4. DISKS

4.1 Statical conditions

Equilibruim + boundary

4.2 Geometrical conditions

Strain-displ
Compatibility

4.3 Virtual Work

4.4 Constitutive equations

4.3 Virtual Work

4.4.1 Plastic strains in disks

Yield condition
Normality rule

$$-(f_{tx}-\sigma_x)(f_{ty}-\sigma_y)+\tau^2{}_{xy}=0 \qquad (2.82)$$
$$-(f_c+\sigma_x)(f_c+\sigma_y)+\tau^2{}_{xy}=0 \qquad (2.83)$$

For $\sigma_y \geq -\eta\sigma_x + \eta f_{tx} - f_c$ for (2.82)

By normality rule

$$\frac{\partial F}{\partial \sigma_x}=\varepsilon_x=\lambda(f_{ty}-\sigma_y)$$

Since

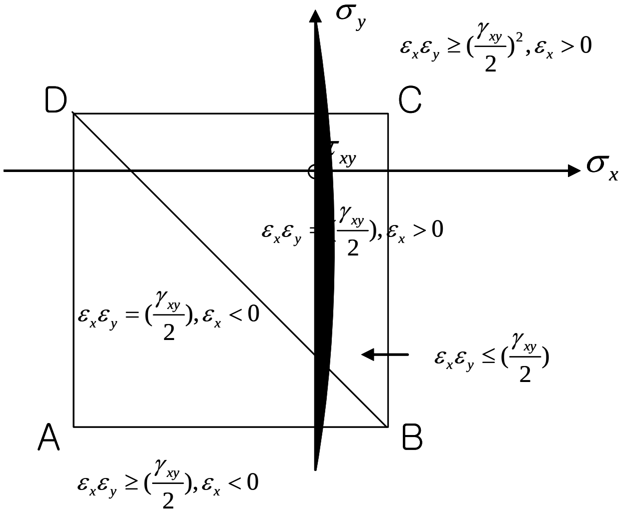
$$f_{ty}=\frac{A_{sy}f_y}{t}=\Phi_y f_c$$
$$\varepsilon_x=\lambda(\Phi_y f_c-\sigma_y)$$

In similar manner

$$\varepsilon_y=\lambda(\Phi_x f_c-\sigma_y)$$
$$r_{xy}=2\varepsilon_{xy}=2\lambda\tau_{xy}$$

For $\sigma_y \leq -\eta\sigma_x + \eta\Phi_x f_c - f_c$, we have

$\varepsilon_x = -\lambda(f_c + \sigma_y)$
$\varepsilon_y = -\lambda(f_c + \sigma_x)$
$r_{xy} = 2\varepsilon_{xy} = 2\lambda\tau_{xy}$



$\varepsilon_x\varepsilon_y = \varepsilon^2{}_{xy}$, one of $\varepsilon_{1,3}$ is zero

$$\varepsilon_{1,3} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon^2{}_{xy}}$$

$$\frac{1}{2}\varepsilon_x\varepsilon_y = -\frac{1}{2}\varepsilon_x\varepsilon_y + \varepsilon^2{}_{xy}$$

$$\varepsilon^2{}_{xy} = \varepsilon_x\varepsilon_y \qquad -(*)$$

Using (*) we have

$$\lambda^2(\Phi_y f_c - \sigma_y)(\Phi_x f_c - \sigma_x) = \lambda^2\tau^2{}_{xy}$$

$$(\Phi_y f_c - \sigma_y)(\Phi_x f_c - \sigma_x) - \tau^2{}_{xy} = 0$$

Along BD

$\varepsilon_x = \lambda_1\varepsilon_x$ of BCD + $\lambda_2\varepsilon_x$ of ABD

$= \lambda_1(\Phi_y f_c - \sigma_y) + \lambda_2(-f_c - \sigma_y)$

In the same way

$\varepsilon_y = \lambda_1(\Phi_x f_c - \sigma_x) + \lambda_2(-f_c - \sigma_x)$

$r_{xy} = 2\lambda_1\tau_{xy} + 2\lambda_2\tau_{xy}$

$\varepsilon_x\varepsilon_y = \lambda_1{}^2(\Phi_y f_c - \sigma_y)(\Phi_x f_c - \sigma_x)$

$-\lambda_1\lambda_2(\Phi_y f_c - \sigma_y)(f_c + \sigma_x)$

$-\lambda_1\lambda_2(\Phi_x f_c - \sigma_x)(f_c + \sigma_x)$

$+\lambda_2{}^2(f_c + \sigma_x)(f_c + \sigma_y)$

$$(\frac{1}{2}r_{xy})^2 = \lambda_1^{\,2}\tau_{xy}^{\;2} + 2\lambda_1\lambda_2\tau_{xy}^{\;2} + \lambda_2^{\,2}\tau_{xy}^{\;2}$$

$$(\frac{1}{2}r_{xy})^2 - \varepsilon_x\varepsilon_y = 2\lambda_1\lambda_2\tau_{xy}^{\;2} + \lambda_1\lambda_2[(\Phi_y f_c - \sigma_y)(f_c + \sigma_x) + (\Phi_x f_c - \sigma_x)(f_c + \sigma_y)] > 0$$

$$(\Phi_x f_c - \sigma_x) = \frac{1}{\Phi_y f_c - \sigma_y}(f_c + \sigma_x)(f_c + \sigma_y) \qquad (*)$$

$$(\Phi_y f_c - \sigma_y) = \frac{1}{\Phi_x f_c - \sigma_x}(f_c + \sigma_x)(f_c + \sigma_y) \qquad (*)$$

Substituting (*) into (?) yields

$$\tau_{xy}^{\;2} + (f_c + \sigma_x)(f_c + \sigma_y)[\frac{f_c + \sigma_x}{\Phi_x f_c - \sigma_x} + \frac{f_c + \sigma_y}{\Phi_y f_c - \sigma_y}]$$

Since we know

$$(\Phi_x f_c - \sigma_x)(\Phi_y f_c - \sigma_y) = \tau_{xy}^{\;2}$$

$$(f_c + \sigma_x)(f_c + \sigma_y) = \tau_{xy}^{\;2}$$

$$\tau_{xy}^{\;2}[1 + \frac{\Phi_y f_c - \sigma_y}{f_c + \sigma_y} + \frac{f_c + \sigma_x}{\Phi_y f_c - \sigma_y}] > 0$$

4.4.2 Dissipation formulas

$$W = \sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}r_{xy}$$

In region of (2.8.2), (2.2.41) $\qquad -(f_{tx} - \sigma_x)(f_{ty} - \sigma_y) + \tau_{xy}^{\;2} = 0$

$$W = \sigma_x\lambda(\Phi_y f_c - \sigma_y) + \sigma_y\lambda(\Phi_y f_c - \sigma_x) + 2\lambda\tau_{xy}^{\;2}$$

$$= \sigma_x\lambda(\Phi_y f_c - \sigma_y) + \sigma_y\lambda(\Phi_y f_c - \sigma_x) + 2\lambda(\Phi_x f_c - \sigma_x)(\Phi_x f_c - \sigma_x)$$

$$= \Phi_x f_c\lambda(\Phi_y f_c - \sigma_y) + \Phi_y f_c\lambda(\Phi_x f_c - \sigma_x)$$

$$= \Phi_x f_c\varepsilon_x + \Phi_y f_c\varepsilon_y \qquad (4.21)$$

In region of (2.83)(42) $\qquad -(f_c - \sigma_x)(f_c - \sigma_y) + \tau_{xy}^{\;2} = 0$

$$W = \sigma_x[-\lambda(f_c - \sigma_y)] + \sigma_y[-\lambda(f_c - \sigma_x)] + \tau_{xy}(2\lambda\tau_{xy})$$

$$= -\lambda\sigma_x(f_c + \sigma_y) - \lambda\sigma_y(f_c + \sigma_x) + 2\lambda(f_c + \sigma_x)(f_c + \sigma_y)$$

$$= \lambda(f_c + \sigma_y)(-\sigma_x + f_c + \sigma_x) + \lambda(f_c + \sigma_x)(-\sigma_y + f_c + \sigma_y)$$

$$= \lambda(f_c + \sigma_y)f_c + \lambda(f_c + \sigma_x)f_c$$

$$= f_c\left|\varepsilon_x\right| + f_c\left|\varepsilon_y\right| \qquad (4.22)$$

For an isotropic disk

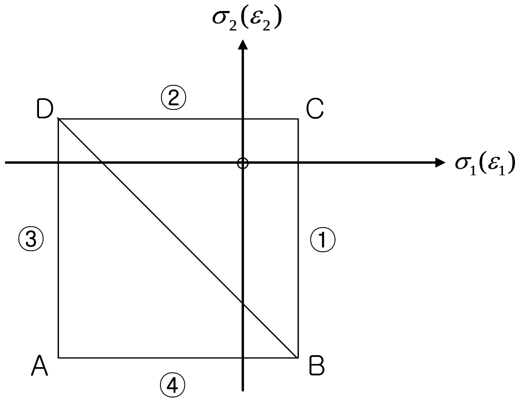
Most approximate yield condition

For $\sigma_x + \sigma_y \geq -(1-\Phi) f_c$

$$-(\Phi f_c - \sigma_x)(\Phi f_c - \sigma_y) + \tau_{xy}{}^2 = 0$$

For $\sigma_x + \sigma_y \geq -(1-\Phi) f_c$

$$-(f_c + \sigma_x)(f_c + \sigma_y) + \tau_{xy}{}^2 = 0$$



① $\sigma_1 - \Phi f_c = 0$

② $\sigma_2 - \Phi f_c = 0$

③ $-\sigma_1 - f_c = 0$

④ $-\sigma_2 - f_c = 0$

By normality rule

$$\varepsilon_i = \lambda \frac{\partial F}{\partial \sigma}$$

Dissipation work

$$W = \varepsilon_1 \sigma_1 + \varepsilon_2 \sigma_2$$

case	Edge of apex			W
1	1	λ	0	$\Phi f_c \varepsilon_1$
2	2	0	λ	$\Phi f_c \varepsilon_2$
3	3	$-\lambda$	0	$-f_c \varepsilon_1$
4	4	0	$-\lambda$	$-f_c \varepsilon_2$
5	1 & 2, C	λ_1	λ_2	$\sigma_1 - \Phi f_c = \sigma_1 - \Phi f_c = 0, \Phi f_c(\varepsilon_1 + \varepsilon_2)$
6	2 & 3, D	$-\lambda_1$	λ_2	$\sigma_2 - \Phi f_c = -\sigma_1 - f_c = 0, -f_c \varepsilon_1 + \Phi f_c$
7	3 & 4, A	$-\lambda_1$	$-\lambda_2$	$-\sigma_1 - f_c = -\sigma_2 - f_c = 0, -f_c(\varepsilon_1 + \varepsilon_2)$
8	4 & 1 B	λ_1	$-\lambda_2$	$-\sigma_2 - f_c = \sigma_1 - \Phi f_c = 0, \Phi f_c \varepsilon_1 + \Phi f_c$

In general form, we have

$$W = \frac{1}{2} f_c[(1+\Phi)(|\varepsilon_1| + |\varepsilon_2|) - (1-\Phi)(\varepsilon_1 + \varepsilon_2)]$$

Case 4

$$W = \frac{1}{2} f_c[(1+\Phi)(-\varepsilon_2) - (1-\Phi)(\varepsilon_2)]$$

$$= \frac{1}{2} f_c(-\varepsilon_2)(1+\Phi+1-\Phi)$$

$$= f_c(-\varepsilon_2)$$

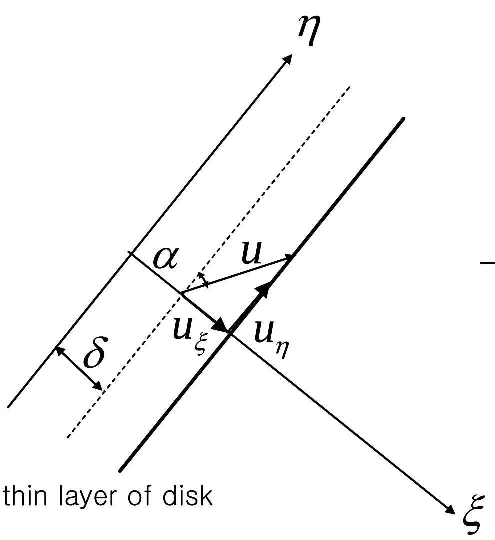
Case 5 ($\varepsilon_{1,2} > 0$)

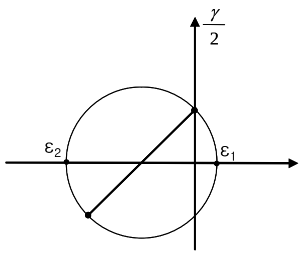
$$W = \frac{1}{2} f_c[(1+\Phi)(\varepsilon_1 + \varepsilon_2) - 1(1-\Phi)(\varepsilon_1 + \varepsilon_2)]$$

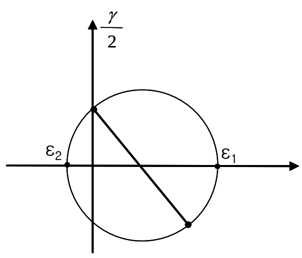
$$= \frac{1}{2} f_c(\varepsilon_1 + \varepsilon_2)2\Phi = \Phi f_c(\varepsilon_1 + \varepsilon_2)$$

Compare with (3.4.58)

$$W = \frac{1}{2} f_c[|\varepsilon_1| + |\varepsilon_2| - (\varepsilon_1 + \varepsilon_2)] \qquad \text{<-} \quad \Phi = 0$$







Special case of a yield line

$$\varepsilon_\xi = \frac{u_\xi}{\xi}$$

$$\varepsilon_\eta = \frac{u_\eta}{\infty} = 0$$

$$r_{\xi\eta} = \frac{u_\eta}{\xi}$$

$$\varepsilon_{1,2} = \frac{1}{2}(\varepsilon_\xi + \varepsilon_\eta) \pm \sqrt{\frac{(\varepsilon_\xi - \varepsilon_\eta)^2}{4} + (\frac{r_{\xi\eta}}{2})^2}$$

Since ε_1 and ε_2 have opposite signs

$$|\varepsilon_1| + |\varepsilon_2| = \sqrt{(\varepsilon_\xi - \varepsilon_\eta)^2 + (r_{\xi\eta})^2}$$

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_\xi + \varepsilon_\eta = \frac{u_\xi}{\delta}$$

$$W = \frac{1}{2} f_c[(1+\Phi)(|\varepsilon_1| + |\varepsilon_2|) - (1-\Phi)(\varepsilon_1 + \varepsilon_2)]$$

$$= \frac{1}{2} f_c[(1+\Phi)\sqrt{(\varepsilon_\xi - \varepsilon_\eta)^2 + (r_{\xi\eta})^2} - (1-\Phi)\frac{u_\xi}{\delta}]$$

$$W_l = W\delta t$$

$$= \frac{1}{2} f_c t[(1+\Phi)\sqrt{u_\xi^2 + u_\eta^2} - (1-\Phi)u_\eta]$$

Since $u_\xi = u\sin\alpha$ and $u_\eta = u\cos\alpha$

$$W_l = \frac{1}{2} f_c tu[(1+\Phi) - (1-\Phi)\sin\alpha]$$

If $\Phi = 0$

$$W_l = \frac{1}{2} f_c tu(1-\sin\alpha)$$

4.6 Lower Bound Solution

4.6.1 Statically Admissible stress Fields

Airy's stress function ψ -> scalar function.

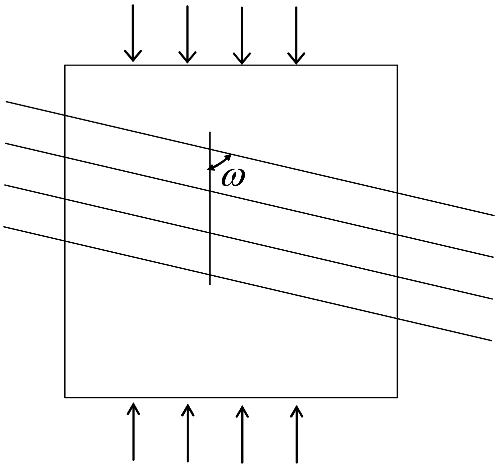
$$\sigma_x = \frac{\partial^2\psi}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2\psi}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2\psi}{\partial x \partial y}$$

4.6 Effective compressive strength of Reinforced Disk

3 main reasons for effective strength in concrete

1) Softening
2) Strength reduction due to internal cracking
3) Strength reduction due to sliding

4.6.1 Strength reduction due to internal cracking



For $w = 90^0$

- Unreinforced disk 0.82 f_c
- Zero stress in tr.reinf: 0.87
- f_y in tr reinf : 0.7 f_c
- Slender, eccentricity, curing
- Plain bar: less internal cracking, less strength reduction
- X parameter: bond, crack, anchorage failure
- Qualitive explanation

$$v = v_0 = 0.7 - \frac{f_c}{200}$$

Vecchio/Collins

$f_v \ (\sigma_x = \sigma_y = 0)$

$\sqrt{f_{tx} f_{ty}}$

$\frac{1}{2} v f_c$

$f_v = 0$

Anchorage failure

-highly orthotropic disks

Crack sliding

-high strength concrete

$$v = v_0 = 0.7 - \frac{f_c}{200} \leq 0.5$$

Takeda, Yamaguchi and Naganuma

$$v = v_0 = \frac{1.9}{f_c^{0.34}} \le 1$$

-X parameter

$$X = \frac{r\sigma_s}{1.41\sqrt{f_c}}$$

For shear stress+biaxial hydrostatic compressive

$$X = \frac{\tau - \sigma}{1.41\sqrt{f'_c}}$$

From Fig. 4.6.3

- overreinforced disks with τ and biaxial hydrostatic compression => higher than pure shear

σ_s decrease => amount of cracking decrease => increase of strength

Fig 4.6.4 =>

V=1.52-0.83X

Fig 4.6.5 =>

Including tension-compression 45′

Fig 4.6.6 =>

0 deg specimen less reduction

More parameters than those combined in the X-value

-cover

-reinforcement diameter

-distribution of reinf.

$$r = 1.52 - 0.83X$$

For pure shear

$$x = \frac{\tau - \sigma}{1.41\sqrt{f_c}}$$

$$\sigma = 0 \quad => \quad x = \frac{\tau}{1.41\sqrt{f_c}}$$

$$v = 1.52 - 0.83\frac{\tau}{1.41\sqrt{f_c}}$$

$$v = 1.52 - 0.83\frac{vf_c}{1.41\sqrt{f_c}2}$$

$$v(1 + 0.294\sqrt{f_c}) = 1.52$$

$$v = \frac{1.52}{1 + 0.294\sqrt{f_c}}$$

For biaxial+shear

$$X = \frac{\tau - \sigma}{1.41\sqrt{f_c}}$$

$$v = 1.52 - \frac{0.83(\tau - \sigma)}{1.41\sqrt{f_c}}$$

$$(v - 1.52)\frac{1.41\sqrt{f_c^{'}}}{0.83} = -(\tau - \sigma)$$

$$\tau = \sigma - 1.374(v - 1.52)\sqrt{f_c^{'}}$$

For low value of σ

$$f_{tx} = f_{ty} = f_t$$

$$\sigma_x = \sigma_y = -\sigma$$

$$-(f_{tx} - \sigma_x)(f_{ty} - \sigma_y) + \tau_{xy}^2 \le 0$$

$$\tau = f_r = f_t + \sigma$$

- the increase of shear capacity due to compression stresses is sometimes important in design
- Vecchio/Collins
- V is a junction of the average strain transverse to the compression direction

$$f_{2,\max} = \frac{f_c^{'}}{0.8 + 170\varepsilon_1} \le f_c^{'}$$

It is not possible to calculate the transverse strain

Only one reinforcement direction

$v = v_0 = 0.7 - \frac{f_c}{200}$, Normal strength

$v = v_0 = \frac{1.9}{f_c^{0.34}}$, High strength

4.6.2 Due to sliding in initial cracks

Test result: compressive failure for a value of the stress in the concrete much less than those using 4.6.1 or 4.6.6

Comlomb type

$$|\tau| = c^{'} + \mu^{'}\sigma$$

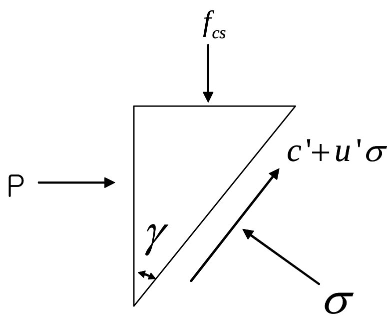
$c^{'}$ and $\mu^{'}$: those in the crack

- Less than or equal to values valid for uncracked concrete
- $\mu^{'} = \mu = 0.75 \quad (\varphi = 37^0)$

- $c^{'}$ strongly reduced

$c^{'} = v_s \frac{1}{4} v_0 f_c$

v_s : sliding reduction factor



$\gamma = \alpha - \beta$

$\frac{1}{4} v_0 f_c$: the cohesion for concrete with compressive strength $v_0 f_c$

$$f_{cs} = \frac{c^{'}}{\cos r \sin r - \mu^{'} \sin r}$$

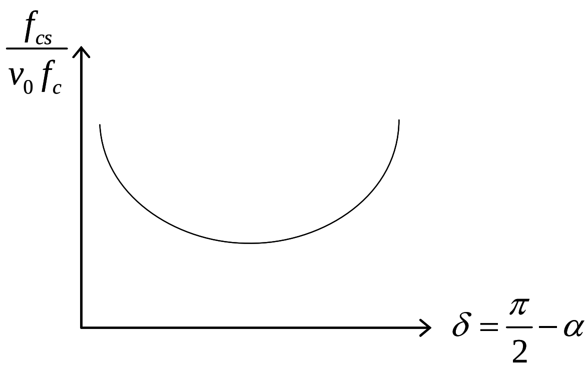
$$\frac{f_{cs}}{v_0 f_c} = \frac{\frac{1}{4} v_s}{\left|\sin r \cos r\right| - 0.75 \sin^2 r} \geq 1$$

The orthotropic disks by Vecchio/Collins

-pure shear

-initial crack angle $\beta = 45^0$

-α (2.4.1), (2.4.2), (2.4.3)



$v = v_0 = 0.7 - \frac{f_c}{200}$

$v_s = 0.5$

Change in crack direction => substantial reduction of the compressive strength

4.6.3 Implications of initial crack sliding on design

- Empirical formulas for the effectiveness factors have been developed on the basis of experiments

- Nonshear R/C beam
  V & a/d

- Crack sliding is unimportant at proportional loading if the reinforcement is designed for a stress field with compression directions not deviating too much from the initial ones

- Upper limit of σ_c
  $v_0 f_c$

- Safe side
  $v = v_s v_0$

- For two crack directions
  $r = \frac{1}{2}(\beta_2 - \beta_1)$

- Crack development
  1) Crack directions corresponding to stress trajectories of uncracked concrete
  2) Some regions with high stress
     → Subsequent cracking
  3) If the reinforcing bar starts yielding a third crack system may develop

- Reinforcement is able to carry a stress in any section which is larger than the effective tensile strength of concrete

- Effective tensile strength
  - stress concentration around rebar
  - shrinkage
  * in homogenous stress field : higher than the standard tensile strength development of a plastic tensile zone

- Necessary reinforcement ratio r
  $rf_y = f_{tef}$

$$f_{tef} = 0.5 f_t$$
$$= 0.5\sqrt{0.1 f_c}$$
$$= 0.16\sqrt{f_c}$$ (Mpa)
$$r f_y = 0.16\sqrt{f_c}$$
$$\phi_{\min} = \frac{0.16}{\sqrt{f_c}}$$

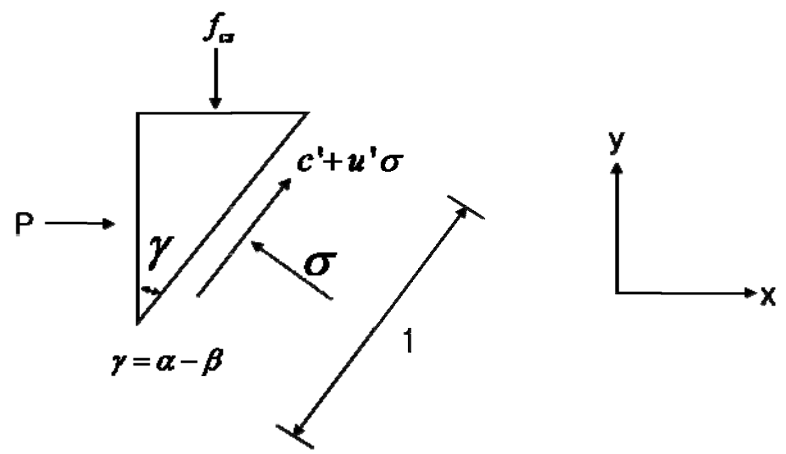
4.6.4 Plastic Solution Taking into Account Initial Crack Sliding

Lower bound – untouched subject
Upper bound – plane strain prob.

$$W_e = \frac{1}{2} f_c u b (l - m \sin\alpha)$$
$$f_t = 0 \ \& \ f_c = v_s v_0 f_c$$

Fig 3.4.12

Left→ non shear R/C Beam
Right→ disk



$$\sum F_x = 0$$
$$P\cos\gamma = -(c' + \mu'\sigma)\sin\gamma + \sigma\cos\gamma$$
$$\sum F_y = 0$$
$$f_{cs}\sin\gamma = (c' + \mu'\sigma)\cos\gamma + \sigma\sin\gamma$$

From Equation (a)

$$\sigma(\cos\gamma - \mu'\sin\gamma) = P\cos\gamma + c'\sin\gamma \to \text{(c)}$$
$$\sigma = \frac{P\cos\gamma + c'\sin\gamma}{\cos\gamma - \mu'\sin\gamma}$$

Sub (c) into (a)

$$f_{cs}\sin\gamma = c'\cos\gamma + \frac{P\cos\gamma + c'\sin\gamma}{\cos\gamma - \mu'\sin\gamma}(\sin\gamma + \mu'\cos\gamma)$$
$$= c'\cos\gamma + \frac{c'\sin\gamma}{\cos\gamma - \mu'\sin\gamma}(\sin\gamma + \mu'\cos\gamma) + \frac{P\cos\gamma(\sin\gamma + \mu'\cos\gamma)}{\cos\gamma - \mu'\sin\gamma}$$
$$= \frac{1}{\cos\gamma - \mu'\sin\gamma}\left[c'\cos^2\gamma - c'\mu'\sin\gamma\cos\gamma + c'\sin^2\gamma + c'\mu'\sin\gamma\cos\gamma + P\cos\gamma\sin\gamma + P\mu'\cos^2\gamma\right]$$

$$f_{cs}\sin\gamma = \frac{c' + P(\cos\gamma\sin\gamma + \mu\cos^2\gamma)}{\cos\gamma - \mu'\sin\gamma}$$
$$f_{cs}\sin\gamma = \frac{c' + P(|\sin\gamma\cos\gamma| + \mu\cos^2\gamma)}{|\cos\gamma\sin\gamma| - \mu'\sin^2\gamma}$$

Then we have

$$P = \frac{f_{cs}(|\cos\gamma\sin\gamma| - \mu'\sin^2\gamma) - c'}{|\sin\gamma\cos\gamma| + \mu\cos^2\gamma}$$

$$\gamma = \frac{\pi}{4} - \frac{\varphi}{2}$$ Dangerous crack direction

$$\varphi = 37°$$
$$\gamma = 27°$$
$$\mu = \mu' = 0.75$$
$$c' = v_s \bullet \frac{1}{4} v_o f_c$$
$$4P = f_{cs} - v_s v_o f_c$$

For prevention of additional reduction in compressive strength by sliding

$$f_{cs} = v_o f_c$$

$$P = \frac{1}{4} v_o f_c (1 - v_s)$$

$$v_s = 0.5 \rightarrow P = \frac{1}{8} v_o f_c$$

(d) Additional reinforcement perpendicular to the compression dir.

$$\gamma f_y = \frac{1}{8} v_o f_c$$

$$\gamma = \frac{1}{8} \frac{v_o f_c}{f_y}$$

$$\Phi = \frac{1}{8} v_o$$

Ex)

$$f_c = 20 Mpa$$

$$f_y = 500 Mpa$$

$$v_o = 0.6$$

$\gamma = 0.3\%$: is about twice $\gamma f_y = 0.16\sqrt{f_c}$

4.6 Lower Bound Solution

4.6.1 Strength reduction due to internal cracking

Airy's stress fn ψ

$$\sigma_x = \frac{\partial^2 \psi}{\partial y^2} \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

Equilibrium conditions

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho f_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho f_y = 0$$

Boundary conditions (=Equil.)

$$P_x = \sigma_x \cos\alpha + \tau_{xy} \sin\alpha$$
$$P_y = \sigma_x \sin\alpha + \tau_{xy} \cos\alpha$$

T_n x-dir.

$$P_x = \sigma_x \frac{dy}{ds} + \tau_{xy} \frac{dx}{ds}$$
$$\rightarrow \begin{bmatrix} P_x = \sigma_x \cos\alpha + \tau_{xy} \sin\alpha \\ P_y = \sigma_x \sin\alpha + \tau_{xy} \cos\alpha \end{bmatrix}$$

A stress field that can be derived from a certain stress fn. ψ and which satisfies the boundary condition. By chain rule for fn. F

$$dF = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$
$$d\left(\frac{d\psi}{dx}\right) = \frac{\partial}{\partial x}\left(\frac{d\psi}{dx}\right)dx + \frac{\partial}{\partial y}\left(\frac{d\psi}{dy}\right)dy$$

Integration of both sides

$$\frac{d\psi}{dx} = \int_{P_o}^{P}\left[\frac{\partial}{\partial x}\left(\frac{d\psi}{dx}\right)dx + \frac{\partial}{\partial y}\left(\frac{d\psi}{dy}\right)dy\right] = \int_{P_o}^{P}\left(\sigma_y dx - \tau_{xy} dy\right)$$

In the same way

$$\frac{d\psi}{dy} = \int_{P_o}^{P}\left(-\tau_{xy} dx + \sigma_x dy\right)$$

We integrate along the boundary from a fixed.

Arbitrarily chosen point P. when we have $\frac{d\psi}{dx} = \frac{d\psi}{dy} = 0$ to a arbitrary point P.

$$\frac{d\psi}{dx} = \int_{P_o}^{P}\left(\sigma_x \frac{dx}{ds} - \tau_{xy}\frac{dy}{ds}\right)ds = -\int_{P_o}^{P} P_y ds$$
$$\frac{d\psi}{dy} = \int_{P_o}^{P}\left(-\tau_{xy}\frac{dx}{ds} + \sigma_x \frac{dy}{ds}\right)ds = \int_{P_o}^{P} P_x ds$$

$$P_x = \int_{P_o}^{P} P_x ds = \frac{d\psi}{dy}$$

Let

$$P_y = \int_{P_o}^{P} P_y ds = -\frac{d\psi}{dx}$$

Finally

$$\psi = \int_{P_o}^{P} \left(\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \right) = \int_{P_o}^{P} \left(\frac{\partial \psi}{\partial x}\frac{dx}{ds} + \frac{\partial \psi}{\partial y}\frac{dy}{ds} \right) ds = \int_{P_o}^{P} \left(-P_y \frac{dx}{ds} + P_x \frac{dy}{ds} \right) ds$$

$$\boxed{\int u'v = uv - \int uv'}$$

$$\psi = -P_y x\Big|_{P_o}^{P} + \int x \frac{dP_y}{ds} ds + P_x y\Big|_{P_o}^{P} - \int y \frac{dP_x}{ds} ds$$

$$= -P_y(P)x_p + P_y(P_o)x_{p_o} + \int x \frac{dP_y}{ds} ds + P_x(P)x_p - P_x(P_o)x_{p_o} - \int y \frac{dP_x}{ds} ds$$

$$= \int_{P_o}^{P} -\frac{dP_y}{ds}(x_P - x) ds + \int_{P_o}^{P} \frac{dP_x}{ds}(y_P - y) ds$$

$$= \int_{P_o}^{P} \left[-P_y (x_P - x) + P_x (y_P - y) \right] ds$$

4.6.3 Implications of initial crack sliding on design

$$\sum x = \sigma_x \Delta y = \frac{\psi_3 - 2\psi_o + \psi_7}{\Delta y}$$

$$\sum y = \sigma_y \Delta x = \frac{\psi_1 - 2\psi_o + \psi_5}{\Delta x}$$

For a element

$$\sum xy = \tau_{xy}\Delta x = \frac{\psi_5 + \psi_7 - (\psi_o + \psi_6)}{\Delta x}$$

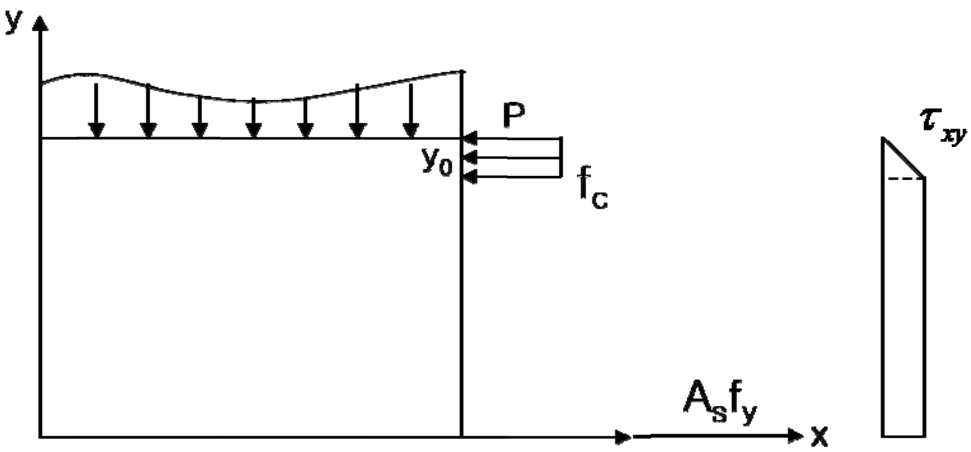
Panel: pure shear

Stringer: axial force

We have thus obtained a system of force which is in equilibrium with the external load and consists of concentrated forces in the net lines and constant shear forces in every mesh rectangles.

4.7.4 Shear Zone Solutions

-Distributed load on the top force



$$y_o = \Phi d$$

$$M_p = (1 - \frac{1}{2}\Phi)\Phi t d^2 f_c = (1 - \frac{\Phi}{2}) A_s f_y d$$

Let $\mu = \dfrac{M_P}{td^2 f_c}$

$$\Phi = 1 - \sqrt{1 - 2\mu}$$

Lower bound solution

$$\sigma_x = -\frac{M}{M_{\max}} f_c \qquad d - y_0 \le y \le d$$

$$0 \qquad 0 < y \le d - y_0$$

$$T = \frac{M}{M_{\max}} A_s f_y = \frac{M}{h_i}$$

When $h_i = d - \dfrac{y_0}{2}$

Equilibrium condition

$$\frac{\partial \tau_{xy}}{\partial y} = -\frac{\partial \sigma_x}{\partial x} = \begin{cases} \dfrac{V}{ty_0 h_i} \\ 0 \end{cases}$$

Boundary condition

$$\tau_{xy} = \begin{cases} -\dfrac{V}{ty_0 h_i}(d - y) & \text{for} \quad d - y_0 \le y \le d \\ -\dfrac{V}{th_i} = -\tau & \text{for} \quad 0 \le y \le d - y_0 \end{cases}$$

Another Equilibrium condition

$$\frac{\partial \sigma_y}{\partial y} = \frac{\partial \tau_{xy}}{\partial x} = \begin{cases} \dfrac{dV}{dx} \dfrac{d - y}{ty_0 h_i} & \text{for} \quad d - y_0 \le y \le d \\ \dfrac{dV}{dx} \dfrac{1}{th_i} & \text{for} \quad 0 \le y \le d - y_0 \end{cases}$$

we know

$$\frac{dV}{dx} = -Pt$$

Boundary condition

$$\sigma_y = \begin{cases} -P & : y = d \\ 0 & : y = 0 \end{cases}$$

$$\sigma_y = \begin{cases} \dfrac{-P}{y_0 h_i}(dy - \dfrac{1}{2} y^2 - \dfrac{1}{2} d^2) - P \\ -\dfrac{P}{h_i} y \end{cases}$$

$$M_p = \frac{1}{2}\frac{\Phi}{1+\Phi}th^2 f_c$$

$$\frac{PL^2}{8} = M_p$$

$$M_p = \Phi f_c th^2$$

$$\Phi f_c = \frac{M_p}{th^2}$$

$$\frac{PL^2}{8} = M_p = \frac{1}{2}\frac{\Phi}{1+\Phi}th^2 f_c$$

$$P = \frac{4\Phi}{(1+\Phi)L^2}h^2 f_c$$

$$\frac{M_p}{\frac{1}{2}th^2 f_c} = \frac{\Phi}{1-\Phi}$$

$$\mu = \frac{\Phi}{1-\Phi}$$