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Concrete Plasticity

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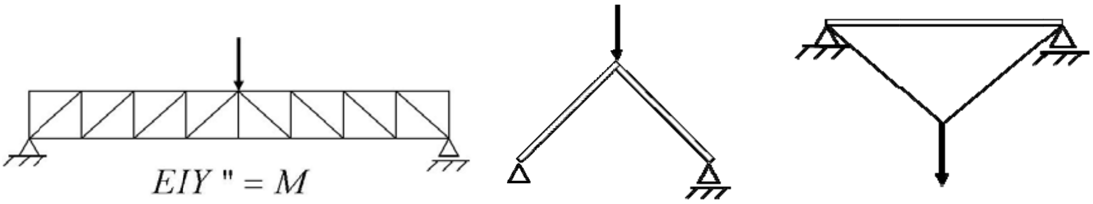
Chapter 1 Theory of Plasticity

1-1 History of truss model

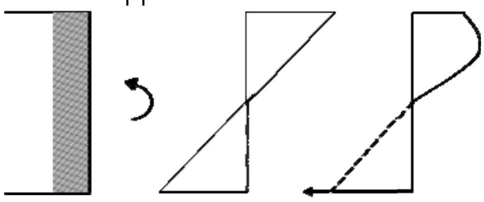
- Ritter & Morsch's 45 degree truss model



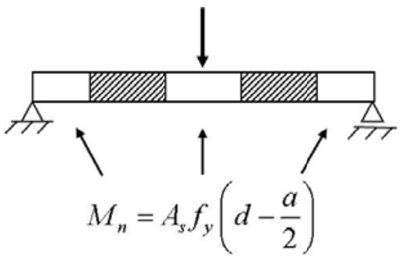
- Franz Leonhardt
  - Use of truss model for detailing of reinforcement.
- B. Thurlimann (ETH), P. Marti, P. Mueller
- Collins & Mitchell – deformation of truss model (compression field theory)
- Vecchio (modified compression field theory)

  → consideration of tensile strength of concrete
- Schlaich
  - Strut-and-tie models

    Practical application of strut-and-tie model for D-regions



Compatibility condition



1.2 Constitutive equations

1-2-1 von Mise's flow rule

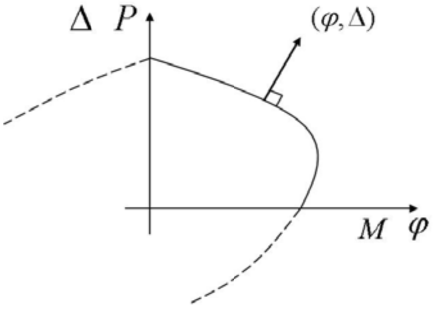
In general mechanics requires three

- stress field(statics) : equilibrium condition
- displacement field : compatibility condition
- stress-strain relationship : ex)hook's law

→ exact solution

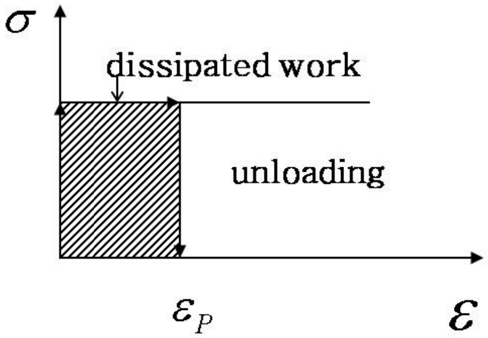
Relationship between strength and plastic deformation vectors

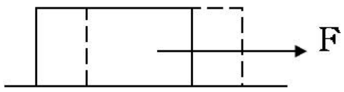
Ex) interaction curve for columns



1-2-2 von Mises flow rule

- Consider a rigid plastic material for dissipation energy : irreversible energy



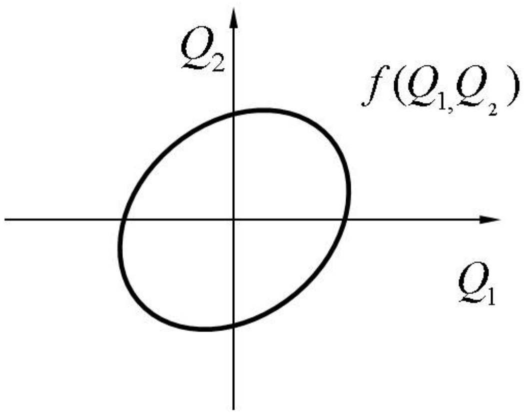


- For given stresses and strain definition of yield surfaces
  - generalized stresses : ← M, V, P (σ, τ ......)
  - generalized strains : q_i ← Φ, Δ (ε, γ .....)
- To explain yield conditions for a uniaxial state and combined stresses

  $\sigma \le f_y$ (for a uniaxial stress state) → $f(Q_i \le 0)$ (generalized form)

  For combined stress states a boundary represents

  $f(Q_1, Q_2, ..., Q_n) = 0$: yield surface



$f < 0$ → safe

$f \ge 0$ → yielding

On the yield surface we have the dissipation work

$$D = \int_v (Q_1 q_1 + \ldots\ldots + Q_n q_n) dv = \int w\, dv$$

- Von Mises's hypothesis on Maximum work

  The stresses corresponding to a given strain field assume such values that W becomes large as possible

  $\overline{\varepsilon} = (q_1, q_2, \ldots\ldots, q_n)$

  $\overline{\sigma} = (Q_1, Q_2, \ldots\ldots, Q_n)$

  $\rightarrow\ W = \overline{\sigma} \bullet \overline{\varepsilon}$

  If $\overline{\varepsilon}$ is assumed $\overline{\sigma}$ is to be determined so that W becomes as large as possible

  • f is convex

  • A closed surface contains Q(0 , 0 , 0 ,.....)

  We need to prove the convexity and the normality rule

  $W = \overline{\sigma} \cdot \overline{\varepsilon} = \Sigma \cdot \sigma \cdot \varepsilon \cdot \cos\theta$

  if $\overline{\varepsilon}$ is assumed given $\overline{\sigma}$ is to be determined such that W is as large as possible

  $f(\overline{\sigma}) = 0$

$$\delta W = W(Q_i + \delta Q_i) - W(Q_i)$$

$$= W(Q_i) + \frac{\partial w}{\partial \theta_i}\delta Q_i + \frac{\partial^2 w}{\partial \theta_i^2}(\delta Q_i)^2 + \ldots\ldots + (-W(Q_i))$$

$$= \delta Q_i\, q_i = 0 \quad \rightarrow \quad ⓐ$$

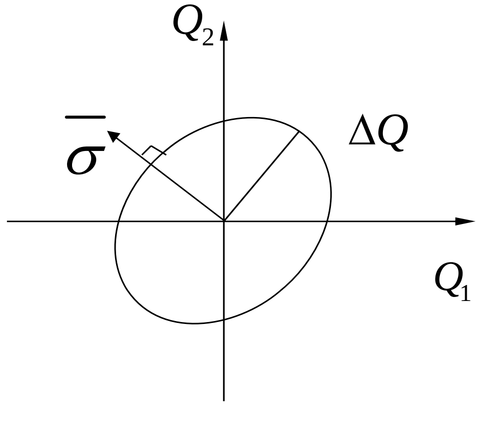
$$\delta f = f(\theta_i + \delta\theta_i) - f(\theta_i)$$

$$= \frac{\partial f}{\partial \theta_i}\delta\theta_i = 0 \quad \rightarrow \quad ⓑ$$

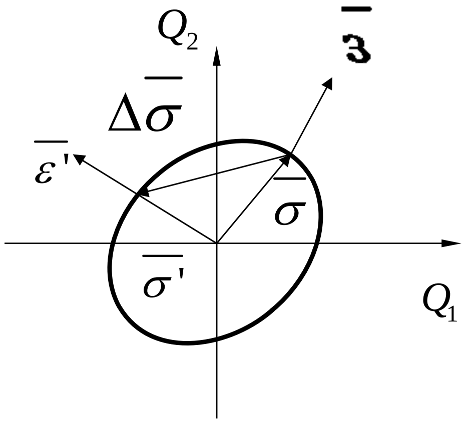
From Eq's ⓐ and ⓑ

$$q_i = \lambda \frac{\partial f}{\partial \theta_i}$$

(normality rule)



$$W(Q+\Delta Q) = W + \Delta\bar{\sigma}\cdot\bar{\varepsilon}$$
$$= W + |\Delta\bar{\sigma}||\bar{\varepsilon}|\cos\theta$$
$$W(Q+\Delta Q) - W = |\Delta\bar{\sigma}||\bar{\varepsilon}|\cos\theta < 0$$
$$\therefore \theta > 90°$$



$w' \leq w \quad \rightarrow \quad convex$

Plastic strain increments or equivalently, plastic strain rates are of primary interest at collapse. A failure mechanism is determined by the ratios of the plastic strain increments; their absolute values are irrelevant since only infinitesimal deformations are considered. In order to avoid complex formulations, the notation without the superscript for the plastic strain rates will be used. The strain rates do not represent differentiation
of the strains with respect to the physical time *t*; rather, *t* is just a scalar, and products are thus termed work or dissipation rather than power or rate of dissipation.
Fig. 3.1 (a) shows a yield surface determined by the yield condition enclosing an a plastic domain . For states of stress below the yield limit, the body remains rigid, while for stress combinations at the yield surface,
, plastic flow may occur. Assuming that the a plastic domain is convex and that the plastic strain increments at failure are perpendicular to the yield surface, i.e., that the associated flow or normality rule is valid,

ε ·= κ·gradΦ (3.1)

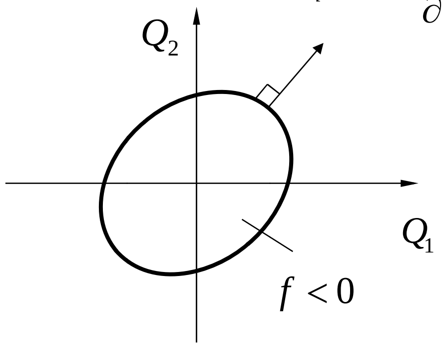
where κ denotes an arbitrary non-negative factor, one obtains

σ σ* – ( )εε · · ≥ 0 (3.2)

where is the actual stress state at the yield surface corresponding to , and is any
other stress state at or within the yield surface, Fig. 3.1 (a). Rearranging Eq. (3.2) one obtains
(3.3)
which is the principle of maximum energy dissipation postulated by von Mises [102].
According to this principle, the (virtual) dissipation per unit volume done on a given plastic strain rate assumes a maximum for the associated (or compatible) state of

1-3 Extreme principles for rigid-plastic materials

$$q_i = \lambda \frac{\partial f}{\partial Q_i} \quad on \;\; f = 0$$

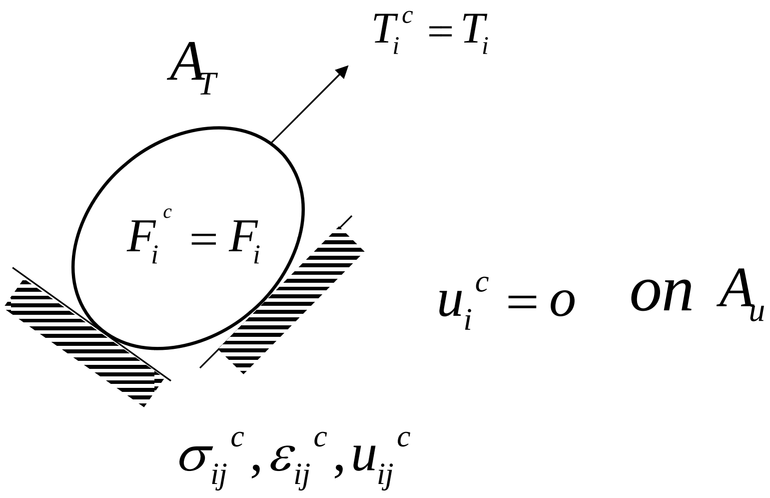


1-3-1 The lower bound theorem

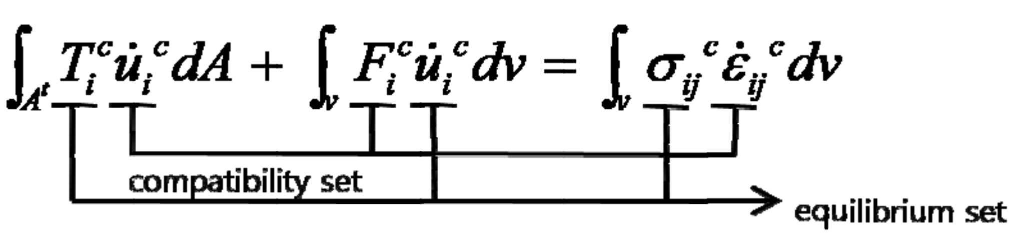
If an equilibrium distribution of stress $\sigma_{ij}{}^{E}$ can be found which balance the body force F_i in v and the applied load T_i on stress boundaries A_T and is everywhere below the yield surface $f\left(\sigma_{ij}{}^{E}\right) < 0$, then the body at load T_i, F_i will not collapse.

(proof)

If the body at T_i, F_i collapse, then a collapse pattern with actual stresses $\sigma_{ij}{}^{c}$, strain rates $\varepsilon_{ij}{}^{c}$, and displacement rate $\dot{u}_i^{c}$ exist.



two equilibrium systems exist.



→ principle of virtual work

$$\int_{A^t} T_i^c \dot{u}_i^c dA + \int_v F_i^c \dot{u}_i^c dv = \int_v \sigma_{ij}{}^{c} \dot{\varepsilon}_{ij}{}^{c} dv$$

$$\int_v \left(\sigma_{ij}{}^c - \dot{\sigma}_{ij}{}^E\right)\underline{\underline{\dot{\varepsilon}_{ij}{}^c}}\, dv = 0$$

↑ plastic

$\left(\sigma_{ij}{}^c - \dot{\sigma}_{ij}{}^E\right)\dot{\varepsilon}_{ij}{}^c > 0 \qquad for\ \ \dot{\sigma}_{ij}{}^E$ (*)

(*) cannot be true Q,E,D

$w' \le w$

1-3-2 the upper bound theorem

theorem

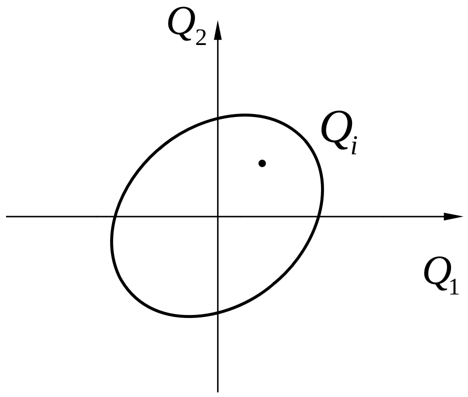
A load p_i for which

$\Sigma\, p_i v_i > \int w\, dv = \int \bar{\sigma} \cdot \bar{\varepsilon}\, dv$

that is a load performing work greater than P, cannot be carried by the body.

(proof)

Assume the load can be carried by the body. If so a statically admissible distribution Q_i' corresponding to stress on or within the yield surface can be found for the load P_i



$f(Q) < 0 \rightarrow$ equilibrium condition

By the principle of virtual work

$\Sigma p_i v_i = \int_v \bar{\sigma}' \cdot \bar{\varepsilon}\, dv$

It is not certain that Q_i' corresponding to the flow rule corresponds to the strain $\bar{\varepsilon}$

$p v_i > \int \bar{\sigma} \cdot \bar{\varepsilon}\, dv$

$\Sigma p_i v_i > \bar{\sigma}' \cdot \bar{\varepsilon}\, dv$

$\int_v \bar{\sigma} \cdot \bar{\varepsilon}\, dv \ge \int \bar{\sigma}' \cdot \bar{\varepsilon}\, dv$

$\int \bar{\sigma}' \cdot \bar{\varepsilon}\, dv \ge \Sigma p_i v_i$ → contradict

Test shows

$v = \gamma_0 = 1.52 - 0.83x \geq 1$

For pure shear and normal strength concrete

$v = \gamma_0 = 0.7 - \frac{f_c}{200}$

Higher strengths lead to higher sensitivity to load induced micro-cracking

Effect of macro-cracks on strength

- Crack directions may change several times
- Sliding reduction factor γ_s

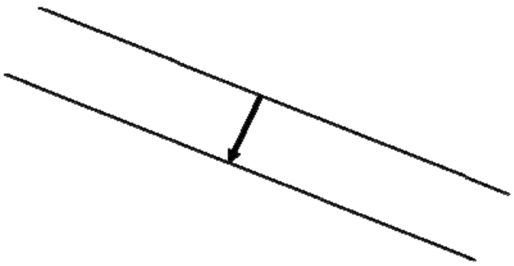
$$f_{ce} = \gamma_s \gamma_o f_c$$

micro

macro

-
- Cohesion : 50% reduction

Tensile strength is zero when w = 0.1mm $\varphi = 37^\circ$

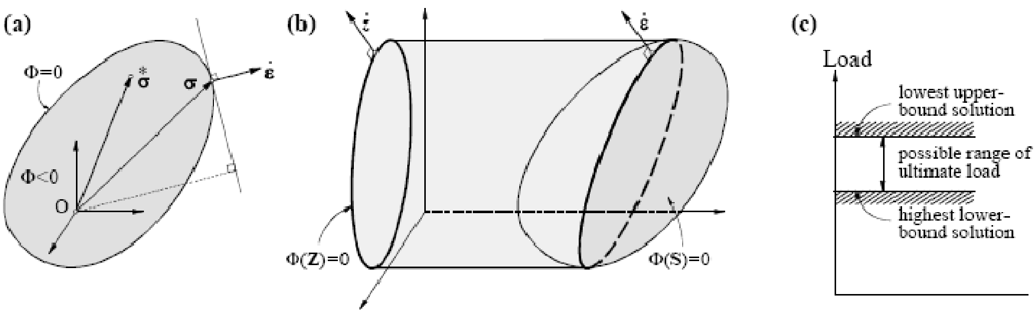


1-3-3 the uniqueness theorem

Between lower and upper bounds we have definitely an exact solution that is complete solution

• by lower solution $Q_s \leq Q_R$

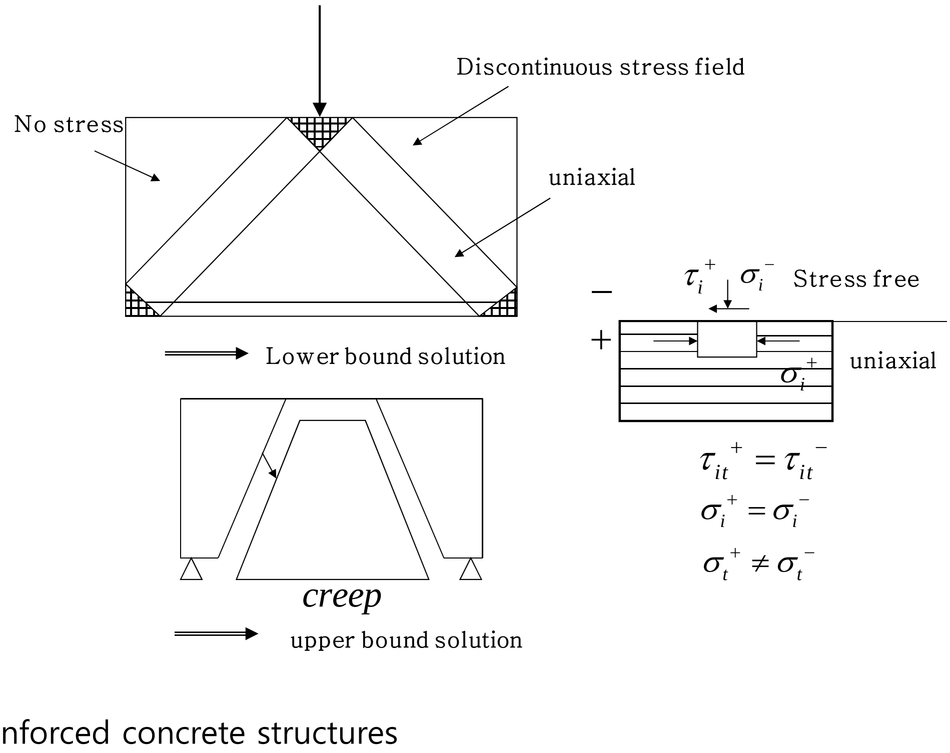
• by upper bound solution c



1-4 the solutions of plasticity problems

- lower bound solution (discontinuous stress field)
- upper bound solution (discontinuous displacement field)

The static or lower-bound method of the theory of plasticity is based on the lower bound theorem. Starting from statically admissible states of stress or stress fields everywhere at or below yield, one attempts to maximize the associated ultimate load. According to the lower-bound theorem, the ultimate load is equal to or higher than the highest load found in this way and hence, the static method yields safe or lower-bound solutions for the actual ultimate load. Note that a state of stress obtained from a linear elastic analysis represents a statically admissible stress field since equilibrium and static boundary conditions are satisfied. Hence, although the elastically determined state of stress normally deviates from the actual state of stress in the structure, Chapter 3.1, design "based on the theory of elasticity" can be justified by the lower-bound theorem of limit analysis.
The kinematic or upper-bound method of the theory of plasticity is based on the upper-bound theorem. Starting from kinematically admissible states of deformation or failure mechanisms, one attempts to minimize the associated ultimate load. According to the upper-bound theorem, the ultimate load is equal to or lower than the lowest load found in this way and hence, the kinematic method yields unsafe or upper-bound solutions for the actual ultimate load.



1-5 reinforced concrete structures

Lower bound theorem : design

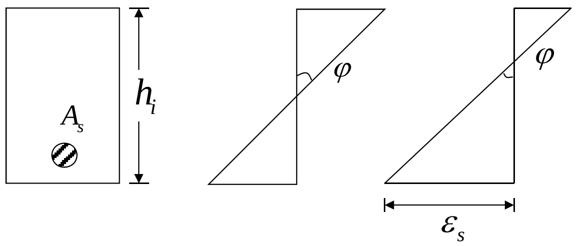
ex) η moment or forces may be chosen how to select the redundant moment

- minimizing the total amount of bending reinforcement

$$\int \frac{M^2}{EI} dx \rightarrow \min$$

For a fully cracked member

$$\sigma_y \cong \frac{|M|}{h_i A_s} \ \text{- ①}$$



Curvature

$$|k| \cong E_s h_i^2 A_s$$

From ① $\quad f_Y \cong \frac{|M|}{h_i A_s}, \ \ A_s \cong \frac{|M|}{h_i f_Y}$

$$EI \cong E_s h_i^2 \left(\frac{|M|}{h_i f_Y} \right) = \frac{E_s |M| h_i}{f_Y}$$

$$\int \frac{M^2}{EI} dx = \frac{E_s |M| h_i}{f_Y} \ \ M^2 = \frac{f_Y}{E_s} \int \frac{|M|}{h_i} =$$