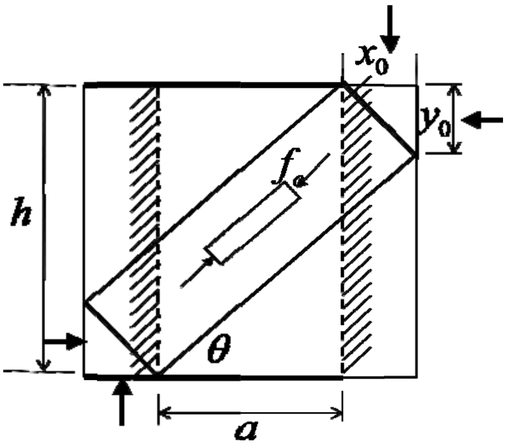
4.8 Strut and Tie models

4.8.1 Introduction

-Mörsch. Johansen
-Schlaich in Stuttgart

4.8.2 The single strut



Plane hydrostatic pressure

$$\tan\theta = \frac{x_0}{y_0} = \frac{h-y_0}{a+x_0}$$

$$\begin{cases} P = x_0 t f_c \\ C = y_0 t f_c \end{cases}$$

$$\frac{x_0}{h} = \frac{P}{thf_c} = \frac{\tau}{f_c} \quad \text{(shear strength)}$$

From

$$\frac{y_0}{h} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{x_0}{h}\left(\frac{a}{h} + \frac{x_0}{h}\right)}$$

Maximum value of $\frac{y_0}{h}$: $\frac{1}{2}$

$$\frac{1}{4} - \frac{x_0}{h}\left(\frac{a}{h} + \frac{x_0}{h}\right) = 0 \Rightarrow \frac{x_0}{h} = \frac{1}{2}\left[\sqrt{1+\left(\frac{a}{h}\right)^2} - \frac{a}{h}\right]$$

So the maximum value of the average shear stress

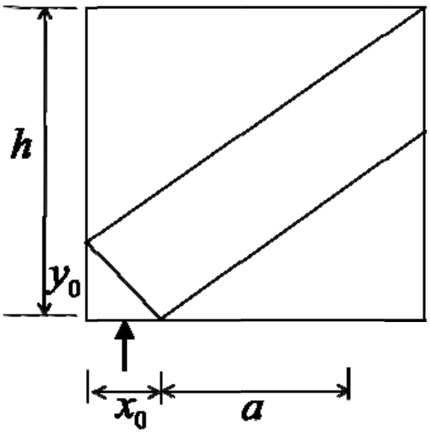
$$\frac{\tau}{f_c} = \frac{1}{2}\left[\sqrt{1+\left(\frac{a}{h}\right)^2} - \frac{a}{h}\right]$$

$$\frac{y_0}{h} = \frac{1}{2} \Rightarrow y_0 = \frac{h}{2}$$

If y_0 is given

$$\frac{x_0}{h} = \frac{1}{2}\left[\sqrt{4\frac{y_0}{h}(1-\frac{y_0}{h}) + \left(\frac{a}{h}\right)^2} - \frac{a}{h}\right]$$

Strut action in a corbel



$$A_s f_y = y_0 t f_c$$

$$\Phi = \frac{A_s f_y}{thf_c} = \frac{y_0}{h} \le \frac{1}{2} \Rightarrow (4.8.8)$$

$$\frac{\tau}{f_c} = \frac{P}{thf_c} = \frac{x_0}{h} = \frac{1}{2}\left[\sqrt{4\Phi(1-\Phi) + \left(\frac{a}{h}\right)^2} - \frac{a}{h}\right]$$

4.8.4 Effectiveness factor

Roikjaer

$$\nu = f_1(f_c)\, f_2(h)\, f_3(\gamma)\, f_4(\frac{a}{h})$$

f_1: ν decreases when f_c increases

f_2 : size effect

$f_3 : 0.15\gamma + 0.58$

$f_4 : \frac{a}{h}$

$f_2, f_3 \rightarrow softening$

Chen

$$\nu = \frac{0.6(2-0.4\frac{a}{h})(\gamma+2)(1-0.25h)}{\sqrt{f_c}}$$

$\frac{a}{h} < 2.5$

$\gamma < 2\%$

$h < 1.0m$

Strut sol. for $\frac{a}{h} < 2$

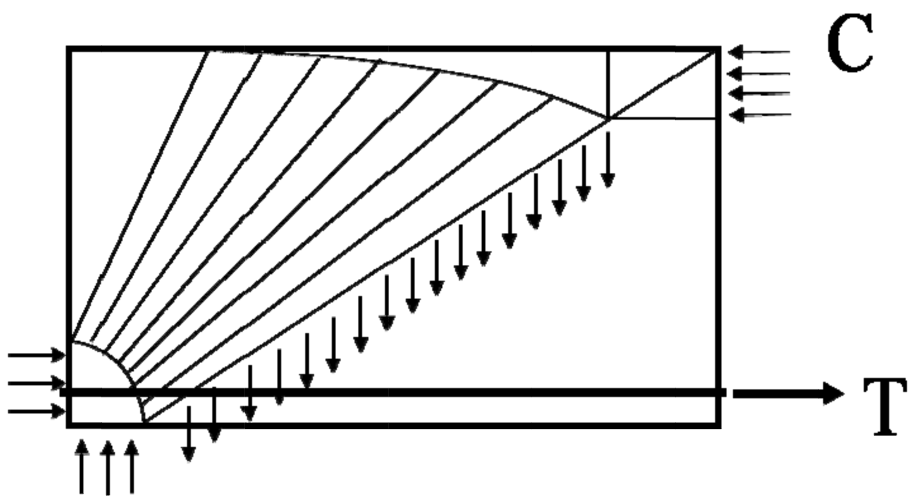
$\nu = \nu_0$ for $\frac{a}{h}$ <0.75

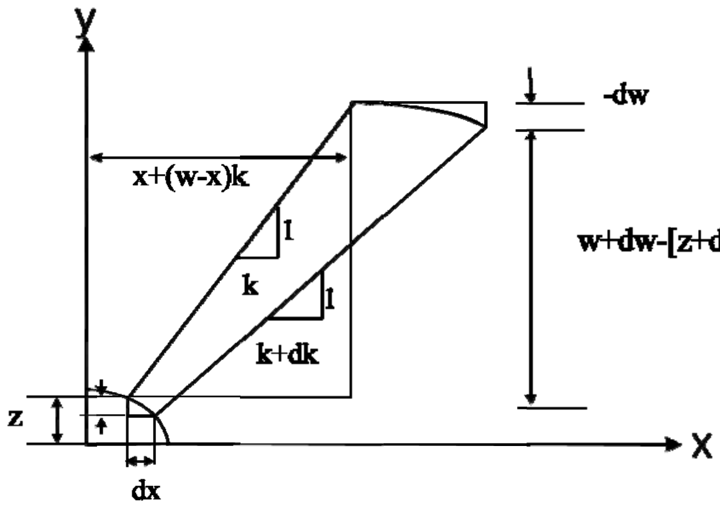
$\nu = \nu_s \nu_0$ for $0.75 < \frac{a}{h} < 2.0$

The low values of ν for $\frac{a}{h}$ around 2.5 due to sliding in initial cracks

CAB more dangerous than yield line CB.

Fan-shaped Stress Fields





x+dx+(w+dw-z-dz)(k+dk)-[x+(w-z)k]

=x+dx+(w-z)k+(dw-dz)k+(w-z)dk+(dw-dz)dk-x-(w-z)k

=dx+(dw-dz)k+(w-z)dk

d¡à=dx+(dw-dz)k+(w-z)dk

$$\text{d□}\bullet\rho\bullet k = dw\bullet s$$

$$-\frac{1}{k} = \frac{\rho}{s}\bullet\frac{\text{d□}}{dw} \rightarrow (d)$$

$$\sum F_X = 0$$

$$\gamma dz = sdw \rightarrow (a)$$

$$\sum F_y = 0$$

$$rdz\frac{1}{k} = qdx$$

$$-\frac{1}{k} = \frac{q}{r}\frac{dx}{dz} \rightarrow (c)$$

$$qdx = \rho[dx + (dw - dz)k + (w - z)dk) \rightarrow (b)$$

Let, $m = w - z$

$$\frac{dm}{dx} = \frac{dw}{dx} - \frac{dz}{dx} \rightarrow (e)$$

Eq(b) is rewritten as

$$\frac{dm}{dx} = \frac{dw}{dx} - \frac{dz}{dx} \rightarrow (e)$$

$$\frac{q}{p} = 1 + \frac{dm}{dx}k + m\frac{dk}{dx} \rightarrow (f)$$

$$(\frac{q}{p} - 1) = \frac{d}{dx}(mk)$$

Integration of the above

$$(\frac{q}{p}-1)x-C_1=mk\rightarrow(g)$$

Sub. Eq(c) into eq(g) yields

$$(\frac{q}{p}-1)x-C_1=m\left(\frac{r}{q}\frac{dz}{dx}\right)$$

$$\frac{q}{r}(1-\frac{q}{p})x+C_1'=m\left(\frac{dz}{dx}\right)\rightarrow(h)$$

From eq (a)

$$\frac{dw}{dx}=\frac{r}{s}\frac{dz}{dx}\rightarrow(a)'$$

Sub. eq $(a)'$ into eq(h) yields

$$\frac{dm}{dx}=\left(\frac{r}{s}-1\right)\frac{dz}{dx}\rightarrow(e)'$$

Sub. eq $(e)'$ into eq(h) yields

$$\frac{q}{r}(1-\frac{q}{p})x+C_1'=\frac{m}{\frac{r}{s}-1}\frac{dm}{dx}$$

$$\frac{q}{r}\left(\frac{r}{s}-1\right)\left(1-\frac{q}{p}\right)x+C_1''=m\frac{dm}{dx}$$

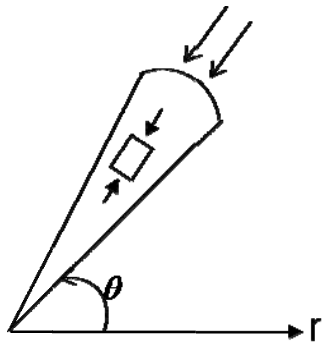
$$\frac{1}{2}\frac{q}{r}\left(\frac{r}{s}-1\right)\left(1-\frac{q}{p}\right)x^2+C_1''x+C_2=\frac{1}{2}m^2$$

$$m=\sqrt{\frac{q}{r}\left(\frac{r}{s}-1\right)\left(1-\frac{q}{p}\right)x^2+C_1''x+C_2}$$

Another Fan-shaped Stress fields

4.8.5 Fan-shaped compression stress

1) Concentric fan



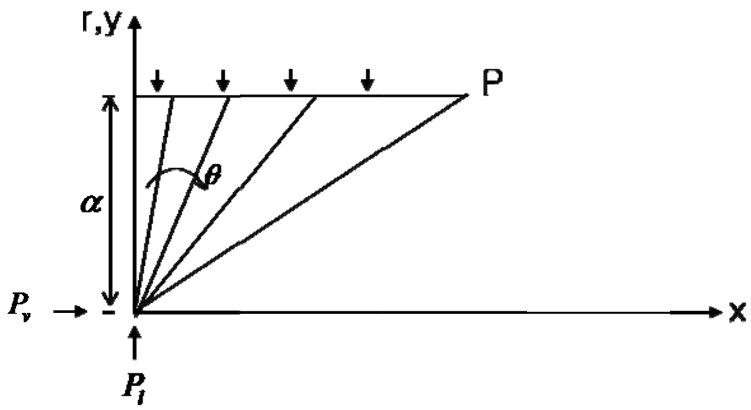
$$\sigma_\theta = \tau_{r\theta} = 0$$

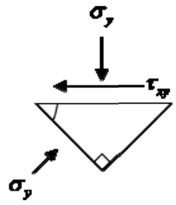
$$\sigma_r = \frac{c}{r} \rightarrow (a)$$ c: negative

Eq(a) written alongs…

$$\sigma_r = \frac{c}{a}\cos\theta$$

$$\sigma_r = \frac{Pa}{r}$$





$$\begin{cases} \sigma_r \sin\theta\cos\theta = \tau_{xy} \\ \sigma_r \cos^2\theta = \sigma_y \end{cases}$$

$$\sigma_r = \frac{\sigma_y}{\cos^2\theta} = \frac{P}{\cos^2\theta}$$

From eq(a)

$$Pa = c$$

Since $\sigma_y = P$ along $y = a$

$$\sigma_r = \frac{c}{r} \rightarrow (a)$$

$$\sigma_r = \frac{c}{a}\cos\theta$$

$$\sigma_r = \frac{\sigma_y}{\cos^2\theta} \rightarrow (b)$$

$$\frac{P}{\cos^2\theta} = \frac{c}{r}$$

$$c = \frac{\mathrm{Pr}}{\cos^2\theta}$$

Hence

$$\sigma_r = \frac{c}{r} = -\frac{\mathrm{Pr}}{\cos^2\theta}\frac{1}{r} = -\frac{\mathrm{Pr}}{\cos^2\theta}\frac{\cos\theta}{a} = -\frac{\mathrm{P}a}{\cos\theta}\frac{r}{a^2} = -\frac{\mathrm{P}a}{\cos^3\theta}\frac{1}{r}$$

Along any other horizontal line

y=b

$$\sigma_y = \sigma_r\cos^2\theta = -\frac{\mathrm{P}a}{\cos^3\theta}\frac{\cos\theta}{b}\cos^2\theta = -\frac{Pa}{b}$$

Since

$$\tau_{xy} = \sigma_r\cos\theta\sin\theta$$

$$\tau_{xy} = \sigma_y\tan\theta$$

Along y=a

$$\tau_{xy} = -P\tan\theta$$

Along y=b

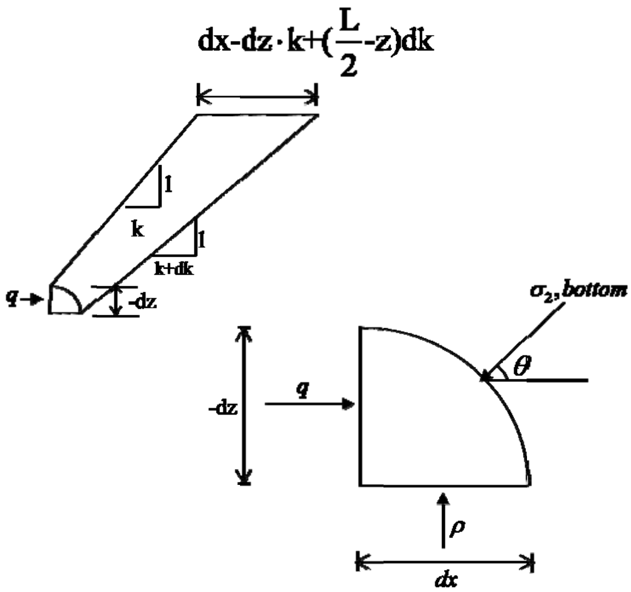
$$\tau_{xy} = -\frac{Pa}{b}\tan\theta$$

Resultants: vertical

$$P_l = Pa\tan\theta : horizental$$

$$P_v = \frac{1}{2}Pa\tan^2\theta$$

Infinitesimal element



Vertical Equilibrium

$$\rho dx = \frac{V_Y}{2k}\left[dx - dzk + \left(\frac{L}{2} - z\right)dk\right]$$

$$k\frac{dz}{dx} + \frac{2P}{V_Y}k - 1 = \left(\frac{L}{2} - z\right)\frac{dk}{dx} \rightarrow (a)$$

$$\frac{dx}{dz} = -\frac{q}{\rho}\bullet\frac{1}{k} \rightarrow (b)$$

Sub. eq(b) into eq(a) yields

$$\frac{P}{q}k^2 - \frac{2P}{V_Y}k + 1 = \left(\frac{L}{2} - z\right)\frac{dk}{dx}$$

Yield condition along bar : bond failure

$$\left(\begin{array}{l} \tau_t = \frac{V_Y}{2} \\ p \,\&\, q \rightarrow f_c \end{array}\right.$$

Let,

$$\zeta = \frac{z}{L/2}$$

$$d\zeta = \frac{dz}{L/2}$$

$$k^2 - \frac{2f_c}{V_Y}k + 1 = -\frac{L}{2}(1 - \frac{z}{L/2})\frac{dk}{dx}$$

$$\frac{dk}{k^2 - \frac{2f_c}{V_Y}k + 1} = \frac{dx}{-\frac{L}{2}(1 - \frac{z}{L/2})} = \frac{-\frac{1}{k}d\zeta}{-(1-\zeta)}$$

$$\boxed{\frac{kdk}{-k^2 + 2\alpha k - 1} = -\frac{d\zeta}{(1-\zeta)}}$$

$$l_n\left(k_{\max} - k\right)\left(k - k_{\min}\right) + \frac{\alpha}{\sqrt{\alpha^2 - 1}} l_n \frac{k_{\max} - k}{k - k_{\min}} + l_n(1 - \zeta^2) = const.$$

Where, $\begin{cases} k_{\max} = \alpha + \sqrt{\alpha^2 - 1} \\ k_{\min} = \alpha - \sqrt{\alpha^2 - 1} \end{cases}$

Equilibrium

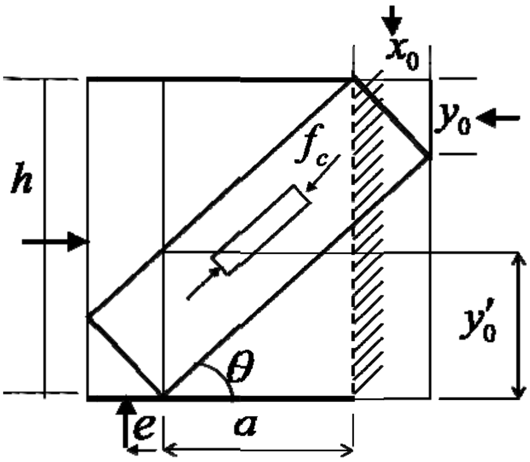
$$(dx + hdk)\sin\theta \cdot \sigma_{2.top} = dx\sin\theta \cdot \sigma_{2.bottom}$$

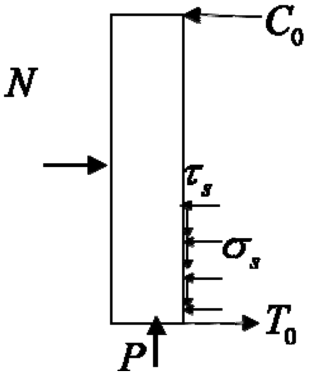
$$\sigma_{2.top} = \frac{dx \cdot \sigma_{2.bottom}}{dx + hdk} = \frac{\sigma_{2.bottom}}{1 + h\dfrac{dk}{dx}}$$

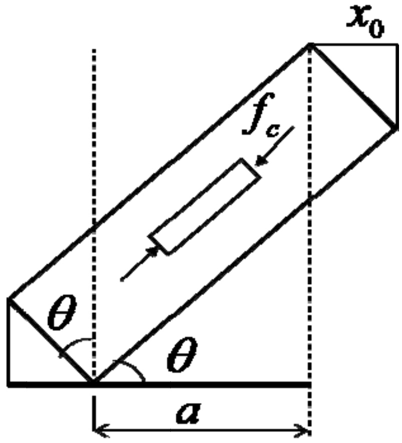
$$\boxed{\sigma_{2.top} = \frac{\sigma_{2.bottom}}{1 + \eta\dfrac{dk}{dx}}}$$

4.9 Shear walls

① Strut action (strut alone)







$$y_0' = \frac{y_0}{\cos^2\theta}$$

$$\tau_s = f_c \cos\theta \sin\theta$$

$$\sigma_s = f_c \cos^2\theta$$

Moment about A

$$T_0 h = \frac{h}{2} N = Pe + \sigma_s y_0' t(h - \frac{y_0'}{2})$$

$$T_0 = \frac{Pe}{h} + \frac{y_0 t}{h} f_c \cos^2\theta (y - \frac{y_0'}{2}) - \frac{1}{2} N$$

Projection

$$C_0 = T_0 - y_0' t f_c \cos^2\theta + N$$

For design

① Minimum reinforcement $\Phi_{\min} = \frac{0.16}{\sqrt{f_c}}$

② strut and diagonal compression field

From single strut model

$$y_0 = \frac{1}{2} h$$

$$\tan\theta = \frac{x_0}{y_0} = \frac{h - y_0}{a + x_0}$$

$$\begin{cases} \sigma_x = -\sigma_c \cos^2\theta = -\tau_t \cot\theta \\ \sigma_y = -\sigma_c \sin^2\theta = -\tau_t \tan\theta \\ \tau_t = \left|\tau_{xy}\right| = \sigma_c \cos\theta \sin\theta \end{cases}$$

$$P_{strut} = \frac{1}{2} t h f_c \left[\sqrt{a + (\frac{a}{h})^2} - \frac{a}{h} \right]$$

The shear stress τ_t which has to be carried by the triangular areas

$$\tau_t = \frac{P - P_s}{t(h - y_0')}$$

Vertical reinforcement ¡æ y-dir.

$$\sigma_y = -\sigma_c \sin^2\theta = -\tau_t \tan\theta$$

$$r_y f_{Y_y} = \tau_t \tan\theta$$

Concrete stress

$$\sigma_c = \frac{\tau_t}{\sin\theta\cos\theta}$$

The maximum force by setting

$$\sigma_c = f_c$$

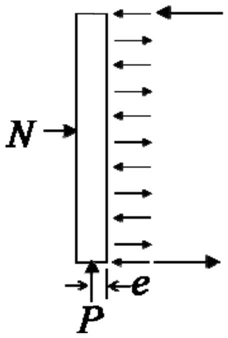
$$P = (h - y_0')tf_c \cos\theta\sin\theta + P_s = atf_c \sin^2\theta + P_s = at\tau_t \tan\theta + P_s$$

$$T_0 \,\&\, C_o$$

$$T_0 = \frac{Pe}{h} + \frac{y_0't}{h}(h - \frac{1}{2}y_0')f_c\cos^2\theta + \frac{1}{2}\frac{(h-y_0')^2 t}{h}\tau_t\cot\theta - \frac{1}{2}N$$

$$C_0 = T_0 - y_0'tf_c\cos^2\theta - (h - y_0')t\tau_t\cot\theta + N$$

In case of yielding of a uniform horizontal reinforcement



$$T_0' = T_0 - \frac{1}{2}r_x f_{Yx} ht$$

$$C_0' = C_0 + r_x f_{Yx} ht$$

Forces in flanges shear stress τ_t is constant

$$C_1 = C_0 + \tau_t at$$

$$T_1 = T_0 + \tau_t at$$

③ diagonal compression field P is maximized

- $\sigma_c = f_c$ with angle θ in the whole wall

$$P = P_{max} = \frac{1}{2}f_c ht$$

$$Eq(4.9.11) < P < \frac{1}{2} f_c ht$$

$$\tau = \frac{P}{ht}$$

$$\tau = f_c \cos\theta \sin\theta = \frac{1}{2} f_c \sin 2\theta$$

$$\sin 2\theta = \frac{2\tau}{f_c}$$

-Vertical reinforcement (꼭 필요)

$r_y f_{Yy} = \tau_t \tan\theta$

Unit load per unit area

-Horizontal reinf (N 으로 불충분)

(4.9.12)¡æ(4.9.14)

T_0 을 경감시킨다.

-Stringer forces

$C_1 = C_0 + \tau_t at$

If a prescribed mini reinforcement is utilized to carry the load

θ→iteration is needed.

Max capacity

$$P = P_{\max} = \frac{1}{2} f_c ht$$

Or

$$P = atf_c \sin^2\theta + P_s = at\tau_t \tan\theta + P_s$$

Between two P_s, determine θ from diagonal compress field.

$\tau = \dfrac{P}{ht}$, $\sigma_c = f_c$

$$\tau = f_c \cos\theta \sin\theta = \frac{1}{2} f_c \sin 2\theta$$

$$\sin 2\theta = \frac{2\tau}{f_c}$$

Necessary reinforcement

($r_y f_{Yy} = \tau_t \tan\theta$)

④ Max P

$$T_0 = \frac{Pe}{h} + \frac{1}{2} ht\tau \cot\theta - \frac{1}{2} r_x f_{Yx} ht - \frac{1}{2} N$$

$$C_0 = T_0 - ht\tau \cot\theta + r_x f_{Yx} ht + N$$

When, $N = 0 \quad and \quad e = 0$

$r_x f_{Yx} = \tau \cot\theta$

$$P = at\tau_t \tan\theta + P_s$$

$$\tau = \frac{P}{ht} \qquad \psi = r_y \frac{f_{Yy}}{f_c}$$

($r_y f_{Yy} = \tau_t \tan\theta$)→(4.9.20)

$$P = at\frac{r_y f_{Yy}}{\tan\theta}\cdot\tan\theta + P_s = \psi\frac{f_c}{r_y f_{Yy}} a\cdot t\cdot r_y f_{Yy} + P_s$$

$$\frac{P}{ht} = \frac{1}{f_c} = \frac{\tau}{f_c} = \frac{1}{2}\left[\sqrt{1+\left(\frac{a}{h}\right)^2} - \frac{a}{h}\right] + \psi\frac{a}{h}$$

Diagonal compression field

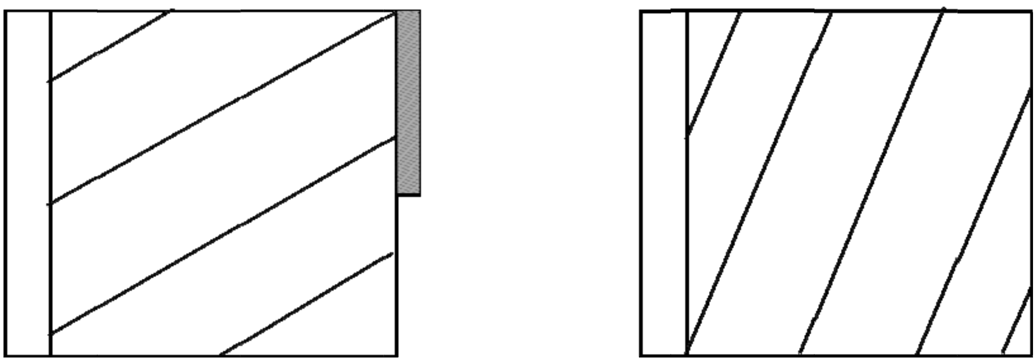
σ_c approaches to f_c

$r_y f_{Yy} = -f_c \sin^2\theta$ → (4.9.6)

$\sin\theta = \sqrt{\psi}$

$\tau = f_c\cos\theta\sin\theta = f_c\sqrt{1-\sin^2\theta}\sin\theta = f_c\sqrt{\psi(1-\psi)}$

4.9.3 Diagonal compression field solution



→ steeper : economical vertical reinforcement

Homogeneous stress economical vertical reinforcement Field solution
assuming horizontal web reinforcement ¡æ only one redundant normal stress in the web.

Min. reinforcement

$N \le \tau ht \quad \sigma_0 = \tau$

$\tau ht < N \le \sqrt{2}\tau ht \quad \sigma_0 = \frac{N}{ht}$

$\sqrt{2}\tau ht \le N \le 2\tau at + \sqrt{2}\tau ht \qquad \sigma_0 = \sqrt{2}\tau$

* Horizontal reinforcement

1) $N \le \tau ht$

$\sigma_0 = \tau$

$r_x f_Y = \tau - \dfrac{N}{ht}$

*Web reinf.

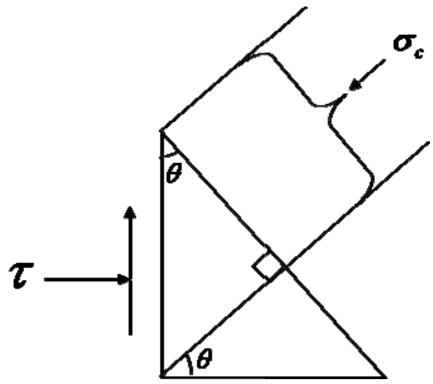
$r_y f_Y = \tau$

$\sigma_0 = 2\tau$

2) Other cases : only vertical reinf. is necessary

$\tan\theta = \dfrac{\tau}{\sigma_0}$

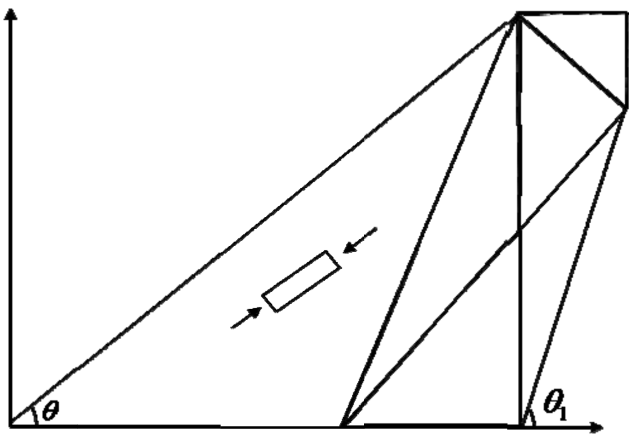
$r_y f_Y = \tau \tan\theta = \dfrac{\tau^2}{\sigma_0}$



$\sigma_c \cos\theta \sin\theta = \tau$

$\sigma_c = \tau \dfrac{1}{\cos\theta\sin\theta} = \sigma_0 \dfrac{1}{\cos^2\theta} = \sigma_0 \left(1 + \left(\dfrac{\tau}{\sigma_0}\right)^2\right)$

$1 + \tan^2\theta = \dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$



$x_0 t f_c = r_y f_{Ys} a' t$

$\tan\theta = \dfrac{h}{a'}$

$\dfrac{x_0}{h} = \psi \cot\theta$

Moment about middle point of a'

$$\frac{1}{2}\left(a'+x_0\right)x_0tf_c = y_0tf_c\left(h-\frac{1}{2}y_0\right)$$

$$\frac{y_0}{h} = 1-\sqrt{1-(1+\psi)\psi\cot^2\theta}$$

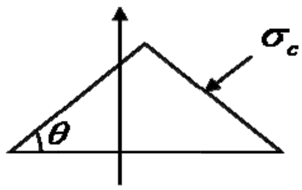
Now

$$\tan\theta_1 = \frac{h-y_0}{x_0}$$

$$r_y f_{Ys}\cot\theta(a'-x)+r_y f_{Ys}\cot\theta_1 x = y_0 f_c$$

$$\frac{x}{h} = \frac{\psi\cot^2\theta-\frac{y_0}{h}}{\psi(\cot\theta-\cot\theta_1)}$$

Region Ⅰ.



$$\frac{\sigma_c}{f_c} = \psi(1+\cot^2\theta)$$

$$\frac{\sigma_c}{f_c} = \psi\frac{1}{\sin^2\theta} = \psi\left(\frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta}\right)$$

Region Ⅱ

$$\frac{\sigma_c}{f_c} = \psi(1+\cot^2\theta_1)$$