SOLUTION METHODS FOR THE EQUATIONS OF RADIATIVE TRANSFER

- **Optically Thin Approximation**
- **Optically Thick Approximation**
- **Exponential Kernel Approximation**
- **Differential Approximations**
  - **Milne-Eddington Approximation**
  - **Spherical Harmonics Approximation (P-N Approximation)**
  - **Shuster-Schwarzchild Approximation (two-flux model)**
  - **Discrete Ordinate Method (S-N Approximation)**

Planar medium with isotropic scattering or non-scattering and diffuse boundaries

Optically Thin Approximation

$$\tau_\lambda \ll 1$$

$$q''_\lambda(\tau_\lambda) = 2\pi\left[i_\lambda^+(0)E_3(\tau_\lambda) + \int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda)E_2(\tau_\lambda - \tau'_\lambda)d\tau'_\lambda \right]$$
$$-2\pi\left[i_\lambda^-(\tau_\lambda)E_3(\tau_{\lambda 0} - \tau_\lambda) + \int_{\tau_\lambda}^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_2(\tau'_\lambda - \tau_\lambda)d\tau'_\lambda \right]$$

$$E_2(\tau) = \int_0^1 e^{-\frac{\tau}{\mu}} d\mu = \int_0^1 \left[1 + O\left(\frac{\tau}{\mu}\right)\right] d\mu = 1 + O(\tau)$$

$$E_3(\tau) = \int_0^1 \mu e^{-\frac{\tau}{\mu}} d\mu = \frac{1}{2} - \tau + O(\tau^2)$$

$$q''_\lambda = 2\pi\left[i^+_\lambda(0)\left(\frac{1}{2} - \tau_\lambda\right) + \int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda)\,d\tau'_\lambda \right]$$

$$-2\pi\left[i^-_\lambda(\tau_{\lambda 0})\left(\frac{1}{2} - \tau_{\lambda 0} + \tau_\lambda\right) + \int_0^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)\,d\tau'_\lambda \right]$$

neglecting terms of $O(\tau_{\lambda 0})$,

$$q''_\lambda = \pi\left[i^+_\lambda(0) - i^-_\lambda(\tau_{\lambda 0}) \right]$$

$$i^+_\lambda(0) = \varepsilon_{\lambda 1} i_{\lambda b1} + \left(1 - \varepsilon_{\lambda 1}\right) G_{\lambda 1}$$

$$i^-_\lambda(\tau_{\lambda 0}) = \varepsilon_{\lambda 2} i_{\lambda b2} + \left(1 - \varepsilon_{\lambda 2}\right) G_{\lambda 2}$$

boundary surface intensities-diffuse surface

$$G_{\lambda 1} = \int_{\omega=2\pi} i_\lambda^-(0,-\mu')\cos\theta' d\omega'$$

$$i_\lambda^-(\tau_\lambda,-\mu) = i_\lambda^-(\tau_{\lambda 0},-\mu)\exp\left[-\frac{1}{\mu}(\tau_{\lambda 0}-\tau_\lambda)\right]$$

$$+\int_{\tau_\lambda}^{\tau_{\lambda 0}} \frac{1}{\mu} S(\tau'_\lambda,-\mu)\exp\left[-\frac{1}{\mu}(\tau'_\lambda-\tau_\lambda)\right] d\tau'_\lambda, \quad \mu > 0$$

isotropic scattering and diffuse boundary

$$i_\lambda^-(0,-\mu) = i_\lambda^-(\tau_{\lambda 0})\exp\left(-\frac{\tau_{\lambda 0}}{\mu}\right)$$

$$+\int_0^{\tau_{\lambda 0}} \frac{1}{\mu} S(\tau'_\lambda)\exp\left(-\frac{\tau'_\lambda}{\mu}\right) d\tau'_\lambda$$

$$G_{\lambda 1} = \int_{\omega=2\pi} \left\{ i_\lambda^-(\tau_{\lambda 0}) \exp\left(-\frac{\tau_{\lambda 0}}{\mu'} \right) + \int_0^{\tau_{\lambda 0}} \frac{1}{\mu} S(\tau'_\lambda) \exp\left(-\frac{\tau'_\lambda}{\mu} \right) d\tau'_\lambda \right\} \cos\theta' d\omega'$$

$$\int_{\omega=2\pi} i_\lambda^-(\tau_{\lambda 0}) \exp\left(-\frac{\tau_{\lambda 0}}{\mu'} \right) \cos\theta' d\omega'$$

$$= 2\pi i_\lambda^-(\tau_{\lambda 0}) \int_0^1 \mu' \exp\left(-\frac{\tau_{\lambda 0}}{\mu'} \right) d\mu' = 2\pi i_\lambda^-(\tau_{\lambda 0}) E_3(\tau_{\lambda 0})$$

$$\int_{\omega=2\pi} \int_0^{\tau_{\lambda 0}} \frac{1}{\mu'} S(\tau'_\lambda) \exp\left(-\frac{\tau'_\lambda}{\mu'} \right) d\tau'_\lambda \cos\theta' d\omega'$$

$$= \int_0^{\tau_{\lambda 0}} S(\tau'_\lambda) \left\{ \int_{\omega=2\pi} \exp\left(-\frac{\tau'_\lambda}{\mu} \right) d\omega' \right\} d\tau'_\lambda = 2\pi \int_0^{\tau_{\lambda 0}} S(\tau'_\lambda) E_2(\tau'_\lambda) d\tau'_\lambda$$

$$G_{\lambda 1} = 2\pi i_\lambda^-(\tau_{\lambda 0}) E_3(\tau_{\lambda 0}) + 2\pi \int_0^{\tau_{\lambda 0}} S(\tau'_\lambda) E_2(\tau'_\lambda) d\tau'_\lambda$$

$$i_\lambda^+(0) = \varepsilon_{\lambda 1} i_{\lambda b1} + 2(1-\varepsilon_{\lambda 1}) \left[i_\lambda^-(\tau_{\lambda 0}) E_3(\tau_{\lambda 0}) + \int_0^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda) E_2(\tau'_\lambda) d\tau'_\lambda \right]$$

$$i_\lambda^-(\tau_{\lambda 0}) = \varepsilon_{\lambda 2} i_{\lambda b2} + 2(1-\varepsilon_{\lambda 2}) \left[i_\lambda^+(0) E_3(\tau_{\lambda 0}) + \int_0^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda) E_2(\tau_{\lambda 0} - \tau'_\lambda) d\tau'_\lambda \right]$$

optically thin, neglecting the terms of order τ_λ

$$i_\lambda^+(0) = \varepsilon_{\lambda 1} i_{\lambda b1} + (1-\varepsilon_{\lambda 1}) i_\lambda^-(\tau_{\lambda 0}) \qquad \left(E_3(0) = \frac{1}{2} \right)$$

$$i_\lambda^-(\tau_{\lambda 0}) = \varepsilon_{\lambda 2} i_{\lambda b2} + (1-\varepsilon_{\lambda 2}) i_\lambda^+(0)$$

$$i_\lambda^+(0) = \frac{\varepsilon_{\lambda 1} i_{\lambda b1} + (1-\varepsilon_{\lambda 1})\varepsilon_{\lambda 2} i_{\lambda b2}}{1-(1-\varepsilon_{\lambda 1})(1-\varepsilon_{\lambda 2})}$$

$$i_\lambda^-(\tau_{\lambda 0}) = \frac{\varepsilon_{\lambda 2} i_{\lambda b2} + (1-\varepsilon_{\lambda 2})\varepsilon_{\lambda 1} i_{\lambda b1}}{1-(1-\varepsilon_{\lambda 1})(1-\varepsilon_{\lambda 2})}$$

$$q''_\lambda = \pi\left[i_\lambda^+(0) - i_\lambda^-(\tau_{\lambda 0})\right]$$

In the optically thin limit: $\tau_\lambda \to 0$

$$q''_{\lambda 1} = \frac{\pi\left[i_{\lambda b1}(T_1) - i_{\lambda b2}(T_2)\right]}{\frac{1}{\varepsilon_{\lambda 1}} + \frac{1}{\varepsilon_{\lambda 2}} - 1} = \frac{e_{\lambda b1} - e_{\lambda b2}}{\frac{1}{\varepsilon_{\lambda 1}} + \frac{1}{\varepsilon_{\lambda 2}} - 1}$$

Optically Thick Approximation

Rosseland or diffusion approximation

$$q''_\lambda(\tau_\lambda) = 2\pi\left[i^+_\lambda(0)E_3(\tau_\lambda) + \int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda)E_2(\tau_\lambda - \tau'_\lambda)d\tau'_\lambda \right]$$

$$-2\pi\left[i^-_\lambda(\tau_{\lambda 0})E_3(\tau_{\lambda 0} - \tau_\lambda) + \int_{\tau_\lambda}^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_2(\tau'_\lambda - \tau_\lambda)d\tau'_\lambda \right]$$

Let $z = \tau_\lambda - \tau'_\lambda$ **then** $\tau'_\lambda = \tau_\lambda - z,\ d\tau'_\lambda = -dz$

$$\int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda)E_2(\tau_\lambda - \tau'_\lambda)d\tau'_\lambda = \int_0^{\tau_\lambda} S_\lambda(\tau_\lambda - z)E_2(z)dz$$

Let $z = \tau'_\lambda - \tau_\lambda$ **then** $\tau'_\lambda = \tau_\lambda + z,\ d\tau'_\lambda = dz$

$$\int_{\tau_\lambda}^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_2(\tau'_\lambda - \tau_\lambda)d\tau'_\lambda = \int_0^{\tau_{\lambda 0} - \tau_\lambda} S_\lambda(\tau_\lambda + z)E_2(z)dz$$

$$q''_\lambda(\tau_\lambda) = 2\pi\left[i^+_\lambda(0)E_3(\tau_\lambda) + \int_0^{\tau_\lambda} S_\lambda(\tau_\lambda - z)E_2(z)dz \right]$$

$$-2\pi\left[i^-_\lambda(\tau_{\lambda 0})E_3(\tau_{\lambda 0} - \tau_\lambda) + \int_0^{\tau_{\lambda 0}-\tau_\lambda} S_\lambda(\tau_\lambda + z)E_2(z)dz \right]$$

For a large optical distance away from the wall $\tau_\lambda >> 1, \quad \tau_{\lambda 0} - \tau_\lambda >> 1$

$$\lim_{x\to\infty} E_n(x) = 0$$

$$q''_\lambda(\tau_\lambda) = 2\pi\int_0^{\tau_\lambda} S_\lambda(\tau_\lambda - z)E_2(z)dz$$

$$-2\pi\int_0^{\tau_{\lambda 0}-\tau_\lambda} S_\lambda(\tau_\lambda + z)E_2(z)dz$$

expand $S_\lambda(\tau_\lambda \pm z)$ **into a Taylor series**

$$S_\lambda(\tau_\lambda \pm z) = S_\lambda(\tau_\lambda) \pm \frac{dS_\lambda}{d\tau_\lambda} z + \frac{1}{2}\frac{d^2 S_\lambda}{d\tau_\lambda^2} z^2 \pm \cdots$$

when $\tau_\lambda >> 1, \quad \tau_{\lambda 0} - \tau_\lambda >> 1$

$$q''_\lambda(\tau_\lambda) = 2\pi\left[S_\lambda(\tau_\lambda)\int_0^\infty E_2(z)dz - \frac{dS_\lambda}{d\tau_\lambda}\int_0^\infty zE_2(z)dz + \cdots\right]$$

$$-2\pi\left[S_\lambda(\tau_\lambda)\int_0^\infty E_2(z)dz + \frac{dS_\lambda}{d\tau_\lambda}\int_0^\infty zE_2(z)dz + \cdots\right]$$

$$= -4\pi\frac{dS_\lambda}{d\tau_\lambda}\int_0^\infty zE_2(z)dz + O\left(\frac{1}{\tau_\lambda^3}\right) \quad \left(\int_0^\infty zE_2(z)dz = \frac{1}{3}\right)$$

$$= -\frac{4\pi}{3}\frac{dS_\lambda}{d\tau_\lambda}$$

For a non-scattering medium

$$S_\lambda = i_{\lambda b}$$

$$q''_\lambda(\tau_\lambda) = -\frac{4\pi}{3}\frac{dS_\lambda}{d\tau_\lambda} = -\frac{4\pi}{3}\frac{di_{\lambda b}}{d\tau_\lambda} = -\frac{4}{3}\frac{de_{\lambda b}}{d\tau_\lambda}$$

For an isotropic scattering medium

$$G_\lambda(\tau_\lambda) = 2\pi\left[\int_0^1 i_\lambda^+(\tau_\lambda,\mu)d\mu + \int_0^1 i_\lambda^-(\tau_\lambda,-\mu)d\mu\right]$$

$$i_\lambda^+(\tau_\lambda,\mu) = i_\lambda^+(0)\exp\left(-\frac{\tau_\lambda}{\mu}\right) + \int_0^{\tau_\lambda}\frac{1}{\mu}S(\tau'_\lambda)\exp\left[-\frac{1}{\mu}(\tau_\lambda - \tau'_\lambda)\right]d\tau'_\lambda$$

$$i_{\lambda}^{-}(\tau_{\lambda},-\mu)=i_{\lambda}^{-}(\tau_{\lambda 0})\exp\left[-\frac{1}{\mu}(\tau_{\lambda 0}-\tau_{\lambda})\right]$$

$$+\int_{\tau_{\lambda}}^{\tau_{\lambda 0}}\frac{1}{\mu}S(\tau_{\lambda}')\exp\left[-\frac{1}{\mu}(\tau_{\lambda}'-\tau_{\lambda})\right]d\tau_{\lambda}'$$

$$\int_{0}^{1}i_{\lambda}^{+}(\tau_{\lambda},\mu)d\mu=\int_{0}^{1}i_{\lambda}^{+}(0)\exp\left(-\frac{\tau_{\lambda}}{\mu}\right)d\mu$$

$$+\int_{0}^{1}\int_{0}^{\tau_{\lambda}}\frac{1}{\mu}S(\tau_{\lambda}')\exp\left[-\frac{1}{\mu}(\tau_{\lambda}-\tau_{\lambda}')\right]d\tau_{\lambda}'d\mu$$

$$=i_{\lambda}^{+}(0)E_{2}(\tau_{\lambda})+\int_{0}^{\tau_{\lambda}}S(\tau_{\lambda}')E_{1}(\tau_{\lambda}-\tau_{\lambda}')d\tau_{\lambda}'$$

$$\int_0^1 i_\lambda^-(\tau_\lambda,-\mu)d\mu = \int_0^1 i_\lambda^-(\tau_{\lambda 0})\exp\left[-\frac{1}{\mu}(\tau_{\lambda 0}-\tau_\lambda)\right]d\mu$$

$$+\int_0^1\int_{\tau_\lambda}^{\tau_{\lambda 0}}\frac{1}{\mu}S(\tau'_\lambda)\exp\left[-\frac{1}{\mu}(\tau'_\lambda-\tau_\lambda)\right]d\tau'_\lambda d\mu$$

$$= i_\lambda^-(\tau_{\lambda 0})E_2(\tau_{\lambda 0}-\tau_\lambda)+\int_{\tau_\lambda}^{\tau_{\lambda 0}} S(\tau'_\lambda)E_1(\tau'_\lambda-\tau_\lambda)d\tau'_\lambda$$

$$G_\lambda(\tau_\lambda) = 2\pi\left[i_\lambda^+(0)E_2(\tau_\lambda)+\int_0^{\tau_\lambda} S(\tau'_\lambda)E_1(\tau_\lambda-\tau'_\lambda)d\tau'_\lambda\right]$$

$$+2\pi\left[i_\lambda^-(\tau_{\lambda 0})E_2(\tau_{\lambda 0}-\tau_\lambda)+\int_{\tau_\lambda}^{\tau_{\lambda 0}} S(\tau'_\lambda)E_1(\tau'_\lambda-\tau_\lambda)d\tau'_\lambda\right]$$

when $\tau_\lambda >> 1, \quad \tau_{\lambda 0} - \tau_\lambda >> 1$

$$G_\lambda(\tau_\lambda) = 2\pi \int_0^{\tau_\lambda} S(\tau'_\lambda) E_1(\tau_\lambda - \tau'_\lambda) d\tau'_\lambda$$

$$+2\pi \int_{\tau_\lambda}^{\tau_{\lambda 0}} S(\tau'_\lambda) E_1(\tau'_\lambda - \tau_\lambda) d\tau'_\lambda$$

$$z = \tau_\lambda - \tau'_\lambda$$

$$\int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda) E_1(\tau_\lambda - \tau'_\lambda) d\tau'_\lambda = \int_0^{\tau_\lambda} S_\lambda(\tau_\lambda - z) E_1(z) dz$$

$$z = \tau'_\lambda - \tau_\lambda$$

$$\int_{\tau_\lambda}^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda) E_1(\tau'_\lambda - \tau_\lambda) d\tau'_\lambda = \int_0^{\tau_{\lambda 0} - \tau_\lambda} S_\lambda(\tau_\lambda + z) E_1(z) dz$$

$$G_\lambda(\tau_\lambda) = 2\pi \left[S_\lambda(\tau_\lambda) \int_0^\infty E_1(z) dz - \frac{dS_\lambda}{d\tau_\lambda} \int_0^\infty z E_1(z) dz + \cdots \right]$$

$$+2\pi \left[S_\lambda(\tau_\lambda) \int_0^\infty E_1(z) dz + \frac{dS_\lambda}{d\tau_\lambda} \int_0^\infty z E_1(z) dz + \cdots \right]$$

$$G_\lambda(\tau_\lambda) = 4\pi S_\lambda(\tau_\lambda)\int_0^\infty E_1(z)dz + O\left(\frac{1}{\tau_\lambda^2}\right)$$

$$\int_0^\infty E_1(z)dz = 1$$

$$G_\lambda(\tau_\lambda) = 4\pi S_\lambda(\tau_\lambda)$$

$$S_\lambda(\tau_\lambda) = \left(1-\omega_{0\lambda}\right)i_{\lambda b} + \frac{\omega_{0\lambda}}{4\pi}G_\lambda = \left(1-\omega_{0\lambda}\right)i_{\lambda b} + \omega_{0\lambda}S_\lambda$$

$$S_\lambda = i_{\lambda b}$$

$$q''_\lambda(\tau_\lambda) = -\frac{4\pi}{3}\frac{dS_\lambda}{d\tau_\lambda} = -\frac{4\pi}{3}\frac{di_{\lambda b}}{d\tau_\lambda} = -\frac{4}{3}\frac{de_{\lambda b}}{d\tau_\lambda}$$

total radiative heat flux

$$q'' = -\frac{4}{3}\pi\int_0^\infty \frac{di_{\lambda b}}{d\tau_\lambda}d\lambda$$

$$\tau_\lambda = \int_0^x \kappa_\lambda(x')dx' \rightarrow d\tau_\lambda = \kappa_\lambda(x)dx$$

$$q''(x) = -\frac{4\pi}{3}\int_0^\infty \frac{1}{\kappa_\lambda}\frac{di_{\lambda b}(x)}{dx}d\lambda$$

but $$\frac{di_{\lambda b}}{dx} = \frac{di_{\lambda b}}{di_b}\frac{di_b}{dx}$$

$$q''(x) = -\frac{4\pi}{3}\frac{di_b}{dx}\int_0^\infty \frac{1}{\kappa_\lambda}\frac{di_{\lambda b}}{di_b}d\lambda$$

Rosseland mean extinction coefficient :

$$\frac{1}{K_R} \equiv \int_0^\infty \frac{1}{\kappa_\lambda}\frac{di_{\lambda b}}{di_b}d\lambda = \int_0^\infty \frac{1}{\kappa_\lambda}\frac{de_{\lambda b}}{de_b}$$

$$q''(x) = -\frac{4\pi}{3K_R}\frac{di_b}{dx} = -\frac{4\sigma}{3K_R}\frac{d\left(n^2T^4\right)}{dx} \quad \left(i_b = \frac{n^2\sigma T^4}{\pi}\right)$$

$$= -\frac{16n^2\sigma T^3}{3K_R}\frac{dT}{dx}$$

radiative conductivity: $\frac{16n^2\sigma T^3}{3K_R}$

in general, $q''_\lambda = -\frac{4\pi}{3\kappa_\lambda}\nabla i_{\lambda b}, \; q'' = -\frac{4\pi}{3K_R}\nabla i_b$

Radiation slip : temperature jump condition at walls
modified diffusion approximation

$$q''_\lambda(\tau_\lambda) = 2\pi\left[i^+_\lambda(0)E_3(\tau_\lambda) + \int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda)E_2(\tau_\lambda - \tau'_\lambda)d\tau'_\lambda \right]$$

$$-2\pi\left[i^-_\lambda(\tau_{\lambda 0})E_3(\tau_{\lambda 0} - \tau_\lambda) + \int_{\tau_\lambda}^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_2(\tau'_\lambda - \tau_\lambda)d\tau'_\lambda \right]$$

at $\tau_\lambda = 0$ $\quad E_3(\tau) = \int_0^1 \mu \exp\left(-\frac{\tau}{\mu}\right)d\mu \rightarrow E_3(0) = \frac{1}{2}$

$$q''_\lambda(0) = \pi i^+_\lambda(0) - 2\pi\left[i^-_\lambda(\tau_{\lambda 0})E_3(\tau_{\lambda 0}) + \int_0^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_2(\tau'_\lambda)d\tau'_\lambda \right]$$

boundary surface intensity

$$i_\lambda^+(0) = \varepsilon_{\lambda 1} i_{\lambda b1}$$

$$+2(1-\varepsilon_{\lambda 1})\left[i_\lambda^-(\tau_{\lambda 0})E_3(\tau_{\lambda 0}) + \int_0^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_2(\tau'_\lambda)d\tau'_\lambda \right]$$

$$q''_\lambda(0) = \pi i_\lambda^+(0) - 2\pi\left[i_\lambda^-(\tau_{\lambda 0})E_3(\tau_{\lambda 0}) + \int_0^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_2(\tau'_\lambda)d\tau'_\lambda \right]$$

$$q''_\lambda(0) = \pi\varepsilon_{\lambda 1}\left[i_{\lambda b1} - 2i_\lambda^-(\tau_{\lambda 0})E_3(\tau_{\lambda 0}) - 2\int_0^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_2(\tau'_\lambda)d\tau'_\lambda \right]$$

for optically thick medium

$E_3(\tau_{\lambda 0}) \to 0$ **as** $\tau_{\lambda 0} \to \infty$ **and** $S_\lambda(\tau_\lambda) = i_{\lambda b}(\tau_\lambda)$

$$q''_\lambda(0) = -\frac{4\pi}{3}\left.\frac{di_{\lambda b}}{d\tau_\lambda}\right|_{\tau_\lambda = 0}$$

$$q''_\lambda(0) = -\frac{4\pi}{3}\frac{di_{\lambda b}}{d\tau_\lambda}\bigg|_{\tau_\lambda=0}$$

$$= \pi\varepsilon_{\lambda 1}\left[i_{\lambda b1} - 2\int_0^{\tau_{\lambda 0}} i_{\lambda b}(\tau'_\lambda)E_2(\tau'_\lambda)d\tau'_\lambda\right]$$

$$i_{\lambda b}(\tau'_\lambda) = i_{\lambda b}(0) + \tau'_\lambda\frac{di_{\lambda b}}{d\tau_\lambda}\bigg|_{\tau_\lambda=0} + \frac{\tau'^2_\lambda}{2!}\frac{d^2 i_{\lambda b}}{d\tau^2_\lambda}\bigg|_{\tau_\lambda=0} + \cdots$$

for large τ

$$\int_0^\infty E_2(\tau')d\tau' = \frac{1}{2},\ \int_0^\infty \tau' E_2(\tau')d\tau' = \frac{1}{3},\ \int_0^\infty \tau'^2 E_2(\tau')d\tau' = \frac{1}{2}$$

$$-\frac{4\pi}{3}\frac{di_{\lambda b}}{d\tau_\lambda}\bigg|_{\tau_\lambda=0} = \pi\varepsilon_{\lambda 1}\left[i_{\lambda b1} - i_{\lambda b}(0) - \frac{2}{3}\frac{di_{\lambda b}}{d\tau_\lambda}\bigg|_{\tau_\lambda=0} - \frac{1}{2}\frac{d^2 i_{\lambda b}}{d\tau^2_\lambda}\bigg|_{\tau_\lambda=0} - \cdots\right]$$

neglecting second and higher order terms

$$-\frac{4\pi}{3}\frac{di_{\lambda b}}{d\tau_\lambda}\bigg|_{\tau_\lambda=0} = \pi\varepsilon_{\lambda 1}\left[i_{\lambda b1} - i_{\lambda b}(0) - \frac{2}{3}\frac{di_{\lambda b}}{d\tau_\lambda}\bigg|_{\tau_\lambda=0}\right]$$

$$= \pi\varepsilon_{\lambda 1}\left[i_{\lambda b1} - i_{\lambda b}(0)\right] - \pi\varepsilon_{\lambda 1}\frac{2}{3}\frac{di_{\lambda b}}{d\tau_\lambda}\bigg|_{\tau_\lambda=0}$$

$$\left(-\frac{4\pi}{3} + \pi\varepsilon_{\lambda 1}\frac{2}{3}\right)\frac{di_{\lambda b}}{d\tau_\lambda}\bigg|_{\tau_\lambda=0} = \pi\varepsilon_{\lambda 1}\left[i_{\lambda b1} - i_{\lambda b}(0)\right]$$

$$\pi\left[i_{\lambda b1} - i_{\lambda b}(0)\right] = -\left(\frac{1}{\varepsilon_{\lambda 1}} - \frac{1}{2}\right)\frac{4\pi}{3}\frac{di_{\lambda b}}{d\tau_\lambda}\bigg|_{\tau_\lambda=0}$$

or $$\pi\left[i_{\lambda b1} - i_{\lambda b}(0)\right] = \left(\frac{1}{\varepsilon_{\lambda 1}} - \frac{1}{2}\right)q''_\lambda(0)$$

similarly

$$\pi\left[i_{\lambda b}(\tau_{\lambda 0}) - i_{\lambda b2}\right] = -\left(\frac{1}{\varepsilon_{\lambda 2}} - \frac{1}{2}\right)\frac{4\pi}{3}\frac{di_{\lambda b}}{d\tau_\lambda}\bigg|_{\tau_0}$$

or $$\pi\left[i_{\lambda b}(\tau_{\lambda 0}) - i_{\lambda b2}\right] = \left(\frac{1}{\varepsilon_{\lambda 2}} - \frac{1}{2}\right) q''_\lambda(\tau_{\lambda 0})$$

total quantities

$$\sigma\left[T_1^4 - T^4(0)\right] = \left(\frac{1}{\varepsilon_1} - \frac{1}{2}\right) q''$$

$$\sigma\left[T^4(\tau_0) - T_2^4\right] = \left(\frac{1}{\varepsilon_2} - \frac{1}{2}\right) q''$$

radiative equilibrium

$$q'' = -\frac{4\pi}{3}\frac{di_b}{d\tau} = \text{constant}$$

$$\int_0^{\tau_0} q'' d\tau = \int_0^{\tau_0} -\frac{4\pi}{3} di_b$$

$$\tau_0 q'' = -\frac{4\pi}{3}\left[i_b(\tau_0) - i_b(0)\right]$$

when $i_b(0) = i_{b1}(T_1), \; i_b(\tau_0) = i_{b2}(T_2)$

$$q'' = \frac{4\pi}{3\tau_0}\left[i_{b1}(T_1) - i_{b2}(T_2)\right] = \frac{4\pi}{3\tau_0}\left(\sigma T_1^4 - \sigma T_2^4\right)$$

$$\frac{q''}{\sigma\left(T_1^4 - T_2^4\right)} = \frac{4}{3}\frac{1}{\tau_0}$$

with slip condition

$$\tau_0 q'' = -\frac{4\pi}{3}\left[i_b(\tau_0) - i_b(0)\right]$$

$$q'' = -\frac{4\sigma}{3\tau_0}\left[T^4(\tau_0) - T^4(0)\right]$$

$$\sigma\left[T_1^4 - T^4(0)\right] = \left(\frac{1}{\varepsilon_1} - \frac{1}{2}\right)q''$$

$$\sigma\left[T^4(\tau_0) - T_2^4\right] = \left(\frac{1}{\varepsilon_2} - \frac{1}{2}\right)q''$$

$$\frac{q''}{\sigma\left(T_1^4 - T_2^4\right)} = \frac{1}{\frac{3}{4}\tau_0 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

Temperature distribution

$$q'' = -\frac{4\pi}{3}\frac{di_b}{d\tau} = -\frac{4\sigma}{3}\frac{dT^4}{d\tau}$$

$$\int_{\tau}^{\tau_0} q'' d\tau' = -\frac{4\sigma}{3}\int_{\tau}^{\tau_0} dT^4, \quad q''\left(\tau_0 - \tau\right) = \frac{4\sigma}{3}\left[T^4(\tau) - T^4(\tau_0)\right]$$

$$\sigma\left[T^4(\tau_0) - T_2^4\right] = \left(\frac{1}{\varepsilon_2} - \frac{1}{2}\right)q'', \; T^4(\tau_0) = T_2^4 + \frac{1}{\sigma}\left(\frac{1}{\varepsilon_2} - \frac{1}{2}\right)q''$$

$$q''\left(\tau_0 - \tau\right) = \frac{4\sigma}{3}\left[T^4(\tau) - T_2^4\right] - \frac{4}{3}\left(\frac{1}{\varepsilon_2} - \frac{1}{2}\right)q''$$

$$\sigma\left[T^4(\tau) - T_2^4\right] = \left(\frac{1}{\varepsilon_2} - \frac{1}{2}\right)q'' + \frac{3}{4}q''\left(\tau_0 - \tau\right)$$

$$\sigma\left[T^4(\tau)-T_2^4\right]=\left(\frac{1}{\varepsilon_2}-\frac{1}{2}\right)q''+\frac{3}{4}q''\left(\tau_0-\tau\right)$$

$$\frac{q''}{\sigma\left(T_1^4-T_2^4\right)}=\frac{1}{\frac{3}{4}\tau_0+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1}$$

$$\frac{T^4(\tau)-T_2^4}{T_1^4-T_2^4}=\frac{\frac{1}{\varepsilon_2}-\frac{1}{2}+\frac{3}{4}\left(\tau_0-\tau\right)}{\frac{3}{4}\tau_0+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1}$$

$$T^4(0)-T_2^4=\left(\frac{1}{\varepsilon_2}-\frac{1}{2}+\frac{3}{4}\tau_0\right)\frac{T_1^4-T_2^4}{\frac{3}{4}\tau_0+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1}$$

$$T^4(0)-T_2^4=\left(\frac{1}{\varepsilon_2}-\frac{1}{2}+\frac{3}{4}\tau_0\right)\frac{T_1^4-T_2^4}{\frac{3}{4}\tau_0+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1}$$

$$T^4(0)-T_1^4=-\left(T_1^4-T_2^4\right)+\left(\frac{1}{\varepsilon_2}-\frac{1}{2}+\frac{3}{4}\tau_0\right)\frac{T_1^4-T_2^4}{\frac{3}{4}\tau_0+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1}$$

$$=\left[\frac{1}{\varepsilon_2}-\frac{1}{2}+\frac{3}{4}\tau_0-\left(\frac{3}{4}\tau_0+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1\right)\right]\frac{T_1^4-T_2^4}{\frac{3}{4}\tau_0+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1}$$

$$T^4(0)-T_1^4=\left(\frac{1}{2}-\frac{1}{\varepsilon_1}\right)\frac{T_1^4-T_2^4}{\frac{3}{4}\tau_0+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1}$$

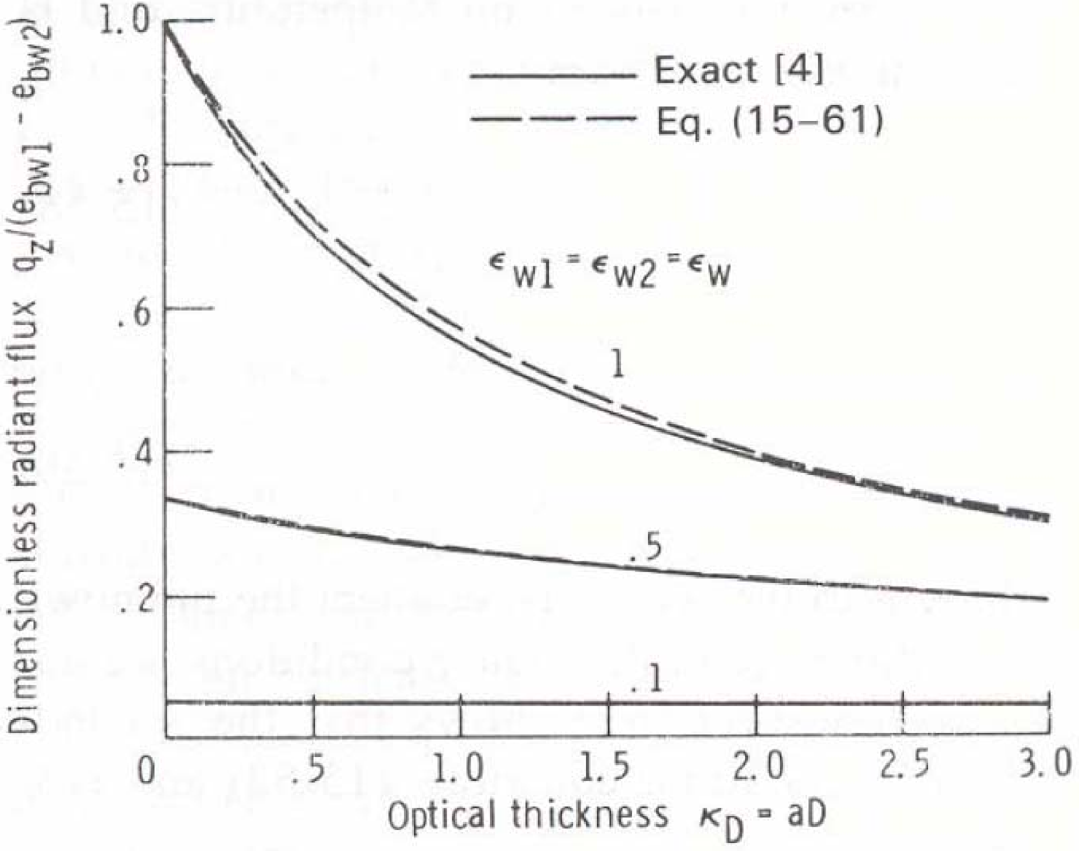


Figure 15-6 Validity of diffusion solution for energy transfer through gray gas between parallel gray plates.

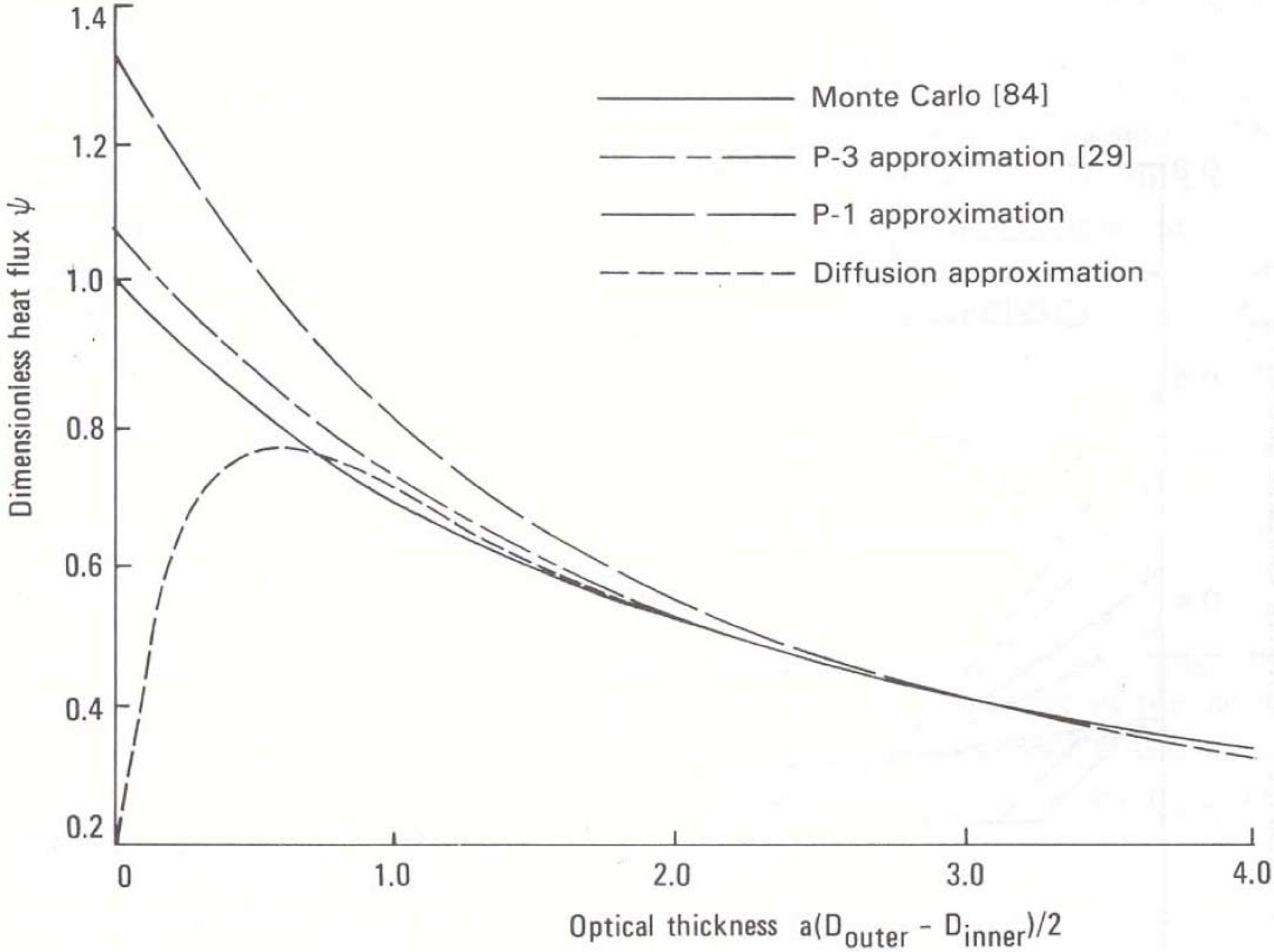


Figure 15-10 Comparison of solutions of energy transfer between infinitely long concentric black cylinders enclosing gray medium; $D_{inner}/D_{outer} = 0.5$.

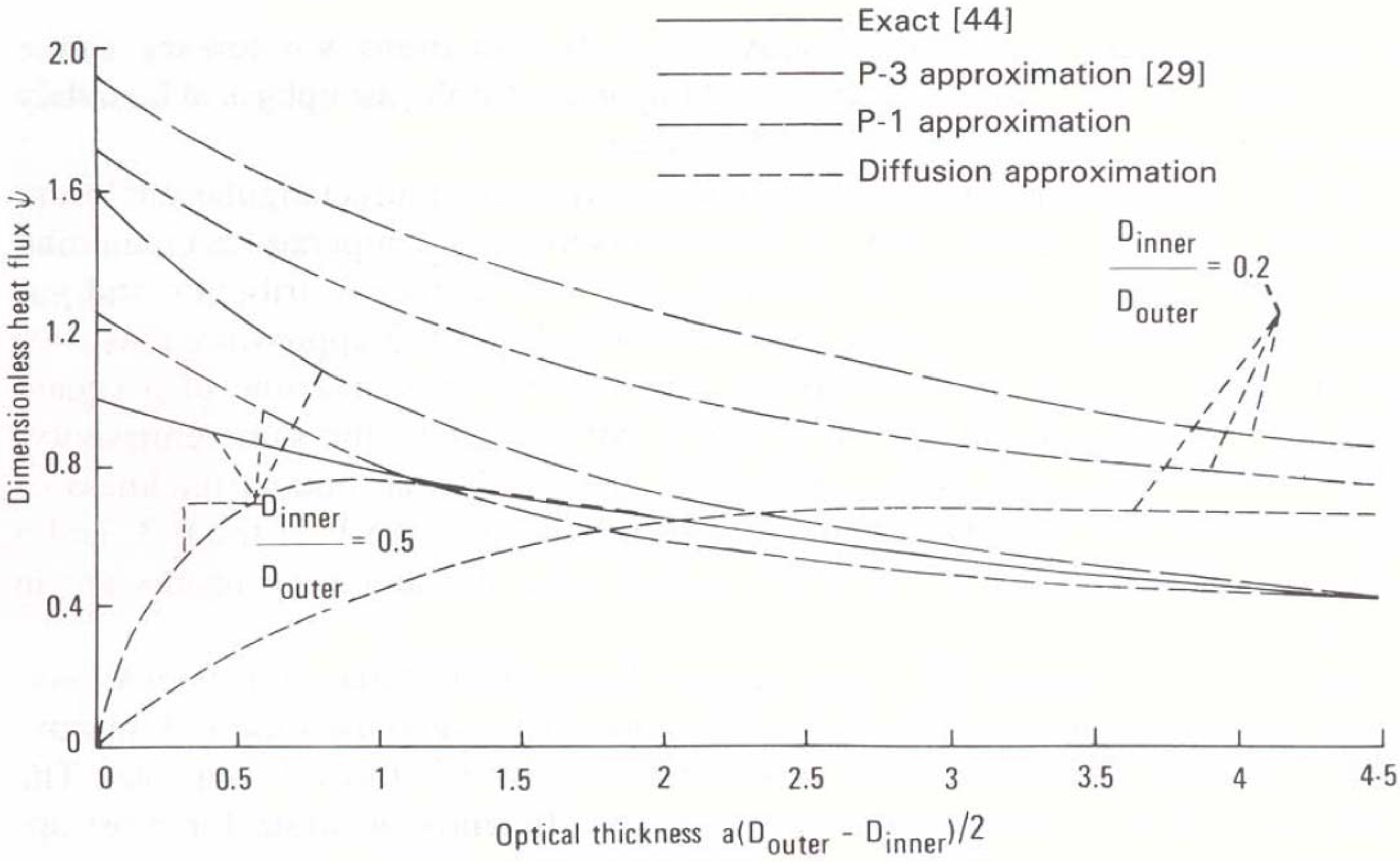


Figure 15-11 Comparison of solutions of energy transfer between black concentric spheres enclosing a gray medium.

Exponential Kernel Approximation

Krook, 1955

exponential integral function $E_2(\tau) \cong \sum_{j=1}^{n} a_j e^{-b_j \tau}$

one-term representation

$$E_2(\tau) = ae^{-b\tau}, \quad E_3(\tau) = -\int E_2(\tau)d\tau = \frac{a}{b}e^{-b\tau}$$

a, b **can be determined by equating area and the first moment over** $\tau = 0$ **to** $\tau = \infty$

Lick: $E_2(\tau) \cong e^{-\frac{3}{2}\tau}, \; E_3(\tau) \cong \frac{1}{2}e^{-\frac{3}{2}\tau}$

Vicenti and Baldwin: $E_2(\tau) = 0.813e^{-1.562\tau}$

Eddington approx.: $E_2(\tau) \approx e^{-\sqrt{3}\tau}$

for an absorbing and emitting medium, diffuse boundaries

$$q''(\tau) = 2\pi\left[i^+(0)E_3(\tau) - i^-(\tau_0)E_3(\tau_0-\tau)\right]$$
$$+2\pi\left[\int_0^{\tau} i_b(\tau')E_2(\tau-\tau')d\tau' - \int_{\tau}^{\tau_0} i_b(\tau')E_2(\tau'-\tau)d\tau'\right]$$

using kernel approximation

$$q''(\tau) = 2\pi\frac{a}{b}\left[i^+(0)e^{-b\tau} - i^-(\tau_0)e^{-b(\tau_0-\tau)}\right]$$
$$-2\pi a\left[\int_0^{\tau} i_b(\tau')e^{-b(\tau-\tau')}d\tau' - \int_{\tau}^{\tau_0} i_b(\tau')e^{-b(\tau'-\tau)}d\tau'\right]$$

differentiation yields

$$\frac{d^2q''(\tau)}{d\tau^2} = 4\pi a\frac{di_b(\tau)}{d\tau} + b^2q''(\tau) = 4a\sigma\frac{dT^4(\tau)}{d\tau} + b^2q''(\tau)$$

for radiative equilibrium $\frac{dq''}{d\tau} = 0$

$$\frac{d^2 q''(\tau)}{d\tau^2} = 4a\sigma \frac{dT^4(\tau)}{d\tau} + b^2 q''(\tau)$$

with $a = \frac{3}{4},\ b = \frac{3}{2},\ \ 3\sigma \frac{dT^4}{d\tau} + \frac{9}{4} q'' = 0,\ \ \frac{d}{d\tau}\left(\sigma T^4\right) = -\frac{3}{4} q''$

In dimensionless form

$$\Phi(\tau) = \frac{\sigma T^4(\tau) - J_2}{J_1 - J_2},\ \ \psi = \frac{q''}{J_1 - J_2}$$

$$\frac{d\Phi}{d\tau} = \frac{1}{J_1 - J_2} \frac{d}{d\tau}\left(\sigma T^4\right) = \frac{1}{J_1 - J_2}\left(-\frac{3}{4} q''\right)$$

$$= \frac{1}{J_1 - J_2}\left(-\frac{3}{4}\right)\left(J_1 - J_2\right)\psi = -\frac{3}{4}\psi \quad \rightarrow \Phi(\tau) = C - \frac{3}{4}\psi\tau$$

$$q''(\tau) = 2\pi \frac{a}{b}\left[i^{+}(0)e^{-b\tau} - i^{-}(\tau_0)e^{-b(\tau_0-\tau)} \right]$$

$$-2\pi a\left[\int_0^{\tau} i_b(\tau')e^{-b(\tau-\tau')}d\tau' - \int_{\tau}^{\tau_0} i_b(\tau')e^{-b(\tau'-\tau)}d\tau' \right]$$

$$a = \frac{3}{4},\ \ b = \frac{3}{2}$$

$$q''(\tau) = J_1 e^{-\frac{3}{2}\tau} - J_2 e^{-\frac{3}{2}(\tau_0-\tau)}$$

$$-\frac{3}{2}\int_0^{\tau} \sigma T^4(\tau')e^{-\frac{3}{2}(\tau-\tau')}d\tau' + \frac{3}{2}\int_{\tau}^{\tau_0} \sigma T^4(\tau')e^{-\frac{3}{2}(\tau'-\tau)}d\tau'$$

$$\psi = e^{-\frac{3}{2}\tau} + \frac{3}{2}e^{-\frac{3}{2}\tau}\int_0^{\tau} \Phi(\tau')e^{\frac{3}{2}\tau'}d\tau' - \frac{3}{2}e^{\frac{3}{2}\tau}\int_{\tau}^{\tau_0} \Phi(\tau')e^{-\frac{3}{2}\tau'}d\tau'$$

$$\psi = e^{-\frac{3}{2}\tau} + \frac{3}{2} e^{-\frac{3}{2}\tau} \int_0^{\tau} \Phi(\tau') e^{\frac{3}{2}\tau'} d\tau' - \frac{3}{2} e^{\frac{3}{2}\tau} \int_{\tau}^{\tau_0} \Phi(\tau') e^{-\frac{3}{2}\tau'} d\tau'$$

$$\Phi(\tau) = C - \frac{3}{4}\psi\tau$$

at $\tau = 0$

$$\psi = 1 - \frac{3}{2}\int_0^{\tau_0} \left(C - \frac{3}{4}\psi\tau' \right) e^{-\frac{3}{2}\tau'} d\tau'$$

$$= 1 + Ce^{-\frac{3}{2}\tau_0} - C - \frac{3\psi}{4}\tau_0 e^{-\frac{3}{2}\tau_0} - \frac{\psi}{2} e^{-\frac{3}{2}\tau_0} + \frac{\psi}{2}$$

at $\tau = \tau_0$

$$\psi = e^{-\frac{3}{2}\tau_0} + \frac{3}{2} e^{-\frac{3}{2}\tau_0} \int_0^{\tau_0} \left(C - \frac{3}{4}\psi\tau' \right) e^{\frac{3}{2}\tau'} d\tau'$$

$$= e^{-\frac{2}{3}\tau_0} + C - Ce^{-\frac{3}{2}\tau_0} - \frac{3\psi}{4}\tau_0 + \frac{\psi}{2} - \frac{\psi}{2} e^{-\frac{3}{2}\tau_0}$$

$$C = 1 - \frac{\psi}{2}$$

$$\psi = e^{-\frac{2}{3}\tau_0} + C - Ce^{-\frac{3}{2}\tau_0} - \frac{3\psi}{4}\tau_0 + \frac{\psi}{2} - \frac{\psi}{2}e^{-\frac{3}{2}\tau_0}$$

$$= 1 - \frac{3}{4}\tau_0\psi$$

$$\psi = \frac{1}{1 + \frac{3}{4}\tau_0} \quad \textbf{or} \quad \frac{q''}{J_1 - J_2} = \frac{1}{1 + \frac{3}{4}\tau_0}$$

$$\Phi(\tau) = C - \frac{3}{4}\psi\tau$$

$$\Phi(\tau) = \frac{\sigma T^4(\tau) - J_2}{J_1 - J_2} = \left(1 - \frac{\psi}{2}\right) - \frac{3}{4}\psi\tau$$

$$J_1 = \sigma T_1^4 - \frac{(1-\varepsilon_1)}{\varepsilon_1} q''$$

$$J_2 = \sigma T_2^4 + \frac{(1-\varepsilon_2)}{\varepsilon_2} q''$$

$$J_1 - J_2 = \sigma\left(T_1^4 - T_2^4\right) - \left[\frac{(1-\varepsilon_1)}{\varepsilon_1} + \frac{(1-\varepsilon_2)}{\varepsilon_2}\right] q''$$

$$= \sigma\left(T_1^4 - T_2^4\right) - \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 2\right) q''$$

$$q'' = \frac{J_1 - J_2}{1 + \frac{3}{4}\tau_0} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{1 + \frac{3}{4}\tau_0} - \frac{\left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)}{1 + \frac{3}{4}\tau_0} q''$$

$$q'' = \frac{J_1 - J_2}{1 + \frac{3}{4}\tau_0} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{1 + \frac{3}{4}\tau_0} - \frac{\left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)}{1 + \frac{3}{4}\tau_0} q''$$

$$\left[1 + \frac{\left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)}{1 + \frac{3}{4}\tau_0}\right] q'' = \frac{\sigma\left(T_1^4 - T_2^4\right)}{1 + \frac{3}{4}\tau_0}$$

$$\frac{q''}{\sigma\left(T_1^4 - T_2^4\right)} = \frac{1}{1 + \frac{3}{4}\tau_0 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 2} = \frac{1}{\frac{3}{4}\tau_0 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

Temperature distribution

$$\Phi(\tau) = \frac{\sigma T^4(\tau) - J_2}{J_1 - J_2} = 1 - \frac{\psi}{2} - \frac{3}{4}\psi\tau,$$

$$\psi = \frac{q''}{J_1 - J_2} = \frac{1}{1 + \frac{3}{4}\tau_0} \quad \rightarrow J_1 - J_2 = \frac{q''}{\psi}$$

$$J_2 = \sigma T_2^4 + \frac{(1-\varepsilon_2)}{\varepsilon_2} q''$$

$$\sigma T^4(\tau) - J_2 = (J_1 - J_2)\left(1 - \frac{\psi}{2} - \frac{3}{4}\psi\tau\right)$$

$$\sigma T^4(\tau) - \sigma T_2^4 - \frac{(1-\varepsilon_2)}{\varepsilon_2} q'' = \frac{q''}{\psi}\left(1 - \frac{\psi}{2} - \frac{3}{4}\psi\tau\right)$$

$$\sigma T^4(\tau) - \sigma T_2^4 = \frac{(1-\varepsilon_2)}{\varepsilon_2} q'' + q'' \left(\frac{1}{\psi} - \frac{1}{2} - \frac{3}{4}\tau \right)$$

$$\frac{q''}{\sigma\left(T_1^4 - T_2^4\right)} = \frac{1}{\frac{3}{4}\tau_0 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1} \equiv \alpha$$

$$\frac{T^4(\tau) - T_2^4}{T_1^4 - T_2^4} = \left(\frac{1}{\varepsilon_2} - \frac{3}{2} + \frac{1}{\psi} - \frac{3}{4}\tau \right)\alpha, \quad \psi = \frac{1}{1 + \frac{3}{4}\tau_0}$$

$$= \left(\frac{1}{\varepsilon_2} - \frac{3}{2} + 1 + \frac{3}{4}\tau_0 - \frac{3}{4}\tau \right)\alpha = \left[\frac{1}{\varepsilon_2} - \frac{1}{2} + \frac{3}{4}(\tau_0 - \tau) \right]\alpha$$

$$T^4(0) - T_1^4 = \left(\frac{1}{2} - \frac{1}{\varepsilon_1} \right) \frac{T_1^4 - T_2^4}{\frac{3}{4}\tau_0 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

Milne-Eddington Approximation

RTE for an isotropic scattering planar medium

$$\mu\frac{di(\tau,\mu)}{d\tau}+i(\tau,\mu)=\left(1-\omega_0\right)i_b(\tau)+\frac{\omega_0}{2}\int_{-1}^{1}i(\tau,\mu')d\mu'$$

apply both sides by the operator $2\pi\int_{-1}^{1}d\mu$

$$2\pi\int_{-1}^{1}\mu\frac{di(\tau,\mu)}{d\tau}d\mu=\frac{d}{d\tau}\left(2\pi\int_{-1}^{1}\mu i(\tau,\mu)d\mu\right)=\frac{dq''(\tau)}{d\tau}$$

$$2\pi\int_{-1}^{1}i(\tau,\mu)d\mu=G(\tau)$$

$$2\pi\int_{-1}^{1}i_b(\tau)d\mu=i_b(\tau)2\pi\int_{-1}^{1}d\mu\ =4\pi i_b(\tau)$$

$$2\pi\int_{-1}^{1}\int_{-1}^{1}i(\tau,\mu')d\mu'd\mu=\int_{-1}^{1}G(\tau)d\mu\ =2G(\tau)$$

$$\frac{dq''(\tau)}{d\tau} + G(\tau) = 4\pi\left(1-\omega_0\right)i_b(\tau) + \omega_0 G(\tau)$$

or $$\frac{dq''(\tau)}{d\tau} = \left(1-\omega_0\right)\left[4\pi i_b(\tau) - G(\tau)\right]$$

next apply by the operator $2\pi\int_{-1}^{1}\mu d\mu$

$$2\pi\int_{-1}^{1}\mu^2\frac{di(\tau,\mu)}{d\tau}d\mu = \frac{d}{d\tau}\left(2\pi\int_{-1}^{1}\mu^2 i(\tau,\mu)d\mu\right)$$

$$= c\frac{dP_r(\tau)}{d\tau}$$

radiative pressure

$$P_r(\tau) = \frac{1}{c}2\pi\int_{-1}^{1}\mu^2 i(\tau,\mu)d\mu$$

$$2\pi\int_{-1}^{1}\mu i(\tau,\mu)d\mu = q''(\tau)$$

$$2\pi\int_{-1}^{1}\mu i_b(\tau)d\mu = i_b(\tau)2\pi\int_{-1}^{1}\mu d\mu = 0$$

$$2\pi\int_{-1}^{1}\mu\left(\int_{-1}^{1} i(\tau,\mu')d\mu'\right)d\mu = G(\tau)2\pi\int_{-1}^{1}\mu d\mu = 0$$

then, $c\dfrac{dP_r(\tau)}{d\tau} = -q''(\tau)$

now we have three unknowns $q''(\tau), G(\tau), P_r(\tau)$, **but have only two equations.**

additional relation $\Rightarrow$ Milne-Eddington approximation

separating intensities

$$G(\tau) = 2\pi\left[\int_{-1}^{0} i^{-}(\tau,\mu)d\mu + \int_{0}^{1} i^{+}(\tau,\mu)d\mu\right]$$

$$q''(\tau) = 2\pi\left[\int_{-1}^{0} \mu i^{-}(\tau,\mu)d\mu + \int_{0}^{1} \mu i^{+}(\tau,\mu)d\mu\right]$$

$$P_r(\tau) = \frac{1}{c}2\pi\left[\int_{-1}^{0} \mu^2 i^{-}(\tau,\mu)d\mu + \int_{0}^{1} \mu^2 i^{+}(\tau,\mu)d\mu\right]$$

Assume that the intensity components are independent of direction.

$$i^{+}(\tau,\mu) = i^{+}(\tau),\;\; i^{-}(\tau,\mu) = i^{-}(\tau)$$

then, $$G(\tau) = 2\pi\left[i^+(\tau) + i^-(\tau)\right]$$

$$q''(\tau) = \pi\left[i^+(\tau) - i^-(\tau)\right]$$

$$P_r(\tau) = \frac{2\pi}{3c}\left[i^+(\tau) + i^-(\tau)\right]$$

Any two of these three relations can be used.

for example,

$$P_r(\tau) = \frac{1}{3c}G(\tau) \Rightarrow \frac{dP_r(\tau)}{d\tau} = \frac{1}{3c}\frac{dG(\tau)}{d\tau} = -\frac{1}{c}q''(\tau)$$

$$\Rightarrow \frac{1}{3}\frac{dG(\tau)}{d\tau} = -q''(\tau)$$

remark: $$c\frac{dP_r(\tau)}{d\tau} = -q''(\tau)$$

$$\frac{dq''(\tau)}{d\tau} = (1-\omega_0)\left[4\pi i_b(\tau) - G(\tau)\right]$$

Differentiating w.r.t. τ

$$\frac{d^2q''(\tau)}{d\tau^2} = (1-\omega_0)\left[4\pi \frac{di_b(\tau)}{d\tau} - \frac{dG(\tau)}{d\tau}\right]$$

$$\frac{1}{3}\frac{dG(\tau)}{d\tau} = -q''(\tau)$$

Milne-Eddington approximation

$$\frac{d^2q''(\tau)}{d\tau^2} = (1-\omega_0)\left[4\pi \frac{di_b(\tau)}{d\tau} + 3q''(\tau)\right]$$

or for $G(\tau)$ $$\frac{1}{3}\frac{dG(\tau)}{d\tau} = -q''(\tau)$$

$$\frac{dq''(\tau)}{d\tau} = \left(1-\omega_0\right)\left[4\pi i_b(\tau) - G(\tau)\right]$$

$$\frac{d^2G(\tau)}{d\tau^2} = -3\frac{dq''(\tau)}{d\tau} = 3\left(1-\omega_0\right)\left[G(\tau) - 4\pi i_b(\tau)\right]$$

Note: neglecting second order term from Milne-Eddington approximation

$$\frac{d^2q''(\tau)}{d\tau^2} = (1-\omega_0)\left[4\pi\frac{di_b(\tau)}{d\tau} + 3q''(\tau)\right]$$

$$q''(\tau) = -\frac{4\pi}{3}\frac{di_b(\tau)}{d\tau}$$ **: diffusion approximation**

Spherical Harmonics Approximation

P-N Approximation

Spherical harmonics

in spherical coordinate system

$$\nabla^2 F(r,\theta,\phi) = 0$$

separation of variables $F(r,\theta,\phi) = R(r)G(\theta,\phi)$

eigenfunction solution : $G(\theta,\phi)$ **surface harmonics**
(associated Legendre function)

that is, $G(\theta,\phi) = \Theta(\theta)\Phi(\phi)$

$$\Phi'' + m^2\Phi = 0$$

$$\left(1-x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left[n\left(n+1\right) - \frac{m^2}{1-x^2}\right]\Theta = 0,$$

$$x = \cos\theta$$

solution

$$Y_n^m(\theta,\phi) = \left[\frac{2n+1}{4\pi}\frac{\left(n-m\right)!}{\left(n+m\right)!}\right]^{1/2} e^{im\phi} P_n^m(\cos\theta)$$

$$= (-1)^{(n+|m|)/2}\left[\frac{2n+1}{4\pi}\frac{\left(n-m\right)!}{\left(n+m\right)!}\right]^{1/2} e^{im\phi} P_n^{|m|}(\cos\theta)$$

conjugate of $Y_n^m(\theta,\phi)$

$$Y_n^{m*}(\theta,\phi) = (-1)^m Y_n^{-m}(\theta,\phi)$$

orthogonality

$$\int_{4\pi} Y_n^m(\omega) Y_r^{s*}(\omega) d\omega = \delta_{nr}\delta_{ms}$$

P-N approximation

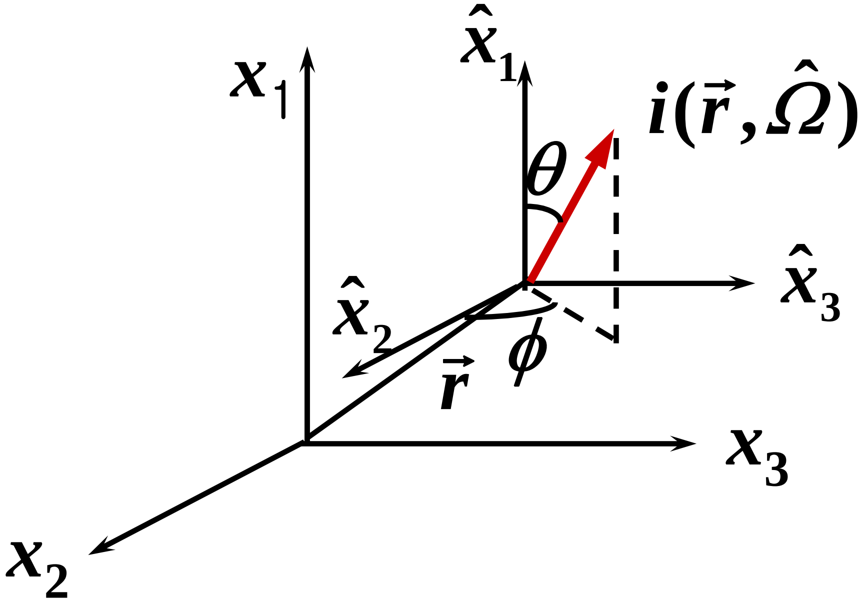
$$i(\vec{r},\hat{\Omega}) = \sum_{n=0}^{N} \sum_{m=-n}^{n} A_n^m(\vec{r}) Y_n^m(\hat{\Omega})$$

Using orthogonality

$$A_n^m(\vec{r}) = \int_{4\pi} (-1)^m Y_n^{-m}(\hat{\Omega}) i(\vec{r},\hat{\Omega}) d\omega$$

$$= (-1)^{(n+|m|)/2} \left[\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right]^{1/2} \int_{4\pi} i(\vec{r},\hat{\Omega}) e^{-im\phi} P_n^{|m|}(\cos\theta) d\omega$$

P-1 approximation



$$i(\vec{r},\hat{\Omega}) = \sum_{n=0}^{N}\sum_{m=-n}^{n} A_n^m(\vec{r})Y_n^m(\hat{\Omega})$$

$$i(\vec{r},\hat{\Omega}) = A_0^0Y_0^0 + A_1^{-1}Y_1^{-1} + A_1^0Y_1^0 + A_1^1Y_1^1$$

$$A_0^0Y_0^0 = \frac{1}{4\pi}I_0, \quad A_1^0Y_1^0 = \frac{3}{4\pi}I_1\ell_1$$

$$A_1^{-1}Y_1^{-1} + A_1^1Y_1^1 = \frac{3}{4\pi}\left[I_2\ell_2 + I_3\ell_3\right]$$

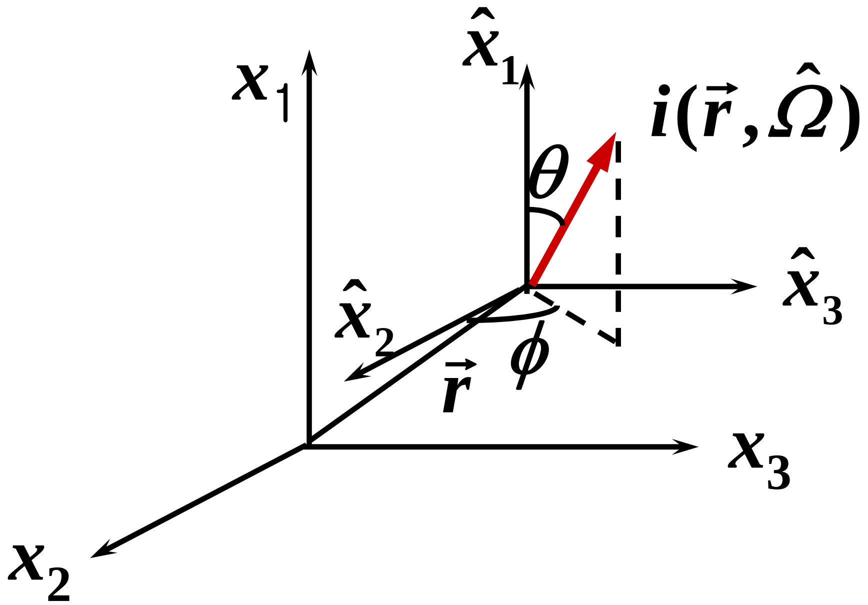
$$i(\vec{r},\Omega) = \frac{1}{4\pi}\left[I_0(\vec{r}) + 3\left(I_1(\vec{r})\ell_1 + I_2(\vec{r})\ell_2 + I_3(\vec{r})\ell_3\right)\right]$$

direction cosines

$$\ell_1 = \cos\theta,$$

$$\ell_2 = \sin\theta\cos\phi,$$

$$\ell_3 = \sin\theta\sin\phi$$



moments of intensity

$$I_0 = \int_{4\pi} i(\vec{r},\hat{\Omega})d\omega, \quad I_i = \int_{4\pi} i(\vec{r},\hat{\Omega})\ell_i d\omega,$$

$$I_{ij} = \int_{4\pi} i(\vec{r},\hat{\Omega})\ell_i \ell_j d\omega, \ \cdots$$

physical meaning of moments

$$I_0 = G, \ \ I_i = q_i''$$

Example: non-scattering planar medium with diffuse boundaries

$$\mu\frac{\partial i}{\partial \tau}+i=i_b,\quad \mu=\cos\theta=\ell_1$$

$$\int_{4\pi}\mu\frac{\partial i}{\partial \tau}d\omega+\int_{4\pi}i\,d\omega=\int_{4\pi}i_b\,d\omega \rightarrow \frac{dI_1}{d\tau}+I_0=4\pi i_b$$

$$\int_{4\pi}\mu^2\frac{\partial i}{\partial \tau}d\omega+\int_{4\pi}\mu i\,d\omega=\int_{4\pi}\mu i_b\,d\omega \rightarrow \frac{dI_{11}}{d\tau}+I_1=0$$

closure condition

$$i(\vec{r},\Omega)=\frac{1}{4\pi}\left[I_0(\vec{r})+3\left(I_1(\vec{r})\ell_1+I_2(\vec{r})\ell_2+I_3(\vec{r})\ell_3\right)\right]$$

$$\int_{4\pi}\ell_1^2 i(\vec{r},\hat{\Omega})d\omega$$

$$=\frac{1}{4\pi}\int_{4\pi}\ell_1^2\left[I_0(\vec{r})+3\left\{I_1(\vec{r})\ell_1+I_2(\vec{r})\ell_2+I_3(\vec{r})\ell_3\right\}\right]d\omega$$

$$\int_{4\pi} \ell_1^2 i(\vec{r},\hat{\Omega}) d\omega = I_{11}$$

$$= \frac{1}{4\pi} \int_{4\pi} \ell_1^2 \left[I_0(\vec{r}) + 3\left\{ I_1(\vec{r})\ell_1 + I_2(\vec{r})\ell_2 + I_3(\vec{r})\ell_3 \right\} \right] d\omega$$

$$\frac{1}{4\pi} \int_{4\pi} \ell_1^2 I_0(\vec{r}) d\omega = \frac{1}{2} I_0 \int_0^{\pi} \cos^2\theta \sin\theta d\theta$$

$$= \frac{1}{2} I_0 \int_{-1}^{1} t^2 dt = \frac{1}{2} I_0 \frac{2}{3} = \frac{1}{3} I_0$$

$$\int_{4\pi} \ell_1^3 I_1(\vec{r}) d\omega = \int_{4\pi} \ell_1^2 \ell_2 I_2(\vec{r}) d\omega = \int_{4\pi} \ell_1^2 \ell_3 I_3(\vec{r}) d\omega = 0$$

$$I_{11} = \frac{1}{3} I_0$$

$$I_{11} = \frac{1}{3} I_0$$

$$\frac{dI_1}{d\tau} + I_0 = 4\pi i_b, \quad \frac{dI_{11}}{d\tau} + I_1 = 0$$

therefore, $\frac{1}{3}\frac{dI_0}{d\tau} + I_1 = 0$ **or** $I_1 = -\frac{1}{3}\frac{dI_0}{d\tau}$

finally

$$\frac{d^2 I_0}{d\tau^2} - 3I_0 = -12\pi i_b$$

$$q''(\tau) = I_1(\tau) = -\frac{1}{3}\frac{dI_0}{d\tau},$$

$$\frac{dq''}{d\tau} = -\frac{1}{3}\frac{d^2 I_0}{d\tau^2} = \frac{dI_1}{d\tau} = 4\pi i_b - I_0$$

Marshak's boundary condition

$$i(0,\mu) = f_1(\mu) \quad \mu > 0, \quad i(\tau_0,\mu) = f_2(\mu) \quad \mu < 0$$

$$\int_0^1 i(0,\mu)\mu^{2i-1} d\mu = \int_0^1 f_1(\mu)\mu^{2i-1} d\mu$$

$$\int_{-1}^0 i(\tau_0,\mu)\mu^{2i-1} d\mu = \int_{-1}^0 f_2(\mu)\mu^{2i-1} d\mu$$

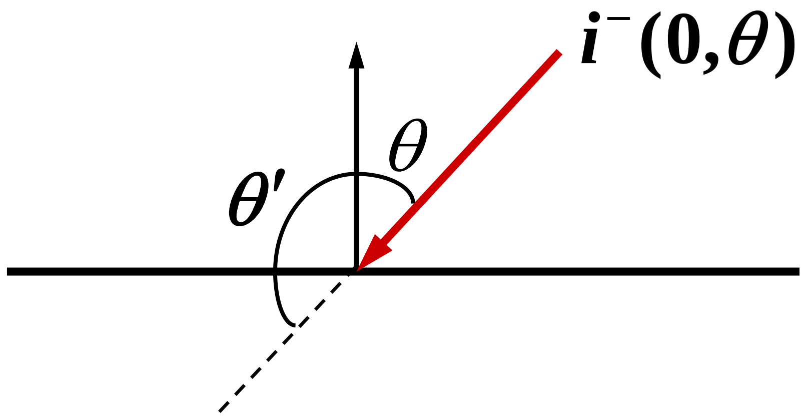
$$i = 1,2,3,\cdots,\frac{1}{2}(N+1)$$

P-1 approximation

$$\int_0^1 i(0,\mu)\mu d\mu = \int_0^1 f_1(\mu)\mu d\mu$$

$$\int_{-1}^0 i(\tau_0,\mu)\mu d\mu = \int_{-1}^0 f_2(\mu)\mu d\mu$$

at $\tau = 0$



$$J_1 = \varepsilon_1 e_{b1} + \rho_1 G_1 \qquad \textbf{or} \qquad \pi i^{+}(0) = \varepsilon_1 \pi i_{b1} + \rho_1 G_1$$

$$i^{+}(0) = \varepsilon_1 i_{b1} + \frac{\rho_1}{\pi}\int_{\omega=2\pi} i^{-}(0,\theta)\cos\theta \, d\omega$$

$$\int_{\omega=2\pi} i^{-}(0,\theta)\cos\theta \, d\omega = 2\pi\int_0^{\pi/2} i^{-}(0,\theta)\cos\theta\sin\theta \, d\theta$$

let $\theta = \pi - \theta'$

$$2\pi\int_0^{\pi/2} i^{-}(0,\theta)\cos\theta\sin\theta \, d\theta$$
$$= 2\pi\int_{\pi}^{\pi/2} i^{-}(0,\theta')\cos(\pi-\theta')\sin(\pi-\theta')\left(-d\theta'\right)$$

$$= 2\pi \int_{\pi}^{\pi/2} i^{-}(0,\theta')\cos\theta' \sin\theta' d\theta'$$

$$= 2\pi \int_{\pi/2}^{\pi} i^{-}(0,\theta')\cos\theta' d\left(\cos\theta'\right)$$

$$i(\vec{r},\hat{\Omega}) = \frac{1}{4\pi}\left[I_0(\vec{r}) + 3\left(I_1(\vec{r})\ell_1 + I_2(\vec{r})\ell_2 + I_3(\vec{r})\ell_3 \right)\right]$$

$$= 2\pi \int_{\pi/2}^{\pi} \frac{1}{4\pi}\left[I_0(0) + 3\left(I_1(0)\cos\theta' \right.\right.$$

$$\left.\left. + I_2(0)\sin\theta' \cos\phi' + I_3(0)\sin\theta' \sin\phi' \right)\right]\cos\theta' d\left(\cos\theta'\right)$$

$$= \frac{1}{2}\int_{0}^{-1} \left(I_0 + 3I_1\mu \right)\mu d\mu$$

$$= \frac{1}{2}\left[I_0 \frac{\mu^2}{2} + I_1\mu^3 \right]_0^{-1} = \frac{1}{2}\left(\frac{1}{2} I_0 - I_1 \right)$$

$$i^+(0) = \varepsilon_1 i_{b1} + \frac{\rho_1}{\pi}\frac{1}{2}\left(\frac{1}{2}I_0 - I_1\right)$$

apply Marshak's boundary condition

$$\int_0^1 i(0,\mu)\mu d\mu = \int_0^1 f_1(\mu)\mu d\mu$$

$$\int_0^1 \frac{1}{4\pi}\left[I_0 + 3\left(I_1\cos\theta + I_2\sin\theta\cos\phi + I_3\sin\theta\sin\phi\right)\right]\mu d\mu$$

$$= \int_0^1 \left[\varepsilon_1 i_{b1} + \frac{\rho_1}{\pi}\frac{1}{2}\left(\frac{1}{2}I_0 - I_1\right)\right]\mu d\mu$$

$$\frac{1}{4\pi}\int_0^1 \left(I_0 + 3I_1\mu\right)\mu d\mu = \left[\varepsilon_1 i_{b1} + \frac{\rho_1}{2\pi}\left(\frac{1}{2}I_0 - I_1\right)\right]\int_0^1 \mu d\mu$$

$$\frac{1}{4\pi}\left[I_0\frac{\mu^2}{2}+I_1\mu^3\right]_0^1=\frac{1}{2}\left[\varepsilon_1 i_{b1}+\frac{\rho_1}{2\pi}\left(\frac{1}{2}I_0-I_1\right)\right]$$

$$\frac{1}{2}I_0+I_1=2\pi\varepsilon_1 i_{b1}+\frac{\rho_1}{2}I_0-\rho_1 I_1$$

$$\frac{\varepsilon_1}{2}I_0+\left(1+\rho_1\right)I_1=2\pi\varepsilon_1 i_{b1}$$

or $$I_0+2\frac{1+\rho_1}{\varepsilon_1}I_1=4\pi i_{b1}$$

let $$\frac{1+\rho_1}{\varepsilon_1}=\frac{1+\left(1-\varepsilon_1\right)}{\varepsilon_1}=\frac{2-\varepsilon_1}{\varepsilon_1}\equiv 1+2\lambda_1,$$

$$\lambda_1=\frac{\rho_1}{\varepsilon_1}=\frac{1-\varepsilon_1}{\varepsilon_1}$$

then, $I_0 + 2(1 + 2\lambda_1) I_1 = 4\pi i_{b1}$

but $I_1 = -\frac{1}{3}\frac{dI_0}{d\tau}$

thus, at $\tau = 0$,

$$I_0 - \frac{2}{3}(1 + 2\lambda_1)\frac{dI_0}{d\tau} = 4\pi i_{b1}$$

similarly at $\tau = \tau_0$,

$$I_0 + \frac{2}{3}(1 + 2\lambda_2)\frac{dI_0}{d\tau} = 4\pi i_{b2}$$

radiative equilibrium

$$\frac{dq''}{d\tau} = 4\pi i_b - I_0 = 0 \ \rightarrow i_b = \frac{I_0}{4\pi} \quad \text{or} \quad I_0(\tau) = 4\sigma T^4(\tau)$$

$$\frac{d^2 I_0}{d\tau^2} - 3I_0 = -12\pi i_b \qquad \text{thus,} \quad \frac{d^2 I_0}{d\tau^2} = 0$$

general solution $I_0(\tau) = A\tau + B$

boundary conditions $$I_0 - \frac{2}{3}\left(1 + 2\lambda_1\right)\frac{dI_0}{d\tau} = 4\pi i_{b1}$$

$$I_0 + \frac{2}{3}\left(1 + 2\lambda_2\right)\frac{dI_0}{d\tau} = 4\pi i_{b2}$$

$$B - \frac{2}{3}\left(1 + 2\lambda_1\right)A = 4\sigma T_1^4,$$

$$A\tau_0 + B + \frac{2}{3}\left(1 + 2\lambda_2\right)A = 4\sigma T_2^4, \ \ \lambda_1 = \frac{1}{\varepsilon_1} - 1$$

solution $$\frac{q''}{\sigma\left(T_1^4 - T_2^4\right)} = \frac{1}{\frac{3}{4}\tau_0 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

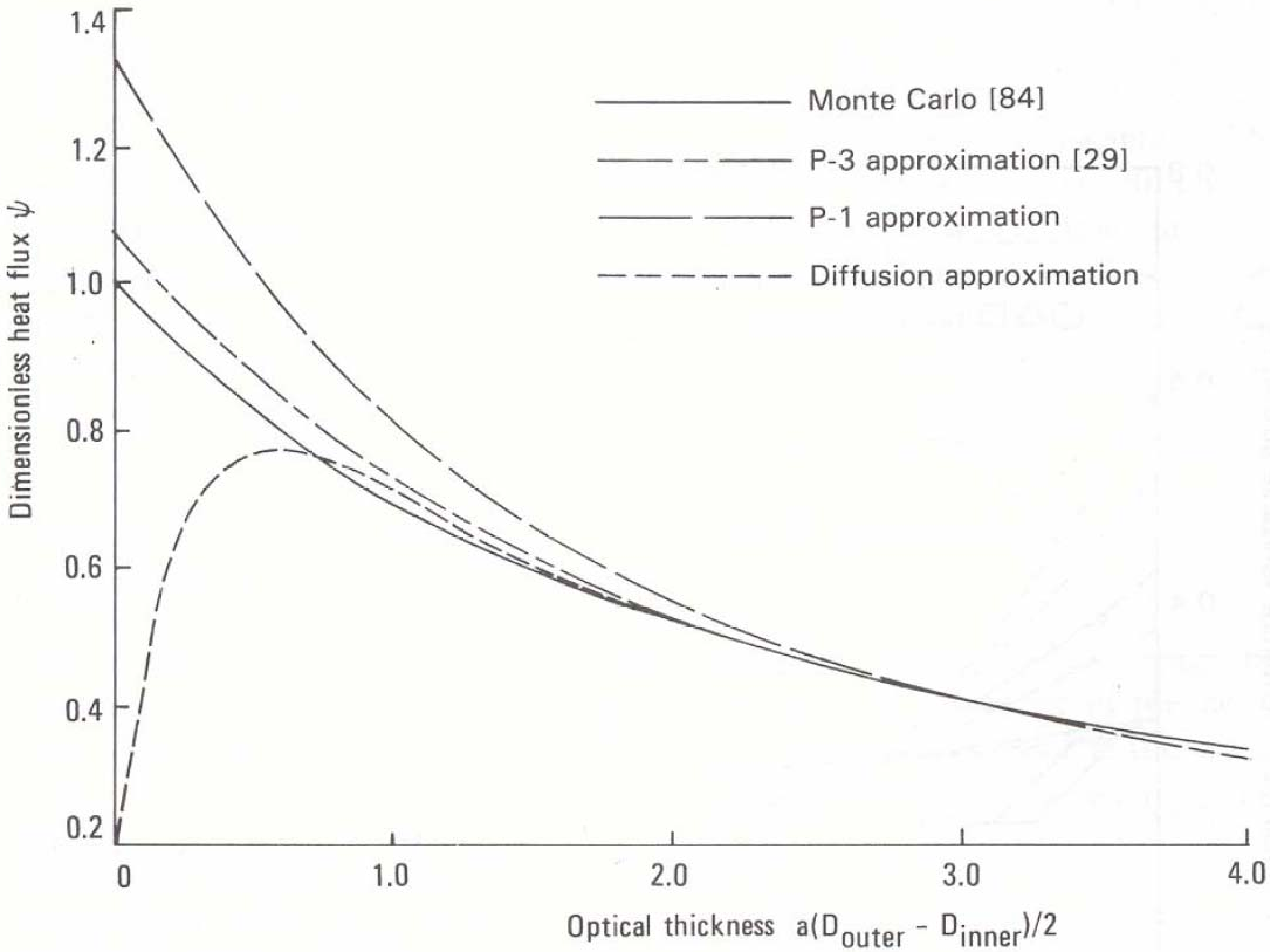


Figure 15-10 Comparison of solutions of energy transfer between infinitely long concentric black cylinders enclosing gray medium; $D_{\text{inner}}/D_{\text{outer}} = 0.5$.

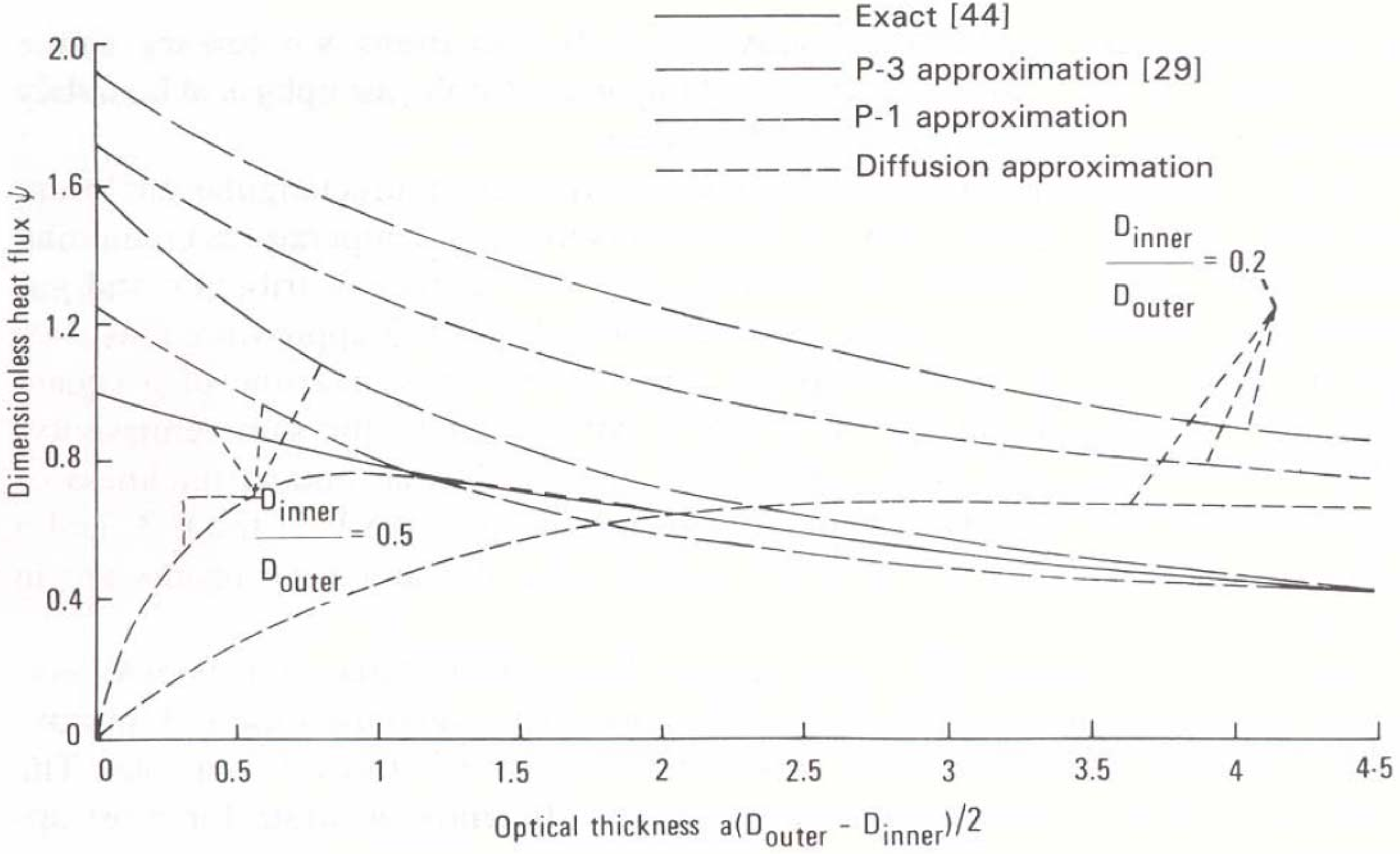


Figure 15-11 Comparison of solutions of energy transfer between black concentric spheres enclosing a gray medium.

Shuster-Schwarzchild Approximation

two-flux model

radiation stream

$$j^{+}(\tau) \equiv \int_{0}^{1} i^{+}(\tau,\mu)d\mu, \quad 0 \le \mu \le 1$$

$$j^{-}(\tau) \equiv \int_{-1}^{0} i^{-}(\tau,\mu)d\mu, \quad -1 \le \mu \le 0$$

$$\mu \frac{di(\tau,\mu)}{d\tau} + i(\tau,\mu) = (1-\omega_0) i_b(\tau) + \frac{\omega_0}{2}\int_{-1}^{1} i(\tau,\mu')d\mu'$$

$$\mu \frac{di^{+}(\tau,\mu)}{d\tau} + i^{+}(\tau,\mu) = (1-\omega_0) i_b(\tau) + \frac{\omega_0}{2}\left(j^{+}(\tau) + j^{-}(\tau)\right) \quad (1)$$

$$\mu \frac{di^{-}(\tau,\mu)}{d\tau} + i^{-}(\tau,\mu) = (1-\omega_0) i_b(\tau) + \frac{\omega_0}{2}\left(j^{+}(\tau) + j^{-}(\tau)\right) \quad (2)$$

operate eq.(1) by $\int_0^1 d\mu$ and eq.(2) by $\int_{-1}^0 d\mu$

$$\mu \frac{di^+(\tau,\mu)}{d\tau} + i^+(\tau,\mu) = \left(1-\omega_0\right) i_b(\tau) + \frac{\omega_0}{2}\left(j^+(\tau) + j^-(\tau)\right) \quad \textbf{(1)}$$

$$\frac{d}{d\tau}\left[\int_0^1 \mu i^+(\tau,\mu) d\mu\right] + j^+(\tau)$$

$$= \left(1-\omega_0\right) i_b(\tau) + \frac{\omega_0}{2}\left(j^+(\tau) + j^-(\tau)\right)$$

$$\mu \frac{di^-(\tau,\mu)}{d\tau} + i^-(\tau,\mu) = \left(1-\omega_0\right) i_b(\tau) + \frac{\omega_0}{2}\left(j^+(\tau) + j^-(\tau)\right) \quad \textbf{(2)}$$

$$\frac{d}{d\tau}\left[\int_{-1}^0 \mu i^-(\tau,\mu) d\mu\right] + j^-(\tau)$$

$$= \left(1-\omega_0\right) i_b(\tau) + \frac{\omega_0}{2}\left(j^+(\tau) + j^-(\tau)\right)$$

Shuster-Schwartzchild approximation

$$\int_0^1 \mu i^+(\tau,\mu)d\mu \cong \frac{1}{2}\int_0^1 i^+(\tau,\mu)d\mu = \frac{1}{2}j^+(\tau)$$

$$\int_{-1}^0 \mu i^-(\tau,\mu)d\mu \cong -\frac{1}{2}\int_{-1}^0 i^-(\tau,\mu)d\mu = -\frac{1}{2}j^-(\tau)$$

$$\frac{d}{d\tau}\left[\int_0^1 \mu i^+(\tau,\mu)d\mu\right] = \frac{1}{2}\frac{dj^+}{d\tau}$$

$$\frac{1}{2}\frac{dj^+}{d\tau} + j^+ = \left(1-\omega_0\right)i_b + \frac{\omega_0}{2}\left(j^+ + j^-\right)$$

$$\frac{d}{d\tau}\left[\int_{-1}^0 \mu i^-(\tau,\mu)d\mu\right] = -\frac{1}{2}\frac{dj^-}{d\tau}$$

$$-\frac{1}{2}\frac{dj^-}{d\tau} + j^- = \left(1-\omega_0\right)i_b + \frac{\omega_0}{2}\left(j^+ + j^-\right)$$

Heat flux

$$q''(\tau) = 2\pi\int_{-1}^{1}\mu i(\tau,\mu)d\mu$$

$$= 2\pi\left[\int_{0}^{1}\mu i^{+}(\tau,\mu)d\mu + \int_{-1}^{0}\mu i^{-}(\tau,\mu)d\mu\right]$$

$$= \pi\left(j^{+} - j^{-}\right)$$

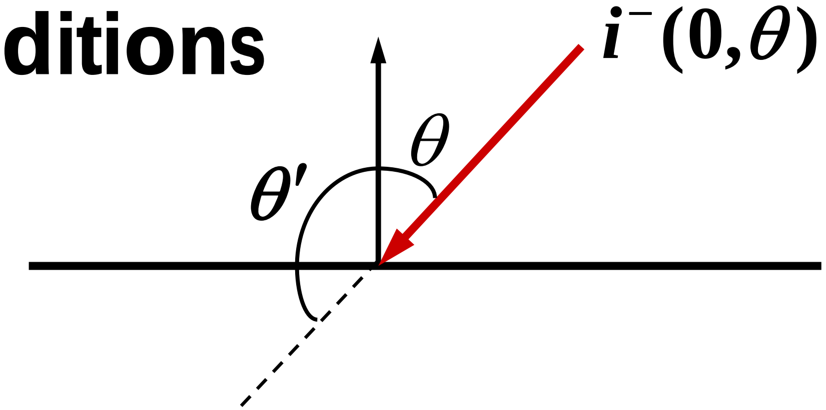
$$\frac{dq''}{d\tau} = \left(1-\omega_{0}\right)\left[4\pi i_{b} - G\right]$$

$$G = 2\pi\int_{-1}^{1} i(\tau,\mu')d\mu' = 2\pi\left(j^{+} + j^{-}\right)$$

$$\frac{dq''}{d\tau} = 2\pi\left(1-\omega_{0}\right)\left[2i_{b} - \left(j^{+} + j^{-}\right)\right]$$

boundary conditions

at $\tau = 0$



$$J_1 = \varepsilon_1 \sigma T_1^4 + (1-\varepsilon_1) G_1$$

$$\pi i^+(0) = \varepsilon_1 \sigma T_1^4 + (1-\varepsilon_1)\int_{\omega=2\pi} i^-(0,\theta)\cos\theta d\omega$$

$$\int_{\omega=2\pi} i^-(0,\theta)\cos\theta d\omega = 2\pi\int_0^{\pi/2} i^-(0,\theta)\cos\theta\sin\theta d\theta$$

let $\theta = \pi - \theta'$

$$2\pi\int_0^{\pi/2} i^-(0,\theta)\cos\theta\sin\theta d\theta$$

$$= 2\pi\int_{\pi}^{\pi/2} i^-(0,\theta')\cos(\pi-\theta')\sin(\pi-\theta')(-d\theta')$$

$$= 2\pi\int_{\pi}^{\pi/2} i^{-}(0,\theta')\cos\theta'\sin\theta' d\theta'$$

$$= -2\pi\int_{\pi}^{\pi/2} i^{-}(0,\theta')\cos\theta' d\left(\cos\theta'\right)$$

$$= -2\pi\int_{-1}^{0} i^{-}(0,\mu)\mu d\mu$$

$$\pi i^{+}(0) = \varepsilon_1\sigma T_1^4 + \left(1-\varepsilon_1\right)\int_{\omega=2\pi} i^{-}(0,\theta)\cos\theta d\omega$$

$$i^{+}(0) = \frac{\varepsilon_1\sigma T_1^4}{\pi} - 2\left(1-\varepsilon_1\right)\int_{-1}^{0}\mu i^{-}(0,\mu)d\mu$$

similarly

$$i^{-}(\tau_0) = \frac{\varepsilon_2\sigma T_2^4}{\pi} + 2\left(1-\varepsilon_1\right)\int_{0}^{1}\mu i^{+}(\tau_0,\mu)d\mu$$

$$i^+(0) = \frac{\varepsilon_1 \sigma T_1^4}{\pi} - 2(1-\varepsilon_1)\int_{-1}^{0} \mu i^-(0,\mu) d\mu$$

$$i^-(\tau_0) = \frac{\varepsilon_2 \sigma T_2^4}{\pi} + 2(1-\varepsilon_1)\int_{0}^{1} \mu i^+(\tau_0,\mu) d\mu$$

introducing approximation

$$i^+(0) = \frac{\varepsilon_1 \sigma T_1^4}{\pi} + (1-\varepsilon_1) j^-(0)$$

$$i^-(\tau_0) = \frac{\varepsilon_2 \sigma T_2^4}{\pi} + (1-\varepsilon_2) j^+(\tau_0)$$

operate $\int_0^1 d\mu$ **and** $\int_{-1}^0 d\mu$

$$j^+(0) = \frac{\varepsilon_1 \sigma T_1^4}{\pi} + (1-\varepsilon_1) j^-(0)$$

$$j^-(\tau_0) = \frac{\varepsilon_2 \sigma T_2^4}{\pi} + (1-\varepsilon_2) j^+(\tau_0)$$

Example: non-scattering planar medium with diffuse boundaries

RTE

$$\frac{1}{2}\frac{dj^+}{d\tau}+j^+=(1-\omega_0)i_b+\frac{\omega_0}{2}\left(j^++j^-\right)\rightarrow\ \frac{1}{2}\frac{dj^+}{d\tau}+j^+=i_b$$

$$-\frac{1}{2}\frac{dj^-}{d\tau}+j^-=(1-\omega_0)i_b+\frac{\omega_0}{2}\left(j^++j^-\right)\rightarrow\ -\frac{1}{2}\frac{dj^-}{d\tau}+j^-=i_b$$

radiative equilibrium

$$\frac{dq''}{d\tau}=2\pi(1-\omega_0)\left[2i_b-\left(j^++j^-\right)\right]$$

$$\frac{dq''}{d\tau}=0\Rightarrow 2i_b-\left(j^++j^-\right)=0\Rightarrow i_b=\frac{1}{2}\left(j^++j^-\right)$$

$$\frac{1}{2}\frac{dj^+}{d\tau} + j^+ = i_b \rightarrow \frac{1}{2}\frac{dj^+}{d\tau} + j^+ = \frac{1}{2}\left(j^+ + j^-\right)$$

$$\frac{dj^+}{d\tau} + j^+ = j^-$$

$$-\frac{1}{2}\frac{dj^-}{d\tau} + j^- = i_b \rightarrow -\frac{1}{2}\frac{dj^-}{d\tau} + j^- = \frac{1}{2}\left(j^+ + j^-\right)$$

$$-\frac{dj^-}{d\tau} + j^- = j^+$$

general solution

$$j^+ = A\tau + B, \quad j^- = A\tau + (A + B)$$

boundary conditions

$$j^{+}(0) = \frac{\varepsilon_1 \sigma T_1^4}{\pi} + \left(1 - \varepsilon_1\right) j^{-}(0)$$

$$j^{-}(\tau_0) = \frac{\varepsilon_2 \sigma T_2^4}{\pi} + \left(1 - \varepsilon_2\right) j^{+}(\tau_0)$$

$$j^{+} = A\tau + B, \quad j^{-} = A\tau + (A + B)$$

$$B = \frac{\varepsilon_1 \sigma T_1^4}{\pi} + \left(1 - \varepsilon_1\right)\left(A + B\right)$$

$$A\tau_0 + A + B = \frac{\varepsilon_2 \sigma T_2^4}{\pi} + \left(1 - \varepsilon_2\right)\left(A\tau_0 + B\right)$$

solution

$$q''(\tau) = 2\pi\int_{-1}^{1}\mu i(\tau,\mu)d\mu = \pi\left(j^{+} - j^{-}\right)$$

$$\frac{q''}{\sigma\left(T_1^4 - T_2^4\right)} = \frac{1}{\tau_0 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

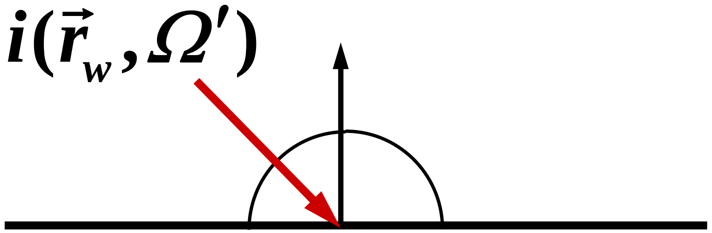
Discrete Ordinate Method

S-N Approximation

a discrete representation of the directional variation of the radiative intensity: a finite differencing of the directional dependence of the equation of transfer

$$\frac{di}{ds} = \hat{\Omega}\cdot\nabla i(\vec{r},\hat{\Omega}) = -\kappa(\vec{r})i(\vec{r},\hat{\Omega}) + a(\vec{r})i_b(\vec{r})$$
$$+\frac{\sigma_s(\vec{r})}{4\pi}\int_{4\pi} i(\vec{r},\hat{\Omega}')P(\vec{r},\hat{\Omega}',\hat{\Omega})d\omega'$$

for diffusely emitting and reflecting walls



$$i(\vec{r}_w) = \varepsilon(\vec{r}_w)i_b(\vec{r}_w) + \frac{\rho(\vec{r}_w)}{\pi}\int_{\cap} i(\vec{r}_w,\hat{\Omega}')\,|\,\hat{\Omega}'\cdot\hat{n}\,|\,d\omega'$$

Discrete ordinate equations

for a set of n different directions $\hat{\Omega}_i, \; i = 1, 2, \cdots, n$

$$\int_{4\pi} f(\hat{\Omega}) d\omega \simeq \sum_{i=1}^{n} w_i f(\hat{\Omega}_i)$$

w_i: quadrature weights associated with the direction $\hat{\Omega}_i$

$$\vec{\Omega}_i \cdot \nabla i(\vec{r}, \hat{\Omega}_i) = -\kappa(\vec{r}) i(\vec{r}, \hat{\Omega}_i) + a(\vec{r}) i_b(\vec{r}) + \frac{\sigma_s(\vec{r})}{4\pi} \sum_{j=1}^{n} w_j i(\vec{r}, \hat{\Omega}_j) P(\vec{r}, \hat{\Omega}_i, \hat{\Omega}_j)$$

$$i(\vec{r}_w) = \varepsilon(\vec{r}_w) i_b(\vec{r}_w) + \frac{\rho(\vec{r}_w)}{\pi} \sum_{\hat{n}\cdot\hat{\Omega}_j < 0} w_j i(\vec{r}_w, \hat{\Omega}_j) \left| \hat{n} \cdot \Omega_j \right|, \; \hat{n} \cdot \hat{\Omega}_i > 0$$

n simultaneous, first order, linear differential equations for the unknown $i_i(\vec{r}) = i(\vec{r}, \hat{\Omega}_i)$

$\hat{\Omega}_i$ **intersects the enclosure twice: emanates from the wall** $(\hat{n} \cdot \hat{\Omega}_i > 0)$**, and strikes the wall to be absorbed or reflected** $(\hat{n} \cdot \hat{\Omega}_i < 0)$

$$\vec{q}''(\vec{r}) = \int_{4\pi} i(\vec{r}, \hat{\Omega})\hat{\Omega}\, d\omega \simeq \sum_{i=1}^{n} w_i i_i(\vec{r})\hat{\Omega}_i$$

$$G(\vec{r}) = \int_{4\pi} i(\vec{r}, \hat{\Omega})\, d\omega \simeq \sum_{i=1}^{n} w_i i_i(\vec{r})$$

Selection of discrete ordinate directions

the choice of quadrature scheme: arbitrary

different sets of ordinates may results in considerably different accuracy

directions $\hat{\Omega}_i$ and quadrature weights w_i

customary conditions

1) completely symmetric (sets that are invariant after any rotation of $90°$): symmetry requirement

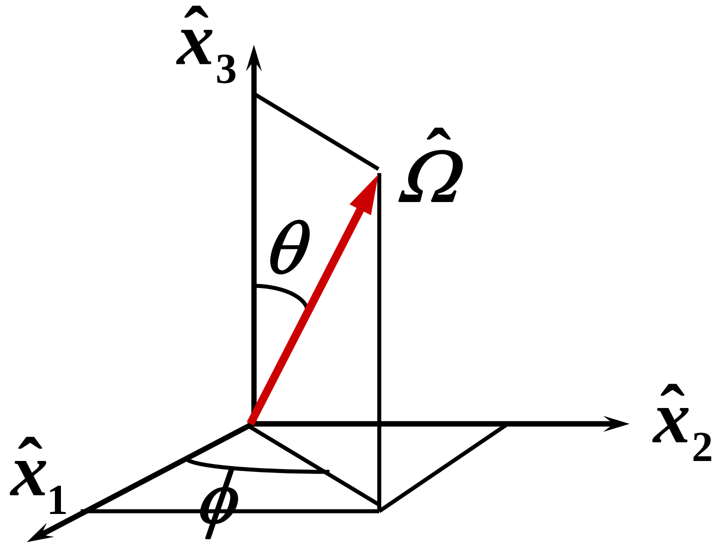
2) zeroth, first and second moments: moment equation

3) half-moment equation: a half range of 2π at the walls

$$\int_{4\pi} d\omega = 4\pi = \sum_{i=1}^{n} w_i$$

$$\int_{4\pi} \hat{\Omega} d\omega = \vec{0} = \sum_{i=1}^{n} w_i \hat{\Omega}_i$$

$$\int_{4\pi} \hat{\Omega}\hat{\Omega} d\omega = \frac{4\pi}{3}\underset{\sim}{\delta} = \sum_{i=1}^{n} w_i \hat{\Omega}_i \hat{\Omega}_i$$



$$\ell_1 = \sin\theta\cos\phi, \quad \ell_2 = \sin\theta\sin\phi, \quad \ell_3 = \cos\theta$$

$$\int_{4\pi} (\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}) d\omega$$

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta\cos\phi\hat{i}\sin\theta d\theta d\phi = 0 \quad \cdots$$

$$\ell_1 = \sin\theta\cos\phi, \quad \ell_2 = \sin\theta\sin\phi, \quad \ell_3 = \cos\theta$$

$$\int_{4\pi} \hat{\Omega}\hat{\Omega} d\omega$$

$$= \int_0^{2\pi}\int_0^{\pi} \begin{pmatrix} \sin^2\theta\cos^2\phi & \sin^2\theta\sin\phi\cos\phi & \sin\theta\cos\theta\cos\phi \\ \sin^2\theta\sin\phi\cos\phi & \sin^2\theta\sin^2\phi & \sin\theta\cos\theta\sin\phi \\ \sin\theta\cos\theta\cos\phi & \sin\theta\cos\theta\sin\phi & \cos^2\theta \end{pmatrix} \sin\theta d\theta d\phi$$

$$= \int_0^{\pi} \begin{pmatrix} \pi\sin^2\theta & 0 & 0 \\ 0 & \pi\sin^2\theta & 0 \\ 0 & 0 & 2\pi\cos^2\theta \end{pmatrix} \sin\theta d\theta$$

$$= \frac{4\pi}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{4\pi}{3} \underset{\sim}{\delta}$$

$$\int_0^{\pi} \sin^3\theta d\theta = \frac{4}{3}, \quad \int_0^{\pi} \cos^2\theta\sin\theta d\theta = \frac{2}{3}$$

first moment over a half range at walls

$$\int_{\hat{n}\cdot\hat{\Omega}<0} \left|\hat{n}\cdot\hat{\Omega}\right| d\omega = \int_{\hat{n}\cdot\hat{\Omega}>0} \hat{n}\cdot\hat{\Omega}\, d\omega = \pi = \sum_{\hat{n}\cdot\hat{\Omega}_i>0} w_i \hat{n}\cdot\hat{\Omega}_i$$

in Table

$$\hat{\Omega}_i = \left(\hat{\Omega}_i\cdot\hat{i}\right)\hat{i} + \left(\hat{\Omega}_i\cdot\hat{j}\right)\hat{j} + \left(\hat{\Omega}_i\cdot\hat{k}\right)\hat{k} = \xi_i\hat{i} + \eta_i\hat{j} + \mu_i\hat{k}$$

Table: covering one eighth of the total range of solid angle 4π

→ each row of ordinates contains 8 different directions

Since symmetric S_2-approximation does not satisfy the half-moment condition, a non-symmetric S_2-approximation is also included.

TABLE

Discrete ordinates for the S_N-approximation (N =2, 4, 6, 8)

Order of Approximation	Ordinates ξ	η	μ	Weights w
S_2 (symmetric)	0.5773503	0.5773503	0.5773503	1.5707963
S_2 (nonsymmetric)	0.5000000	0.7071068	0.5000000	1.5707963
S_4	0.2958759	0.2958759	0.9082483	0.5235987
	0.2958759	0.9082483	0.2958759	0.5235987
	0.9082483	0.2958759	0.2958759	0.5235987
S_6	0.1838670	0.1838670	0.9656013	0.1609517
	0.1838670	0.6950514	0.6950514	0.3626469
	0.1838670	0.9656013	0.1838670	0.1609517
	0.6950514	0.1838670	0.6950514	0.3626469
	0.6950514	0.6950514	0.1838670	0.3626469
	0.9656013	0.1838670	0.1838670	0.1609517
S_8	0.1422555	0.1422555	0.9795543	0.1712359
	0.1422555	0.5773503	0.8040087	0.0992284
	0.1422555	0.8040087	0.5773503	0.0992284
	0.1422555	0.9795543	0.1422555	0.1712359
	0.5773503	0.1422555	0.8040087	0.0992284
	0.5773503	0.5773503	0.5773503	0.4617179
	0.5773503	0.8040087	0.1422555	0.0992284
	0.8040087	0.1422555	0.5773503	0.0992284
	0.8040087	0.5773503	0.1422555	0.0992284
	0.9795543	0.1422555	0.1422555	0.1712359

$$4\pi = 12.5663706$$

Ex) S_2 approximation (symmetric)

$$\xi = \eta = \mu = 0.5773503$$

$$\hat{\Omega}_1 = 0.5773503\left(\hat{i} + \hat{j} + \hat{k}\right), \quad \hat{\Omega}_2 = 0.5773503\left(\hat{i} + \hat{j} - \hat{k}\right)$$

$$\hat{\Omega}_3 = 0.5773503\left(\hat{i} - \hat{j} + \hat{k}\right), \quad \hat{\Omega}_4 = 0.5773503\left(\hat{i} - \hat{j} - \hat{k}\right)$$

$$\hat{\Omega}_5 = 0.5773503\left(-\hat{i} + \hat{j} + \hat{k}\right), \quad \hat{\Omega}_6 = 0.5773503\left(-\hat{i} + \hat{j} - \hat{k}\right)$$

$$\hat{\Omega}_7 = 0.5773503\left(-\hat{i} - \hat{j} + \hat{k}\right), \quad \hat{\Omega}_8 = 0.5773503\left(-\hat{i} - \hat{j} - \hat{k}\right)$$

S_N approximation indicates N different direction cosines are used for each principal direction

number of directions: $n = N(N+2)$

Ex) S_4 : $\xi_i = \pm 0.295876$ **and** ± 0.908248

S_6 : $\xi_i = \pm 0.1838670, \quad \pm 6950514$

and ± 0.9656013

One-dimensional planar medium with isotropic scattering

$$\mu \frac{di(\tau,\mu)}{d\tau} + i(\tau,\mu) = (1-\omega_0)\, i_b(\tau) + \frac{\omega_0}{4\pi}\int_{\omega'=4\pi} i(\tau,\hat{\Omega}')\, d\omega'$$

$$\mu_i \frac{di_i}{d\tau} + i_i = (1-\omega_0)\, i_b + \frac{\omega_0}{4\pi}\sum_{j=1}^{N} w'_j i_j \qquad i = 1,2,\cdots,N$$

summed quadrature weights w'_j

one-dimensional $\hat{\Omega}_i = \xi_i \hat{i} + \eta_i \hat{j} + \mu_i \hat{k} = \mu_i \hat{k}$

$$\int_{4\pi} d\omega = 4\pi = \sum_{i=1}^{N} w'_i$$

$$\int_{4\pi} \hat{\Omega}\, d\omega = \vec{0} = \sum_{i=1}^{N} w'_i \hat{\Omega}_i = \sum_{i=1}^{N} w'_i \mu_i \hat{k}$$

Ex) S_4

Order of Approximation	Ordinates			Weights
	ξ	η	μ	w
S_2 (symmetric)	0.5773503	0.5773503	0.5773503	1.5707963
S_2 (nonsymmetric)	0.5000000	0.7071068	0.5000000	1.5707963
S_4	0.2958759	0.2958759	0.9082483	0.5235987
	0.2958759	0.9082483	0.2958759	0.5235987
	0.9082483	0.2958759	0.2958759	0.5235987

$$\mu = 0.2958759$$

$$w' = 4 \times (0.5235987 + 0.5235987) = 4.1887902$$

$$= \frac{4\pi}{3} (= 4.1987902)$$

$$\mu = 0.9082435$$

$$w' = 4 \times 0.5235987 = 2.0943948$$

$$= \frac{2\pi}{3} (= 2.0943951)$$

TABLE 15.2
Discrete Ordinates for the One-Dimensional S_N-approximation (N =2, 4, 6, 8).

Order of Approximation	Ordinates μ	Weights w'
S_2 (symmetric)	0.5773503	6.2831853
S_2 (nonsymmetric)	0.5000000	6.2831853
S_4	0.2958759	4.1887902
	0.9082483	2.0943951
S_6	0.1838670	2.7382012
	0.6950514	2.9011752
	0.9656013	0.6438068
S_8	0.1422555	2.1637144
	0.5773503	2.6406988
	0.8040087	0.7938272
	0.9795543	0.6849436

$$4\pi = 12.5663706$$

N **different intensities**

$\frac{N}{2}$ **from the wall at** $\tau = 0 \quad (\mu_i > 0)$

$\frac{N}{2}$ **from the wall at** $\tau = \tau_0 \quad (\mu_i < 0)$

let $i_1^+, i_2^+, \cdots, i_{N/2}^+, \quad i_1^-, i_2^-, \cdots, i_{N/2}^-$

$$\mu_i \frac{di_i^+}{d\tau} + i_i^+ = (1-\omega_0) i_b + \frac{\omega_0}{4\pi} \sum_{j=1}^{N/2} w_j \left(i_j^+ + i_j^- \right)$$

$$-\mu_i \frac{di_i^-}{d\tau} + i_i^- = (1-\omega_0) i_b + \frac{\omega_0}{4\pi} \sum_{j=1}^{N/2} w_j \left(i_j^+ + i_j^- \right)$$

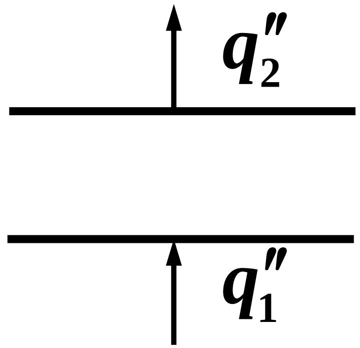
$$i = 1, 2, \cdots, N/2 \qquad \mu_i > 0$$

boundary conditions

$$\int_{\cap} i^{+}\cos\theta d\omega = \varepsilon_1 \int_{\cap} i_{b1}\cos\theta d\omega + \left(1-\varepsilon_1\right)\int_{\cap} i^{-}\cos\theta' d\omega'$$

$$q_1'' = \int_{\cap} i^{+}\cos\theta d\omega - \int_{\cap} i^{-}\cos\theta' d\omega'$$

$$\rightarrow \int_{\cap} i^{-}\cos\theta' d\omega' = \int_{\cap} i^{+}\cos\theta d\omega - q_1''$$



$$\pi i^{+} = \varepsilon_1 \pi i_{b1} + \left(1-\varepsilon_1\right)\left(\pi i^{+} - q_1''\right)$$

at $\tau = 0,\quad i_i^{+} = i_{b1} - \dfrac{1-\varepsilon_1}{\varepsilon_1 \pi} q_1''$

at $\tau = \tau_0,\quad i_i^{-} = i_{b2} + \dfrac{1-\varepsilon_2}{\varepsilon_2 \pi} q_2''$

$$i = 1, 2, \cdots, \frac{N}{2} \qquad \mu_i > 0$$

$$q'' = \sum_{i=1}^{N/2} w_i' \mu_i \left(i_i^{+} - i_i^{-}\right), \quad G = \sum_{i=1}^{N/2} w_i' \left(i_i^{+} + i_i^{-}\right)$$

surface heat flux

$$\underline{q}'' \cdot \hat{n}(\underline{r}_w) = \varepsilon(\underline{r}_w)\left[\pi i_b(\underline{r}_w) - G(\underline{r}_w)\right]$$

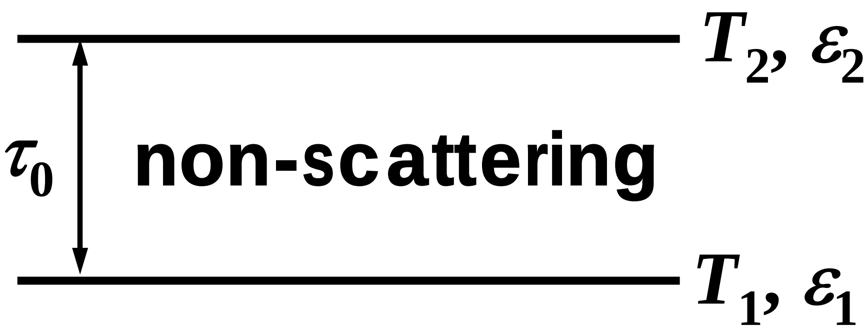
$$\simeq \varepsilon(\underline{r}_w)\left[\pi i_b(\underline{r}_w) - \sum_{\hat{n}\cdot\Omega_i<0} w_i i_i(\underline{r}_w)) \left|\hat{n}\cdot\Omega_i\right|\right]$$

at $\tau = 0,\quad q_1'' = q''(0) = \varepsilon_1\left(e_{b1} - \sum_{i=1}^{N/2} w_i' \mu_i i_i^-\right)$

at $\tau = \tau_0,\quad q_2'' = -q''(\tau_0) = -\varepsilon_2\left(e_{b2} - \sum_{i=1}^{N/2} w_i' \mu_i i_i^+\right)$

Ex) Radiative equilibrium using S_2

$q'' =$ constant



$$i_b = \frac{1}{4\pi} G$$

$$\mu_1 \frac{di_1^+}{d\tau} + i_1^+ = \frac{1}{4\pi} G, \quad -\mu_1 \frac{di_1^-}{d\tau} + i_1^- = \frac{1}{4\pi} G$$

boundary conditions

at $\tau = 0,$ $$i_1^+ = \frac{J_1}{\pi} = i_{b1} - \frac{1-\varepsilon_1}{\varepsilon_1 \pi} q_1''$$

at $\tau = \tau_0,$ $$i_1^- = \frac{J_2}{\pi} = i_{b2} + \frac{1-\varepsilon_2}{\varepsilon_2 \pi} q_2''$$

since $w_i = 2\pi$ for S_2-approximation

$$G = 2\pi\left(i_1^+ + i_1^-\right)$$

$$q'' = 2\pi\mu_1\left(i_1^+ - i_1^-\right)$$

from two equations for i_1^+ and i_1^-
eliminate i_1^+, i_1^-

$$\mu_1 \frac{d}{d\tau}\left(i_1^+ + i_1^-\right) + \left(i_1^+ - i_1^-\right) = 0 \rightarrow \mu_1 \frac{dG}{d\tau} + \frac{1}{\mu_1} q'' = 0$$

or $$\frac{dG}{d\tau} = -\frac{1}{\mu_1^2} q''$$

$$G = A - \frac{1}{\mu_1^2} q'' \tau$$

from equations for G and q''

$$\mu_1 G = 2\pi\mu_1\left(i_1^+ + i_1^-\right)$$

$$q'' = 2\pi\mu_1\left(i_1^+ - i_1^-\right)$$

$$\mu_1 G + q'' = 4\pi\mu_1 i_1^+$$

or $$i_1^+ = \frac{1}{4\pi}\left(G + \frac{q''}{\mu_1}\right)$$

$$\mu_1 G - q'' = 4\pi\mu_1 i_1^-$$

or $$i_1^- = \frac{1}{4\pi}\left(G - \frac{q''}{\mu_1}\right)$$

at walls

at $\tau = 0,\qquad i_1^+ = \frac{1}{4\pi}\left(G + \frac{q''}{\mu_1}\right) = \frac{J_1}{\pi}$

at $\tau = \tau_0,\qquad i_1^- = \frac{1}{4\pi}\left(G - \frac{q''}{\mu_1}\right) = \frac{J_2}{\pi}$

$$G = A - \frac{1}{\mu_1^2} q''\tau$$

at $\tau = 0,\qquad \frac{1}{4\pi}\left(A + \frac{q''}{\mu_1}\right) = \frac{J_1}{\pi}$

at $\tau = \tau_0,\qquad \frac{1}{4\pi}\left(A - \frac{1}{\mu_1^2} q''\tau_0 - \frac{q''}{\mu_1}\right) = \frac{J_2}{\pi}$

$$4J_1 = A + \frac{1}{\mu_1} q''$$

$$4J_2 = A - \frac{1}{\mu_1^2} q'' \tau_0 - \frac{q''}{\mu_1}$$

$$4J_1 - 4J_2 = \frac{1}{\mu_1} q'' + \frac{1}{\mu_1^2} q'' \tau_0 + \frac{q''}{\mu_1} = \left(\frac{2}{\mu_1} + \frac{\tau_0}{\mu_1^2} \right) q''$$

$$q'' = \frac{4\left(J_1 - J_2\right)}{\frac{2}{\mu_1} + \frac{\tau_0}{\mu_1^2}}$$

non-symmetric $\mu = 0.5$

$$q'' = \frac{4(J_1 - J_2)}{4 + 4\tau_0} = \frac{J_1 - J_2}{\tau_0 + 1}$$

$$J_1 = \pi i_{b1} - \frac{1-\varepsilon_1}{\varepsilon_1} q'', \quad J_2 = \pi i_{b2} + \frac{1-\varepsilon_2}{\varepsilon_2} q''$$

$$J_1 - J_2 = \sigma\left(T_1^4 - T_2^4\right) - \left(\frac{1-\varepsilon_1}{\varepsilon_1} + \frac{1-\varepsilon_2}{\varepsilon_2}\right) q''$$

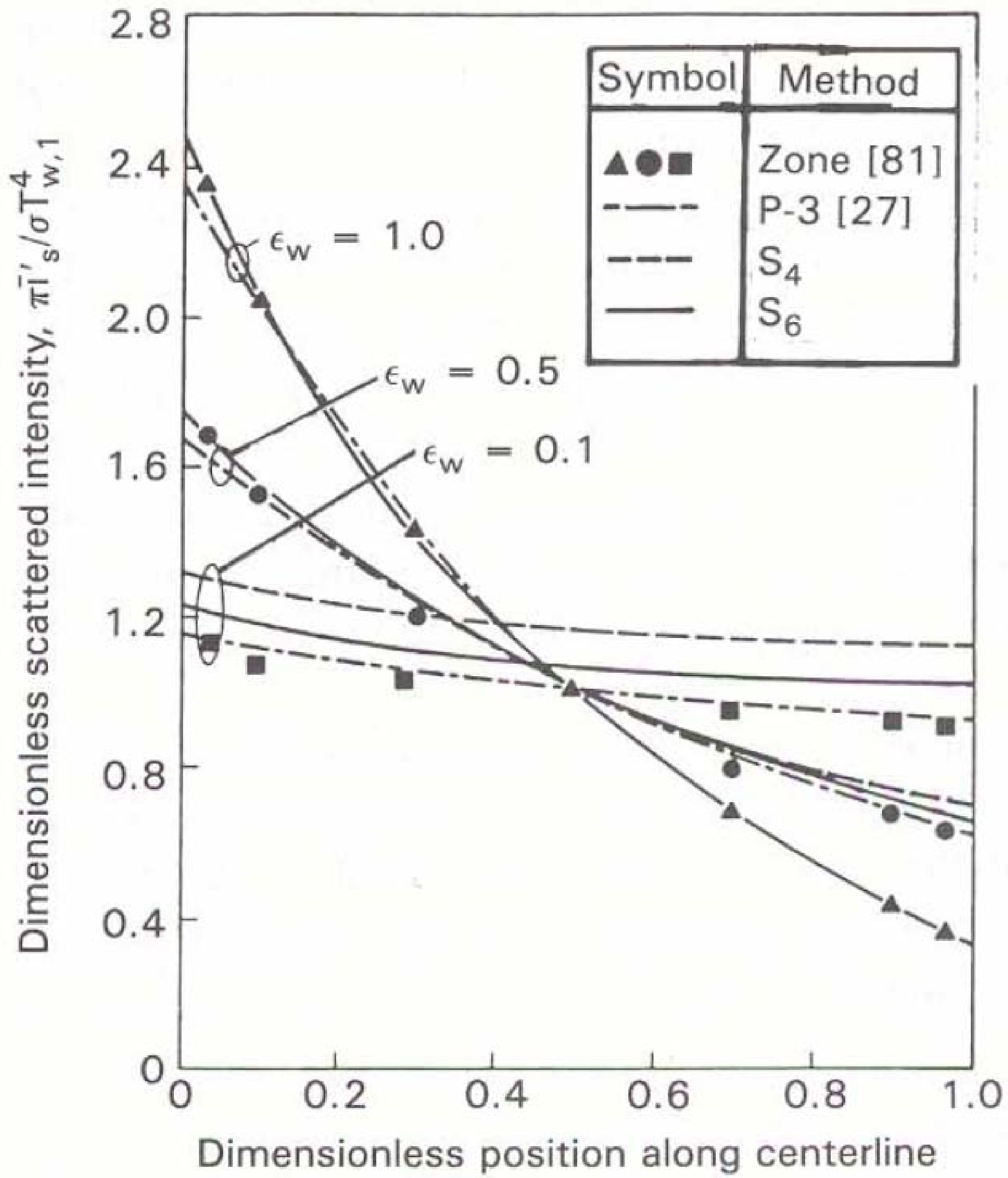
$$= \sigma\left(T_1^4 - T_2^4\right) - \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 2\right) q''$$

$$q'' = \frac{\sigma\left(T_1^4 - T_2^4\right) - \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 2\right) q''}{\tau_0 + 1}$$

$$\left(\tau_0 + 1\right) q'' = \sigma\left(T_1^4 - T_2^4\right) - \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 2\right) q''$$

$$\left(\tau_0 + 1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 2\right) q'' = \sigma\left(T_1^4 - T_2^4\right)$$

$$\frac{q''}{\sigma\left(T_1^4 - T_2^4\right)} = \frac{1}{\tau_0 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$



Symbol
Method
Zone [81]
P-3 [27]
S_4
S_6
$\epsilon_w = 1.0$
$\epsilon_w = 0.5$
$\epsilon_w = 0.1$
Dimensionless scattered intensity, $\pi \bar{I}'_s/\sigma T^4_{w,1}$
Dimensionless position along centerline
0
0.2
0.4
0.6
0.8
1.0
0.4
0.8
1.2
1.6
2.0
2.4
2.8

(a)

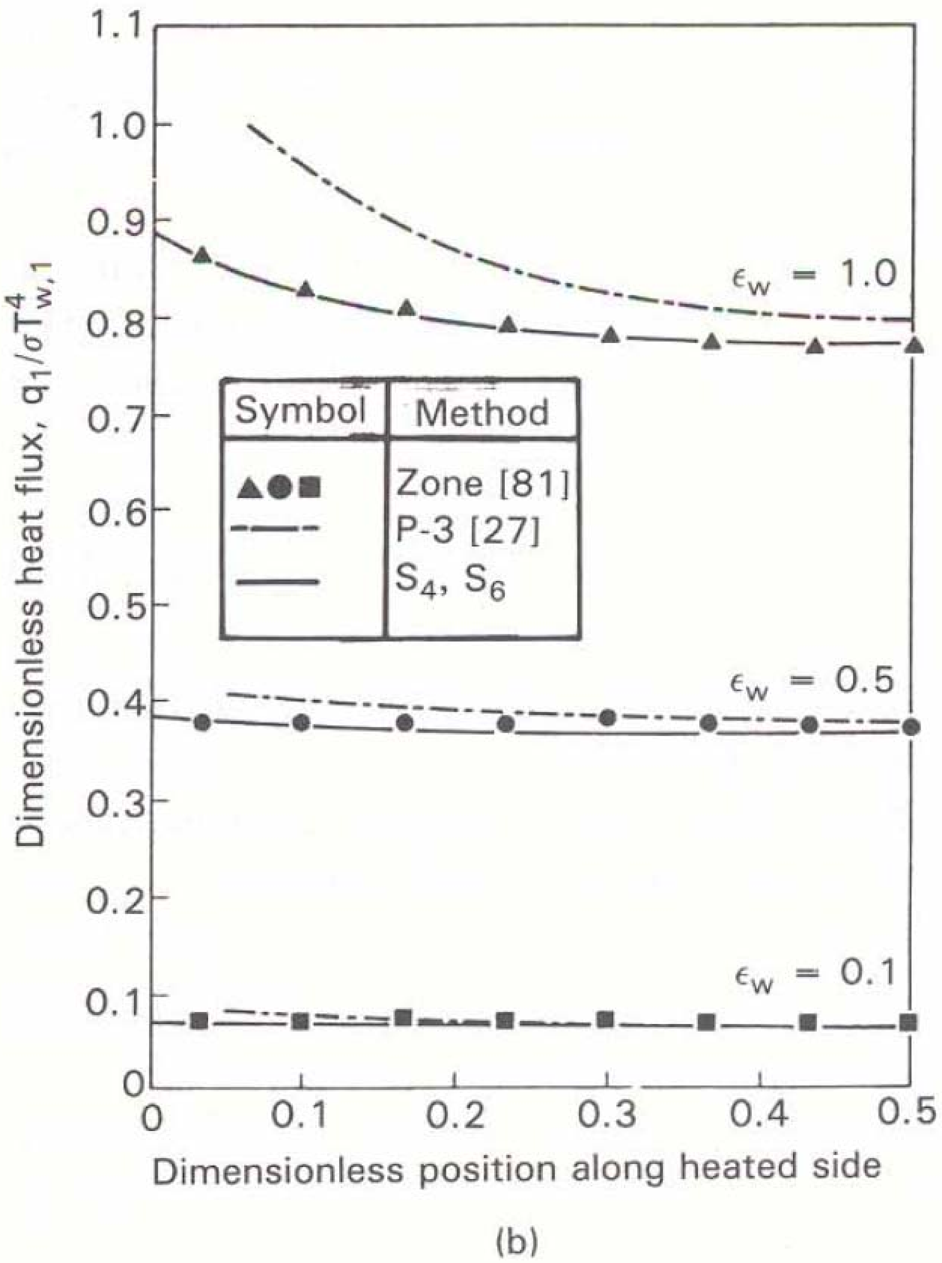


Figure 15-14 Radiative behavior in a gray square enclosure with wall emissivity ϵ_w, filled with a medium that scatters only. One side of the enclosure is at uniform temperature $T_{w,1}$; the other three sides are at zero temperature [67]. (a) Average incident scattered intensity along centerline; (b) Local heat transfer rate at the hot surface.