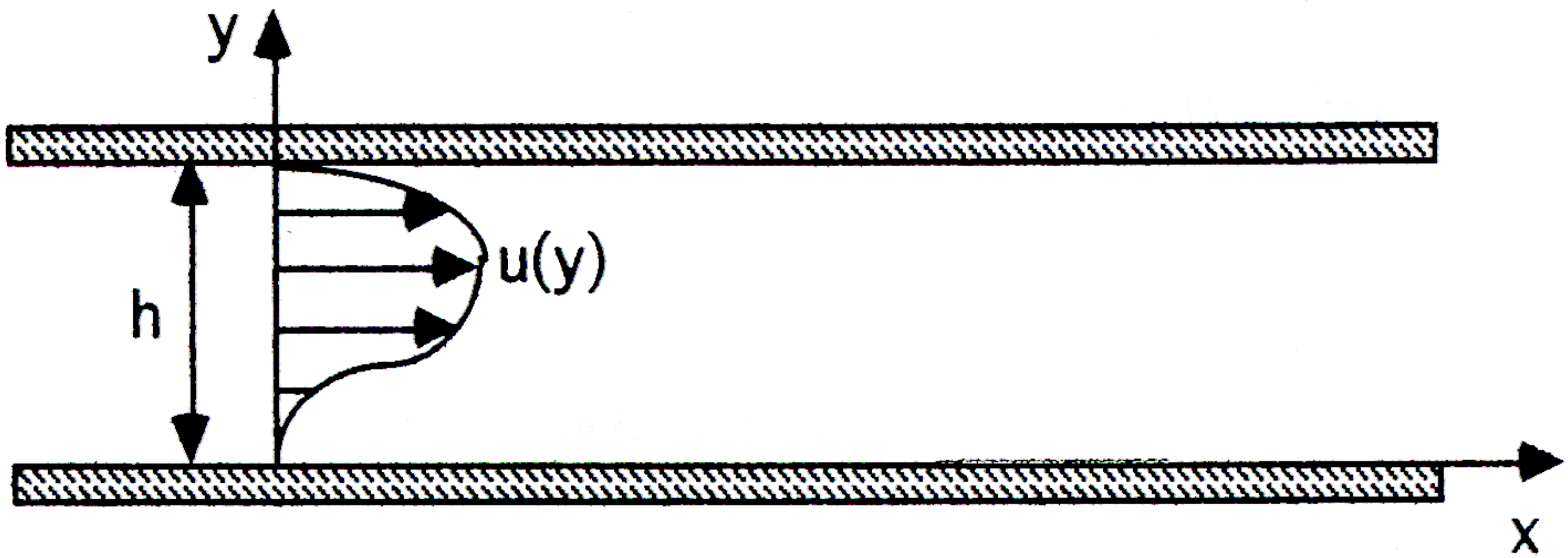
$x > x_0 \rightarrow$ 1-D dispersion model can be applied

4.2.3 Dispersion in Turbulent Shear Flow

Cross-sectional velocity profile in turbulent motion in the channel is different than in a laminar flow.

Consider unidirectional turbulent flow between parallel plates

4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model

Begin with 2-D turbulent diffusion equation

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=\frac{\partial}{\partial x}\left(\varepsilon_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon_y\frac{\partial C}{\partial y}\right) \qquad \text{(a)}$$

where, $C,u,v=$ time mean values; the cross-sectional mixing coefficient $\varepsilon(y)$ is function of cross-sectional position.

Let $v=0,$ turbulent fluctuation $v^{'}\neq 0$

Assume $u\frac{\partial C}{\partial x}\gg\frac{\partial}{\partial x}\varepsilon_x\frac{\partial C}{\partial x}$

4.2 Fickian Dispersion Model

Then (a) becomes

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial C}{\partial y}\right) \qquad \text{(b)}$$

Now, decompose C and u into cross-sectional mean and deviation

$$\frac{\partial(\bar{C} + C^{'})}{\partial t} + \left(\bar{u} + u^{'}\right)\frac{\partial}{\partial x}(\bar{C} + C^{'}) = \frac{\partial}{\partial y}\varepsilon_y \frac{\partial}{\partial y}\left(\cancel{\bar{C}} + C^{'}\right) \qquad \text{(c)}$$

Transform coordinate system into moving coordinate according to $\bar{u}$

$$\cancel{\frac{\partial \bar{C}}{\partial \tau}} + \cancel{\frac{\partial C^{'}}{\partial \tau}} + u^{'}\frac{\partial \bar{C}}{\partial \xi} + \cancel{u^{'}\frac{\partial C^{'}}{\partial \xi}} = \frac{\partial}{\partial y}\varepsilon_y \frac{\partial C^{'}}{\partial y}$$

4.2 Fickian Dispersion Model

Now, introduce Taylor's assumptions (discard three terms)

$$u'\frac{\partial \overline{C}}{\partial \xi} = \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial C'}{\partial y}\right) \qquad (4.28)$$

Solution of Eq. (4.28) can be derived by integrating twice w.r.t. y

$$\boxed{C' = \frac{\partial \overline{C}}{\partial \xi}\int_0^y \frac{1}{\varepsilon_y}\int_0^y u'\,dy\,dy + C'(0)} \qquad (4.29)$$

Mass transport in streamwise direction is

$$\dot{M} = \int_0^h u'C'\,dy = \frac{\partial \overline{C}}{\partial \xi}\int_0^h u'\int_0^y \frac{1}{\varepsilon_y}\int_0^y u'\,dy\,dy\,dy$$

4.2 Fickian Dispersion Model

$$q = \frac{\dot{M}}{h} = -K \frac{\partial \bar{C}}{\partial \xi}$$

$$\boxed{K = -\frac{1}{h}\int_0^h u' \int_0^y \frac{1}{\varepsilon_y} \int_0^y u' \, dy\,dy\,dy} \qquad (4.30)$$

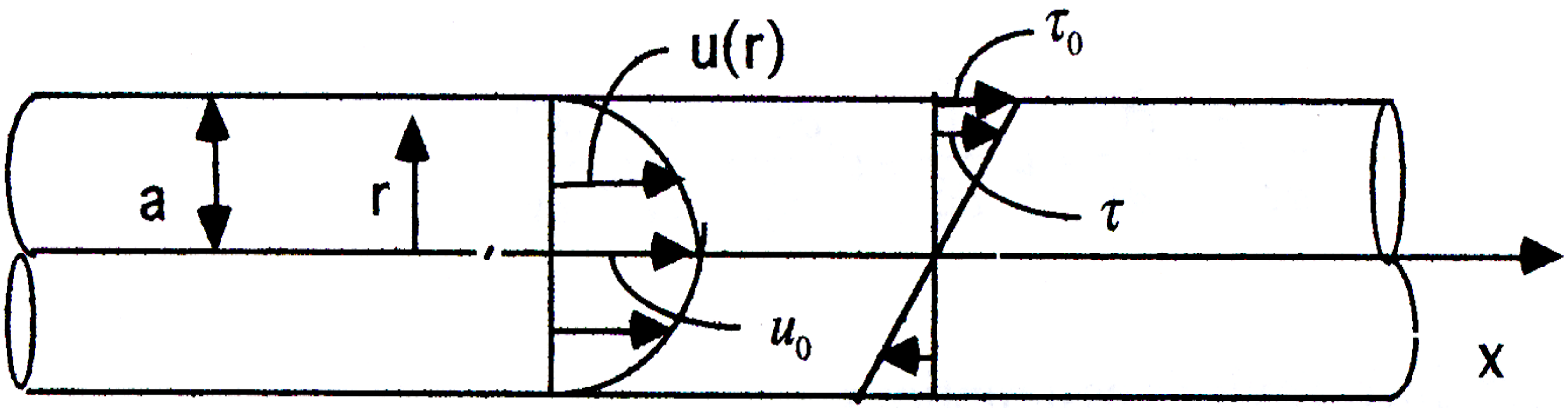
4.2.4 Taylor's analysis of turbulent flow in pipe (1954)

Set $z = \frac{r}{a} \rightarrow \frac{dz}{dr} = \frac{1}{a}$

Then, velocity profile is

$$u(z) = u_0 - u^* f(z) \qquad \text{(a)} \qquad (4.31)$$

4.2 Fickian Dispersion Model



4.2 Fickian Dispersion Model

where $u^* = \text{shear velocity} = \sqrt{\dfrac{\tau_0}{\rho}}$ (4.32)

$f(z)$ = logarithmic function

[Re] velocity defect law [Eq. (1.27)]

$$u = \overline{u} + \frac{3}{2}\frac{u^*}{\kappa} + \frac{2.30}{\kappa} u^* \log_{10}\frac{\zeta}{a} \tag{4.33}$$

in which κ = von Karman's constant ≈ 0.4

ς = distance from the wall

4.2 Fickian Dispersion Model

$$u = \overline{u} + 3.75u^{*} + 5.75u^{*}\log_{10}\frac{\zeta}{a}$$

$$\frac{u-\overline{u}}{u^{*}} = 3.75 + 2.5\ln\frac{\zeta}{a} \qquad (4.34)$$

The cross-sectional mixing coefficient can be obtained from Reynolds analogy.

→ The mixing coefficients for momentum and mass transports are the same.

i) momentum flux through a surface

$$\frac{\tau}{\rho} = -\varepsilon\frac{\partial u}{\partial r}$$

☜ Daily & Harleman (p. 56)

kinematic eddy viscosity

4.2 Fickian Dispersion Model

ii) mass flux - Fickian behavior

$$q = -\varepsilon \frac{\partial C}{\partial r}$$

$$\therefore \varepsilon = \frac{q}{-\dfrac{\partial C}{\partial r}} = \frac{\tau}{-\rho \dfrac{\partial u}{\partial r}} \qquad \text{(b)} \qquad (4.35)$$

For turbulent flow in pipe, shear stress is given

$$\tau = \tau_0 \frac{r}{a} = z\tau_0 \qquad \text{(c)} \qquad (4.36)$$

4.2 Fickian Dispersion Model

Differentiate (a) w.r.t. r

$$\frac{\partial u}{\partial r} = -u^{*}\frac{df(z)}{dz}\frac{dz}{dr} = -u^{*}\frac{df}{dz}\frac{1}{a} \qquad \text{(d)} \qquad (4.37)$$

Divide (c) by (d)

$$\frac{\tau}{\dfrac{\partial u}{\partial r}} = \frac{z\tau_0}{-u^{*}\dfrac{df}{dz}\dfrac{1}{a}} \qquad \text{(e)}$$

4.2 Fickian Dispersion Model

Substitute (e) into (b)

$$\therefore \varepsilon(r) = -\frac{\tau}{\rho \dfrac{\partial u}{\partial r}} = \frac{z\tau_0}{\rho u^* \dfrac{df}{dz}\dfrac{1}{a}} = \frac{az\overbrace{(\tau_0/\rho)}^{\boxed{(u^*)^2}}}{u^* \dfrac{df}{dz}} = \frac{azu^*}{\dfrac{df}{dz}}$$

Now, it is possible to tabulate $u^{'}(r) = u(r) - \bar{u}, \quad \varepsilon(r)$ (f)

And, numerically integrate Eq. (4.39) [Taylor's equation in radial coordinates] to obtain $C^{'}(r)$ using $\varepsilon(r)$ obtained in (f)

$$u^{'}\frac{\partial \bar{C}}{\partial \xi} = \varepsilon\left[\frac{\partial^2 C^{'}}{\partial r^2} + \frac{1}{r}\frac{\partial C^{'}}{\partial r}\right] \tag{4.39}$$

4.2 Fickian Dispersion Model

Again, numerically integrate Eq. (4.30) to find K

$$\boxed{K = 10.1 a u^*} \tag{4.40}$$

$$u^* \propto \frac{a u_0^2}{D}$$

in which a = pipe radius

[Cf] For laminar flow in a tube, $K = \dfrac{a^2 u_0^2}{192 D}$

4.2 Fickian Dispersion Model

4.2.5 Elder's application of Taylor's method (1959) in open flows

Consider turbulent flow down an infinitely wide inclined plane of depth d assuming von Karman logarithmic velocity profile

$$u'(y) = \frac{u^*}{\kappa}(1 + \ln y') \qquad \text{(a)} \qquad (4.41)$$

where $u' = u - \bar{u} \quad \rightarrow \quad \frac{du}{dy} = \frac{u^*}{\kappa}\frac{1}{y'}\frac{1}{d}$ (b)

$$y' = y/d$$

$$\boxed{\frac{d\bar{u}}{dy} = 0}$$

4.2 Fickian Dispersion Model

For open channel flow, shear stress is given

$$\tau = \rho\varepsilon\frac{du}{dy} = \tau_0(1-y^{'}) \qquad \text{(c)} \qquad (4.42)$$

Parabolic profile

$$\varepsilon(y) = \frac{\tau_0}{\rho}\frac{(1-y^{'})}{\dfrac{du}{dy}} = \frac{\tau_0}{\rho}\frac{(1-y^{'})}{\dfrac{u^*}{\kappa}\dfrac{1}{y^{'}}\dfrac{1}{d}} = \kappa y'(1-y^{'})du^* \qquad \text{(d)} \qquad (4.43)$$

Substitute Eq. (a) and Eq. (d) into Eqs. (4.29) and (4.30) and integrate

$$C^{'} = \frac{\partial\bar{C}}{\partial x}\frac{d}{\kappa^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{d-y}{d}\right)^n - 0.648\right) \qquad (4.44)$$

$$K = \frac{0.404}{\kappa^3}du^* \qquad (4.45)$$

4.2 Fickian Dispersion Model

Input $\kappa = 0.41$

$$\boxed{K = 5.93du^{*}} \tag{4.46}$$

4.2.6 General form for the longitudinal dispersion coefficient

Introduce dimensionless quantities

$$y' = \frac{y}{h} \rightarrow y = hy',\ dy = hdy' \qquad \text{(a)}$$

$$u'' = \frac{u'}{\sqrt{\overline{u'^2}}} \rightarrow u' = u''\sqrt{\overline{u'^2}} \qquad \text{(b)}$$

$$\varepsilon' = \frac{\varepsilon}{E} \rightarrow \varepsilon = \varepsilon' E \qquad \text{(c)}$$

4.2 Fickian Dispersion Model

3. Shear Flow Dispersion

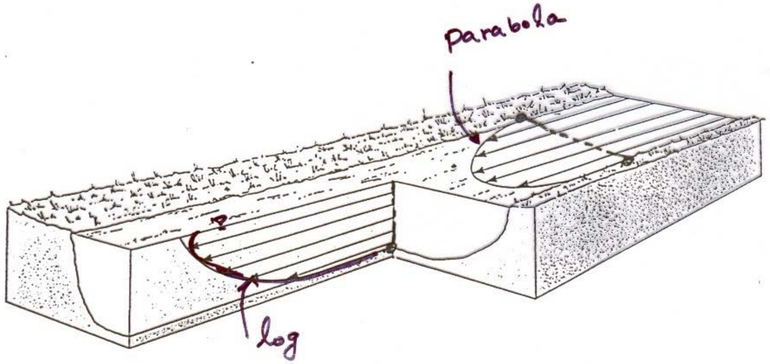
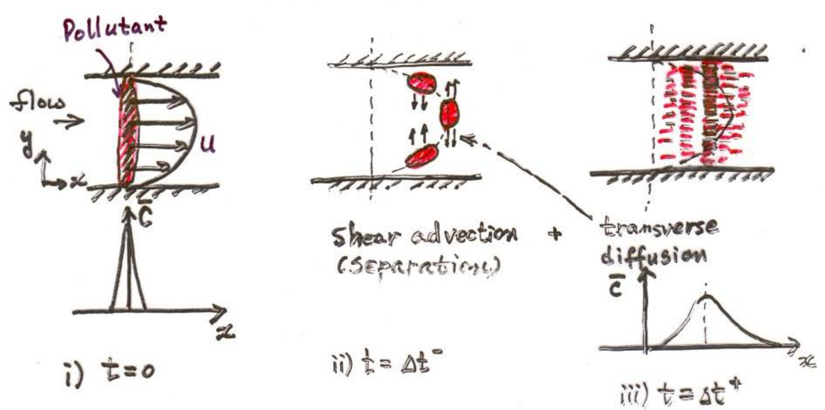


Figure 10.5
Variations in the velocity of flow in natural stream channels occur both horizontally and vertically. Friction reduces the velocity along the floor and sides of the channels. The maximum velocity in a straight channel is near the top and center of the channel.



4.2 Fickian Dispersion Model

where E = cross-sectional average of ε

$u^{'}$ = velocity deviation from cross-sectional mean velocity

$$\sqrt{\overline{u^{'2}}} = \left\{ \frac{1}{h} \int_0^h (u^{'})^2 \, dy \right\}^{\frac{1}{2}}$$

= intensity of the velocity deviation (different from turbulent intensity)

~ measure of how much the turbulent averaged velocity deviates throughout the cross section from its cross-sectional mean

4.2 Fickian Dispersion Model

Substitute (a) ~ (c) into Eq. (4.30)

$$K = -\frac{1}{h}\int_0^1 u'' \sqrt{\overline{u'^2}} \int_0^{y'} \frac{1}{\varepsilon' E} \int_0^{y'} u'' \sqrt{\overline{u'^2}} h^3 dy' dy' dy'$$

$$= -\frac{1}{h}\sqrt{\overline{u'^2}}\frac{1}{E}\sqrt{\overline{u'^2}} h^3 \int_0^1 u'' \int_0^{y'} \frac{1}{\varepsilon'} \int_0^{y'} u'' dy' dy' dy'$$

$$= \frac{\overline{u'^2} h^2}{E}\left(-\int_0^1 u'' \int_0^{y'} \frac{1}{\varepsilon'} \int_0^{y'} u'' dy' dy' dy' \right) \qquad \text{(d)}$$

Set $I = -\int_0^1 u'' \int_0^{y'} \frac{1}{\varepsilon'} \int_0^{y'} u'' dy' dy' dy'$ (4.47)

4.2 Fickian Dispersion Model

Then (d) becomes

$$\boxed{K = \frac{h^2 \overline{u'^2}}{E} I} \quad (4.48)$$

$$I = 0.054 \sim 0.10 \quad \rightarrow I \cong 0.10$$

4.2.7 Aris's Analysis

Aris (1956) proposed the <u>concentration moment method</u> in which he obtained Taylor's main results without stipulating the feature of the concentration distribution.

4.2 Fickian Dispersion Model

Flow	Velocity profile	h	I	K
(i)Laminar flow in a tube	$u = u_0(1 - \frac{r^2}{a^2})$	A	0.0625	$\frac{a^2 u_0^2}{192D}$
(ii)Laminar flow at depth down on inclined plane	$u = u_0\left[2\left(\frac{y}{d}\right) - \frac{y^2}{d^2}\right]$	d	0.0952	$\frac{8}{945}\frac{d^2 u_0^2}{D}$
(iii)Laminar flow with a linear velocity profile across a spacing	$u = U\frac{y}{h}$	h	0.10	$\frac{U^2 h^2}{120D}$
(iv)Turbulent flow in a pipe	Empirical	a	0.054	$10.1\ au^*$
(v)Turbulent flow at depth down an inclined plane	$u = \bar{u} + \frac{u^*}{\kappa}(1 + \ln\frac{y}{d})$	d	0.067	$5.93du^*$

4.2 Fickian Dispersion Model

Begin with 2-D advective-diffusion equation in the moving coordinate system to analyze the flow between two plates (Couette flow)

$$\underset{(1)}{\frac{\partial C}{\partial \tau}} + u' \underset{(2)}{\frac{\partial C}{\partial \xi}} = D\left(\underset{(3)}{\frac{\partial^2 C}{\partial \xi^2}} + \underset{(4)}{\frac{\partial^2 C}{\partial y^2}} \right) \tag{4.49}$$

Now, define the p_{th} moments of the concentration distribution

$$C_P(y) = \int_{-\infty}^{\infty} \xi^p C(\xi, y)\, d\xi \tag{4.50}$$

4.2 Fickian Dispersion Model

Take the moment of Eq. (4.49) by applying the operator $\int_{-\infty}^{\infty} \xi^{P} (\quad) d\xi$

$$(1) = \int_{-\infty}^{\infty} \xi^{p} \frac{\partial C}{\partial \tau} d\xi = \frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} \xi^{p} C d\xi = \frac{\partial C_{p}}{\partial \tau} \qquad \leftarrow \text{Leibnitz rule} \qquad (4.52)$$

[Re] Leibnitz formula: $\int_{u_0}^{u_1} \frac{\partial f}{\partial \alpha} dx = \frac{d}{d\alpha} \int_{u_0}^{u_1} f dx$

u

v’

$$(2) = \int_{-\infty}^{\infty} \xi^{p} u' \frac{\partial C}{\partial \xi} d\xi = u' \int_{-\infty}^{\infty} \underline{\xi^{p}} \, \underline{\frac{\partial C}{\partial \xi}} d\xi \qquad \leftarrow \text{integral by parts}$$

$\boxed{C\big|_{\xi=\pm\infty} = 0}$

$$= u' \left\{ \cancel{\left[\xi^{p} C \right]_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} p \xi^{p-1} C \, d\xi \right\}$$

4.2 Fickian Dispersion Model

$$= -pu' \int_{-\infty}^{\infty} \xi^{p-1} C d\xi = -pu' C_{p-1}$$

$$(3) = \int_{-\infty}^{\infty} \xi^{p} D \frac{\partial^2 C}{\partial \xi^2} d\xi = D \int_{-\infty}^{\infty} \overbrace{\xi^{p}}^{u} \overbrace{\frac{\partial}{\partial \xi}\left(\frac{\partial C}{\partial \xi}\right)}^{v'} d\xi$$

← integral by parts

$$= D\left\{ \cancel{\left[\xi^{p} \frac{\partial C}{\partial \xi}\right]_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} \frac{\partial C}{\partial \xi} p \xi^{p-1} d\xi \right\}$$

$$= -Dp \int_{-\infty}^{\infty} \xi^{p-1} \frac{\partial C}{\partial \xi} d\xi$$

$$= -Dp \left\{ \cancel{\left[\xi^{p-1} C\right]_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} C (p-1) \xi^{p-2} d\xi \right\}$$

$$= Dp(p-1) \int_{-\infty}^{\infty} \xi^{p-2} C d\xi = Dp(p-1) C_{p-2}$$

4.2 Fickian Dispersion Model

$$(4) = \int_{-\infty}^{\infty} \xi^p D \frac{\partial^2 C}{\partial y^2} d\xi = D \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} \xi^p C d\xi = D \frac{\partial^2 C_P}{\partial y^2}$$

Applying these terms to Eq. (4.49) yields

$$\frac{\partial C_p}{\partial \tau} - p u' C_{p-1} = D \left\{ p(p-1) C_{p-2} + \frac{\partial^2 C_P}{\partial y^2} \right\} \qquad (4.53)$$

B.C. gives

$$D \frac{\partial C_P}{\partial y} = 0 \; at \;\; y = 0, h$$ ← impermeable boundary

4.2 Fickian Dispersion Model

Take cross-sectional average of Eq. (4.53)

$$\overline{\frac{\partial C_p}{\partial \tau}} - \overline{pu^{'}C_{p-1}} = D\left\{ \overline{p(p-1)C_{p-2}} + \cancel{\overline{\frac{\partial^2 C_P}{\partial y^2}}} \right\}$$

$$\overline{\frac{\partial^2 C_P}{\partial y^2}} = \frac{\partial^2 \overline{C}_P}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \overline{C}_P}{\partial y} \right) = 0$$

$$\boxed{\frac{dM_p}{d\tau} - \overline{pu^{'}C_{p-1}} = p(p-1)DM_{p-2}}$$

Aris' analysis is more general than Taylor's analysis in that it applies for low values of time.

Eq. (4.54) can be solved sequentially for p = 0, 1, 2, …

4.2 Fickian Dispersion Model

	Equation	Consequences as $t \rightarrow \infty$
$p=0$	$dM_0 / d\tau = 0$	Mass is conserved
	$M_0 \dfrac{1}{A}\int_A C_0(y)dA = \dfrac{1}{A}\int_A \int_{-\infty}^{\infty} C d\xi dA$	
	(4.53) $\rightarrow$ $\dfrac{\partial C_0}{\partial \tau} = D\dfrac{\partial^2 C_0}{\partial y^2}$	
$p=1$	$\dfrac{dM_1}{dt} = \overline{u' C_0}$	$M_1 \rightarrow consant$
	(4.53) $\rightarrow$ $\dfrac{\partial C_1}{\partial \tau} - u' C_0 = D\dfrac{\partial^2 C_1}{\partial y^2}$	
$p=2$	$\dfrac{dM_2}{dt} = \overline{2u' C_1} + 2D\overline{C_0}$	$\boxed{\dfrac{d\sigma^2}{dt} = 2K + 2D}$

→ molecular diffusion and shear flow dispersion are additive

4.3 Dispersion in Unsteady Shear Flow

Real environmental flows are often unsteady flow.

- reversing flow in a tidal estuary; wind driven flow in a lake caused by a passing storm

Suppose that unsteady flow = steady component + oscillatory component

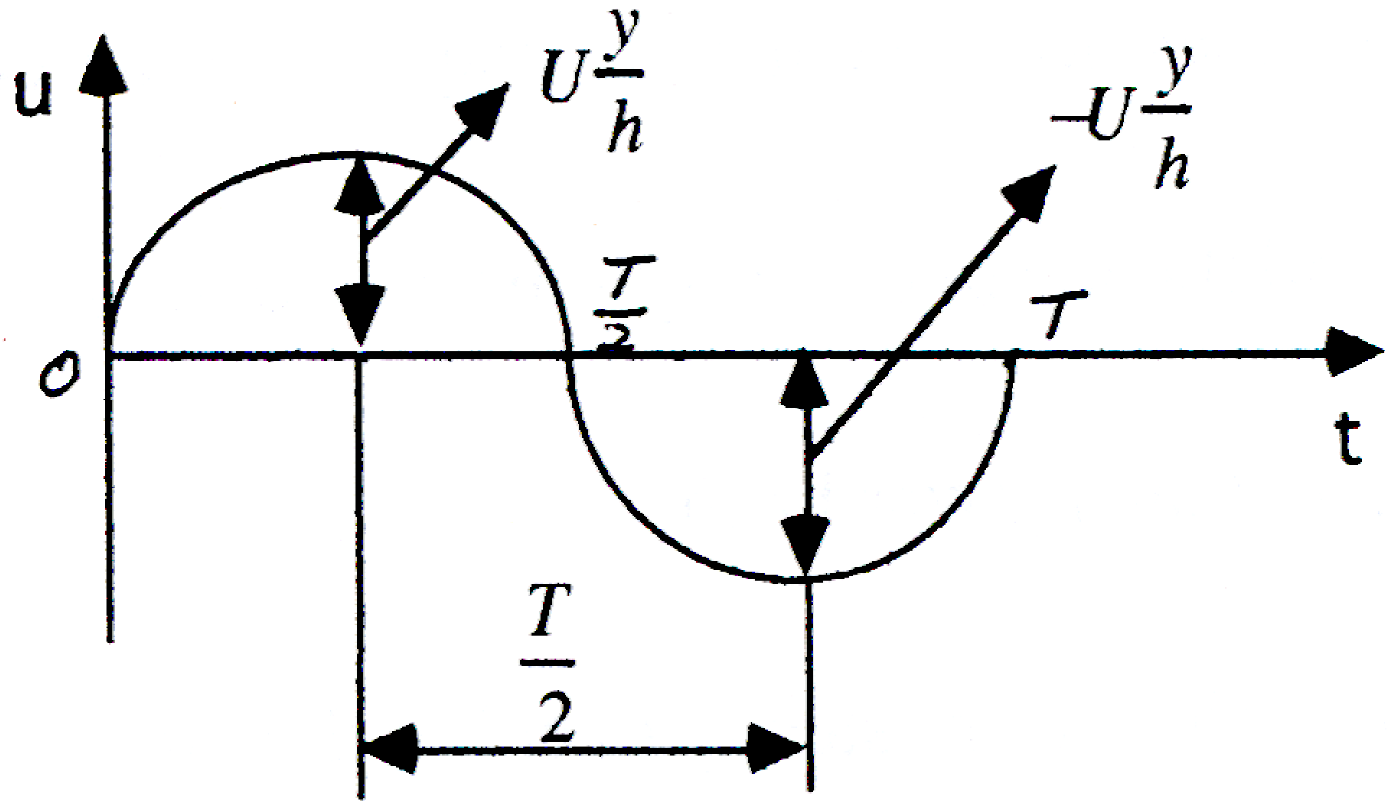
Application of Taylor's analysis to an oscillatory shear flow

(A) Linear velocity profile with a sinusoidal oscillation

$$u = U\frac{y}{h}\sin\left(\frac{2\pi t}{T}\right) \tag{1}$$

where T = period of oscillation

4.3 Dispersion in Unsteady Shear Flow



4.3 Dispersion in Unsteady Shear Flow

- 'flip-flop' sort of flow

- reversing instantaneously between $u = U\dfrac{y}{h}$ and $-u = U\dfrac{y}{h}$ after time interval $\dfrac{T}{2}$

→ after each reversal the concentration profile has to be reversed

→ substitute - y for y in Eq. (4.21)

→ but enough time bigger than mixing time $(T_c \approx h^2 / D)$ is required before the concentration profile is completely adopted to a new velocity profile.

(1) $T >> T_C$

- concentration profile will have sufficient time to adopt itself to the velocity profile in each direction

4.3 Dispersion in Unsteady Shear Flow

- time required for to reach the profile given by Eq.(4.21) is short compared to the time during which has that profile.

→ dispersion coefficient will be <u>the same as that in a steady flow</u>

→ dispersion as if flow were steady in either direction

(2) $T << T_C$

- period of reversal is very short compared to the cross-sectional mixing time

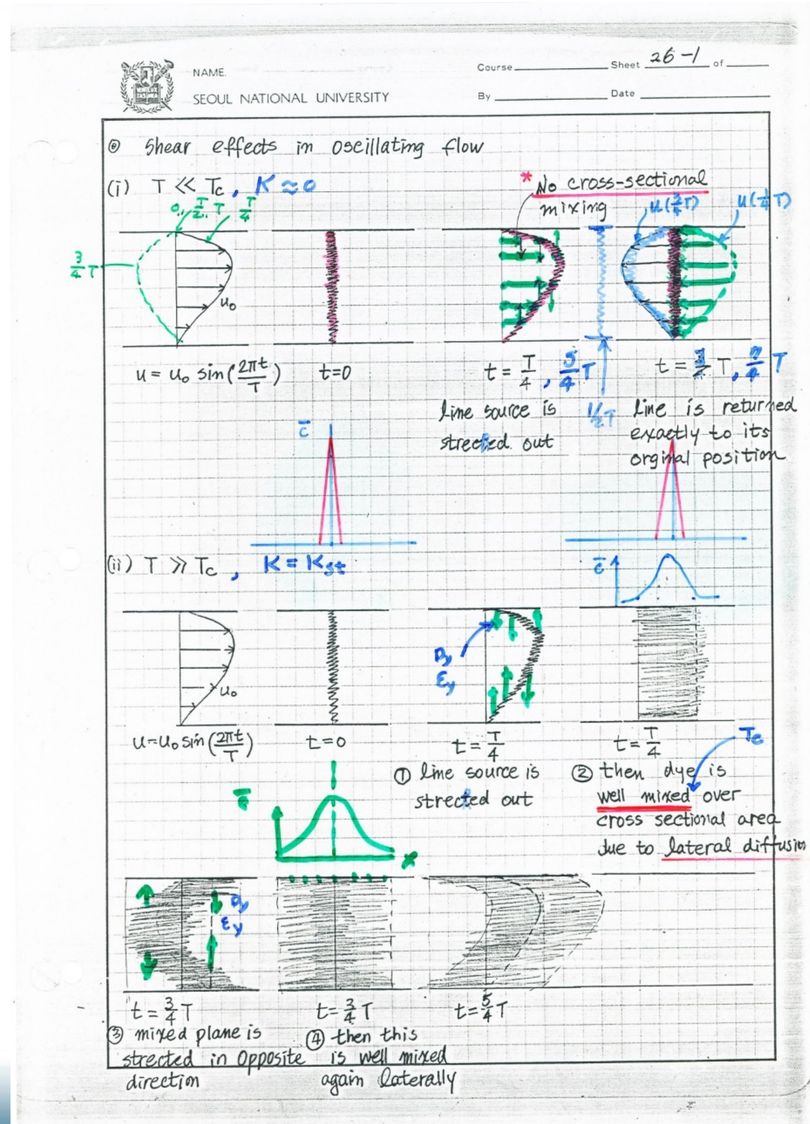
- concentration profile does <u>not have time to respond to the velocity profile</u>

- $C^{'}$ will oscillate around the mean of the symmetric limiting profiles, which is $C^{'} = 0$.

→ dispersion coefficient tends toward zero

→ <u>no dispersion due to the velocity profile</u>

4.3 Dispersion in Unsteady Shear Flow



4.3 Dispersion in Unsteady Shear Flow

- Fate of an instantaneous line source when $T << T_C$

Solution of Taylor's equation by Carslaw and Jaeger (1959)

unsteady source term

$$\frac{\partial C'}{\partial \tau} - D\frac{\partial^2 C'}{\partial y^2} = -u'\frac{\partial \overline{C}}{\partial \xi} \qquad \text{(a)}$$

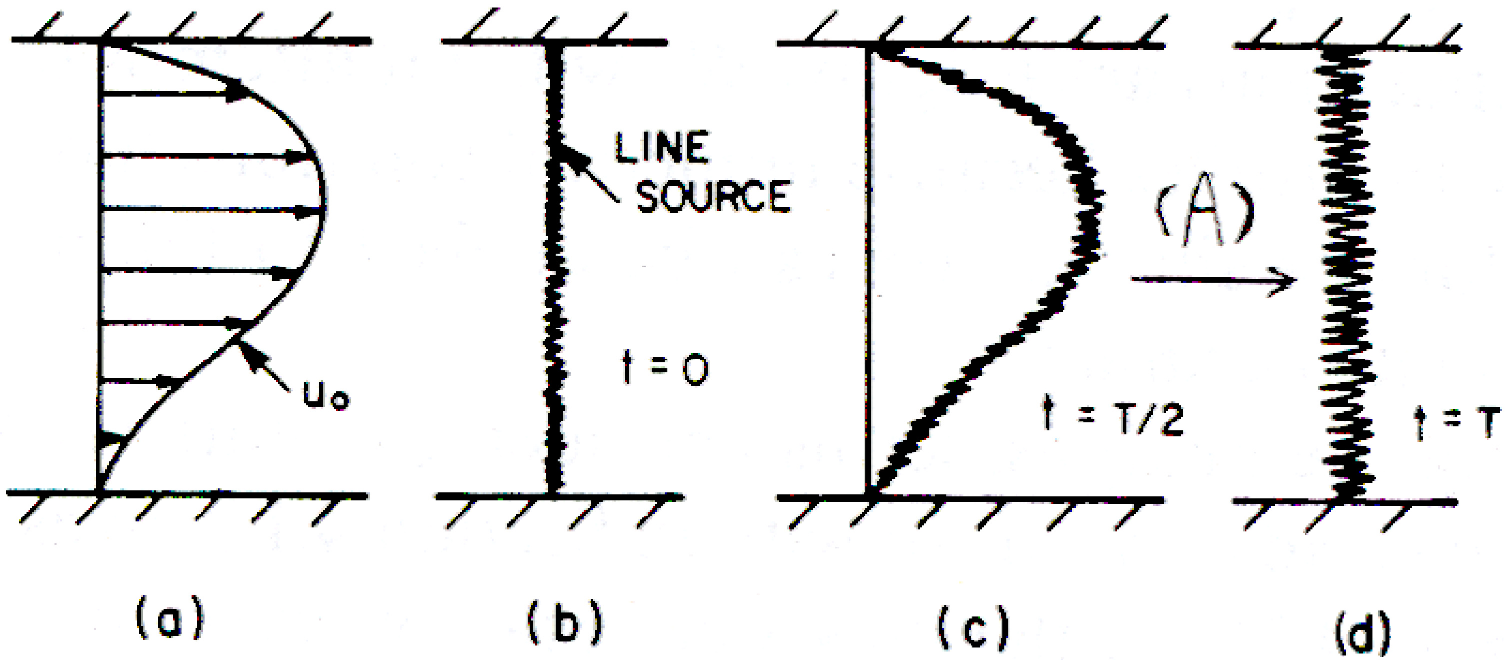
Taylor's equation for unsteady flow

$$u = u' = U\frac{y}{h}\sin\frac{2\pi t}{T} \quad (\because \overline{u} = 0) \qquad \text{(1)}$$

B.C. $$\frac{\partial C'}{\partial y} = 0 \ at \ y = \pm\frac{h}{2} \qquad \text{(b)}$$

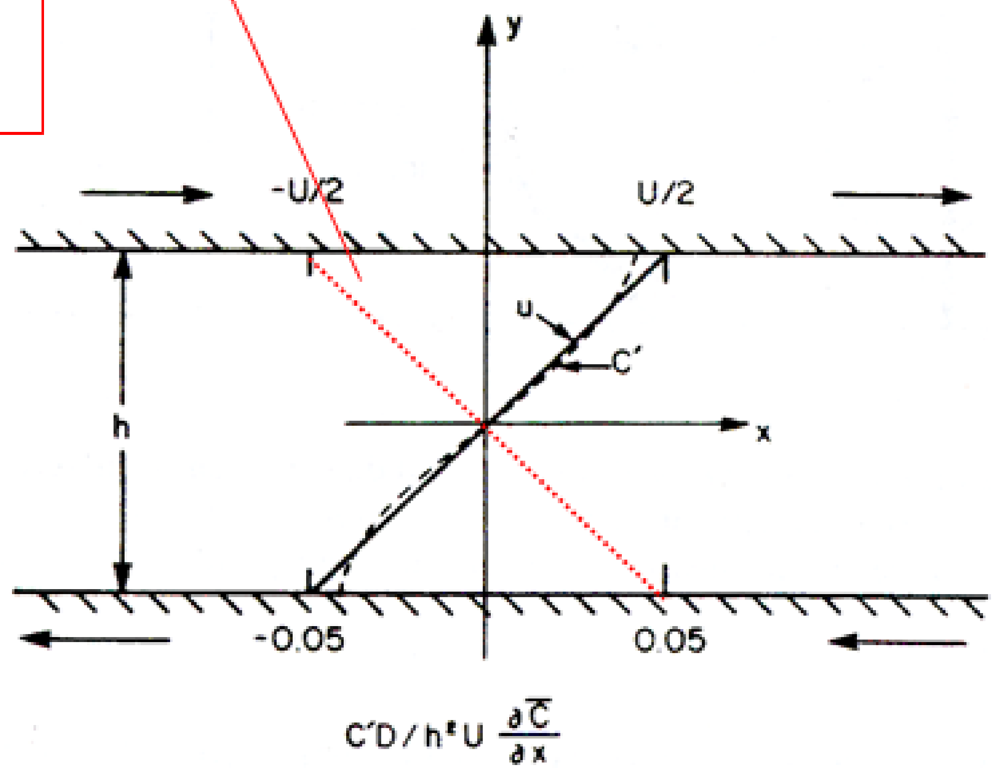
I.C. $$C'(y,0) = 0 \qquad \text{(c)}$$

4.3 Dispersion in Unsteady Shear Flow



4.3 Dispersion in Unsteady Shear Flow

$$-u = U\frac{y}{h}$$



4.3 Dispersion in Unsteady Shear Flow

Replace unsteady source term $u'\frac{\partial \bar{C}}{\partial \xi}$ by a source of constant strength by setting $t = t_0$

$$\frac{\partial C^*}{\partial \tau} - D\frac{\partial^2 C^*}{\partial y^2} = -U\frac{y}{h}\frac{\partial \bar{C}}{\partial x}\sin(\frac{2\pi t_0}{T}) \quad \text{(d)}$$

$$\frac{\partial C^*}{\partial y} = 0 \;\; at \;\; y = \pm\frac{h}{2} \quad \text{(e)}$$

$$C^*(y,0) = 0 \quad \text{(f)}$$

where C^*= distribution resulting from a suddenly imposed source distribution of constant strength

4.3 Dispersion in Unsteady Shear Flow

As diagrammed in Fig. 2.8, the solution for a series of sources of variable strength can be obtained by

$$C^{'}(y,t) = \int_0^t \frac{\partial}{\partial t} C^{*}(y, t - t_0; t_0)\, dt_0 \qquad \text{(g)}$$

For large t

$$C^{'}(y,t) = \int_{-\infty}^t \frac{\partial}{\partial t} C^{*}(y, t - t_0; t_0)\, dt_0 \qquad \text{(h)}$$

C^{*} can be expressed by the sum

$$C^{*}(y,t) = u(y) + w(y,t) \qquad \text{(i)}$$

4.3 Dispersion in Unsteady Shear Flow

$w(y,t)$ can be solved by separation of variables and Fourier expansion.

Further integration of the result leads to

$$C' = \frac{2Uh^2}{\pi^3 D}\frac{T}{T_c}\frac{\partial \overline{C}}{\partial x}\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(2n-1\right)^2}\sin(2n-1)\pi\frac{y}{h}$$

$$\times\left[\left(\frac{\pi}{2}(2n-1)\right)^2\frac{T}{T_c}+1\right]^{-\frac{1}{2}}\sin\left(\frac{2\pi t}{T}+\theta_{2n-1}\right)$$

where $\theta_{2n-1} = \sin^{-1}\left(-\left\{\left[\frac{1}{2}\pi\left(2n-1\right)^2\frac{T}{T_c}\right]^2+1\right\}^{-\frac{1}{2}}\right)$

4.3 Dispersion in Unsteady Shear Flow

Average over the period of oscillation of K

$$\bar{K} = \frac{1}{T}\int_0^T \left(-\int_{-\frac{h}{2}}^{\frac{h}{2}} u'C'dy / h \frac{\partial \bar{C}}{\partial x} \right) dt$$

$$= \frac{U^2}{\pi^4}\frac{h^2}{D}\left(\frac{T}{T_c}\right)^2 \sum_{n=1}^{\infty}(2n-1)^{-2}\left\{\left[\frac{\pi}{2}(2n-1)^2\left(\frac{T}{T_c}\right)^2\right]^2 + 1\right\}^{-1} \qquad (4.55)$$

$$\rightarrow \left[\begin{array}{l} T << T_c,\ K \rightarrow 0 \\ T >> T_c,\ \boxed{K_0 = \dfrac{1}{240}\dfrac{U^2h^2}{D}} \end{array} \right. \qquad (4.56)$$

4.3 Dispersion in Unsteady Shear Flow

[Re] Case of $T >> T_c$

For a linear steady velocity profile, $u = U\dfrac{y}{h}\sin\alpha$

$$K_{st} = \frac{1}{120}\frac{U^2h^2}{D}\sin^2\frac{\alpha}{D} \tag{4.57}$$

$\rightarrow K_0 = \dfrac{1}{240}\dfrac{U^2h^2}{D}$ is an ensemble average of K_{st} over all values of α

Intermediate behavior, Eq. (4.55) → Fig.4.18

4.3 Dispersion in Unsteady Shear Flow

$$\frac{T}{T_c} = 0.1 \rightarrow K \approx 0.03 K_0$$

$$\frac{T}{T_c} = 1 \rightarrow K \approx 0.8 K_0 \qquad (4.58)$$

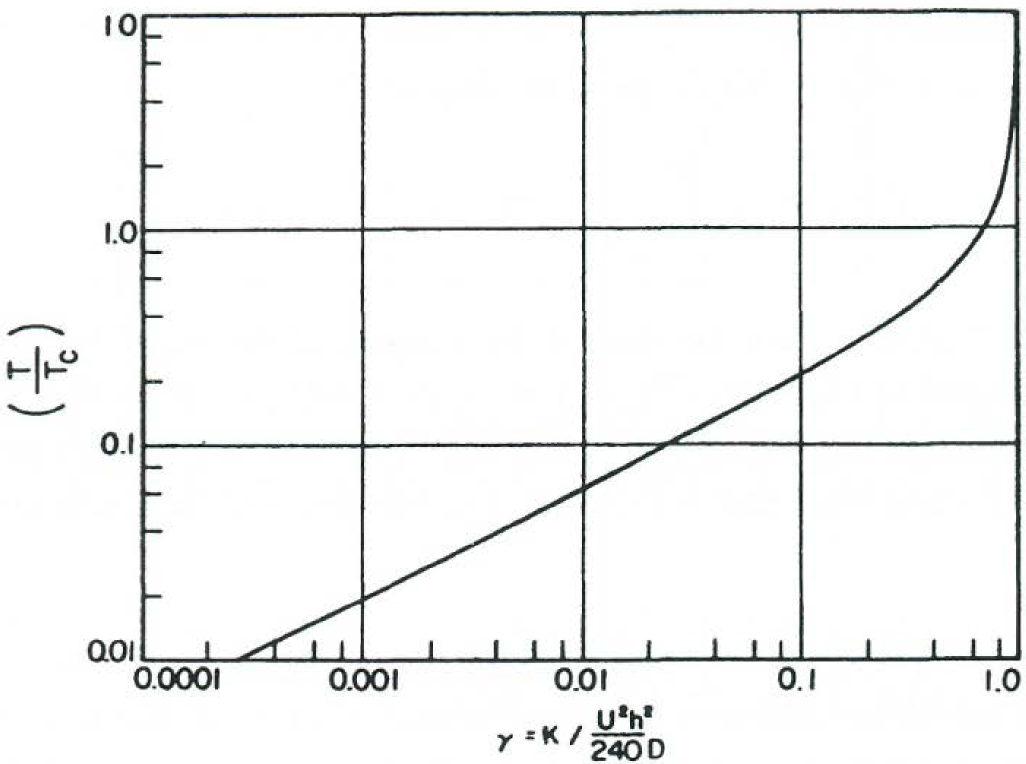
$$\frac{T}{T_c} = 10 \rightarrow K = K_0$$

(B) Flow including oscillating and a steady component

→ pulsating flow found in blood vessel

$$\boxed{\begin{aligned} u(y) &= u_1(y)\sin 2\pi t / T + u_2(y) \\ u_1 &= u_2 = Uy/h \end{aligned}} \qquad (4.59)$$

4.3 Dispersion in Unsteady Shear Flow



4.3 Dispersion in Unsteady Shear Flow

Assume that the results by separate velocity profile are additive.

Let $C' = C_1' + C_2'$ is solution to $\dfrac{\partial C'}{\partial t} + u(t)\dfrac{\partial \overline{C}}{\partial x} = \varepsilon \dfrac{\partial^2 C'}{\partial y^2}$

Then C_1' is solution to the equation

$$\frac{\partial C_1'}{\partial t} + u_1 \sin(2\pi t / T)\frac{\partial \overline{C}}{\partial x} = \varepsilon \frac{\partial^2 C_1'}{\partial y^2}$$

And C_2' is solution to the equation

$$\frac{\partial C_2'}{\partial t} + u_2 \frac{\partial \overline{C}}{\partial x} = \varepsilon \frac{\partial^2 C_2'}{\partial y^2}$$

4.3 Dispersion in Unsteady Shear Flow

cycle-averaged dispersion coefficient

$$\bar{K} = \frac{1}{T}\int_0^T -\frac{1}{h\dfrac{\partial \bar{C}}{\partial x}}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(u_1 \sin\frac{2\pi t}{T} + u_2\right)(C_1' + C_2')\,dy\,dt$$

$$= -\frac{1}{h\dfrac{\partial \bar{C}}{\partial x}}\left[\frac{1}{T}\int_0^T\int_{-\frac{h}{2}}^{\frac{h}{2}} u_1 C_1' \sin\frac{2\pi t}{T}\,dy\,dt + \int_{-\frac{h}{2}}^{\frac{h}{2}} u_2 C_2'\,dy\right] \qquad (4.60)$$

$$= K_1 + K_2$$

4.3 Dispersion in Unsteady Shear Flow

where K_1 = result of oscillatory profile = $f(T/T_c)$ → Fig. 4.18

K_2 = result of steady profile

- Application to tidal rivers and estuaries

Consider shear effects in estuaries and tidal rivers

Flow oscillation - flow goes back and forth.

Consider effect of oscillation on the longitudinal dispersion coefficient

$$K = K_0 f(T') \tag{4.61}$$

where $f(T')$ is plotted in Fig. 4. 18.

$T' = T/T_c$ = dimensionless time scale for cross-sectional mixing

4.3 Dispersion in Unsteady Shear Flow

$T =$ tidal period ~12 hrs

$T_C =$ cross-sectional mixing time = W^2/ε_t

$K_0 =$ dispersion coefficient if $T \gg Tc$

- For wide and shallow cross section with no density effects

$$K_0 = I\overline{u'^2}T_C \tag{4.62}$$

where I = dimensionless triple integral ≈ 0.1 (Table 4.1)

Combine Eq. (4.61) and Eq. (4.62)

$$\boxed{K = 0.1\overline{u'^2}T\left[\left(1/T'\right)f\left(T'\right)\right]} \tag{4.63}$$

4.3 Dispersion in Unsteady Shear Flow

Function $\left[\left(1/T'\right) f\left(T'\right)\right]$ is plotted in Fig. 4.19

i) T_C is small (narrow estuary) $T_C = \dfrac{W^2}{\varepsilon_t}$

$$T' = \frac{T}{T_C} >> 1 \rightarrow \quad K \text{ is small}$$

ii) T_C is very large (very wide estuary)

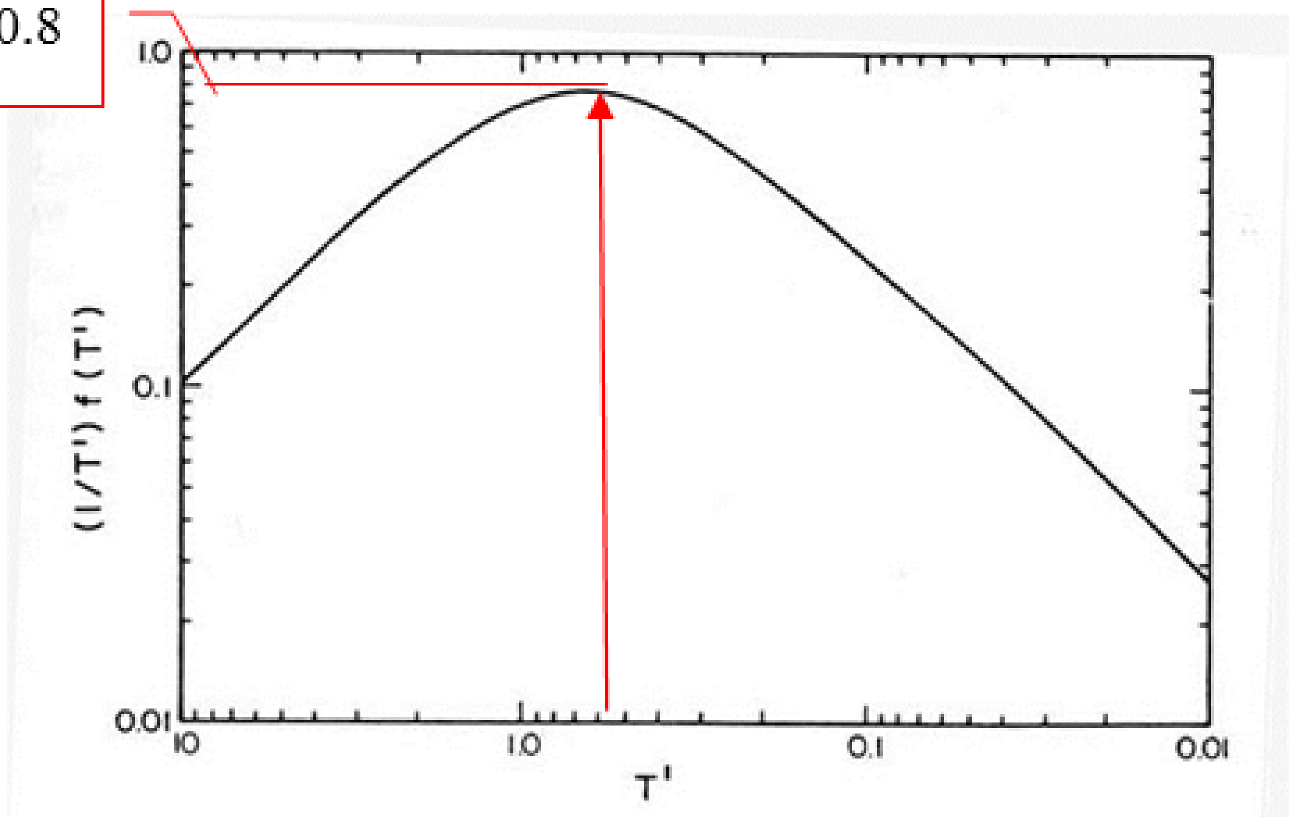
$$T' = \frac{T}{T_C} << 1 \rightarrow \quad K \text{ is smallest}$$

iii) $T' = \dfrac{T_C}{T} \approx 1 \quad : \left[\left(1/T'\right) f\left(T'\right)\right] \approx 0.8$

$$\therefore K_{\max} = 0.08\overline{u'^2}T$$

4.3 Dispersion in Unsteady Shear Flow

Max = 0.8



4.3 Dispersion in Unsteady Shear Flow

[Ex] $T = 12.5\text{ hrs},\quad \bar{u} = 0.3\text{ m/s},\quad \overline{u'^2} = 0.2\bar{u}^2$

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$$K_{\max} = 0.08 \times 0.2(0.3)^2 \times (12.5 \times 3600) \approx 60\ \text{m}^2/\text{s}$$

4.4 Dispersion in Two Dimensions

In many environmental flows velocity vector rotates with depth

$$\vec{u} = \vec{i}u(z) + \vec{j}v(z)$$

where $u =$ component of velocity $\vec{u}$ in the x direction

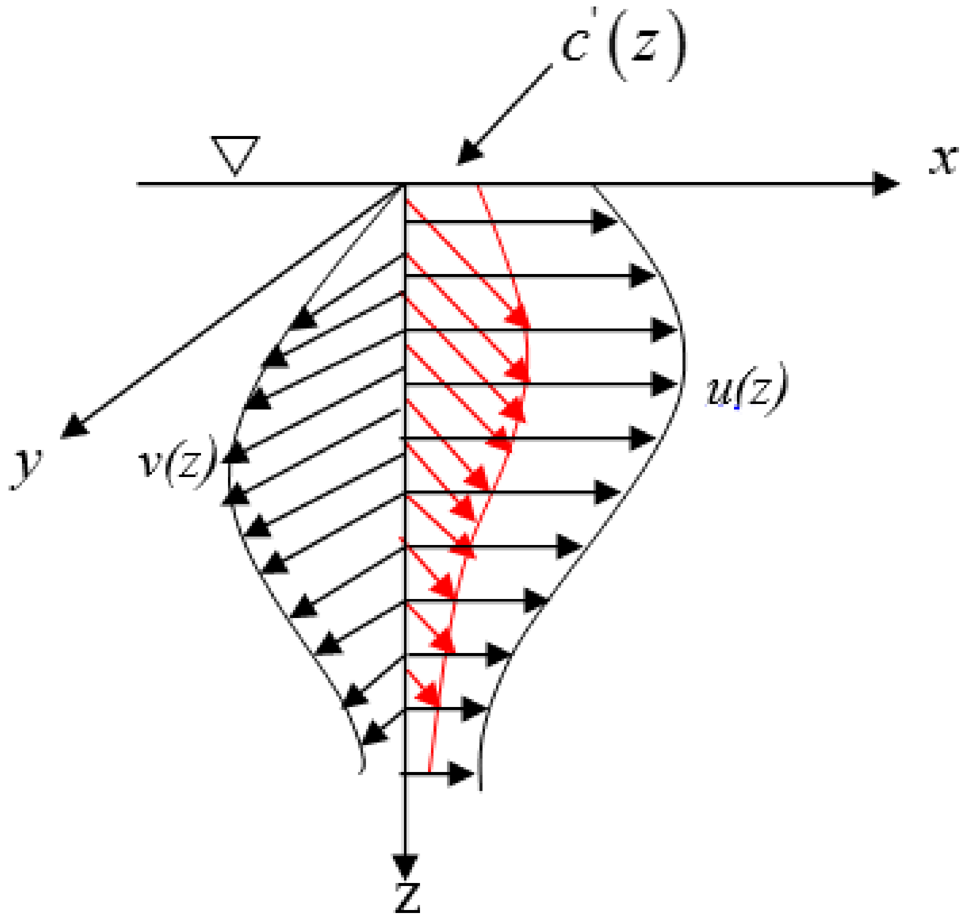
$v =$ component of velocity $\vec{u}$ in the y direction

• Taylor's analysis applied to a skewed shear flow with velocity profiles in two directions

The 2-D form of Eq. (4.10) for turbulent flow is

$$\boxed{u' \frac{\partial \overline{C}}{\partial x} + v' \frac{\partial \overline{C}}{\partial y} = \frac{\partial}{\partial z}\left(\varepsilon \frac{\partial C'}{\partial z} \right)} \qquad (4.64)$$

4.4 Dispersion in Two Dimensions



4.4 Dispersion in Two Dimensions

$$\frac{\partial C'}{\partial z} = 0$$ at $z = 0, h$ (water surface & bottom)

Integrate (4.64) w.r.t. z twice

$$C'(z) = \int_0^z \frac{1}{\varepsilon} \int_0^z \left(u' \frac{\partial \overline{C}}{\partial x} + v' \frac{\partial \overline{C}}{\partial y} \right) dz dz \qquad (4.65)$$

Bulk dispersion tensor can be defined by

$$\dot{M}_x = \int_0^h u' C' dz = -hK_{xx} \frac{\partial \overline{C}}{\partial x} - hK_{xy} \frac{\partial \overline{C}}{\partial y} \qquad (4.66a)$$

$$\dot{M}_y = \int_0^h v' C' dz = -hK_{yx} \frac{\partial \overline{C}}{\partial x} - hK_{yy} \frac{\partial \overline{C}}{\partial y} \qquad (4.66b)$$

4.4 Dispersion in Two Dimensions

Substitute (4.65) into (4.66)

(a): $$\int_0^h u' \int_0^z \frac{1}{\varepsilon} \int_0^z \left(u' \frac{\partial \bar{C}}{\partial x} + v' \frac{\partial \bar{C}}{\partial y} \right) dzdzdz = h \left(-K_{xx} \frac{\partial \bar{C}}{\partial x} - K_{xy} \frac{\partial \bar{C}}{\partial y} \right)$$

$$\boxed{K_{xx} = -\frac{1}{h} \int_0^h u' \int_0^z \frac{1}{\varepsilon} \int_0^z u' dzdzdz} \qquad (4.67a)$$

$$\boxed{K_{xy} = -\frac{1}{h} \int_0^h u' \int_0^z \frac{1}{\varepsilon} \int_0^z v' dzdzdz} \qquad (4.67b)$$

depend on the interaction of the x and y velocity profiles

4.4 Dispersion in Two Dimensions

(b): $$\int_0^h v' \int_0^z \frac{1}{\varepsilon} \int_0^z \left(u' \frac{\partial \bar{C}}{\partial x} + v' \frac{\partial \bar{C}}{\partial y} \right) dzdzdz = h \left(-K_{yx} \frac{\partial \bar{C}}{\partial x} - K_{yy} \frac{\partial \bar{C}}{\partial y} \right)$$

$$\boxed{\begin{aligned} K_{yx} &= -\frac{1}{h} \int_0^h v' \int_0^z \frac{1}{\varepsilon} \int_0^z u' dzdzdz \\ K_{yy} &= -\frac{1}{h} \int_0^h v' \int_0^z \frac{1}{\varepsilon} \int_0^z v' dzdzdz \end{aligned}}$$

(4.67c)

(4.67d)

The velocity gradient in the x direction can produce mass transport in the y direction and vice versa.

K_{xy} = x-dispersion coefficient due to velocity gradient in the y direction

4.4 Dispersion in Two Dimensions

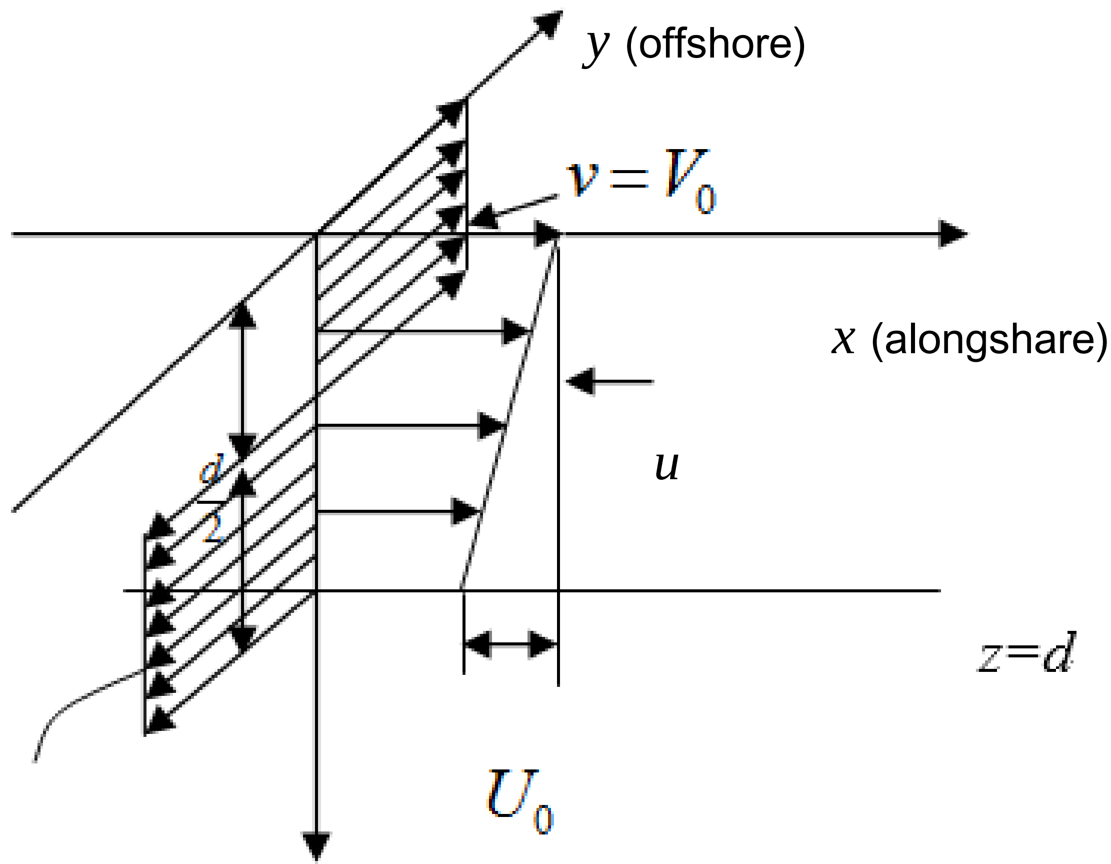
K_{yx} = y-dispersion coefficient due to velocity gradient in the x direction

- Mean flow on a continental shelf (Fischer, 1978)

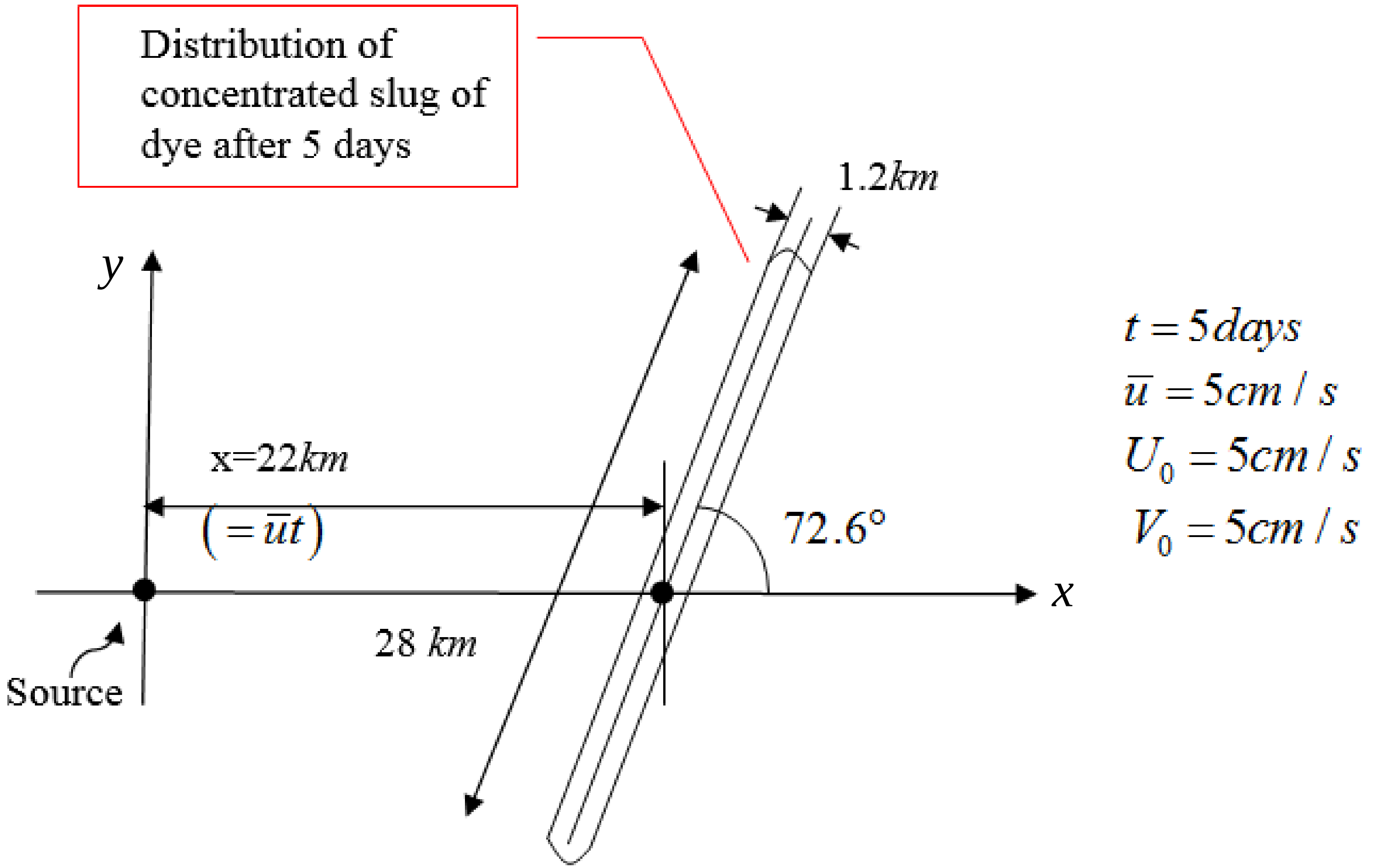
$$v = -V_0$$

$$K = \frac{d^2}{\varepsilon}\begin{pmatrix} U_0^{\ 2}/120 & 5U_0V_0/192 \\ 5U_0V_0/192 & V_0^{\ 2}/120 \end{pmatrix} \tag{4.68}$$

4.4 Dispersion in Two Dimensions



4.4 Dispersion in Two Dimensions



4.4 Dispersion in Two Dimensions

[Re] Derivation of 2-D advection-dispersion equation

(i) Conservation of mass in moving coordinate system is

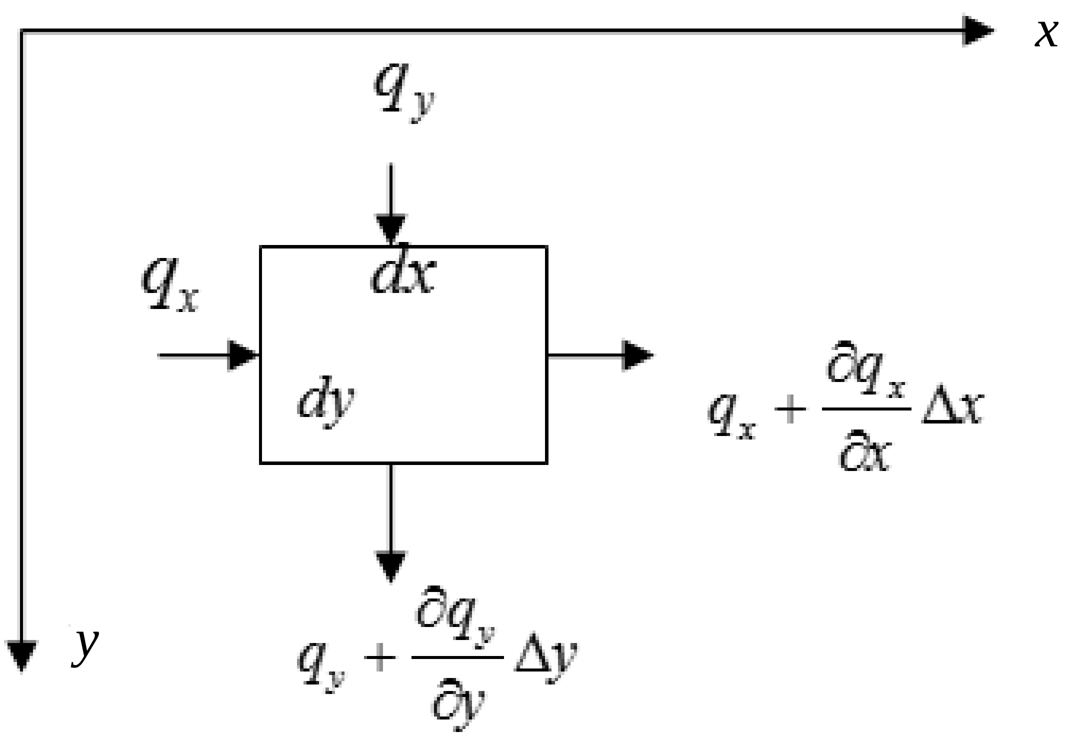
$$\frac{\partial C}{\partial t}\Delta x\Delta y=\left\{q_x-\left(q_x+\frac{\partial q_x}{\partial x}\Delta x\right)\right\}\Delta y+\left\{q_y-\left(q_y+\frac{\partial q_y}{\partial y}\Delta y\right)\right\}\Delta x$$

$$\therefore \quad \frac{\partial C}{\partial t}=-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y} \qquad (1)$$

(ii) Apply Taylor's analysis on 2-D shear flow

$$\dot{q}_x=\dot{M}_x=\left(\overline{u^{'}C^{'}}\right)h=\int_0^h u^{'}C^{'}dz=\int u^{'}\int\frac{1}{\varepsilon}\int\left(u^{'}\frac{\partial \bar{C}}{\partial x}+v^{'}\frac{\partial \bar{C}}{\partial y}\right)dzdzdz$$

4.4 Dispersion in Two Dimensions



4.4 Dispersion in Two Dimensions

$$= -K_{xx}\frac{\partial \overline{C}}{\partial x} - K_{xy}\frac{\partial \overline{C}}{\partial y} \tag{2}$$

$$q_y = \dot{M}_y = \left(\overline{v'c'}\right)h = \int_0^h v'c'dz = \int v' \int \frac{1}{\varepsilon}\int \left(u'\frac{\partial \overline{C}}{\partial x} + v'\frac{\partial \overline{C}}{\partial y} \right) dzdzdz$$

$$= -K_{yx}\frac{\partial \overline{C}}{\partial x} - K_{yy}\frac{\partial \overline{C}}{\partial y} \tag{3}$$

(iii) Substitute (2) & (3) into (1)

$$\frac{\partial \overline{C}}{\partial t} = -\frac{\partial}{\partial x}\left(-K_{xx}\frac{\partial \overline{C}}{\partial x} - K_{xy}\frac{\partial \overline{C}}{\partial y} \right) - \frac{\partial}{\partial y}\left(-K_{yx}\frac{\partial \overline{C}}{\partial x} - K_{yy}\frac{\partial \overline{C}}{\partial y} \right)$$

4.4 Dispersion in Two Dimensions

(iv) Return to fixed coordinate system containing mean advective velocities

$$\frac{\partial \bar{C}}{\partial t}+\bar{u}\frac{\partial \bar{C}}{\partial x}+\bar{v}\frac{\partial \bar{C}}{\partial y}=\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial \bar{C}}{\partial x}+K_{xy}\frac{\partial \bar{C}}{\partial y}\right)+\frac{\partial}{\partial y}\left(K_{yx}\frac{\partial \bar{C}}{\partial x}+K_{yy}\frac{\partial \bar{C}}{\partial y}\right) \quad (4.69)$$

If x-axis is coincident with the flow direction, K_{xy} and K_{yx} can be neglected.

Then, 2-D depth-averaged transport equation becomes

$$\boxed{\frac{\partial \bar{C}}{\partial t}+\bar{u}\frac{\partial \bar{C}}{\partial x}+\bar{v}\frac{\partial \bar{C}}{\partial y}=\frac{\partial}{\partial x}\left(K_{L}\frac{\partial \bar{C}}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{T}\frac{\partial \bar{C}}{\partial y}\right)} \quad (4.70)$$

where $K_L = K_{xx}; K_T = K_{yy}$

4.4 Dispersion in Two Dimensions

[Cf] 2-D depth-averaged scalar transport equation (ASCE, 1988; vol.114, No.9)

$$\frac{\partial\left(H\bar{C}\right)}{\partial t}+\frac{\partial\left(H\bar{u}\bar{C}\right)}{\partial x}+\frac{\partial\left(H\bar{v}\bar{C}\right)}{\partial y}=\frac{1}{\rho}\frac{\partial}{\partial x}\left(H\bar{J}_x\right)+\frac{1}{\rho}\frac{\partial}{\partial y}\left(H\bar{J}_y\right)$$

$$+\underbrace{\frac{1}{\rho}\frac{\partial}{\partial x}\int\rho u^{'}C^{'}dz}_{dispersion}+\underbrace{\frac{1}{\rho}\frac{\partial}{\partial y}\int\rho v^{'}C^{'}dz}_{dispersion}$$

where $\bar{J}_x=\int -\rho\overline{u_x{}^{'}c^{'}}dz$ turbulent diffusion in x-dir

$\bar{J}_y=\int -\rho\overline{u_y{}^{'}c^{'}}dz$ turbulent diffusion in y-dir

4.4 Dispersion in Two Dimensions

$u_x' = u_x - u$ → time fluctuation

$c' = c - C$ → time fluctuation

$u' = u - \overline{u}$ → deviation from depth-averaged value

$C' = C - \overline{C}$ → deviation from depth-averaged value

4.5 Unified View of Diffusion and Dispersion

- Similarities among the various types of diffusion and dispersion are shown.
- Diffusion and dispersion are actually advective transport mechanisms.

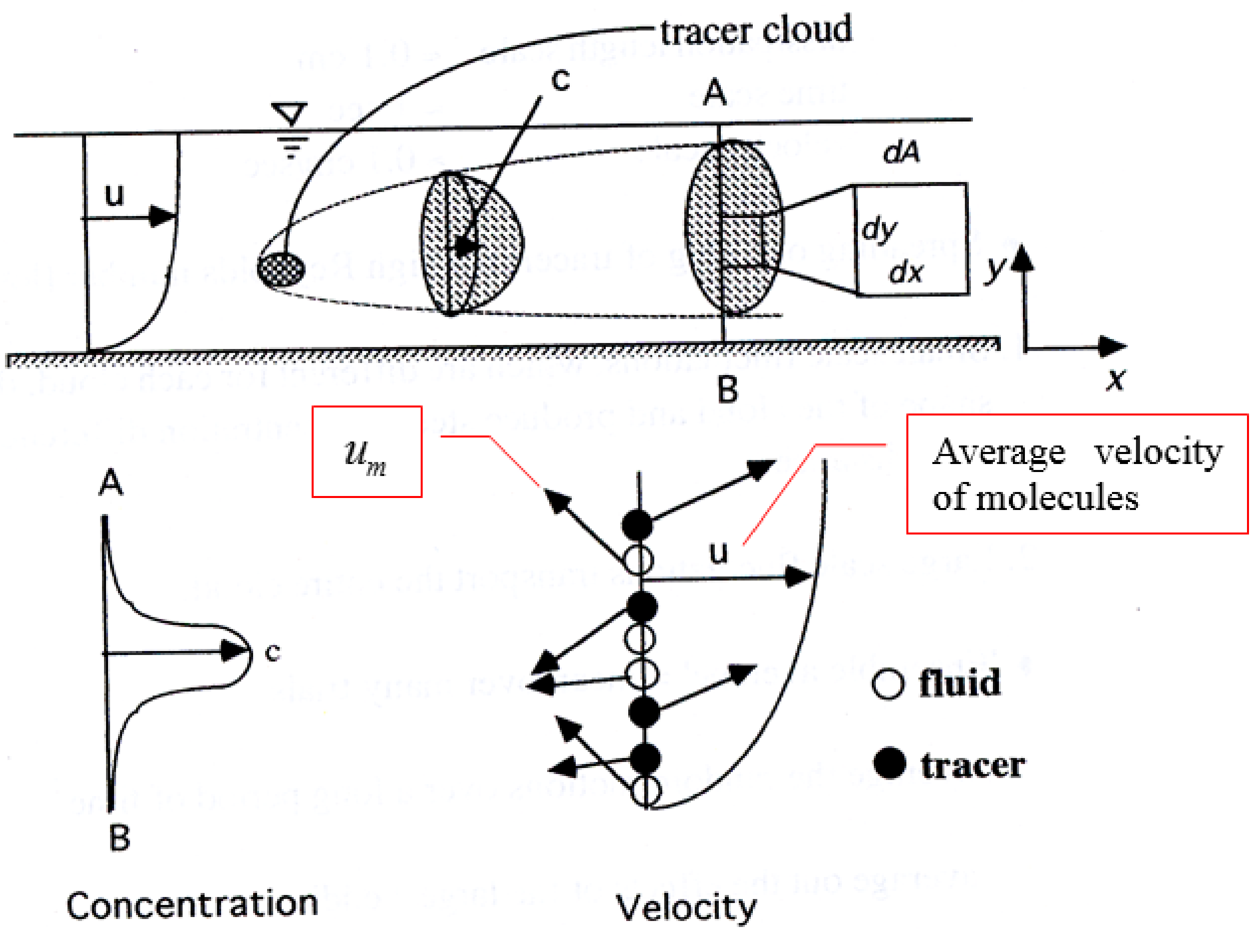
4.5.1 Molecular Diffusion

2-D open-channel flow

To write the mass balance equation, we need to know how many fluid molecules and how many tracer molecules pass through and the direction and spread of each molecule.

→ molecular approach → statistical manner

4.5 Unified View of Diffusion and Dispersion



4.5 Unified View of Diffusion and Dispersion

- Continuum approach

- Assume fluid carries tracer through at a rate depending on the concentration, c, and the fluid velocity, u.

- However, the fluid u, cannot completely represent the tracer movement because the velocity, u, does not account for the movement of the molecules which have directions and speeds different from u.

- Molecular diffusion accounts for the difference between the true molecular motion and the manner chosen to represent the motion. (i.e., by u)

$$\Delta u = u_m - u$$

4.5 Unified View of Diffusion and Dispersion

Thus, mass flux by this **velocity difference** is

$$j = \Delta u \, c$$

Now, apply Fick' law

- transport called molecular diffusion is proportional to the concentration gradient.

$$j_m = \Delta u \, c \propto \frac{\partial c}{\partial x}$$

$$j_m = -D_m \frac{\partial c}{\partial x} \qquad \text{(a)}$$

4.5 Unified View of Diffusion and Dispersion

D_m = constant of proportionality = molecular diffusivity

Now, consider advection by mean motion

$$j_x = cu - D_m \frac{\partial c}{\partial x} \qquad \text{(a)}$$

Then, substituting (a) into 2D mass conservation equation yields 2-D advection-diffusion equation as

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = D_m \frac{\partial^2 c}{\partial x^2} + D_m \frac{\partial^2 c}{\partial y^2} \qquad (4.71)$$

4.5 Unified View of Diffusion and Dispersion

By mean motion

① $\dfrac{\partial c}{\partial t}$ = time rate of change of concentration at a point

② $u\dfrac{\partial c}{\partial x}$ = advection of tracer with the fluid

By velocity fluctuation

③ $D_m\dfrac{\partial^2 c}{\partial x^2},\quad D_m\dfrac{\partial^2 c}{\partial y^2}$ = molecular diffusion

4.5 Unified View of Diffusion and Dispersion

4.5.2 Turbulent Diffusion

Decompose velocity and concentration into mean and fluctuation

$$u = \overline{u} + u'$$

$$c = \overline{c} + c' \qquad \text{(b)}$$

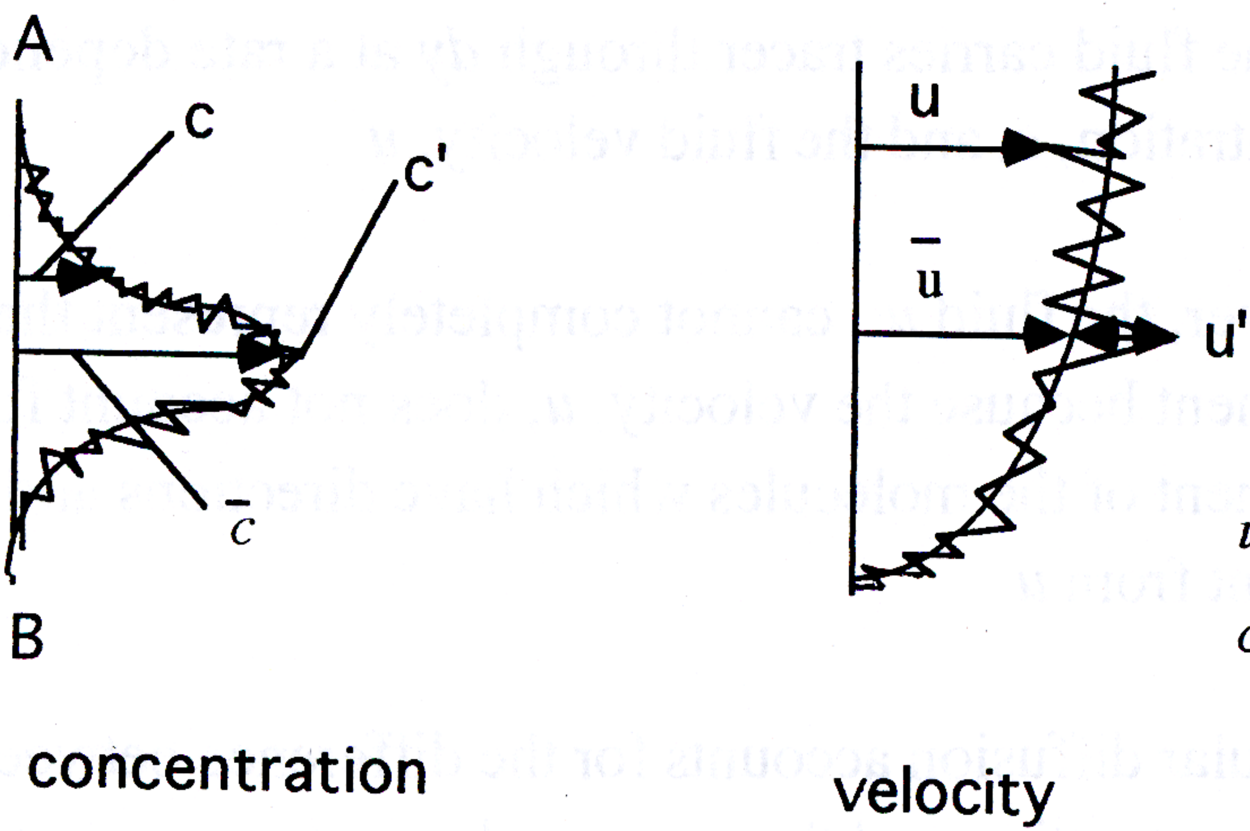
$$v = v'$$ (assume only fluctuation in y-direction)

$\overline{u}$, $\overline{c}$ = time-averaged values of u and c

$$\overline{u} \equiv \frac{1}{T}\int_0^T u\,dt$$

$$\overline{u'} = \overline{v'} = \overline{c'} = 0$$

4.5 Unified View of Diffusion and Dispersion



$$u = \bar{u} + u'$$

$$c = \bar{c} + c'$$

4.5 Unified View of Diffusion and Dispersion

where T = averaging time interval

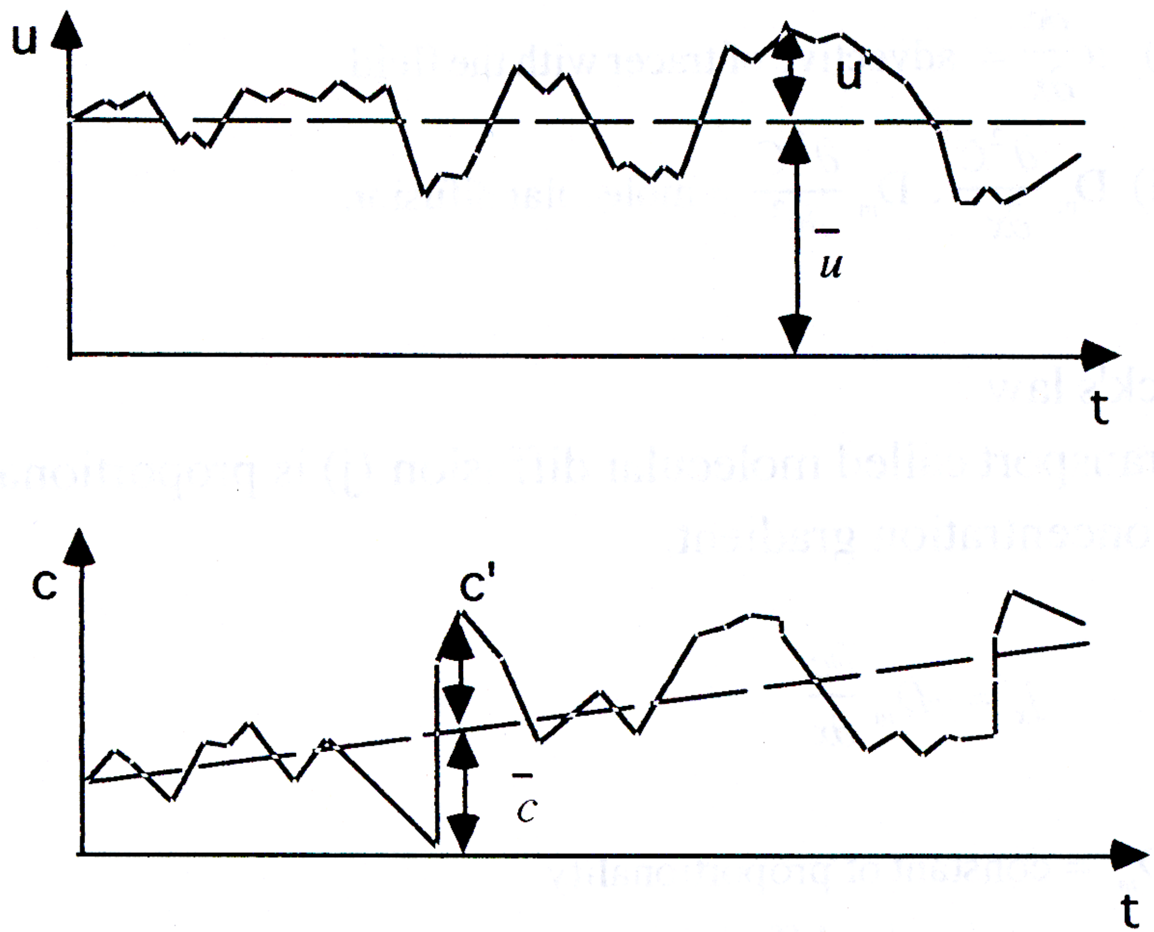
$$\begin{cases} 10^{0} \sim 10^{2}\ \text{sec} & \text{for open channel flow} \\ 10^{-1} \sim 10^{0}\ \text{sec} & \text{for pipe flow} \end{cases}$$

For 2-D flow, the advection-diffusion equation is

$$\frac{\partial c}{\partial t} + \frac{\partial uc}{\partial x} + \frac{\partial vc}{\partial y} = D_m \frac{\partial^2 c}{\partial x^2} + D_m \frac{\partial^2 c}{\partial y^2} \qquad (4.72)$$

Conservative form

4.5 Unified View of Diffusion and Dispersion



4.5 Unified View of Diffusion and Dispersion

Substitute (b) into (4.72), then it becomes

$$\frac{\partial(\bar{c}+c')}{\partial t}+\frac{\partial\left(\bar{u}+u'\right)(\bar{c}+c')}{\partial x}+\frac{\partial v'(\bar{c}+c')}{\partial y}=D_m\frac{\partial^2(\bar{c}+c')}{\partial x^2}+D_m\frac{\partial^2(\bar{c}+c')}{\partial y^2}$$

$$\frac{\partial\bar{c}}{\partial t}+\frac{\partial}{\partial x}\bar{u}\,\bar{c}=D_m\frac{\partial^2\bar{c}}{\partial x^2}+D_m\frac{\partial^2\bar{c}}{\partial y^2}$$

$$-\frac{\partial c'}{\partial t}-\frac{\partial}{\partial x}(\bar{u}\,c')-\frac{\partial}{\partial x}(u'\bar{c})-\frac{\partial}{\partial x}(u'c')-\frac{\partial}{\partial y}(v'\bar{c})-\frac{\partial}{\partial y}(v'c')$$

$$+D_m\frac{\partial^2 c'}{\partial x^2}+D_m\frac{\partial^2 c'}{\partial y^2}$$

4.5 Unified View of Diffusion and Dispersion

Integrate (average) w.r.t. time, and apply Reynolds rule

$$\overline{\frac{\partial \overline{c}}{\partial t}} + \overline{\frac{\partial (\overline{u}\,\overline{c})}{\partial x}} = \overline{D_m \frac{\partial^2 \overline{c}}{\partial x^2}} + \overline{D_m \frac{\partial^2 \overline{c}}{\partial y^2}}$$

$$-\cancel{\overline{\frac{\partial c'}{\partial t}}} + \cancel{\overline{\frac{\partial (\overline{u}\, c')}{\partial x}}} - \cancel{\overline{\frac{\partial (u'\overline{c})}{\partial x}}} - \overline{\frac{\partial u'c'}{\partial x}} - \cancel{\overline{\frac{\partial v'\overline{c}}{\partial y}}} - \overline{\frac{\partial v'c'}{\partial y}}$$

$$+ \cancel{D_m \frac{\partial^2 c'}{\partial x^2}} + \cancel{D_m \frac{\partial^2 c'}{\partial y^2}}$$

4.5 Unified View of Diffusion and Dispersion

[Re] Reynolds rules of averages (Schlichting; p. 460, 371)

$$\overline{\overline{f}} = \overline{f}$$

$$\overline{f + g} = \overline{f} + \overline{g}$$

$$\overline{f\,\overline{g}} = \overline{f}\,\overline{g}$$

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s}$$

$$\overline{\int f ds} = \int \overline{f} ds$$

4.5 Unified View of Diffusion and Dispersion

Drop all zero terms using Reynolds rules of averages

$$\frac{\partial \bar{c}}{\partial t} + \bar{u}\frac{\partial \bar{c}}{\partial x} = D_m \frac{\partial^2 \bar{c}}{\partial x^2} + D_m \frac{\partial^2 \bar{c}}{\partial y^2} + \underbrace{\frac{\partial\left(-\overline{u'c'}\right)}{\partial x} + \frac{\partial\left(-\overline{v'c'}\right)}{\partial y}}_{\substack{\text{advective transport} \\ \text{due to } u', v', \text{and } c'}}$$

It is assumed and confirmed experimentally that transport associated with the turbulent fluctuations is proportional to the gradient of average concentration.

$$\overline{u'c'} \approx \frac{\partial \bar{c}}{\partial x} \rightarrow \boxed{\overline{u'c'} = -\varepsilon_x \frac{\partial \bar{c}}{\partial x}}$$

$$\overline{v'c'} = -\varepsilon_y \frac{\partial \bar{c}}{\partial y}$$

4.5 Unified View of Diffusion and Dispersion

ε_x, ε_y = turbulent diffusion coefficient

$$\frac{\partial}{\partial x}\left(-\overline{u'c'}\right) = \frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial \overline{c}}{\partial x}\right)$$

$$\frac{\partial}{\partial y}\left(-\overline{v'c'}\right) = \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial \overline{c}}{\partial y}\right)$$

Assuming that ε_x and ε_y are constant, the mass balance equation for turbulent flow is given as

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\frac{\partial \overline{c}}{\partial x} = (D_m + \varepsilon_x)\frac{\partial^2 \overline{c}}{\partial x^2} + (D_m + \varepsilon_y)\frac{\partial^2 \overline{c}}{\partial y^2} \qquad (4.73)$$

4.5 Unified View of Diffusion and Dispersion

Drop overbars, and neglect molecular diffusion terms

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = \varepsilon_x \frac{\partial^2 c}{\partial x^2} + \varepsilon_y \frac{\partial^2 c}{\partial y^2} \tag{4.74}$$

For 3-D flow:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = \frac{\partial}{\partial x}(\varepsilon_x \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y}(\varepsilon_y \frac{\partial c}{\partial y}) + \frac{\partial}{\partial z}(\varepsilon_z \frac{\partial c}{\partial z}) \tag{4.75}$$

☞ Remember, $\varepsilon_x \frac{\partial c}{\partial x}, \varepsilon_y \frac{\partial c}{\partial y}, \varepsilon_z \frac{\partial c}{\partial z}$ and are actually advective transport.

4.5 Unified View of Diffusion and Dispersion

4.5.3 Longitudinal Dispersion

After the tracer is essentially completely mixed both vertically and laterally, the primary variation of concentration is in just longitudinal direction.

→ one-dimensional equation

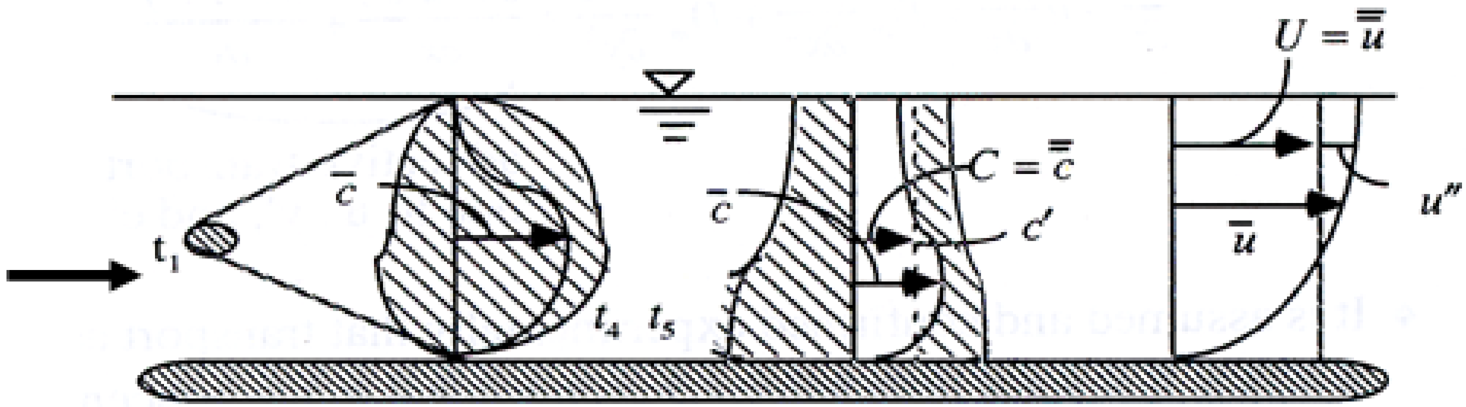
Far-field mixing

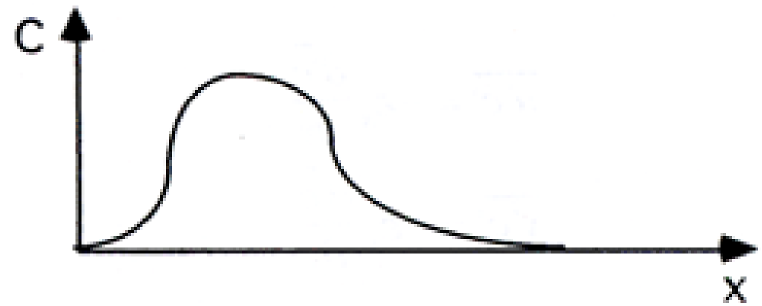
Decompose velocity and concentration into <u>cross-sectional mean</u> and deviation (fluctuation)

$$\bar{u} = U + u'' \qquad \overline{u''} = 0 \qquad \text{(c)}$$

$$\bar{c} = C + c'' \qquad \overline{c''} = 0$$

4.5 Unified View of Diffusion and Dispersion





4.5 Unified View of Diffusion and Dispersion

where U, C = cross-sectional average of the velocity and concentration

After substituting (c) into (4.74), averaging it over the cross-sectional area yields

$$\overline{\overline{\frac{\partial\left(C+c''\right)}{\partial t}}}+\overline{\overline{(U+u'')\frac{\partial\left(C+c''\right)}{\partial x}}}=\overline{\overline{(D_m+\varepsilon_x)\frac{\partial^2\left(C+c''\right)}{\partial x^2}}}+\overline{\overline{(D_m+\varepsilon_y)\frac{\partial^2\left(C+c''\right)}{\partial y^2}}}$$

By Reynolds rule

$$\frac{\partial C}{\partial t}+U\frac{\partial C}{\partial x}=(D_m+\varepsilon_x)\frac{\partial^2 C}{\partial x^2}+(D_m+\varepsilon_y)\frac{\partial^2 C}{\partial y^2}-\frac{\partial\left(\overline{\overline{u''c''}}\right)}{\partial x} \quad (4.76)$$

4.5 Unified View of Diffusion and Dispersion

3. Shear Flow Dispersion

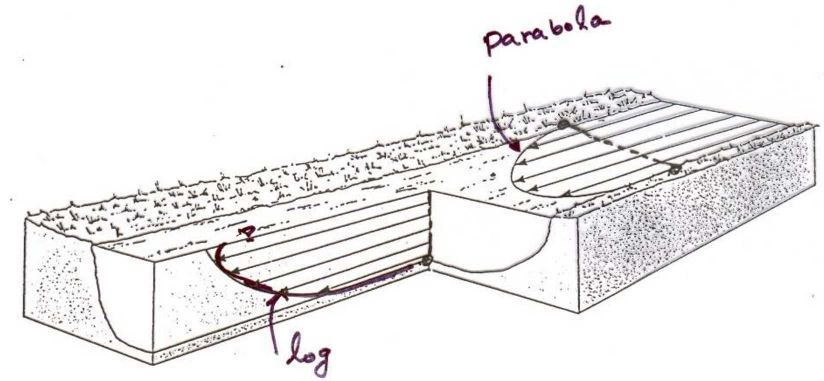
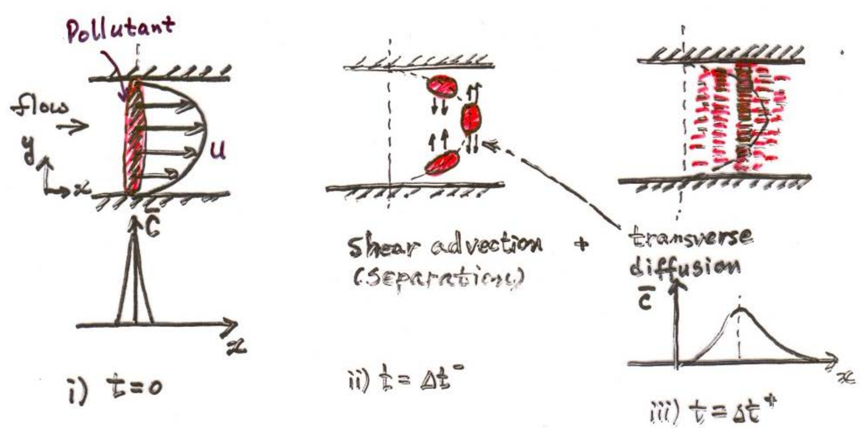


Figure 10.5
Variations in the velocity of flow in natural stream channels occur both horizontally and vertically. Friction reduces the velocity along the floor and sides of the channels. The maximum velocity in a straight channel is near the top and center of the channel.



4.5 Unified View of Diffusion and Dispersion

Then neglect $\frac{\partial^2 C}{\partial y^2}$ because after lateral mixing is completed,

$$\frac{\partial C}{\partial y} \approx 0; \;\; C = \overline{C} \neq f\left(y\right)$$

Then, Eq. (4.76) becomes

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = (D_m + \varepsilon_x)\frac{\partial^2 C}{\partial x^2} + \frac{\partial\left(-\overline{\overline{u''c''}}\right)}{\partial x} \tag{4.77}$$

Taylor (1953, 1954) show that the advective transport associated with u'' is proportional to the longitudinal gradient of C.

$$-\overline{\overline{u''c''}} \propto \frac{\partial C}{\partial x}$$

4.5 Unified View of Diffusion and Dispersion

$$-\overline{\overline{u''c''}} = K\frac{\partial C}{\partial x}$$

$$\frac{\partial}{\partial x}\left(-\overline{\overline{u''c''}}\right) = \frac{\partial}{\partial x}\left(K\frac{\partial C}{\partial x}\right) \quad \rightarrow \text{longitudinal dispersion} \qquad (4.78)$$

K = longitudinal dispersion coefficient

Substituting Eq. (4.78) into Eq. (4.77) yields

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \left(D_m + \varepsilon_x + K\right)\frac{\partial^2 C}{\partial x^2}$$

$$\underset{1\%}{(D_m + \varepsilon_x)\frac{\partial C}{\partial x}} << \underset{99\%}{\overline{-u''c''}}$$

4.5 Unified View of Diffusion and Dispersion

$$\boxed{\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = K\frac{\partial^2 C}{\partial x^2}} \qquad (4.79)$$

→ 1-D Dispersion Equation

Because the velocity distribution influences u'' and the lateral diffusion plays a large role in determining the distribution of c''

→ both velocity distribution and lateral diffusion contribute to longitudinal dispersion.

4.5 Unified View of Diffusion and Dispersion

- Limitation of Taylor's 1D model (Chatwin, 1970)

→ Taylor's model should be applied after initial period.

$$t > \frac{0.4W^2}{\varepsilon_t}$$ → Taylor period

$$x > \frac{0.4UW^2}{\varepsilon_t} \qquad (4.80)$$

4.5 Unified View of Diffusion and Dispersion

4.5.4 Summary

To investigate the relative importance of dispersion, use dimensionless term as

$$H = \frac{dispersion\ rate}{advective\ rate} = \frac{K\dfrac{\partial C}{\partial x}}{UC} = \frac{K}{U}\frac{1}{C}\frac{\partial C}{\partial x} = \frac{K}{U}\frac{\partial(\ln C)}{\partial x}$$

If $H < H_c \approx 0.01 \rightarrow$ dispersive transport may be neglected

1) Diffusion

= transport associated with fluctuating components of molecular action and with turbulent action

4.5 Unified View of Diffusion and Dispersion

= transport in a given direction at a point in the flow due to the differences between the true advection in that direction and the time average of the advection in that direction

2) Dispersion

= transport associated with the deviations (variations) of the velocity across the flow section

= transport in a given direction due to the difference between the true advection in that direction and the spatial average of the advection in that direction

4.5 Unified View of Diffusion and Dispersion

$D_{ij}, cm^2/\sec$	
10^{-6} -	
10^{-5} -	Molecular Diffusion in Water
10^{-4} -	
10^{-3} -	
10^{-2} -	
10^{-1} -	Molecular Diffusion Gases
10^{0} -	
	Eddy Diffusion – Pipes
10^{1} -	
10^{2} -	Eddy Diffusion – Streams
10^{3} -	
10^{4} -	
10^{5} -	Longitudinal Dispersion – Streams (0.1~1,500 m^2/s)
10^{6} -	Longitudinal Tidal Dispersion - Estuaries
10^{7} -	
10^{8} -	
10^{9} -	
10^{10} -	Eddy Diffusion – Atmosphere