# Chap 8. Description of Random Sea Waves

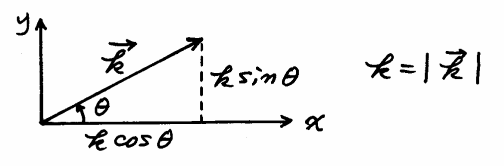
## 8.1 Profiles of Progressive Waves and Dispersion Relationship

## 8.2 Description of Random Sea Waves by Means of Variance Spectrum

Regular wave:

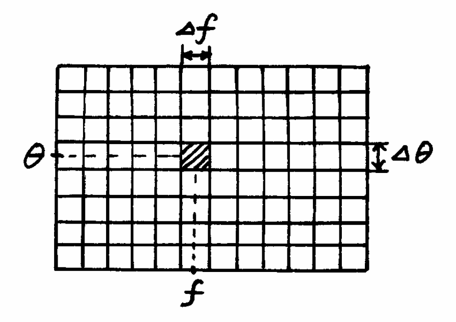
$$\eta = a\cos(kx\cos\theta + ky\sin\theta - 2\pi ft + \varepsilon)$$

$$E = \frac{1}{2}\rho g a^2$$




Random waves: superposition of many regular waves

$$\eta = \eta(x,y,t) = \sum_{n=1}^{\infty} a_n \cos(k_n x\cos\theta_n + k_n y\sin\theta_n - 2\pi f_n t + \varepsilon_n)$$




$$S(f,\theta)\Delta f\Delta\theta = \frac{1}{2}a_n^2$$

↓ one-to-one correspondence between $f$ and $k$ by dispersion relation

$$S(k,\theta)\Delta k\Delta\theta = \frac{1}{2}a_n^2$$

Single point measurement ( $x = y = 0$ ):

$$\eta = \eta(t) = \sum_{n=1}^{\infty} a_n \cos(2\pi f_n t + \varepsilon_n)$$

$$S(f)\Delta f = \frac{1}{2}a_n^2$$

## 8.3 Stochastic Process and Variance Spectrum

$$\eta(t) = \{\eta_1(t), \eta_2(t), \cdots, \eta_j(t), \cdots\} \leftarrow \{\ \} = \text{ensemble}$$

↑ varies randomly with time

Assume

(1) stationality: independent of time (valid for short duration ≤ 20~30 min)

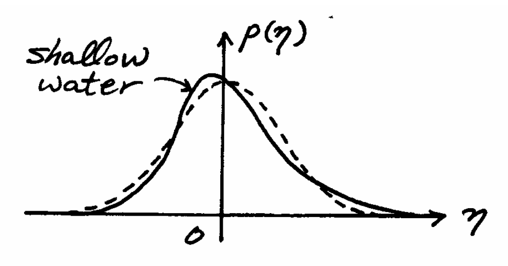
(2) ergodicity: Time-averaged statistics are equal to ensemble-averaged statistics

$$E[\eta(t)] = \overline{\eta_j(t)} = \lim_{t_0 \to \infty} \frac{1}{t_0} \int_0^{t_0} \eta_j(t) dt$$

↑ ensemble-average (from many records)　　↑ time-average (from a single record)

∴ We need only one record

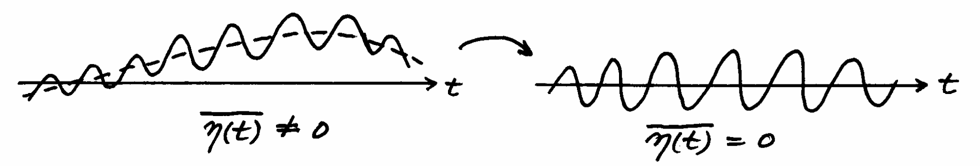
(3) Gaussian process: probability density of $\eta(t)$ is given by Gaussian (normal) distribution, but in shallow water, peaked crests and flatter troughs




Assume zero-mean process:

$$E[\eta(t)] = \overline{\eta(t)} = 0$$

If the wave record includes tidal variation, remove it.



Relation between $\Psi(\tau)$ and $S_0(f)$:

$$\left.\begin{aligned}\Psi(\tau) &= \int_{-\infty}^{\infty} S_0(f) e^{i2\pi f\tau} df \\ S_0(f) &= \int_{-\infty}^{\infty} \Psi(\tau) e^{-i2\pi f\tau} d\tau\end{aligned}\right\}\text{Fourier transform pair (Dean and Dalrymple's book)}$$

Redefining in the range of $0$ to $\infty$ for both $\tau$ and $f$,

$$\left.\begin{aligned}\Psi(\tau) &= \int_{0}^{\infty} S(f)\cos(2\pi f\tau) df \\ S(f) &= 4\int_{0}^{\infty} \Psi(\tau)\cos(2\pi f\tau) d\tau\end{aligned}\right\}\text{Wiener - Khintchine relation}$$

For irregular wave profile,

$$\eta(t) = \sum_{n=1}^{\infty} a_n \cos(2\pi f_n t + \varepsilon_n)$$

$$\begin{aligned}\Psi(\tau) &= E\left[\eta(t+\tau)\eta(t)\right] \\ &= \overline{\eta_j(t+\tau)\eta_j(t)} \\ &= \lim_{t_0\to\infty}\frac{1}{t_0}\int_0^{t_0} \eta_j(t+\tau)\eta_j(t)dt \\ &= \lim_{t_0\to\infty}\frac{1}{t_0}\int_0^{t_0} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \cos\left[2\pi f_n(t+\tau)+\varepsilon_n\right]\cos\left(2\pi f_m t+\varepsilon_m\right)dt \\ &= \lim_{t_0\to\infty}\frac{1}{t_0}\int_0^{t_0} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \left[\cos\left(2\pi f_n t+\varepsilon_n\right)\cos\left(2\pi f_m t+\varepsilon_m\right)\cos\left(2\pi f_n\tau\right)\right. \\ &\qquad\qquad \left. -\sin\left(2\pi f_n t+\varepsilon_n\right)\cos\left(2\pi f_m t+\varepsilon_m\right)\sin\left(2\pi f_n\tau\right)\right]dt \\ &= \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \cos\left(2\pi f_n\tau\right)\end{aligned}$$

Now

$$S(f)=4\int_0^{\infty}\frac{1}{2}\sum_{n=1}^{\infty}a_n^2\cos(2\pi f_n\tau)\cos(2\pi f\tau)d\tau$$
$$=\sum_{n=1}^{\infty}a_n^2\int_0^{\infty}\left[\cos 2\pi(f_n+f)\tau+\cos 2\pi(f_n-f)\tau\right]d\tau$$

$$\int_0^{\infty}(\text{periodic function})d\tau=0$$

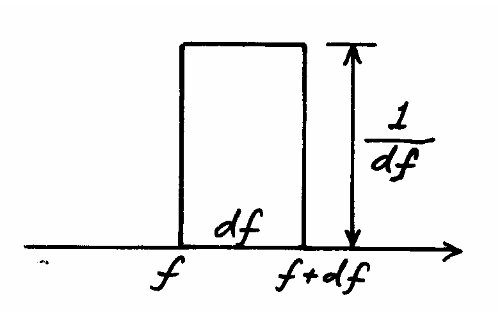
$$\int_0^{\infty}(1)d\tau=\infty \quad \rightarrow \text{Dirac delta function at} \quad f_n=-f \quad \text{and} \quad f_n=f$$

↑

take only this

Note: delta function is defined as the integral over $(-\infty,\infty)$. But we integrate over $(0,\infty)$. Therefore, take 1/2 of delta function.

$$S(f)=\sum_{n=1}^{\infty}\frac{1}{2}a_n^2\delta(f_n-f)$$




$$S(f)=\frac{1}{df}\sum_{f}^{f+df}\frac{1}{2}a_n^2$$

$$m_0=\overline{\eta^2}=\overline{\eta_j(t+0)\eta_j(t)}=\Psi(0)=\int_0^{\infty}S(f)\cos(2\pi f\cdot 0)df=\int_0^{\infty}S(f)df$$