DEFINITIONS

- **Intensity**
- **Radiosity and Irradiation**
- **Emissive Power**
- **Blackbody Radiation**

Intensity (spectral)

the amount of radiation energy
streaming out through a unit area
perpendicular to the direction of
propagation $\hat{\Omega}$,
per unit solid angle around the
direction ω,
per unit wavelength around λ,
and per unit time about t

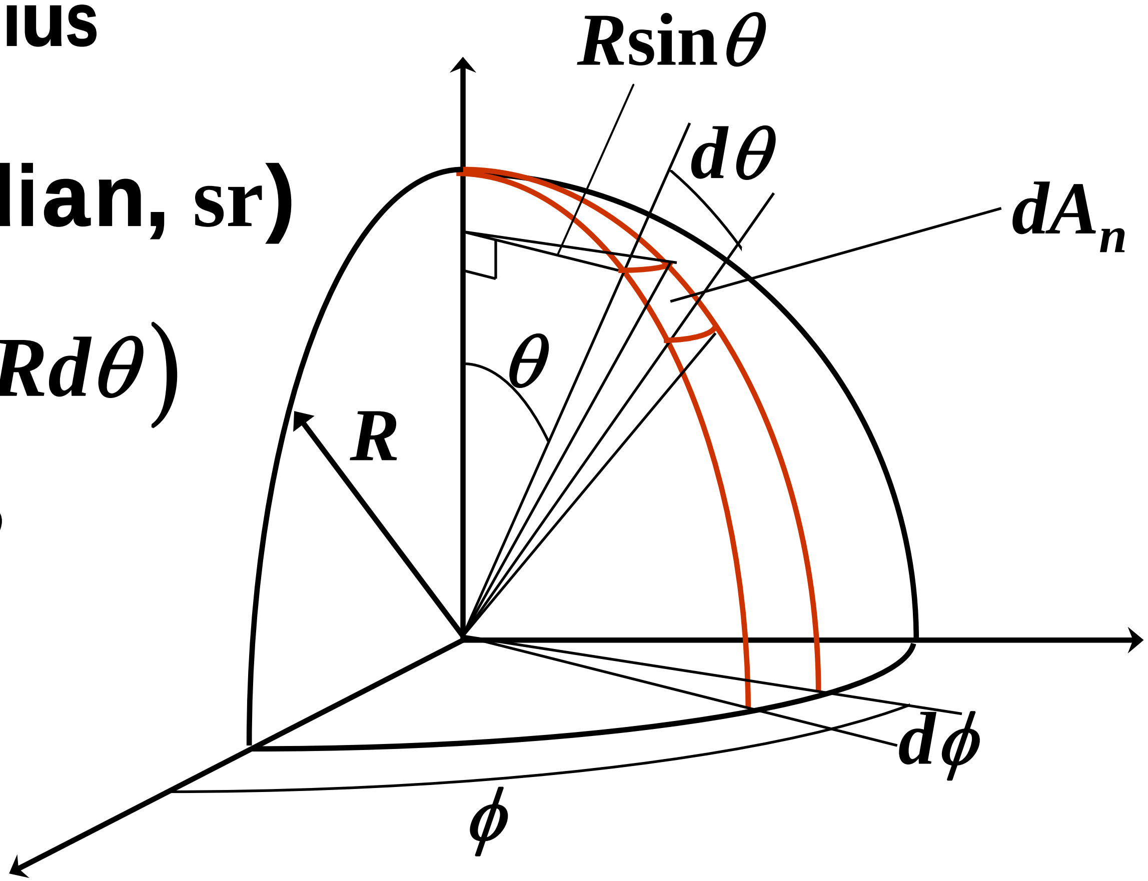
Solid angle: a region between the rays of a sphere and measured as the ratio of the element area dA_n on the sphere to the square of the sphere's radius

$$d\omega = \frac{dA_n}{R^2} \quad \text{(steradian, sr)}$$

$$dA_n = (R\sin\theta d\phi)(Rd\theta)$$

$$= R^2 \sin\theta d\theta d\phi$$

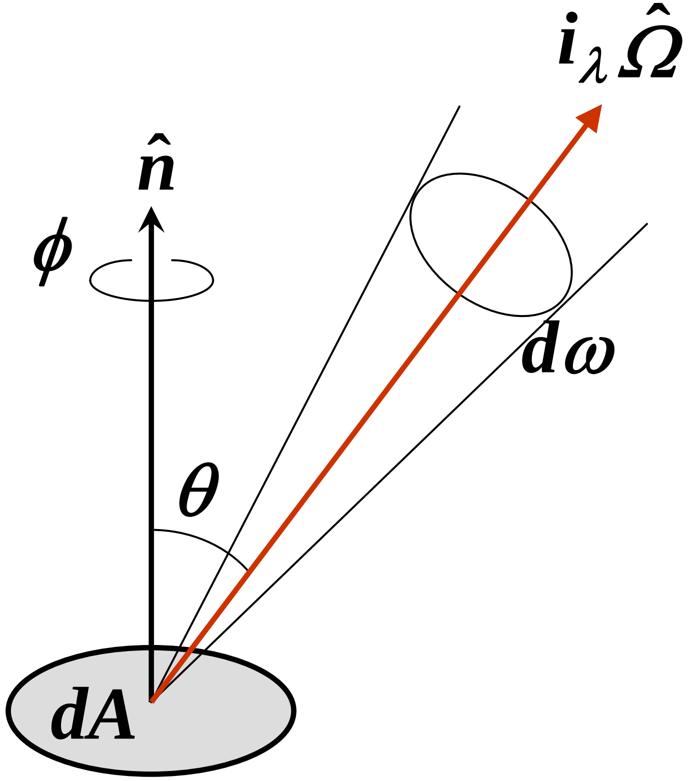
$$d\omega = \sin\theta d\theta d\phi$$

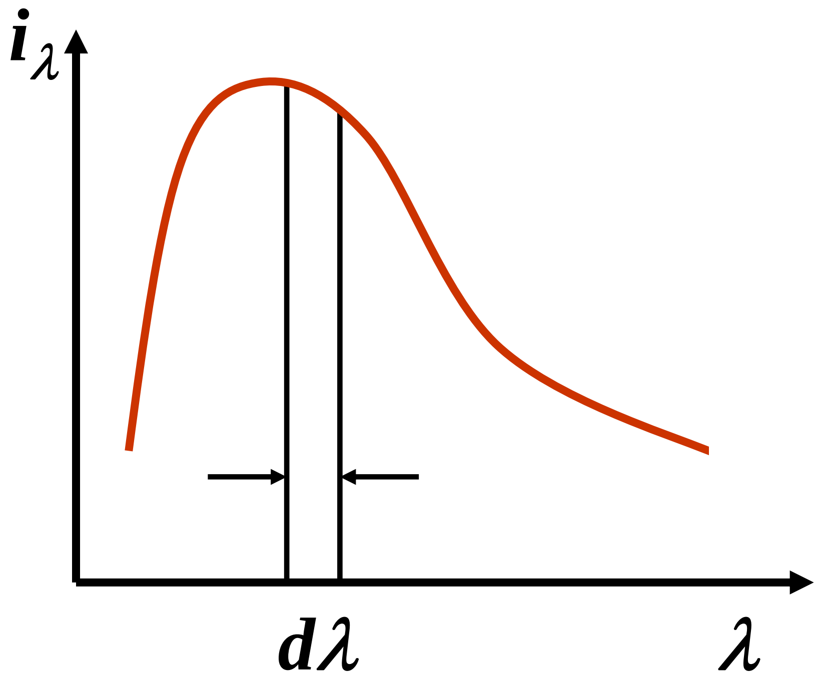


ex) hemisphere:

$$\omega = \int_{\cap} d\omega = \int_0^{2\pi}\int_0^{\pi/2} \sin\theta d\theta d\phi = 2\pi \text{ (sr)}$$

spectral intensity:





$$i_\lambda = \frac{d^4Q}{dA\cos\theta\, d\omega\, d\lambda\, dt}, \quad i_\lambda(\underline{r}, \hat{\Omega}) = i_\lambda(x, y, z, \theta, \phi)$$

total intensity: $i = \int_0^\infty i_\lambda d\lambda$

$$i_\lambda = \frac{d^4Q}{dA\cos\theta\, d\omega\, d\lambda\, dt} \quad [\mathrm{J/m^2 \cdot sr \cdot \mu m \cdot s}]$$

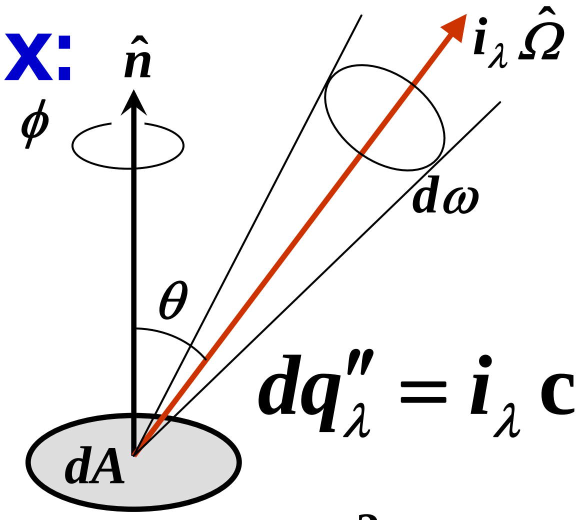
$$d^4Q = i_\lambda dA\cos\theta\, d\omega\, d\lambda\, dt \quad [\mathrm{J}]$$

$$d^3q = \frac{d^4Q}{dt} = i_\lambda dA\cos\theta\, d\omega\, d\lambda \quad [\mathrm{W}]$$

$$d^2q'' = \frac{d^4Q}{dA\,dt} = i_\lambda \cos\theta\, d\omega\, d\lambda \quad [\mathrm{W/m^2}]$$

$$dq''_\lambda = \frac{d^4Q}{dA\,dt\,d\lambda} = i_\lambda \cos\theta\, d\omega \quad [\mathrm{W/m^2 \cdot \mu m}]$$

spectral radiative heat flux:

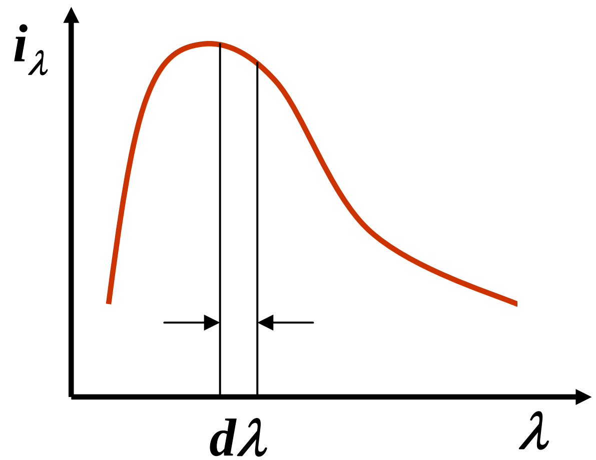


$$q''_\lambda = \int_\cap i_\lambda \cos\theta d\omega$$

$$= \int_\cap i_\lambda \hat{\Omega} \cdot \hat{n} d\omega$$

$$= \int_0^{2\pi} \int_0^{\pi/2} i_\lambda \cos\theta \sin\theta d\theta d\phi \quad [\mathrm{W/m^2 \cdot \mu m}]$$

total radiative heat flux:



$$q'' = \int_0^\infty q''_\lambda d\lambda$$

$$= \int_0^\infty \left(\int_\cap i_\lambda \cos\theta d\omega \right) d\lambda$$

$$= \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} i_\lambda \cos\theta \sin\theta d\theta d\phi d\lambda \quad [\mathrm{W/m^2}]$$

Radiosity and Irradiation

Radiosity: all out-going radiative heat flux

radiosity = emitted heat flux + reflected heat flux

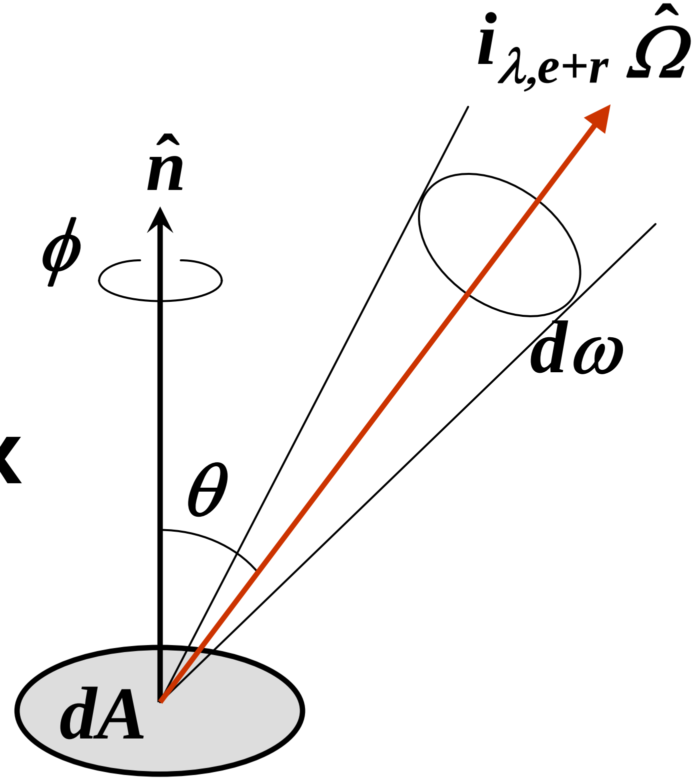
Spectral Quantities

spectral emitted heat flux

$$q''_{\lambda,e} = \int_{\cap} i_{\lambda,e} \cos\theta_e d\omega_e$$

spectral reflected heat flux

$$q''_{\lambda,r} = \int_{\cap} i_{\lambda,r} \cos\theta_r d\omega_r$$



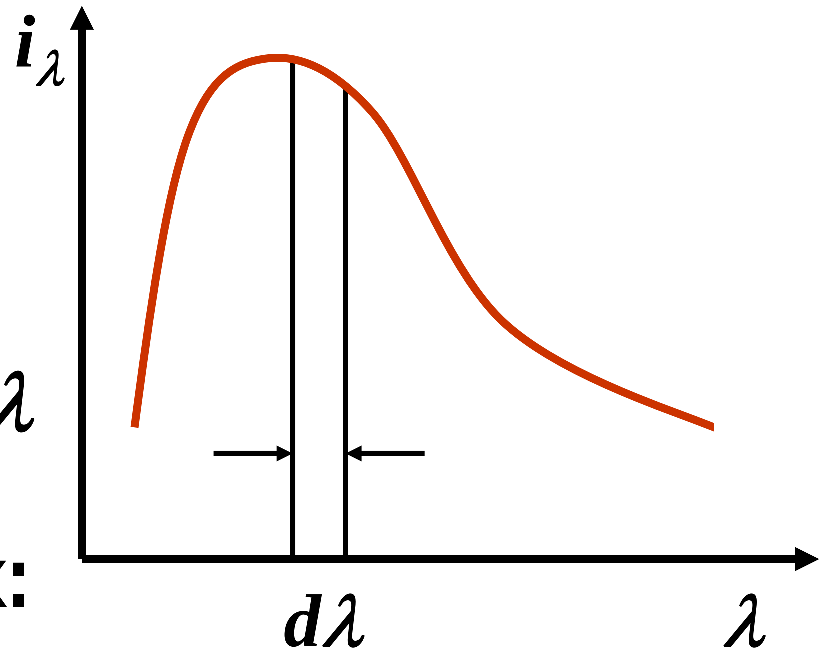
spectral radiosity:

$$J_\lambda = q''_{\lambda,e} + q''_{\lambda,r} = \int_{\cap} i_{\lambda,e} \cos\theta_e d\omega_e + \int_{\cap} i_{\lambda,r} \cos\theta_r d\omega_r$$

$$= \int_{\cap} i_{\lambda,e+r} \cos\theta d\omega$$

Total Quantities

total emitted heat flux



$$q''_e = \int_0^\infty q''_{\lambda,e} d\lambda$$

$$= \int_0^\infty \left(\int_\cap i_{\lambda,e} \cos\theta_e d\omega_e \right) d\lambda$$

total reflected heat flux:

$$q''_r = \int_0^\infty q''_{\lambda,r} d\lambda = \int_0^\infty \int_\cap \left(i_{\lambda,r} \cos\theta_r d\omega_r \right) d\lambda$$

total radiosity:

$$J = \int_0^\infty J_\lambda d\lambda = q''_e + q''_r$$

$$= \int_0^\infty \int_\cap i_{\lambda,e} \cos\theta_e d\omega_e d\lambda + \int_0^\infty \int_\cap i_{\lambda,r} \cos\theta_r d\omega_r d\lambda$$

Irradiation: all incident radiative heat flux through the control surface

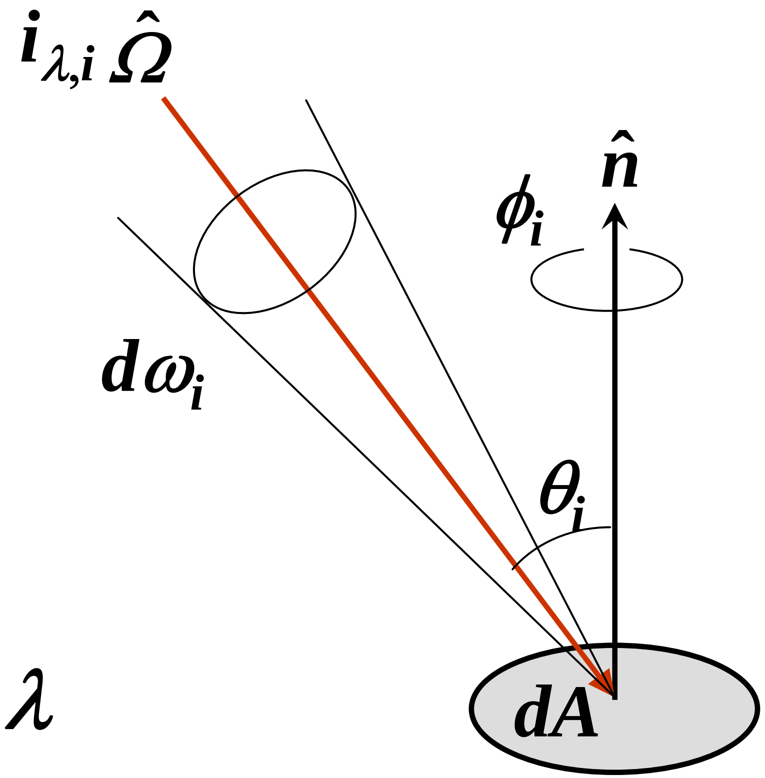
spectral irradiation:

$$G_\lambda = \int_{\cap} i_{\lambda,i} \cos\theta_i d\omega_i$$

total irradiation:

$$G = \int_0^\infty G_\lambda d\lambda$$

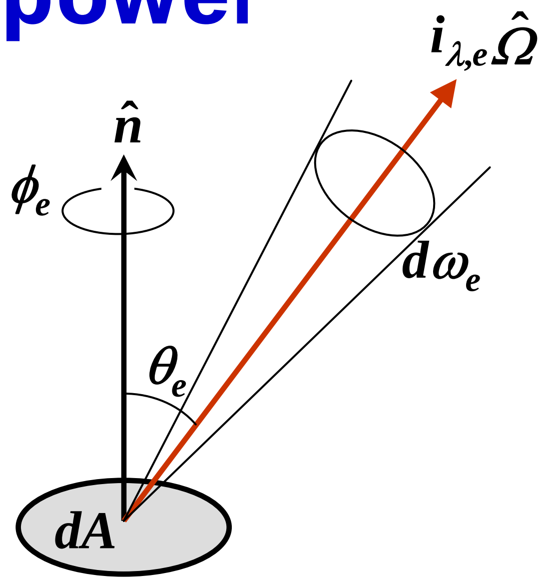
$$= \int_0^\infty \int_{\cap} i_{\lambda,i} \cos\theta_i d\omega_i d\lambda$$



Emissive Power

directional spectral emissive power

$$e'_\lambda = i_{\lambda,e}\cos\theta_e = i_{\lambda,e}\hat{\Omega}\cdot\hat{n}$$



hemispherical spectral emissive power

$$e_\lambda = q''_{\lambda,e} = \int_\cap i_{\lambda,e}(\underline{r},\hat{\Omega})\cos\theta_e d\omega_e$$

$$= \int_0^{2\pi}\int_0^{\pi/2} i_{\lambda,e}(\underline{r},\hat{\Omega})\cos\theta_e \sin\theta_e d\theta_e d\phi_e$$

hemispherical total emissive power

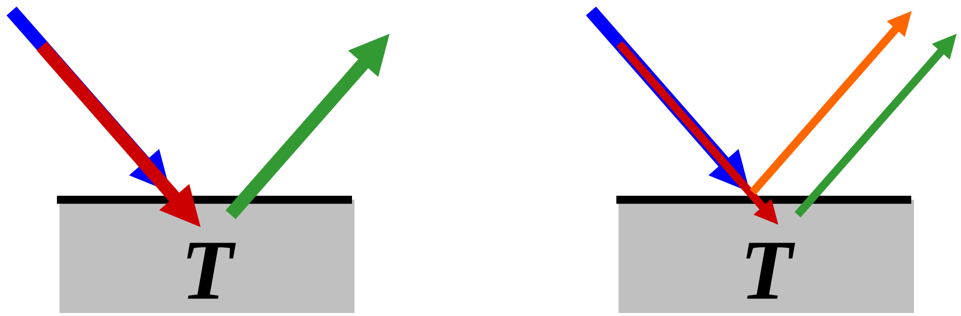
$$e = q''_e = \int_0^\infty e_\lambda d\lambda = \int_0^\infty\int_\cap i_{\lambda,e}(\underline{r},\hat{\Omega})\cos\theta_e d\omega_e d\lambda$$

Blackbody Radiation

a) Blackbody: a perfect absorber for all incident radiation

black: termed based on the visible radiation, so not a perfect description

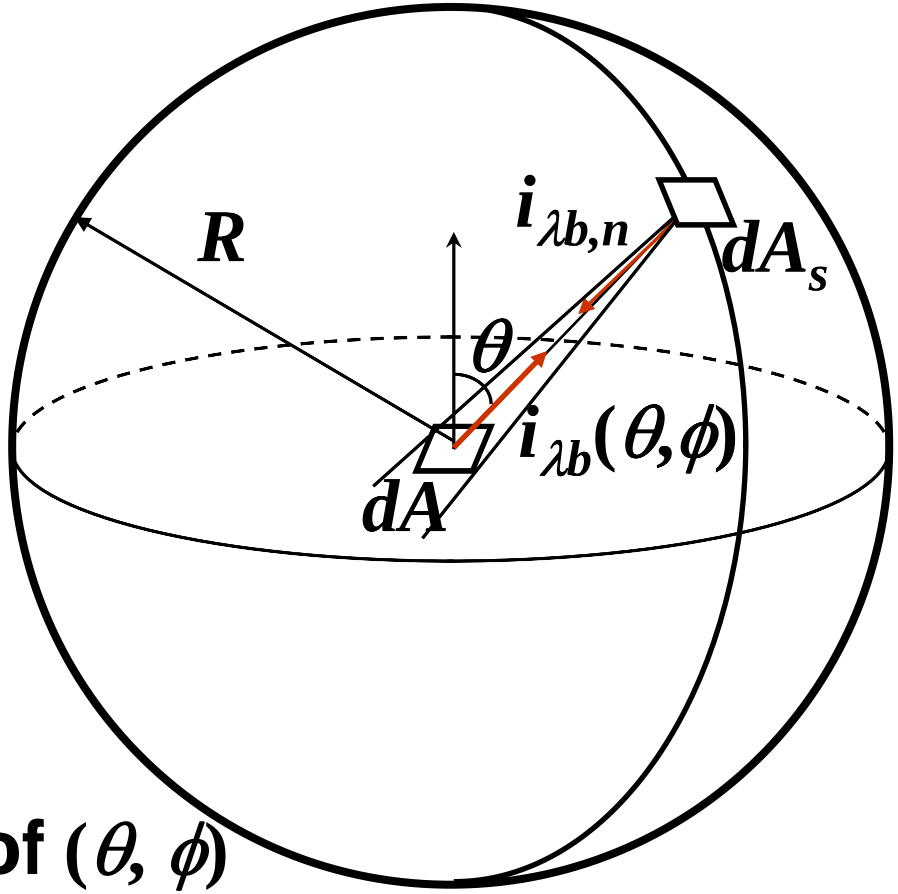
b) Maximum emitter in each direction and at every wavelength



black non-black

c) Emitted intensity from a blackbody is invariant with emission angle.

Proof) Consider energy exchange between an element on a spherical black enclosure, dA_s and a black element at the center of the enclosure, dA. Both elements are in thermal equilibrium.



energy absorbed by dA

$$i_{\lambda b,n} dA_s d\lambda d\omega_s$$

$$= i_{\lambda b,n} dA_s d\lambda \frac{dA\cos\theta}{R^2}$$

energy absorbed by dA_s

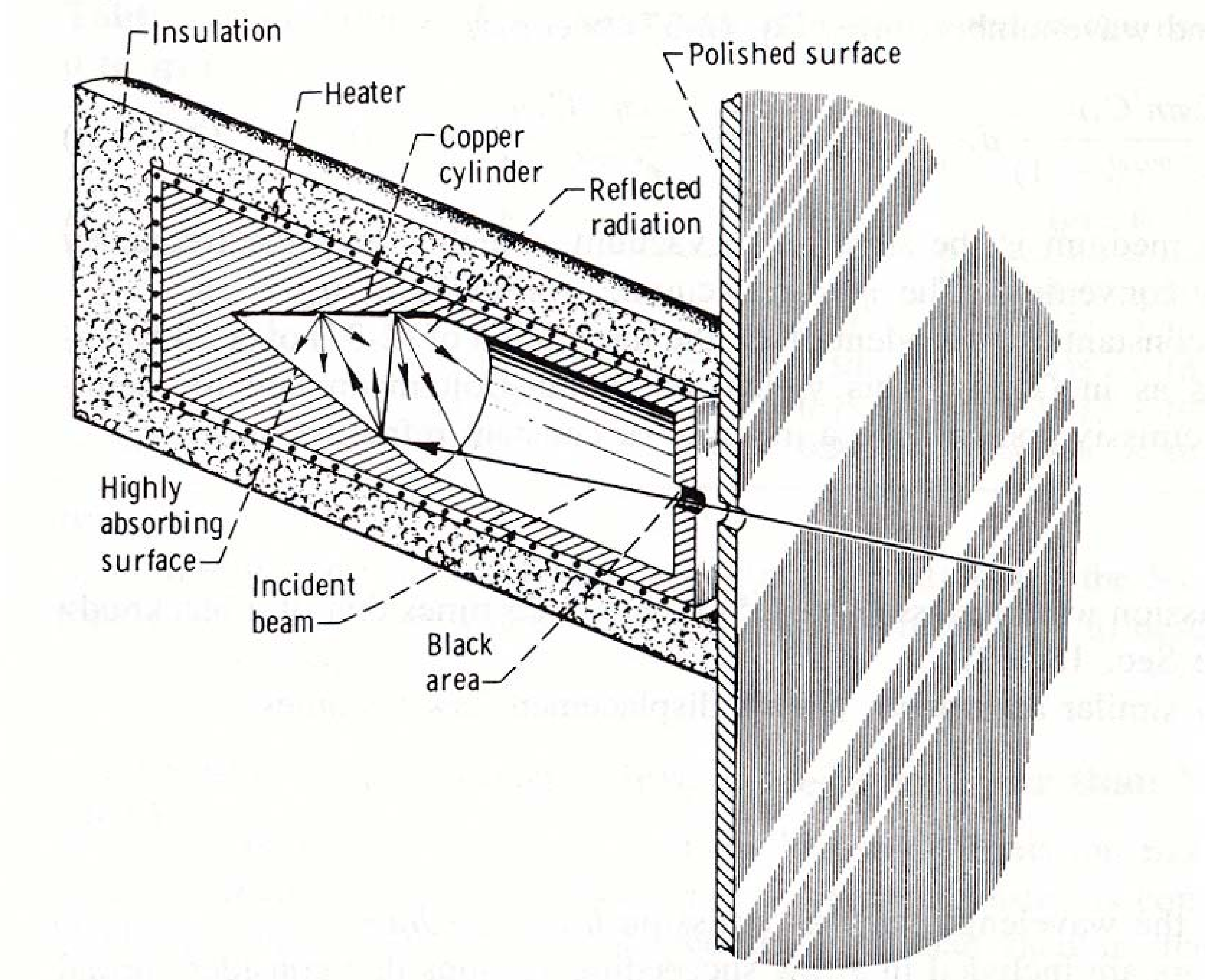
$$i_{\lambda b}(\theta,\phi) dA\cos\theta d\lambda d\omega$$

$$= i_{\lambda b}(\theta,\phi) dA\cos\theta d\lambda \frac{dA_s}{R^2}$$

in equilibrium

$i_{\lambda b}(\theta,\phi) = i_{\lambda b,n} \neq$ function of (θ, ϕ)

and since max at a given temperature



Simulated blackbody

Blackbody hemispherical spectral emissive power

$$e_{\lambda b} = q''_{\lambda b,e} = \int_{\cap} i_{\lambda b}(\underline{r})\cos\theta d\omega$$

$$= \int_0^{2\pi}\int_0^{\pi/2} i_{\lambda b}(\underline{r})\cos\theta \sin\theta d\theta d\phi$$

$$= i_{\lambda b}(\underline{r})\int_0^{2\pi}\int_0^{1}\cos\theta d(\cos\theta) d\phi$$

$$= \pi i_{\lambda b}(\underline{r})$$

Planck's law (The Theory of Heat Radiation, Max Planck, 1901):

spectral distribution of hemispherical emissive power of a blackbody in vacuum

$$e_{\lambda b} = \pi i_{\lambda b} = \frac{2\pi C_1}{\lambda^5\left(e^{C_2/\lambda T} - 1\right)}$$

$$C_1 = hC_0^2,\ C_2 = hC_0/k$$

C_0: speed of light in vacuum

h: Planck constant

k: Boltzmann constant

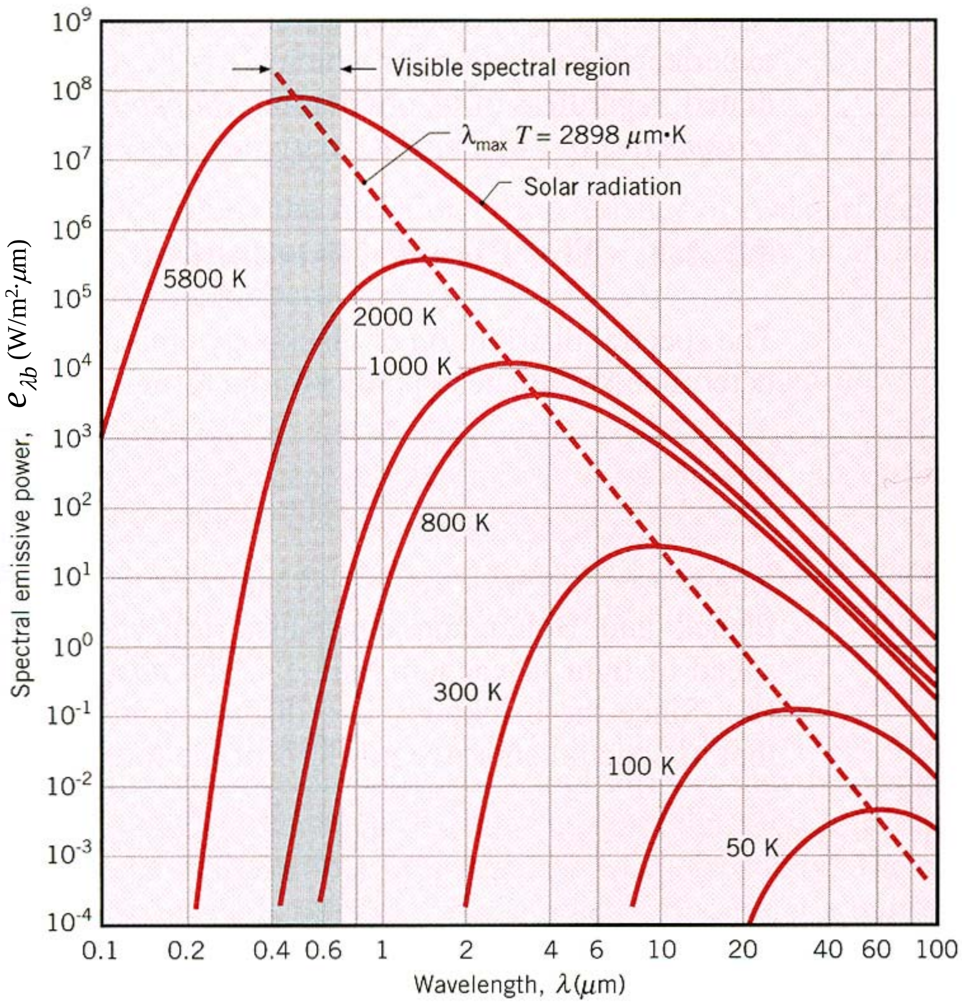
in a medium with a refractive index n:

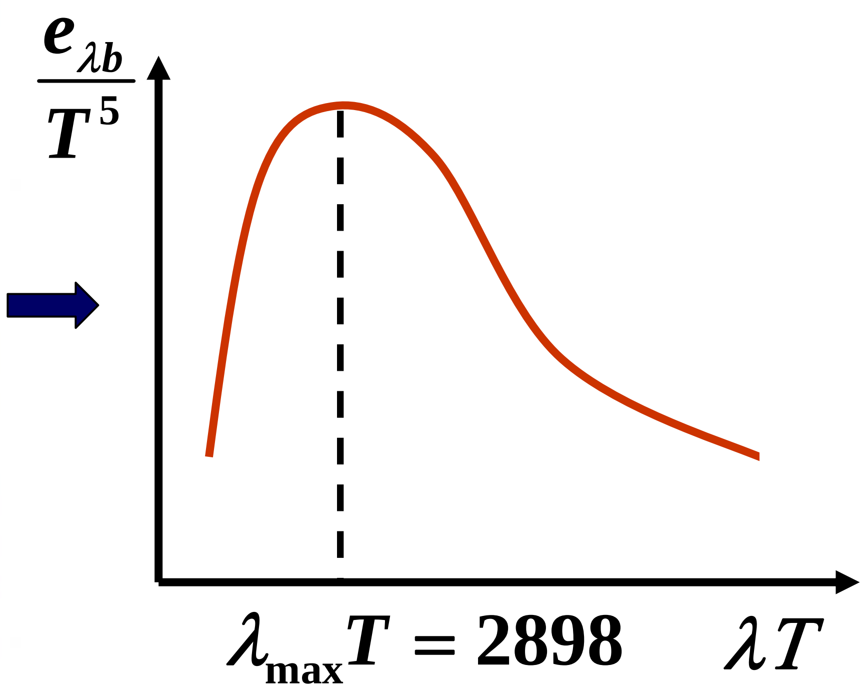
$$e_{\lambda b} = \pi i_{\lambda b} = \frac{2\pi C_1}{n^2 \lambda^5 \left(e^{C_2/\lambda n T} - 1\right)}$$

$n = 1$ in vacuum and $n = 1.00029$ in air at room temperature over the visible spectrum

$$e_{\lambda b} = \pi i_{\lambda b} = \frac{2\pi C_1}{\lambda^5 \left(e^{C_2/\lambda T} - 1\right)}$$

$$\frac{e_{\lambda b}(\lambda, T)}{T^5} = \frac{2\pi C_1}{(\lambda T)^5 \left(e^{C_2/\lambda T} - 1\right)} \equiv E(\lambda T)$$



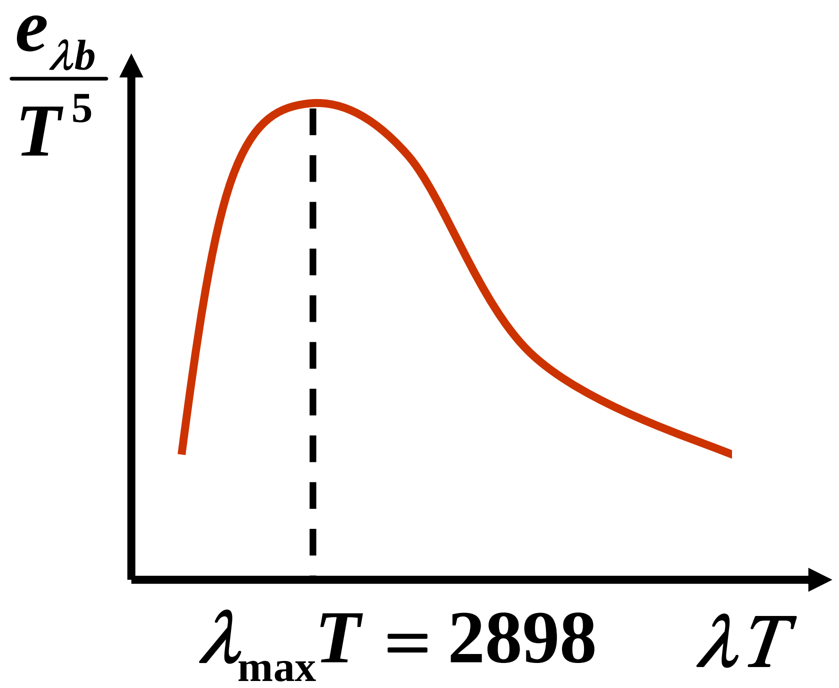


Blackbody spectral emissive power

See Table A-5 (pp. 971-978).

Wien's displacement law (1891)

$\lambda_{\max}$: the wavelength at which $e_{\lambda b}(\lambda, T)$ is maximum



$$\frac{d}{d(\lambda T)}\left(\frac{e_{\lambda b}}{T^5}\right) = 0$$

$$\rightarrow \lambda_{\max} T = \frac{C_2}{5} \frac{1}{1 - e^{-C_2/\lambda_{\max} T}}$$

$$\lambda_{\max} T = C_3 = 2897.8\ \mu\text{m} \cdot \text{K}$$

Blackbody total intensity and total emissive power

$$i_b = \int_0^\infty i_{\lambda b} d\lambda = \int_0^\infty \frac{2C_1}{\lambda^5\left(e^{C_2/\lambda T}-1\right)} d\lambda$$

$$= \frac{2C_1 T^4}{C_2^4}\int_0^\infty \frac{\zeta^3}{e^\zeta - 1} d\zeta = \frac{2C_1 T^4}{C_2^4}\frac{\pi^4}{15} = \frac{\sigma}{\pi} T^4$$

$$\sigma = \frac{2C_1\pi^5}{15C_2^4} = 5.6696\times 10^{-8}\ \text{W/m}^2\cdot\text{K}^4$$

***Stefan-Boltzmann's law*:**

$$e_b = q''_{b,e} = \int_0^\infty e_{\lambda b} d\lambda = \pi i_b = \sigma T^4\ [\text{W/m}^2]$$

Stefan by experiment (1879): $e_b \sim T^4$

Boltzmann by theory (1884): $e_b = \sigma T^4$

Blackbody radiation in a wavelength interval

$$F_{\lambda_1-\lambda_2} = \frac{\int_{\lambda_1}^{\lambda_2} e_{\lambda b}(\lambda,T)\,d\lambda}{\int_0^{\infty} e_{\lambda b}(\lambda,T)\,d\lambda} = \frac{1}{\sigma T^4}\int_{\lambda_1}^{\lambda_2} e_{\lambda b}(\lambda,T)\,d\lambda$$

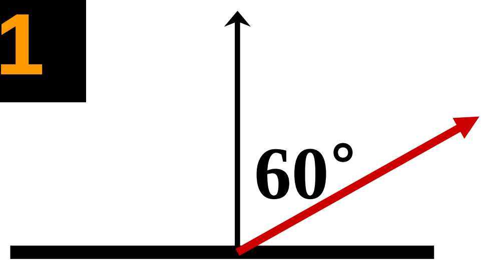
$$= \frac{1}{\sigma T^4}\left[\int_0^{\lambda_2} e_{\lambda b}(\lambda,T)\,d\lambda - \int_0^{\lambda_1} e_{\lambda b}(\lambda,T)\,d\lambda\right]$$

$$= F_{0-\lambda_2} - F_{0-\lambda_1}$$

or $$F_{\lambda_1-\lambda_2} = \frac{1}{\sigma T^4}\int_{\lambda_1}^{\lambda_2} e_{\lambda b}(\lambda,T)\,d\lambda$$

$$= \frac{1}{\sigma}\int_{\lambda_1 T}^{\lambda_2 T} \frac{e_{\lambda b}(\lambda,T)}{T^5}\,d(\lambda T) \equiv F_{\lambda_1 T-\lambda_2 T}$$

Ex 2-1



$$e'_{\lambda b}\left(6\mu\text{m}, 60^{\circ}, 1000^{\circ}\text{C}\right) = ?$$

Black surface element at $T_b = 1000^{\circ}\text{C}$

$$e'_{\lambda b} = i_{\lambda b}\cos\theta$$

$$e_{\lambda b} = \pi i_{\lambda b} = \frac{2\pi C_1}{\lambda^5\left(e^{C_2/\lambda T} - 1\right)} \rightarrow i_{\lambda b} = \frac{2C_1}{\lambda^5\left(e^{C_2/\lambda T} - 1\right)}$$

$$e'_{\lambda b} = \frac{2C_1}{\lambda^5\left(e^{C_2/\lambda T} - 1\right)}\cos\theta = 1373\ \text{W/(m}^2\cdot\mu\text{m}\cdot\text{sr)}$$

The value of $\dfrac{e_{\lambda b}}{T^5}$ is given in Table A-5.

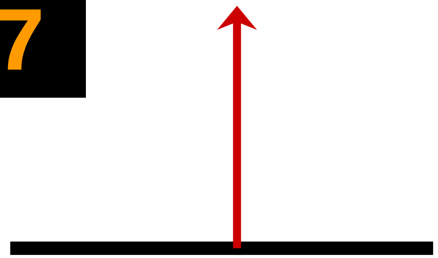
$$e'_{\lambda b} = i_{\lambda b}\cos\theta = \frac{T^5}{\pi}\frac{\pi i_{\lambda b}}{T^5}\cos\theta = \frac{T^5}{\pi}\frac{e_{\lambda b}}{T^5}\cos\theta$$

at $\lambda T = 1273 \times 6 = 7638 \rightarrow \dfrac{e_{\lambda b}}{T^5} = 259.0 \times 10^{-14}$

$$e'_{\lambda b} = \frac{(1273)^5}{\pi} \times 259 \times 10^{-14} \times \cos 60^\circ$$

$$= 1378\ \text{W}/(\text{m}^2 \cdot \mu\text{m} \cdot \text{sr})$$

Ex 2-7



$$e'_b(0^\circ) = 10{,}000\ \text{W}/(\text{m}^2\cdot\text{sr})$$

$$T = ?$$

Black surface element

$$e'_{\lambda b} = i_{\lambda b}\cos\theta$$

$$e'_b = \int_0^\infty i_{\lambda b}\cos\theta\, d\lambda = i_b\cos\theta \rightarrow e'_{b,n} = i_b$$

$$e_b = \int_\cap i_b\cos\theta\, d\omega = \pi i_b = \sigma T^4$$

$$T = \left(\frac{\pi i_b}{\sigma}\right)^{1/4} = \left(\frac{\pi e'_{b,n}}{\sigma}\right)^{1/4} = \left(\frac{\pi\times 10000}{5.67\times 10^{-8}}\right)^{1/4} = 862.7\ \text{K}$$

Ex **What is the fraction contained in the visible range of solar radiation ?**

Sun: blackbody at 5780 K

$$0.4 \le \lambda \le 0.7\ \mu\text{m}$$

$$\lambda_1 T = 0.4 \times 5780 = 2312,\ \lambda_2 T = 0.7 \times 5780 = 4046$$

from Table A-5,

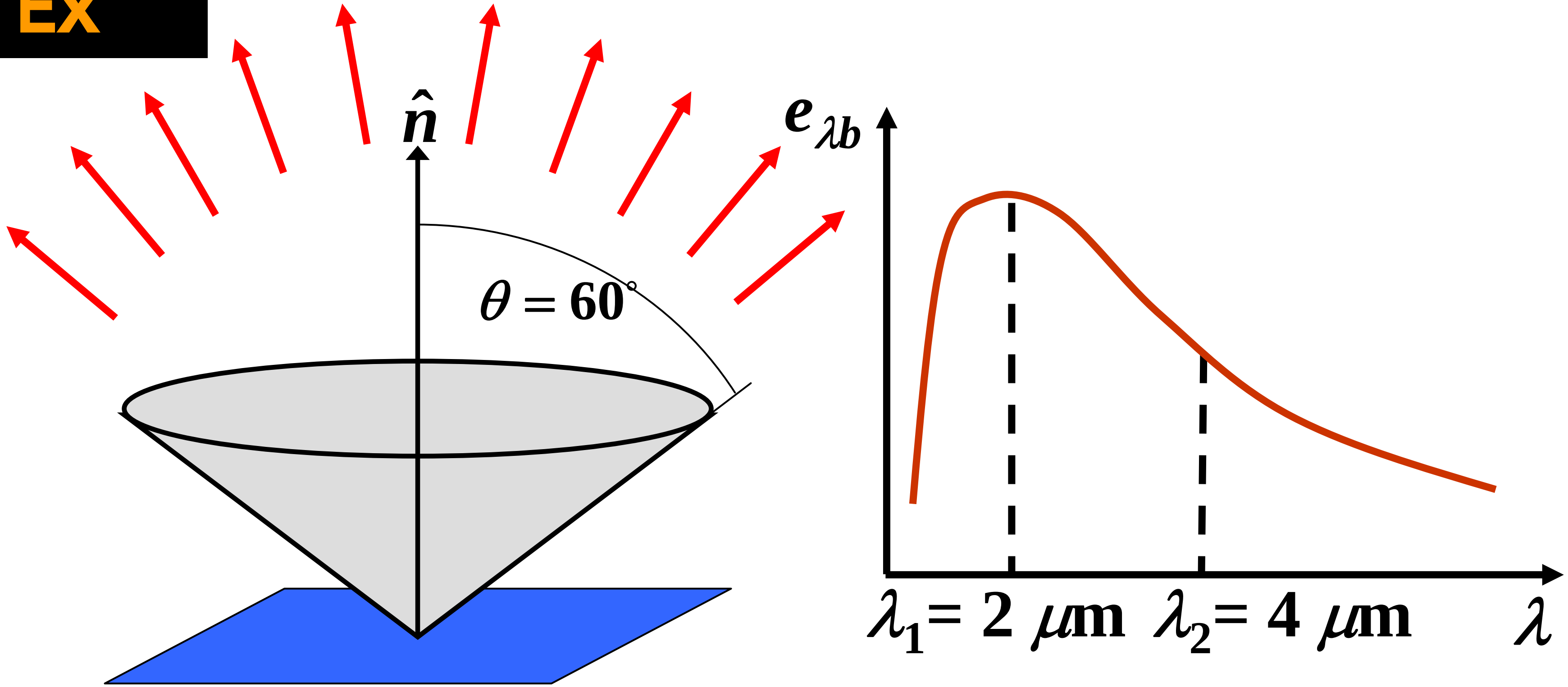
$$2300 \to 0.12003,\ 2325 \to 0.12500 : 2312 \to 0.12242$$

$$4000 \to 0.48087,\ 4050 \to 0.48987 : 4046 \to 0.48915$$

$$F_{\lambda_1 T - \lambda_2 T} = F_{0-\lambda_2 T} - F_{0-\lambda_1 T} = 0.48915 - 0.12242 = 0.36673$$

37% in the visible region
12% in the ultra violet region
51% in the infrared region

Ex



Blackbody at 1500 K

Find:

Rate of emission per unit area in directions

$0^\circ \le \theta \le 60^\circ$, **and in spectral range**

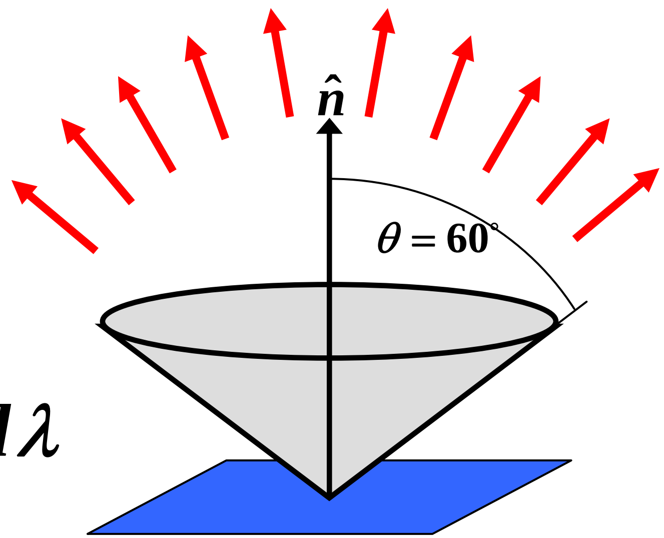
$2\mu m \le \lambda \le 4\mu m$

$$0^\circ \le \theta \le 60^\circ,\ 2\mu m \le \lambda \le 4\mu m$$

$$i_{\lambda b}\cos\theta d\omega d\lambda$$

$$\Delta E = \int_{\lambda_1}^{\lambda_2}\int_{\omega} i_{\lambda b}\cos\theta d\omega d\lambda$$

$$\Delta E = \int_2^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/3} i_{\lambda b}\cos\theta\sin\theta d\theta d\phi d\lambda$$



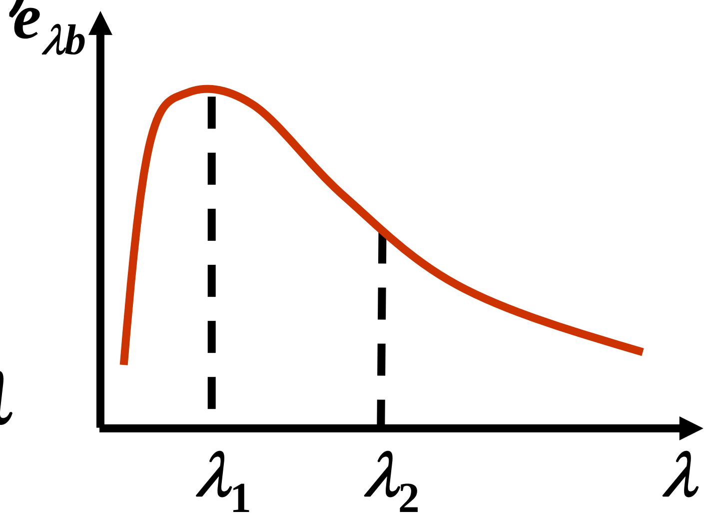
Blackbody at 1500 K

$i_{\lambda b}$: independent of direction

$$\Delta E = \int_2^4 i_{\lambda b}\left(\int_0^{2\pi}\int_0^{\pi/3}\cos\theta\sin\theta d\theta d\phi\right)d\lambda$$

$$\Delta E = \int_2^4 i_{\lambda b}\left(2\pi\frac{\sin^2\theta}{2}\bigg|_0^{\pi/3}\right)d\lambda$$

$$= 0.75\int_2^4 \pi i_{\lambda b} d\lambda = 0.75\int_2^4 e_{\lambda b} d\lambda$$



$$\Delta E = 0.75 e_b \int_2^4 \frac{e_{\lambda,b}}{e_b} d\lambda = 0.75 e_b \left[F_{(0\to 4\,\mu\text{m})} - F_{(0\to 2\,\mu\text{m})} \right]$$

From Table A-5

$$\lambda_1 T = 2\,\mu\text{m} \times 1500\,\text{K} = 3000\,\mu\text{m}\cdot\text{K} \rightarrow F_{(0\to 2\,\mu\text{m})} = 0.27323$$

$$\lambda_2 T = 4\,\mu\text{m} \times 1500\,\text{K} = 6000\,\mu\text{m}\cdot\text{K} \rightarrow F_{(0\to 4\,\mu\text{m})} = 0.73779$$

$$e_b = \sigma T^4$$

$$\Delta E = 0.75 \sigma T^4 \left[F_{(0\to 4\,\mu\text{m})} - F_{(0\to 2\,\mu\text{m})} \right]$$

$$= 0.75 \times 5.67 \times 15^4 \times (0.73779 - 0.27323) = 10^5\,\text{W/m}^2$$