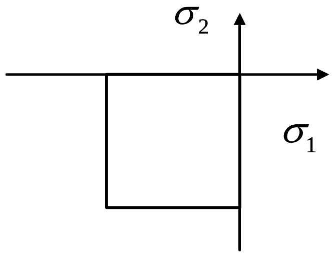
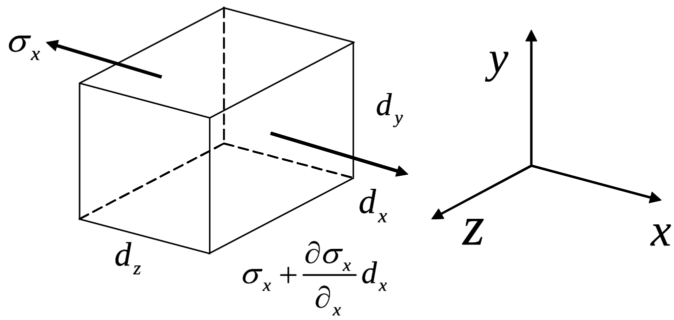
Chap 3. The Theory of PLAIN Concrete

3.1 Statical condition

= equilibrium cond. + boundary cond.

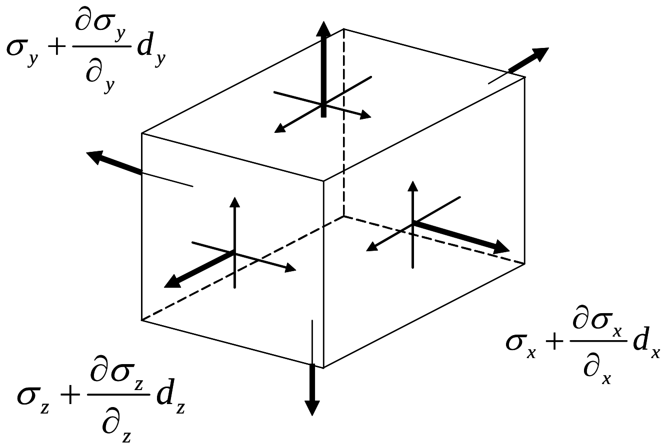




$$\sigma_y + \frac{\partial \sigma_y}{\partial y} dy$$

Taylor's series

$$f(a+h) = f(a) + \frac{f'(a)}{1!} h + \frac{f'(a)}{2!} h^2$$



3.2 Geometrical condition

-> displacement – strain relationship

=compatibility condition

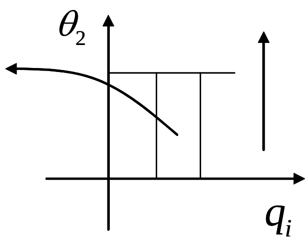
3.3 Virtual Work

The equation in an identity if the statical conditions are fulfilled and is then valid for any displacement field satisfying the geometrical condition

if on the contrary, the equation is know to be valid for any displacement field, the statical conditions are fulfilled.

3.4 Constitutive equations

(by normality rule)



$$q_i = \lambda \frac{\partial f}{\partial \theta_i}$$

3.5 Plastic strains in coulomb material

$$k\sigma_1 - \sigma_3 = f_c = 2c\sqrt{k}$$

Note $\sigma_3 \le \sigma_2 \le \sigma_1$

Where

$$k = (\frac{1+\sin\varphi}{\cos\varphi})^2 = (\frac{1+\sin\varphi}{1-\sin\varphi}) = \tan^2(\frac{\pi}{4}+\frac{\varphi}{2})$$

Six yield conditions

1. $k\sigma_1 - \sigma_3 = f_c \quad : \quad \sigma_3 \le \sigma_2 \le \sigma_1$
2. $k\sigma_3 - \sigma_1 = f_c \quad : \quad \sigma_1 \le \sigma_2 \le \sigma_3$
3. $k\sigma_1 - \sigma_2 = f_c \quad : \quad \sigma_2 \le \sigma_3 \le \sigma_1$
4. $k\sigma_2 - \sigma_1 = f_c \quad : \quad \sigma_1 \le \sigma_3 \le \sigma_2$
5. $k\sigma_2 - \sigma_3 = f_c \quad : \quad \sigma_3 \le \sigma_1 \le \sigma_2$
6. $k\sigma_3 - \sigma_2 = f_c \quad : \quad \sigma_2 \le \sigma_1 \le \sigma_3$

By normality rule

$$q_i = \lambda \frac{\partial f}{\partial Q_i}$$

1. $\varepsilon_1 = \lambda \frac{\partial f}{\partial \sigma_1} = \lambda k \quad , \varepsilon_2 = 0 \quad , \varepsilon_3 = -\lambda$
2. $\varepsilon_1 = -\lambda \quad , \varepsilon_2 = 0 \quad , \varepsilon_3 = \lambda k$
3. $\varepsilon_1 = \lambda k \quad , \varepsilon_2 = -\lambda \quad , \varepsilon_3 = 0$
4. $\varepsilon_1 = -\lambda \quad , \varepsilon_2 = \lambda k \quad , \varepsilon_3 = 0$
5. $\varepsilon_1 = 0 \quad , \varepsilon_2 = \lambda k \quad , \varepsilon_3 = -\lambda$
6. $\varepsilon_1 = 0 \quad , \varepsilon_2 = -\lambda \quad , \varepsilon_3 = \lambda k$

$$\therefore \ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda(k-1)$$

Along the edges, plastic strain vectors

1/5 linear combination of plastic strain on planes 1 & 5

$\varepsilon_1 = \lambda k \qquad \varepsilon_2 = 0 + \lambda_2 k \qquad \varepsilon_3 = -\lambda_1 - \lambda_2$

4/5 edge : $\varepsilon_1 = -\lambda_1 \qquad \varepsilon_2 = (\lambda_1 + \lambda_2) \qquad \varepsilon_3 = -\lambda_2$

2/4 edge : $\varepsilon_1 = -(\lambda_1 + \lambda_2) \qquad \varepsilon_2 = \lambda_2 k \qquad \varepsilon_3 = \lambda_1 k$

2/6 edge : $\varepsilon_1 = -\lambda_1 \qquad \varepsilon_2 = -\lambda_2 \qquad \varepsilon_3 = (\lambda_1 + \lambda_2)k$

3/6 edge : $\varepsilon_1 = -\lambda_1 k \qquad \varepsilon_2 = -(\lambda_1 + \lambda_2) \qquad \varepsilon_3 = -\lambda_2 k$

2/6 edge : $\varepsilon_1 = (\lambda_1 + \lambda_2) \qquad \varepsilon_2 = -\lambda_2 \qquad \varepsilon_3 = -\lambda_1$

At the apex

$$\sigma_1 = \sigma_2 = \sigma_3 = \frac{f_c}{k-1}$$

From (3.4.4)

$$\sum_1^6 \varepsilon_1 = (\lambda_1 + \lambda_3)k - (\lambda_2 + \lambda_4) = \varepsilon_1$$

$$\sum_1^6 \varepsilon_2 = (\lambda_4 + \lambda_5)k - (\lambda_3 + \lambda_6) = \varepsilon_2$$

$$\sum_1^6 \varepsilon_3 = (\lambda_2 + \lambda_6)k - (\lambda_1 + \lambda_5) = \varepsilon_3$$

Except for the apex, on planes

$$\sum \varepsilon^{+} = \lambda k, \qquad \sum \left|\varepsilon^{-}\right| = \lambda k$$

On edges

$$\sum \varepsilon^{+} = (\lambda_1 + \lambda_2)k, \qquad \sum \left|\varepsilon^{-}\right| = \lambda_1 + \lambda_2$$

$$\therefore \frac{\sum \varepsilon^{+}}{\sum \left|\varepsilon^{-}\right|} = k$$

On apex

$$\frac{\sum \varepsilon^{+}}{\sum \left|\varepsilon^{-}\right|} \geq k$$

Volume change $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = ?$

On planes $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda(k-1)$

On edges $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = k(\lambda_1 + \lambda_2) - (\lambda_1 + \lambda_2)$

Since $k = \dfrac{\sum \varepsilon^{+}}{\sum \left|\varepsilon^{-}\right|} = \dfrac{1+\sin\varphi}{1-\sin\varphi}$

Plastic strains of coulomb's material

$$\frac{\sum \varepsilon^{+}}{\sum \left|\varepsilon^{-}\right|} = k \qquad \text{except for apex} \quad k = \frac{1+\sin\varphi}{1-\sin\varphi} \qquad -①$$

Volume change

$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \lambda(k-1)$ on plane -②

$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = k(\lambda_1 + \lambda_2) - (\lambda_1 + \lambda_2)$ on edge -③

From ①

$$k = \frac{\sum \varepsilon^{+}}{\sum \left|\varepsilon^{-}\right|} = \frac{1+\sin\varphi}{1-\sin\varphi}$$

Eq ② is rewritten as

$$\varepsilon_1+\varepsilon_2+\varepsilon_3=\sum\left|\varepsilon^-\right|(k-1)=\sum\left|\varepsilon^-\right|\frac{2\sin\varphi}{1-\sin\varphi}$$

$$\varepsilon_1+\varepsilon_2+\varepsilon_3=\frac{2\sin\varphi}{1-\sin\varphi}\frac{1}{k}k\sum\left|\varepsilon^-\right|$$

$$=\frac{2\sin\varphi}{1-\sin\varphi}\frac{1-\sin\varphi}{1+\sin\varphi}\frac{\sum\left|\varepsilon^+\right|}{\sum\left|\varepsilon^-\right|}\sum\left|\varepsilon^-\right|$$

$$=\frac{2\sin\varphi}{1+\sin\varphi}\sum\left|\varepsilon^+\right|$$

$$(\varepsilon_1+\varepsilon_2+\varepsilon_3)(1-\sin\varphi)=\sum\left|\varepsilon^-\right|2\sin\varphi \qquad -④$$

$$(\varepsilon_1+\varepsilon_2+\varepsilon_3)(1+\sin\varphi)=\sum\left|\varepsilon^+\right|2\sin\varphi \qquad -⑤$$

④ + ⑤ →

$$2(\varepsilon_1+\varepsilon_2+\varepsilon_3)=2\sin\varphi(\sum\left|\varepsilon^-\right|+\sum\left|\varepsilon^+\right|)$$

$$\sin\varphi=\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3}{\left|\varepsilon_1\right|+\left|\varepsilon_2\right|+\left|\varepsilon_3\right|}$$

3.4.2 Dissipation formula for coulomb material along planes

- along the planes

$$W=\sigma_1\varepsilon_1+\sigma_2\varepsilon_2+\sigma_3\varepsilon_3$$
$$=\sigma_1(\lambda k)+0+\sigma_3(-\lambda)$$
$$=\lambda(k\sigma_1-\sigma_3)=\lambda f_c$$

-along the edges

$$W=\sigma_1\varepsilon_1+\sigma_2\varepsilon_2+\sigma_3\varepsilon_3$$
$$W=\sigma_1\lambda_1 k+\sigma_2\lambda_2 k-\sigma_3(\lambda_1+\lambda_2)$$
$$=k(\sigma_1\lambda_1+\sigma_2\lambda_2)-(k\sigma_1-f_c)(\lambda_1+\lambda_2)$$
$$=k\lambda_2(\sigma_1-\sigma_2)+f_c(\lambda_1+\lambda_2)$$

Since $\sigma_1=\sigma_2$

$$W=f_c(\lambda_1+\lambda_2)$$

-at apex

$$W = \sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3$$

Since $\sigma_1 = \sigma_2 = \sigma_3 = \dfrac{f_c}{k-1} = c\cot\psi$

$$W = \frac{f_c}{k-1}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

$$= \frac{f_c}{k-1}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6)$$

=summary

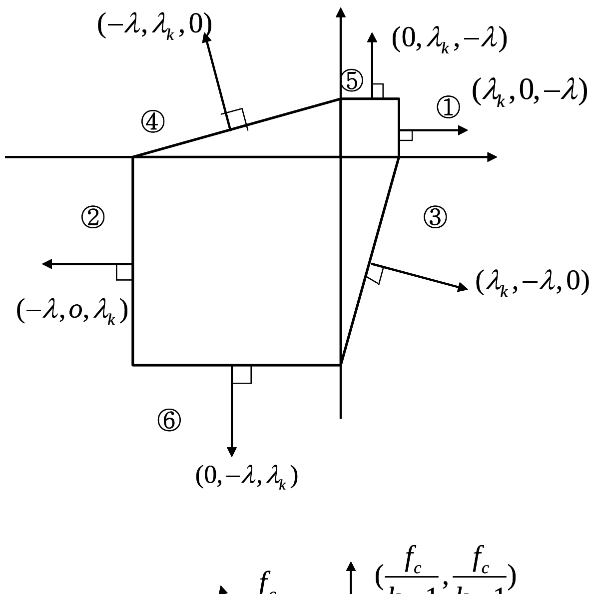
$$k = \frac{\sum \varepsilon^+}{\sum |\varepsilon^-|}, \quad \sum \varepsilon^+ = \lambda k, \quad \sum |\varepsilon^-| = \lambda$$

$$W = \lambda f_c = \frac{\sum \varepsilon^+}{k} f_c = \sum |\varepsilon^-| f_c \qquad \text{for plane}$$

$$W = (\lambda_1 + \lambda_2) f_c = \frac{\sum \varepsilon^+}{k} f_c = \sum |\varepsilon^-| f_c \qquad \text{for edge}$$

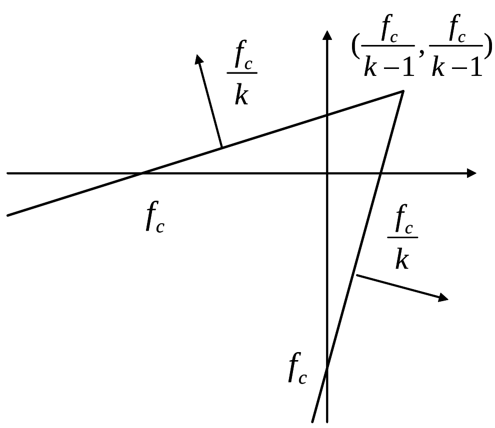
6 yield surfaces

1. $k\sigma_1 - \sigma_3 - f_c = 0$
2. $k\sigma_3 - \sigma_3 - f_c = 0$
3. $k\sigma_1 - \sigma_2 - f_c = 0$
4. $k\sigma_2 - \sigma_1 - f_c = 0$
5. $k\sigma_2 - \sigma_3 - f_c = 0$
6. $k\sigma_3 - \sigma_2 - f_c = 0$



Plane strain → ε_3 =0

ε_1	ε_2	ε_3
1 $(\lambda k,$	$0,$	$-\lambda)$
2 $(-\lambda,$	$0,$	$\lambda k)$
3 $(\lambda k,$	$-\lambda,$	$0)$



4 $(-\lambda, \quad \lambda k, \quad -\lambda)$
5 $(0, \quad \lambda k, \quad -\lambda)$
6 $(0, \quad -\lambda, \quad \lambda k)$

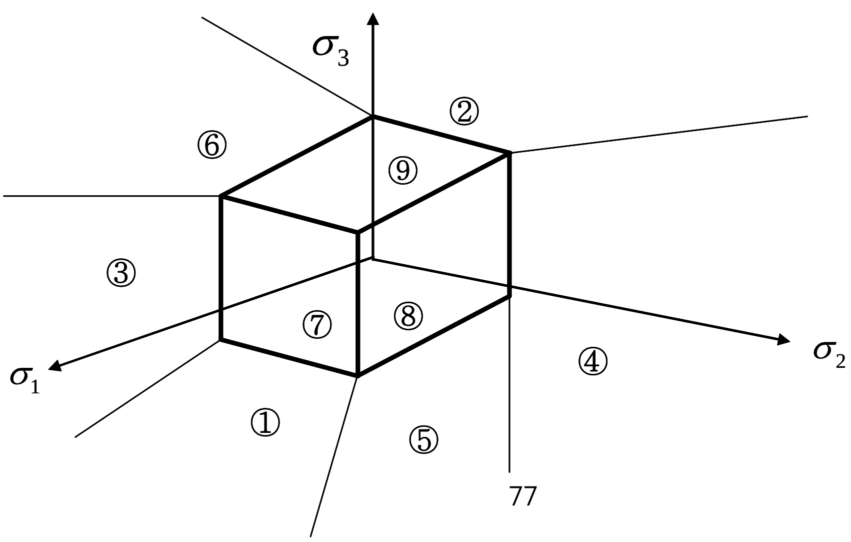
3.4.3 & 3.4.4 Plastic strains & dissipation formulas for modified coulomb material

Plastic strains by normality rule

plane	1	2	3	4	5	6	7	8	9
ε_1	λk	$-\lambda$	λk	$-\lambda$	0	0	λ	0	0
ε_2	0	0	$-\lambda$	λk	λk	$-\lambda$	0	λ	0
ε_3	$-\lambda$	λk	0	0	$-\lambda$	λk	0	0	λ

9 yield surfaces

1. $k\sigma_1 - \sigma_3 - f_c = 0$
2. $k\sigma_3 - \sigma_3 - f_c = 0$
3. $k\sigma_1 - \sigma_2 - f_c = 0$
4. $k\sigma_2 - \sigma_1 - f_c = 0$
5. $k\sigma_2 - \sigma_3 - f_c = 0$
6. $k\sigma_3 - \sigma_2 - f_c = 0$
7. $\sigma_1 - f_t = 0$
8. $\sigma_2 - f_t = 0$
9. $\sigma_3 - f_t = 0$



Edge	1/5	4/5	2/4	2/6	3/6
ε_1	$\lambda_1 k$	$-\lambda_1$	$-(\lambda_1+\lambda_2)$	$-\lambda_1$	$\lambda_1 k$
ε_2	$\lambda_2 k$	$(\lambda_1+\lambda_2)k$	$\lambda_2 k$	λ_2	$-(\lambda_1+\lambda_2)$
ε_3	$-(\lambda_1+\lambda_2)$	$-\lambda_2$	$-\lambda_2$	$(\lambda_1+\lambda_2)k$	$\lambda_2 k$
Edge	**1/3**	**1/7**	**3/7**	**6/9**	**2/7**
ε_1	$(\lambda_1+\lambda_2)k$	$\lambda_1 k+\lambda_2$	$\lambda_1 k+\lambda_2$	0	$-\lambda_1$
ε_2	$-\lambda_2$	0	$-\lambda_1$	$-\lambda_1$	0
ε_3	$-\lambda_1$	$-\lambda_1$	0	$\lambda_1 k$	$\lambda_1 k+\lambda_2$
Edge	**4/8**	**5/8**	**8/7**	**7/9**	**8/9**
ε_1	$-\lambda_1$	0	λ_1	λ_1	0
ε_2	$\lambda_1 k$	$\lambda_1 k+\lambda_2$	λ_2	0	λ_1
ε_3	0	$-\lambda_1$	0	λ_2	λ_2

W along edges

1/7 edge

$$W=\sigma_1(\lambda_1 k+\lambda_2)+\sigma_3(-\lambda_1)=\lambda_1(\sigma_1 k-\sigma_3)+\lambda_2\sigma_1 \;=\lambda_1 f_c+\lambda_2 f_t$$

8/7 edge

$$W=\sigma_1\lambda_1+\sigma_2\lambda_2=(\lambda_1+\lambda_2)f_t$$

Apex	ε_1	ε_2	ε_3
4/5/8	$-\lambda_1$	$k(\lambda_1+\lambda_2)+\lambda_3$	$-\lambda_2$
7/8/5/1	$\lambda_1 k+\lambda_3$	$\lambda_2 k+\lambda_4$	$-(\lambda_1+\lambda_2)$
1/3/7	$k(\lambda_1+\lambda_2)+\lambda_3$	$-\lambda_2$	$-\lambda_1$
3/7/6/9	$\lambda_1 k+\lambda_2$	$-(\lambda_1+\lambda_2)$	$\lambda_2 k+\lambda_4$
2/6/9	$-\lambda_1$	$-\lambda_2$	$k(\lambda_1+\lambda_2)+\lambda_3$
8/9/2/4	$-(\lambda_1+\lambda_2)$	$\lambda_2 k+\lambda_3$	$\lambda_1 k+\lambda_4$
7/8/9	λ_1	λ_2	λ_3

W at apex

4/5/8 apex

$$W = -\sigma_1\lambda_1 + [k(\lambda_1+\lambda_2)+\lambda_3]\sigma_2 + \sigma_3(-\lambda_2)$$
$$= -\lambda_1(\sigma_1 - k\sigma_2) - \lambda_2(\sigma_3 - k\sigma_2) + \sigma_2(\lambda_3)$$
$$= f_c(\lambda_1+\lambda_2) + f_t\lambda_3$$

7/8/5/1 apex

$$W = -\sigma_1(\lambda_1 k + \lambda_3) + (k\lambda_2 + \lambda_4)\sigma_2 + \sigma_3(-\lambda_2 - \lambda_1)$$
$$= \lambda_1(\sigma_1 k - \sigma_3) + \lambda_2(\sigma_2 k - \sigma_3) + \sigma_1(\lambda_3) + \sigma_2(\lambda_4)$$
$$= f_c(\lambda_1+\lambda_2) + f_t(\lambda_3+\lambda_4)$$

1/3/9 apex

$$W = f_c(\lambda_1+\lambda_2) + f_t\lambda_3$$

3/7/6/9 apex

$$W = f_c(\lambda_1+\lambda_2) + f_t(\lambda_3+\lambda_4)$$

$$W = f_c\sum\left|\varepsilon^-\right| + f_t\left(\sum\left|\varepsilon^+\right| - k\sum\left|\varepsilon^-\right|\right)$$ for plane stress

In the apex, we have

$$\frac{\sum \varepsilon^+}{\sum\left|\varepsilon^-\right|} \le k$$

(Proof)

$$\frac{\sum \varepsilon^+}{k} - \sum\left|\varepsilon^-\right| \ge 0$$

At apex for coulomb material

$\varepsilon_1, \varepsilon_2 \ge 0, \quad \varepsilon_3 \le 0$

$\varepsilon_1, \varepsilon_3 \ge 0, \quad \varepsilon_2 \le 0$

$\varepsilon_2, \varepsilon_3 \ge 0, \quad \varepsilon_1 \le 0$

$\varepsilon_1 \le 0, \quad \varepsilon_2, \varepsilon_3 \ge 0$

$\varepsilon_2 \le 0, \quad \varepsilon_3, \varepsilon_1 \ge 0$

$\varepsilon_3 \le 0, \quad \varepsilon_1, \varepsilon_2 \ge 0$

$\Rightarrow \dfrac{\varepsilon_1+\varepsilon_2}{k}+\varepsilon_3$

$$=\lambda_1+\lambda_3+\lambda_4+\lambda_5-\frac{\lambda_2+\lambda_4+\lambda_5+\lambda_6}{k}-(\lambda_1+\lambda_5)+(\lambda_3+\lambda_6)k$$

$$=(\lambda_3+\lambda_4)(1-\frac{1}{k})+(\lambda_2+\lambda_6)(k-\frac{1}{k})\geq 1$$

Dissipation formula

$$W=\sigma_1\varepsilon_1+\sigma_2\varepsilon_2+\sigma_3\varepsilon_3$$

For coulomb material

On plane and edges

$$W=f_c\sum\left|\varepsilon^-\right|=\frac{f_c}{k}\sum\left|\varepsilon^+\right|$$

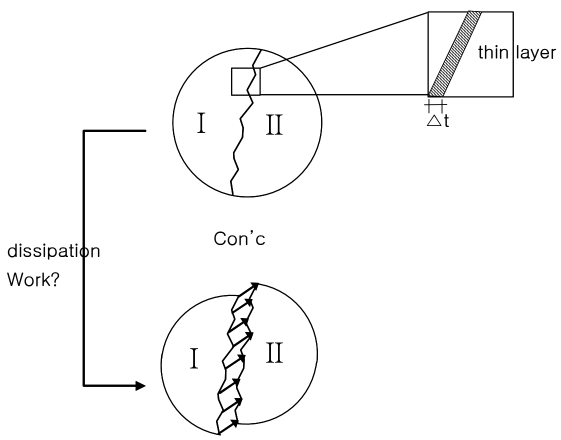
At apex

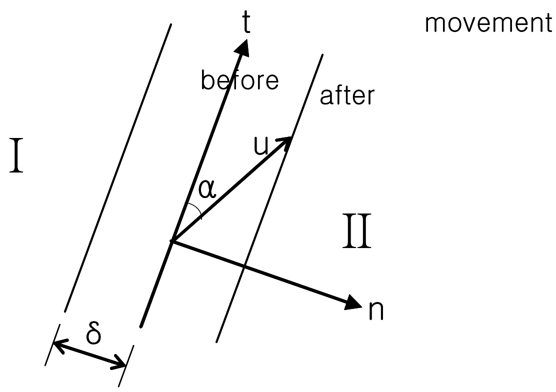
$$W=c\cot\varphi(\varepsilon_1+\varepsilon_2+\varepsilon_3)$$

For modified coulomb material

$$W=f_c\sum\left|\varepsilon^-\right|+f_t(\sum\left|\varepsilon^+\right|-k\sum\left|\varepsilon^-\right|)$$

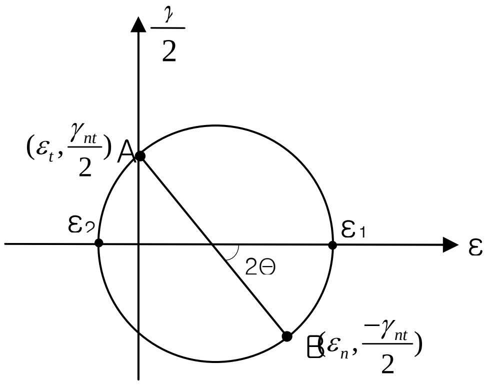
3.4.5 Planes and lines of discontinuity = yield line = failure line





$$u_n = u\sin\alpha$$
$$u_t = u\cos\alpha$$

$$\varepsilon_n = \frac{u_n}{\delta} = \frac{u}{\delta}\sin\alpha$$
$$\varepsilon_t = \frac{u\cos\alpha}{\infty} = 0$$
$$r_{nt} = \frac{u\cos\alpha}{\delta}$$



@A (0, $\frac{u\cos\alpha}{2\delta}$)

@B ($\frac{u\sin\alpha}{\delta}$, $-\frac{u\cos\alpha}{2\delta}$)

$$\varepsilon_1 = \varepsilon_n + R$$
$$\varepsilon_2 = \varepsilon_n - R$$

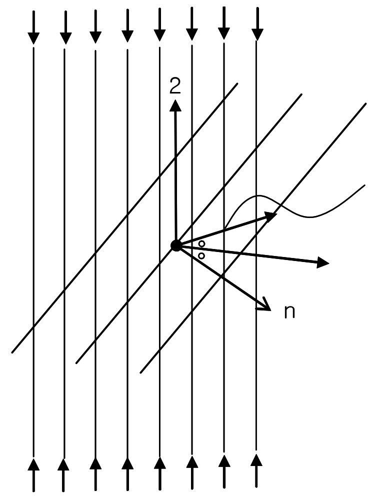
Where $\varepsilon_m = \frac{\varepsilon_1 + \varepsilon_2}{2} = \frac{\varepsilon_n + \varepsilon_t}{2}$

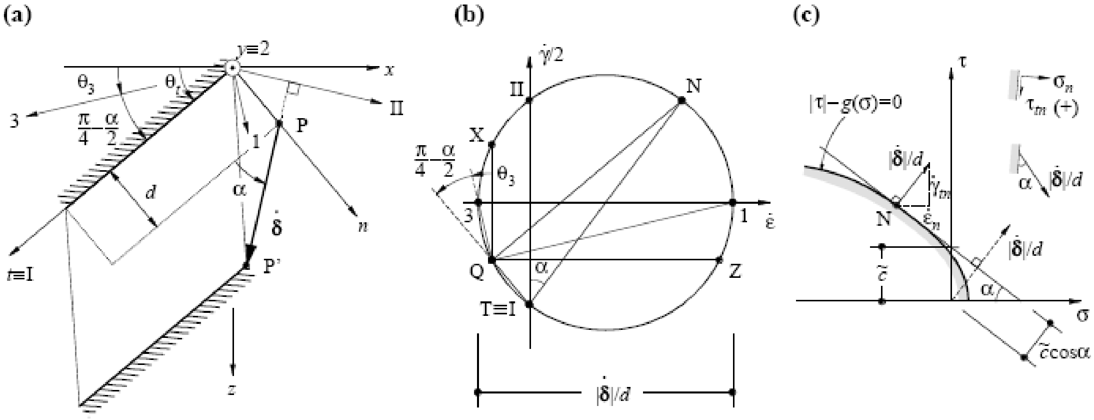
$$R = \sqrt{\varepsilon^2{}_m + (\frac{r_{nt}}{2})^2}$$
$$= \sqrt{\frac{u^2\sin^2\alpha}{4\delta^2} + \frac{u^2\cos^2\alpha}{4\delta^2}} = \frac{u}{2\delta}$$
$$\varepsilon_1 = \frac{u\sin\alpha}{2\delta} + \frac{u}{2\delta}$$
$$\varepsilon_2 = \frac{u\sin\alpha}{2\delta} - \frac{u}{2\delta}$$

→ $\frac{u}{2\delta}(\sin\alpha \pm 1)$

$$\tan 2\theta = \frac{\frac{u\cos\alpha}{2\delta}}{\frac{u\sin\alpha}{2\delta}} = \cot\alpha = \tan(\frac{\pi}{2}-\alpha)$$

If the 1st principal strain coincides with the 1st principal stress

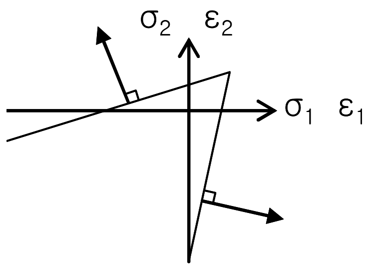




Plane strain

1) Coulomb material

$$\frac{\sum \varepsilon^{+}}{\sum \left|\varepsilon^{-}\right|} = k$$

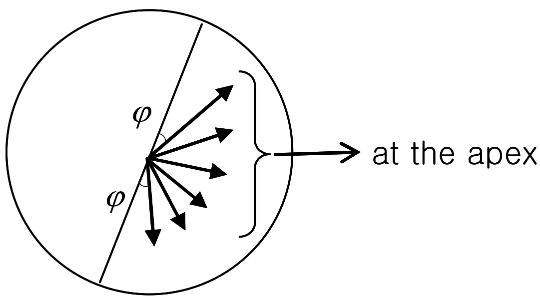


$$\frac{\varepsilon_1}{\left|\varepsilon_2\right|} = \frac{\frac{u}{2\delta}(\sin\alpha + 1)}{-\frac{u}{2\delta}(\sin\alpha - 1)} = k$$

$$k = \frac{1+\sin\alpha}{1-\sin\alpha} = \frac{1+\sin\varphi}{1-\sin\varphi}$$

$$\therefore \tan\alpha = \pm\tan\varphi$$

For $\alpha = \varphi$, $\alpha = \pi - \varphi$



$$W = f_c \sum \left|\varepsilon^{-}\right|$$

$$= f_c \frac{u}{2\delta}(1-\sin\varphi)$$

Dissipation per unit length

$$W_l = W b_\delta$$

$$= f_c \frac{ub}{2}(1-\sin\varphi)$$

$$= f_c \frac{ub}{2k}(1+\sin\varphi)$$

Since $f_c = 2c\sqrt{k}$

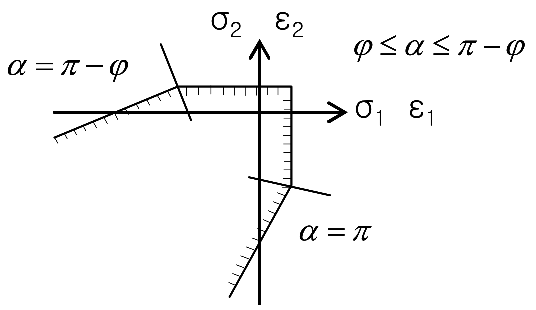
$$W_l = 2c\sqrt{k}\,\frac{ub}{2}(1-\sin\varphi)$$

For $\varphi \le \alpha \le \pi - \varphi$

$$W = c\cot\varphi(\varepsilon_1 + \varepsilon_1 + \varepsilon_1)$$
$$W = c\cot\varphi(\varepsilon_1 + \varepsilon_1 + \varepsilon_1)$$
$$W_l = cub\cot\varphi\sin\alpha$$
$$\alpha = \pi - \varphi$$

Plane strain

2) Modified coulomb material



$$W = f_c\sum\left|\varepsilon^-\right| + f_c(\sum\left|\varepsilon^+\right| - \sum\left|\varepsilon^-\right|)$$
$$= f_c\frac{u}{2\delta}(1-\sin\alpha) + f_c\frac{u}{2\delta}[1+\sin\alpha - k(1-\sin\alpha)]$$
$$= f_c\frac{u}{2\delta}(1-\sin\alpha) + f_c\frac{u}{2\delta}[-(k-1)+(k+1)\sin\alpha)]$$

Where $k = \dfrac{1+\sin\alpha}{1-\sin\alpha} = 4$ for concrete

$$k-1 = \frac{2\sin\alpha}{1-\sin\alpha},\quad k+1 = \frac{2}{1-\sin\alpha}$$

Plane stress

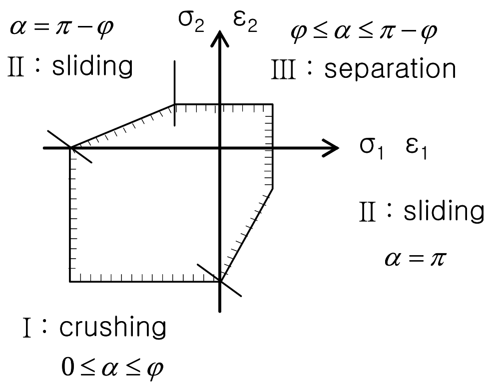
1) Coulomb material

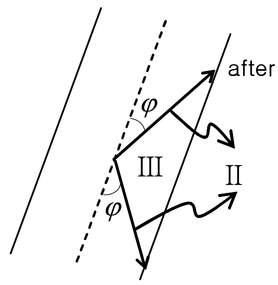
$$\sum\left|\varepsilon^-\right| = \frac{u}{2\delta}(1-\sin\alpha)$$

For $\alpha \le \varphi$, $\alpha \le \pi - \varphi$

$$W = f_c \sum \left| \varepsilon^- \right|$$

$$W_l = \frac{1}{2} f_c ub(1 - \sin\alpha)$$





For $\alpha \le \pi - \varphi$

$$W_l = \frac{1}{2} f_c ub[1 - \sin\alpha + \frac{f_t}{f_c}\{(k-1) + (k+1)\sin\alpha\}]$$