Gaussian Beam

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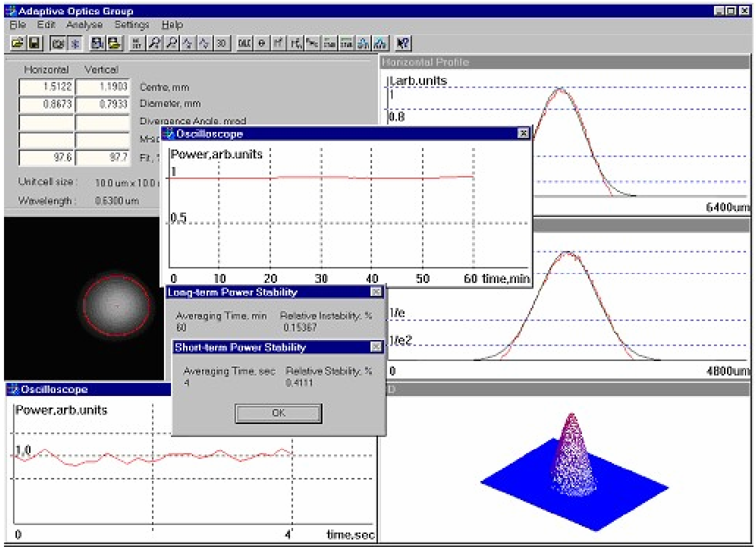
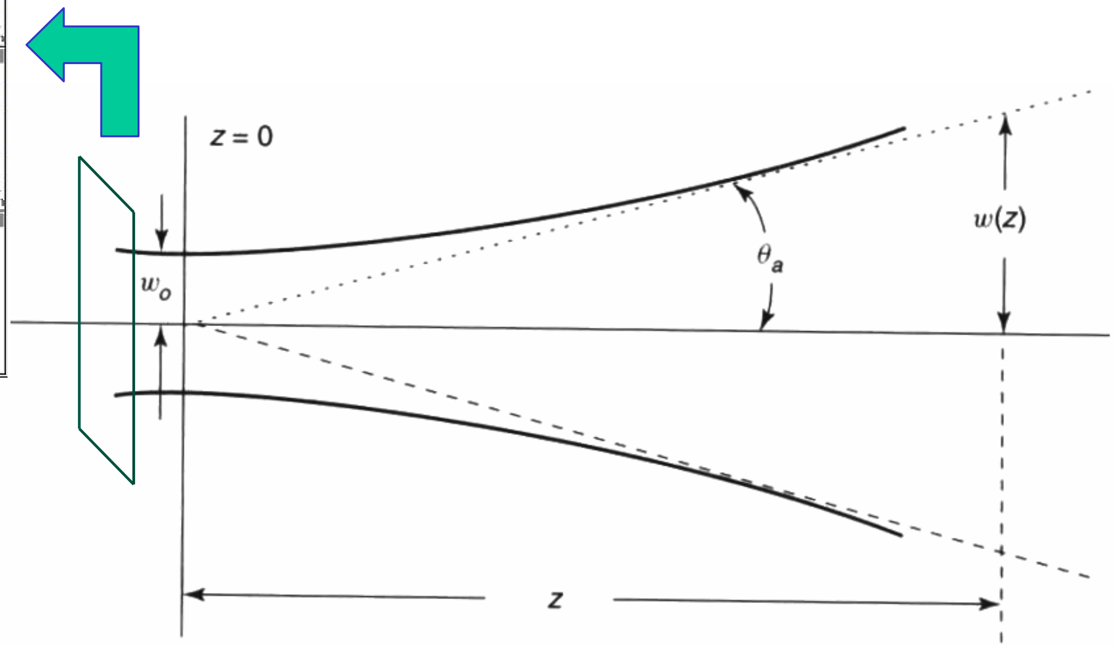
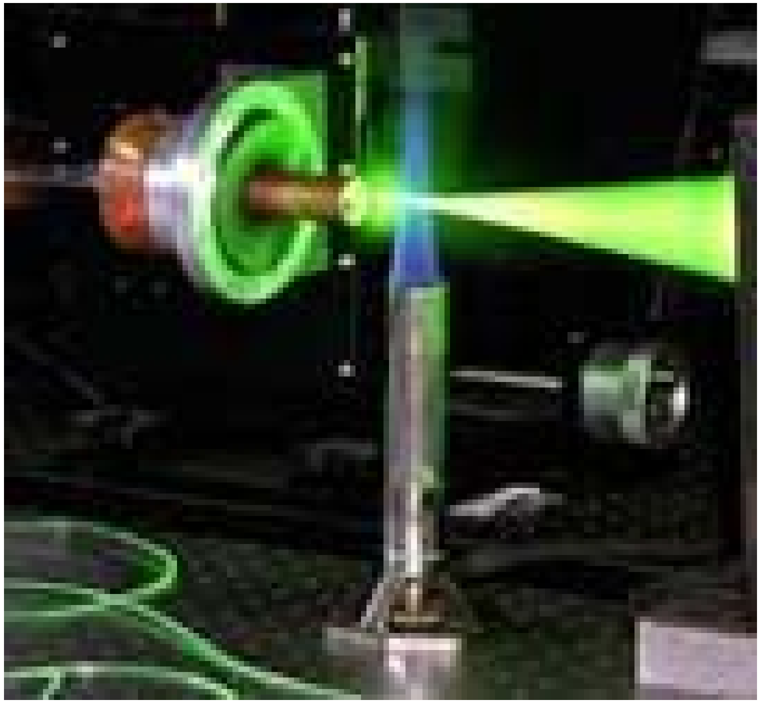
Gaussian Beam



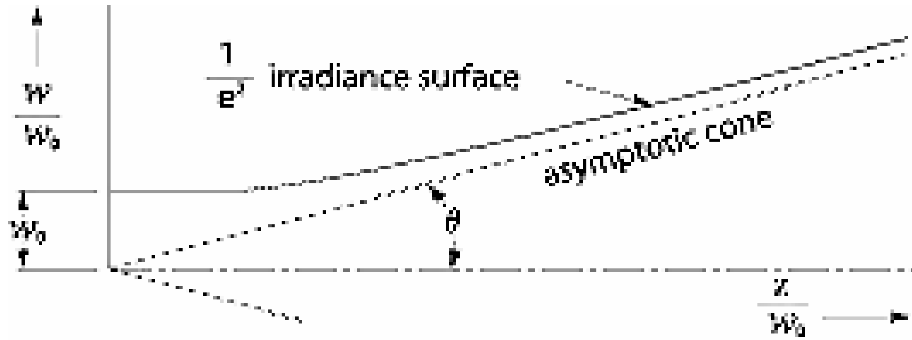


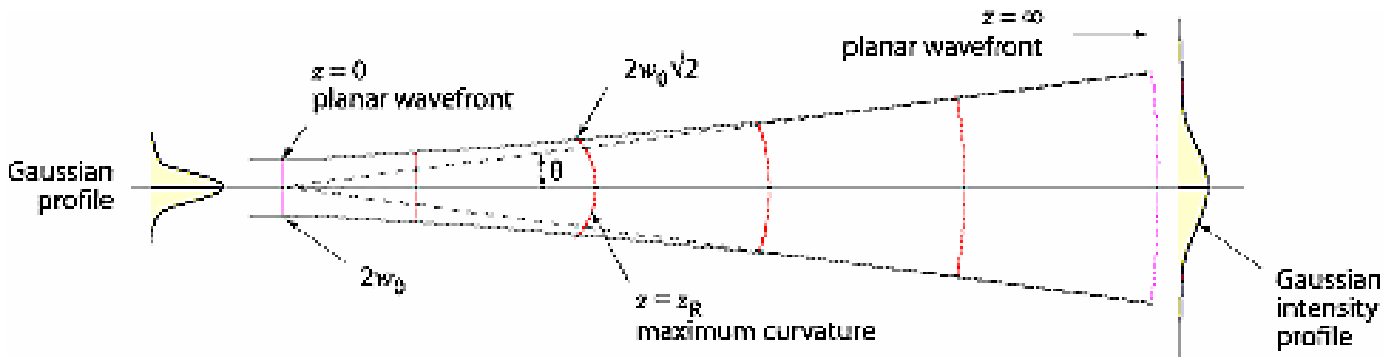
Major characteristics of the Gaussian beam waist $\omega(z)$

Example of Gaussian Beam

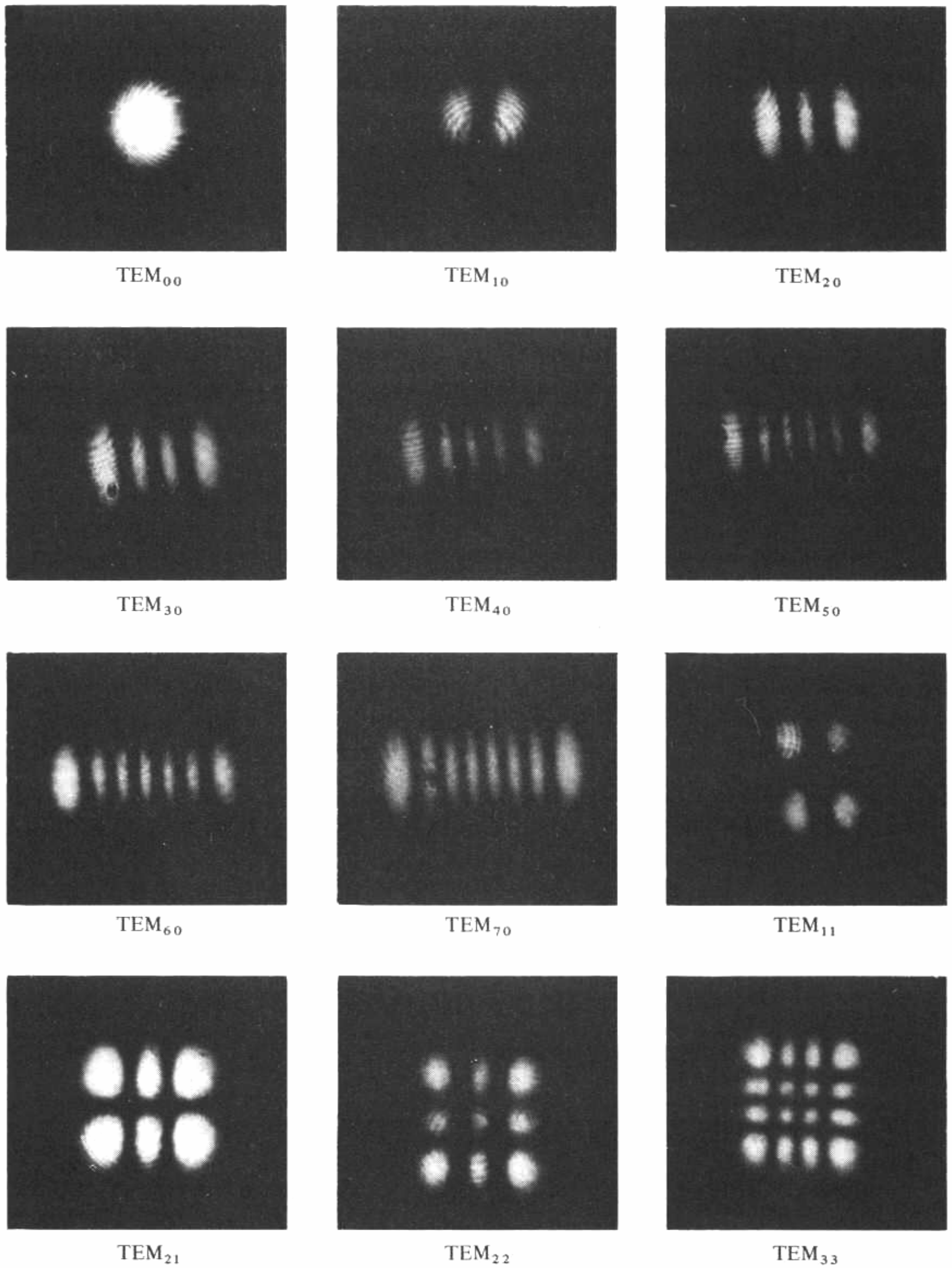


Asymptotic Behavior





Fundamental and Higher Order Gaussian Beams



Intensity photographs of some low-order Gaussian beam modes.

Gaussian Beam Equation

$$\psi(r,z)=\psi_o \exp\left(-j\left[kz-\tan^{-1}\left(\frac{\lambda_0 z}{\pi n w_0^2}\right)\right]\right)\frac{w_o}{w(z)}\exp\left(\frac{-r^2}{w^2(z)}\right)\exp\left(-j\left(\frac{kr^2}{2R(z)}\right)\right)$$

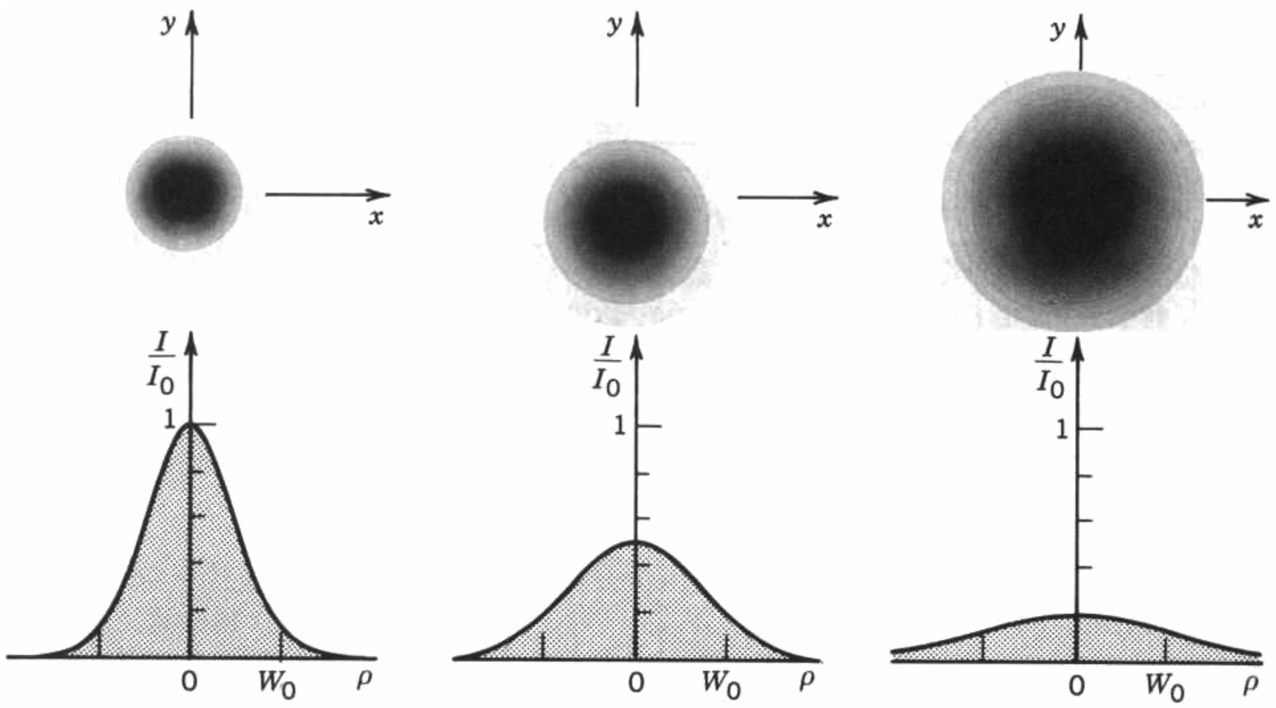
$$w^2(z)=w_0^2\left(1+\left(\frac{\lambda_0 z}{\pi n w_0^2}\right)^2\right)=w_0^2\left(1+\left(\frac{z}{z_R}\right)^2\right)$$

$$\theta_a=\frac{\lambda_0}{\pi n w_0}=\frac{2}{\pi}\frac{\lambda}{2w_0}$$

$$z_R=\frac{\pi n w_o^2}{\lambda_0}=\frac{\pi w_o^2}{\lambda}$$

$$R(z)=z\left(1+\left(\frac{\pi n w_0^2}{\lambda_0 z}\right)^2\right)=z\left(1+\left(\frac{z_R}{z}\right)^2\right)$$

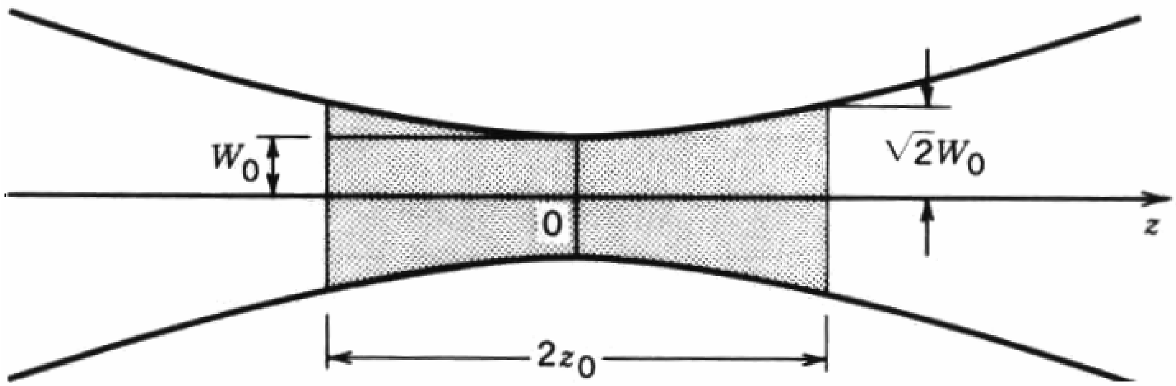
Gaussian Beam Profile

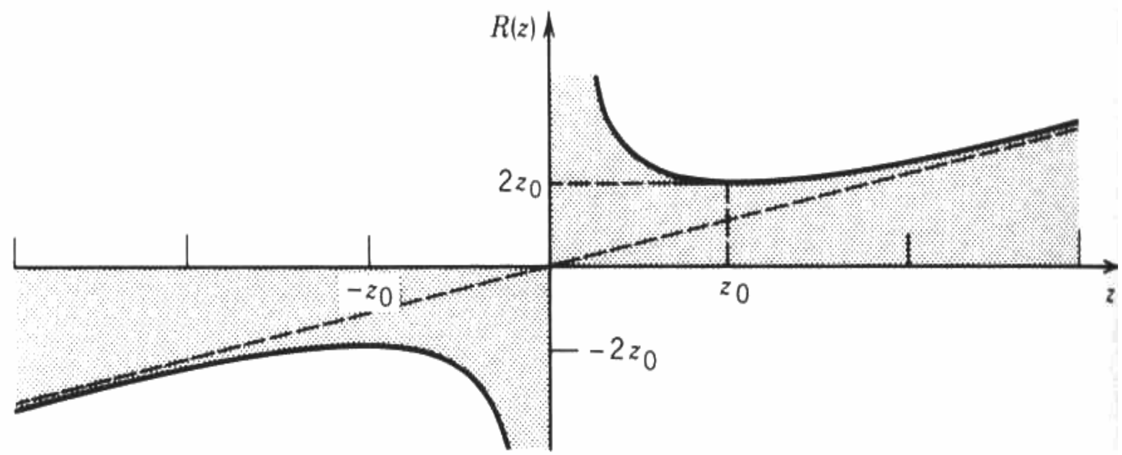


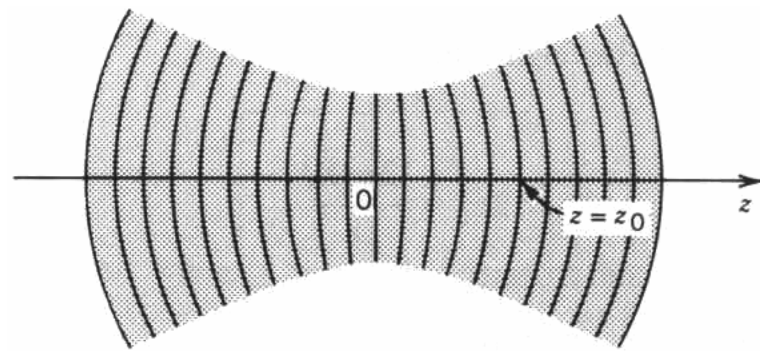
The normalized beam intensity I/I_0 as a function of the radial distance ρ (r) at different axial distances: (a) $z = 0$; (b) $z = z_0$; (c) $z = 2\,z_0$.

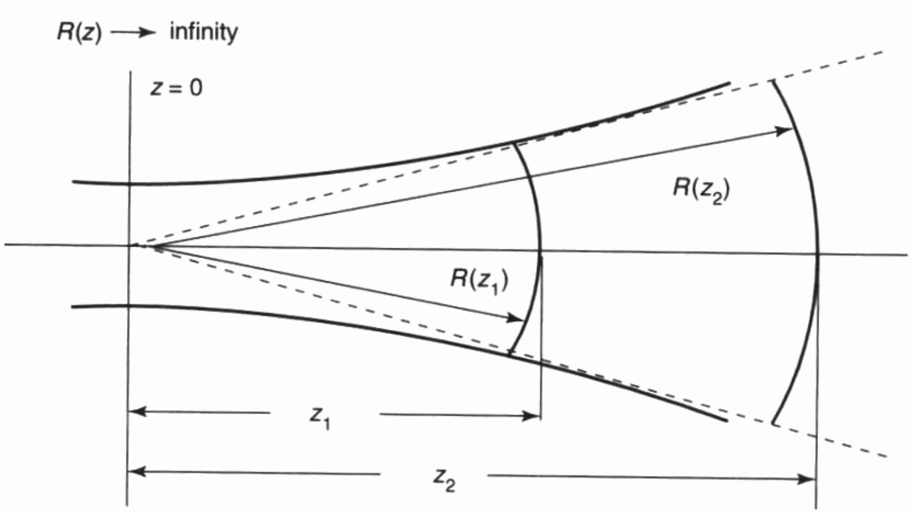
z_0 means z_R.

Gaussian Beam Characteristics

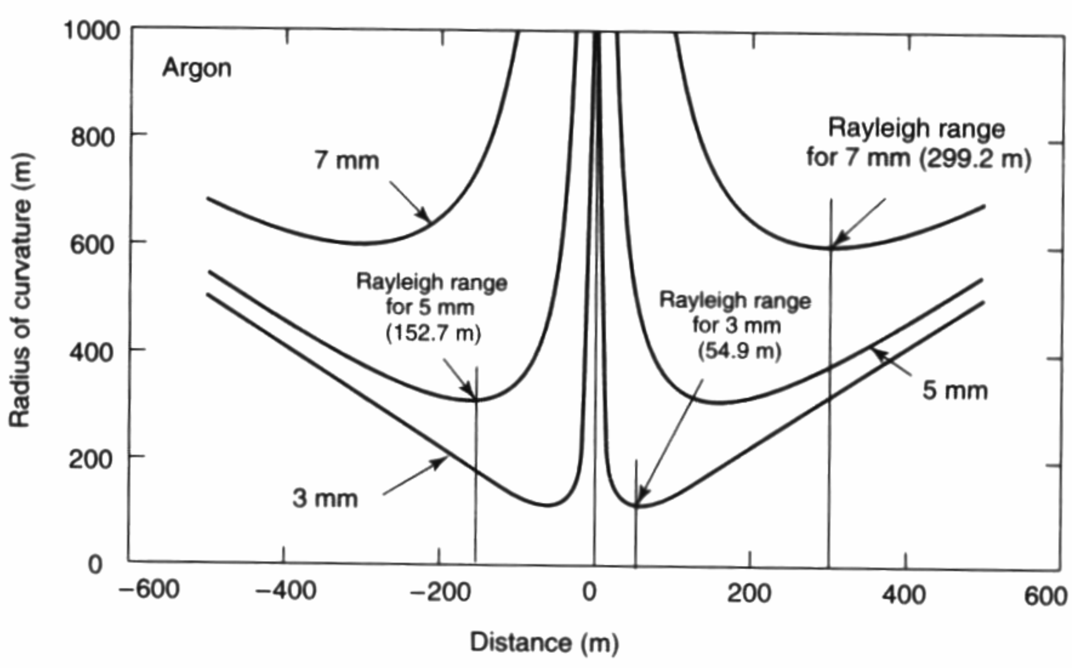








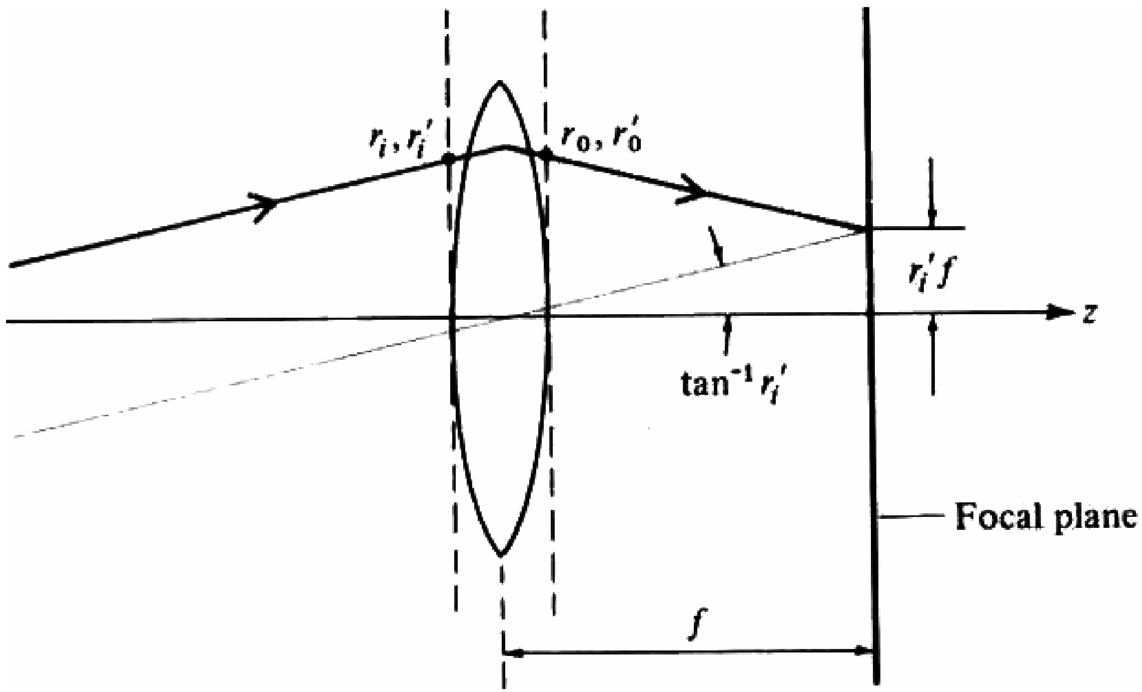
Major characteristics of the Gaussian beam radius of curvature *R(z)*



The Gaussian beam radius of curvature *R(z)* plotted for an argon−ion laser running at λ=514.5nm with three starting beam waists. The beam with a starting waist of ω_0 = 3 mm has a Rayleigh range of 54.955 m, the beam with a starting waist of ω_0 = 5 mm has a Rayleigh range of 152.653 m and the beam with a starting waist of ω_0 = 7 mm has a Rayleigh range of 299.199 m. Notice that the inflection point of the beam radius of curvature *R(z)* corresponds to the Rayleigh range.

Propagation of Rays

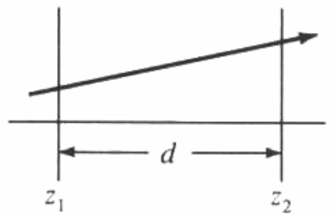
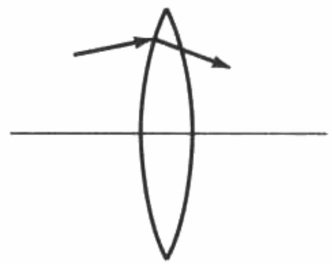
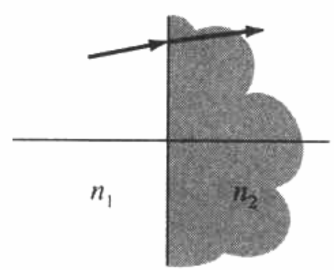
$$\begin{pmatrix} r(z) \\ r'(z) \end{pmatrix}$$



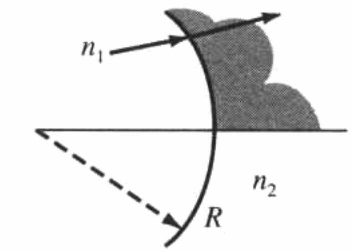
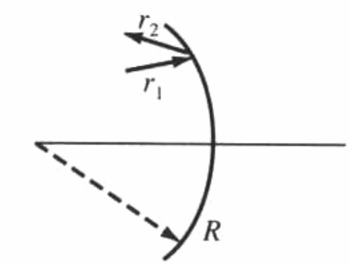
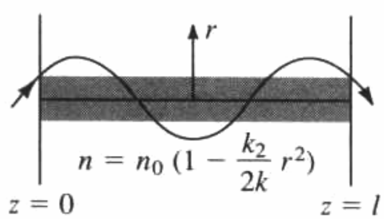
$$\begin{pmatrix} r_{\mathrm{out}} \\ r'_{\mathrm{out}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} r_{\mathrm{in}} \\ r'_{\mathrm{in}} \end{pmatrix}$$

$$\begin{pmatrix} r_{\mathrm{out}} \\ r'_{\mathrm{out}} \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{d}{f}\right) & d \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} r_{\mathrm{in}} \\ r'_{\mathrm{in}} \end{pmatrix}$$

Ray Matrices (I)

(1) Straight Section: Length d		$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$
(2) Thin Lens: Focal length f ($f > 0$, converging; $f < 0$, diverging)		$\begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix}$
(3) Dielectric Interface: Refractive indices n_1, n_2		$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$

Ray Matrices (II)

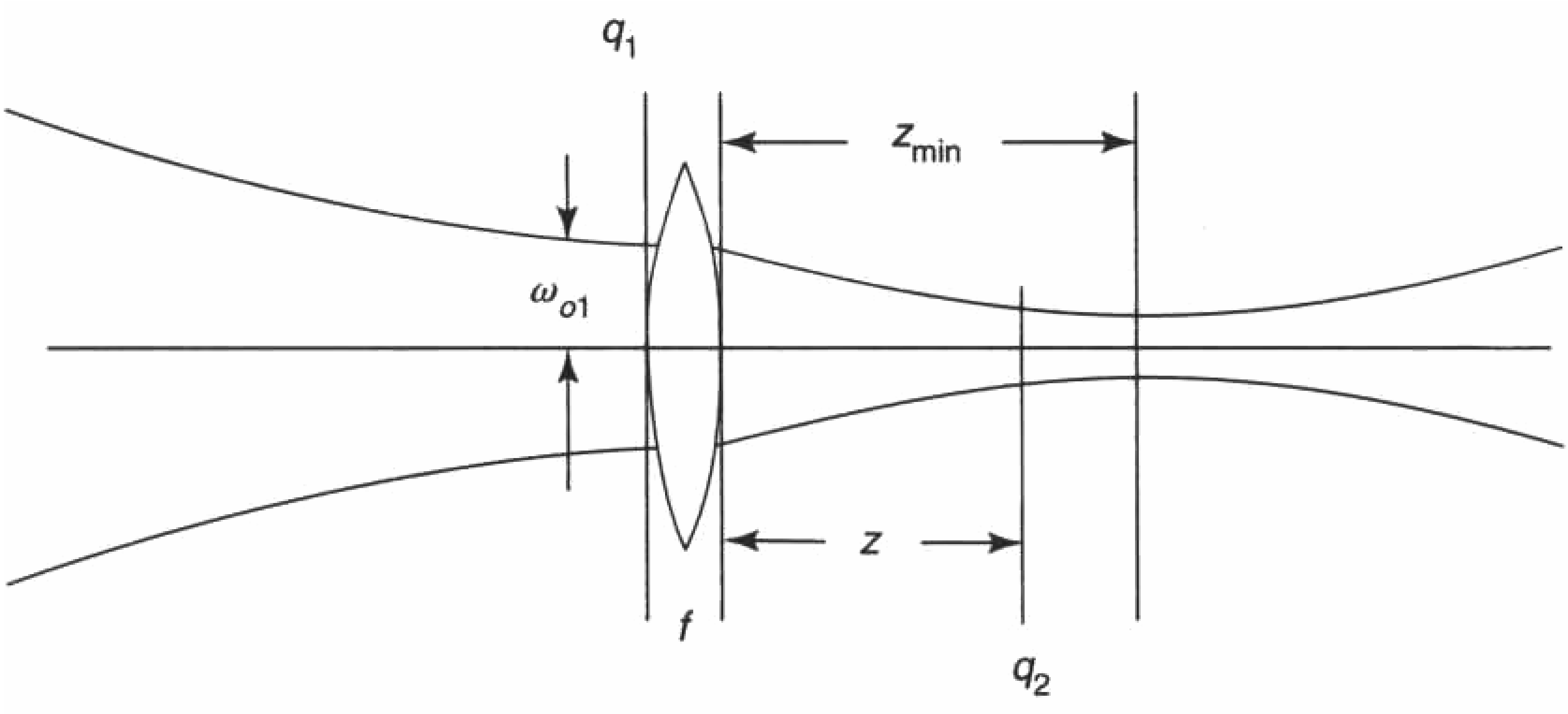
(4) Spherical Dielectric Interface: Radius R		$\begin{bmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$
(5) Spherical Mirror: Radius of curvature R		$\begin{bmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{bmatrix}$
(6) A medium with a quadratic index profile		$\begin{bmatrix} \cos\left(\sqrt{\frac{k_2}{k}}\, l\right) & \sqrt{\frac{k}{k_2}} \sin\left(\sqrt{\frac{k_2}{k}}\, l\right) \\ -\sqrt{\frac{k_2}{k}} \sin\left(\sqrt{\frac{k_2}{k}}\, l\right) & \cos\left(\sqrt{\frac{k_2}{k}}\, l\right) \end{bmatrix}$

ABCD Law

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

$$q_2(z) = \frac{Aq_1(z) + B}{Cq_1(z) + D}$$

Example (I)



A simple lens system composed of a single thin lens. A Gaussian laser beam with a beam waist ω_{o1} is incident on the lens. The incident beam has a perfectly planar wavefront upon incidence on the lens (in order words, $R \to \propto$ at the input surface of the lens).

Example (II)

$$\frac{1}{q_1(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi\omega^2(z)} = \frac{1}{\infty} - j\frac{\lambda_0}{\pi\omega_{o1}^2 n} = -j\frac{\lambda_0}{\pi\omega_{o1}^2 n}$$

$$q_2(z) = \frac{Aq_1(z)+B}{Cq_1(z)+D} = \frac{\left(1-\frac{z}{f}\right)\cdot q_1(z)+z}{\frac{-1}{f}\cdot q_1(z)+1}$$

$$\frac{1}{R(z)} = \frac{\frac{-1}{f} + z\left(\frac{1}{f^2}+\frac{1}{z_{01}^2}\right)}{\left(1-\frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2}$$

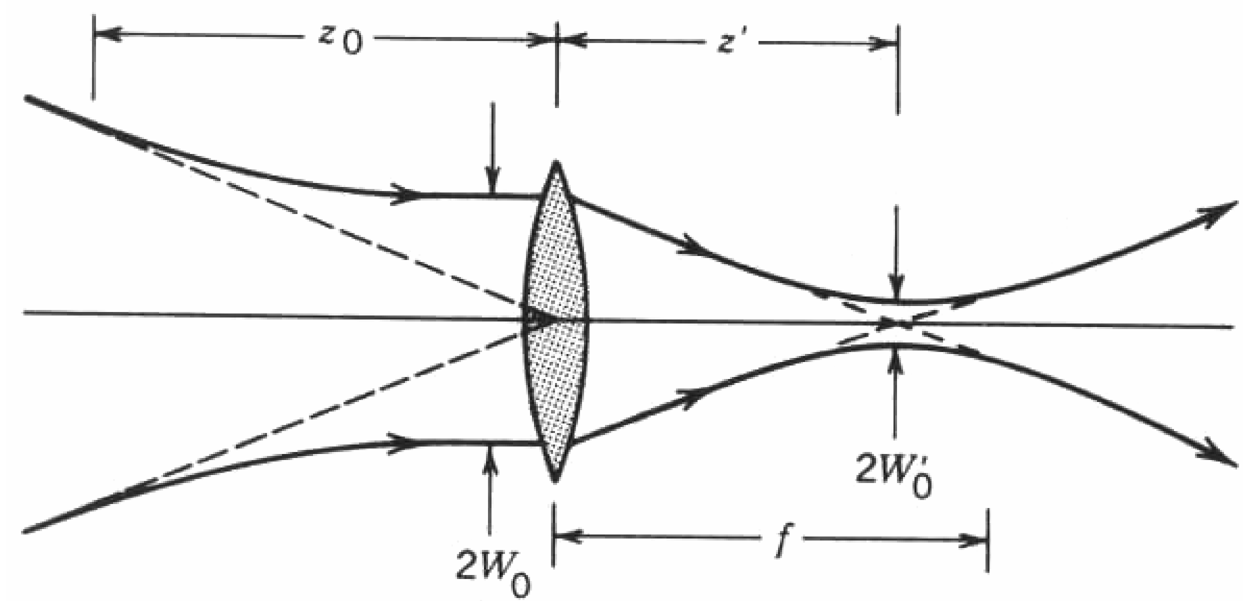
Example (III)

$$\frac{\lambda_0}{\pi\omega^2(z)n} = \frac{\dfrac{1}{z_{01}}}{\left(1-\dfrac{z}{f}\right)^2 + \left(\dfrac{z}{z_{01}}\right)^2}$$

$$z_{01} = \frac{\pi\omega_{01}^2 n}{\lambda_0}$$

$$z_{\min} = \frac{f}{1+\left(\dfrac{f}{z_{01}}\right)^2}$$

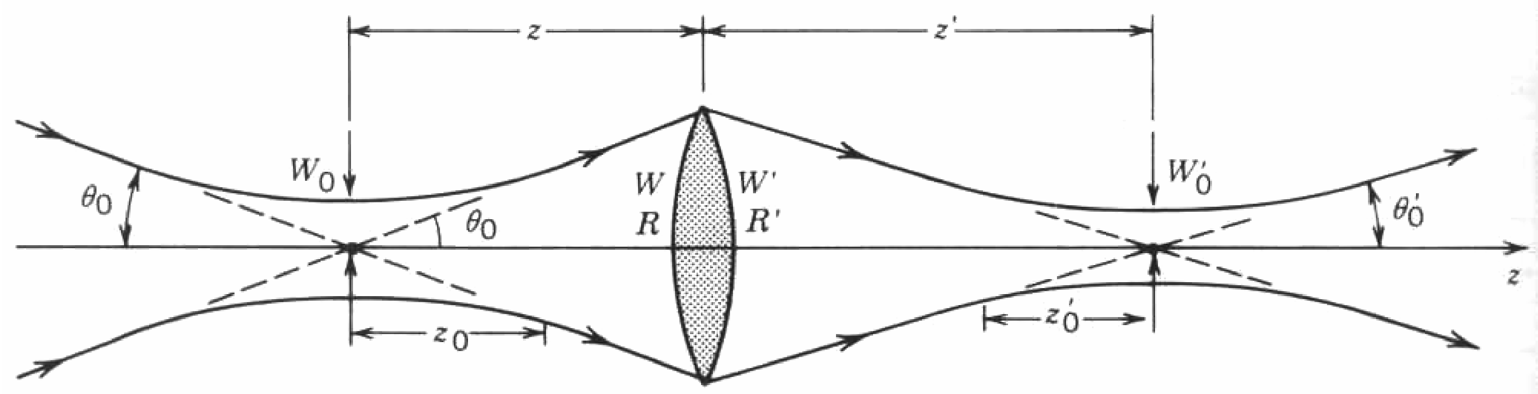
Beam Focusing



$$W_0' = \frac{W_0}{\left[1+\left(z_0/f\right)^2\right]^{1/2}}$$

$$z' = \frac{f}{1+\left(f/z_0\right)^2}$$

Transmission through a Thin Lens



Waist radius $W_0' = MW_0$

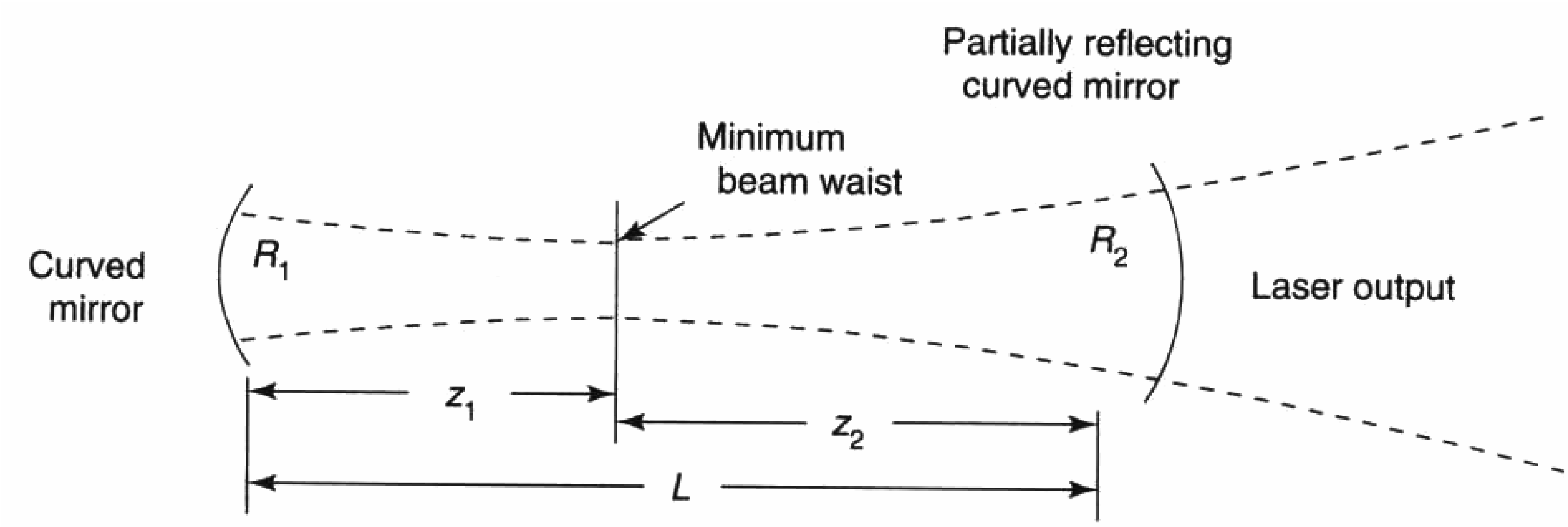
Waist location $(z' - f) = M^2(z - f)$

Depth of focus $2z_0' = M^2(2z_0)$

Divergence $2\theta_0' = \dfrac{2\theta_0}{M}$

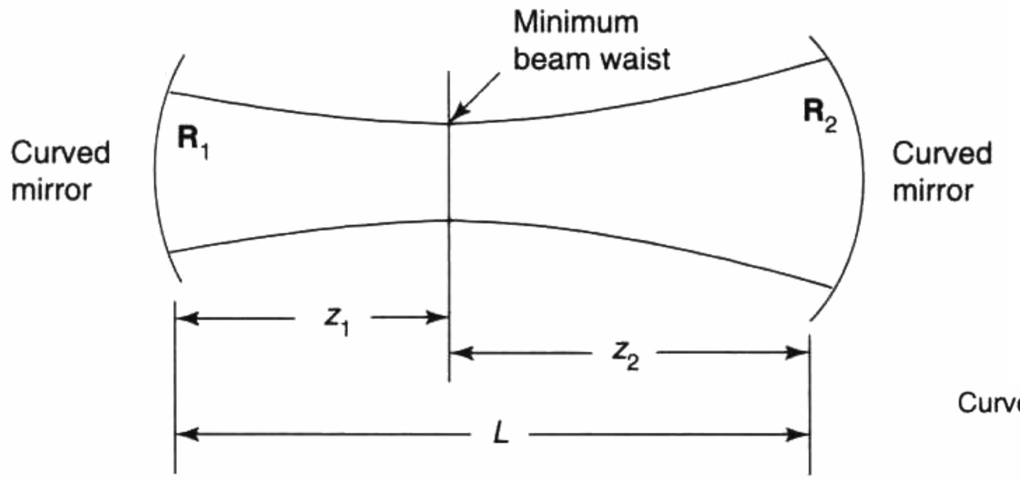
Magnification $M = \dfrac{M_r}{(1 + r^2)^{1/2}}$ $\quad r = \dfrac{z_0}{z - f}, \quad M_r = \left|\dfrac{f}{z - f}\right|$

Laser Cavity

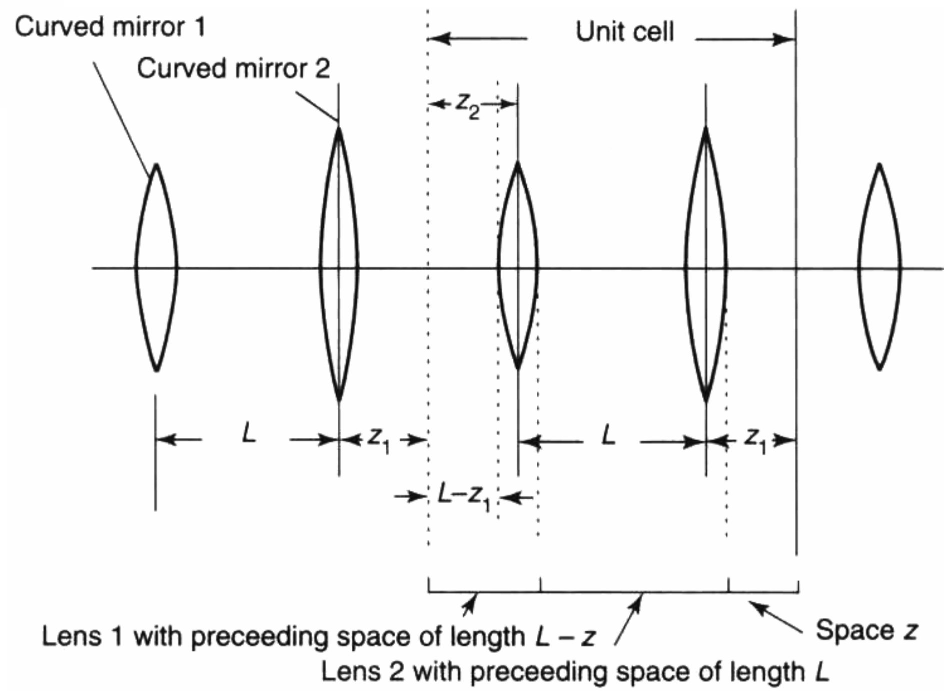


Energy can be extracted from a stable resonator by fabricating one of the end mirrors as a partial reflector.

Periodic Optical Structures

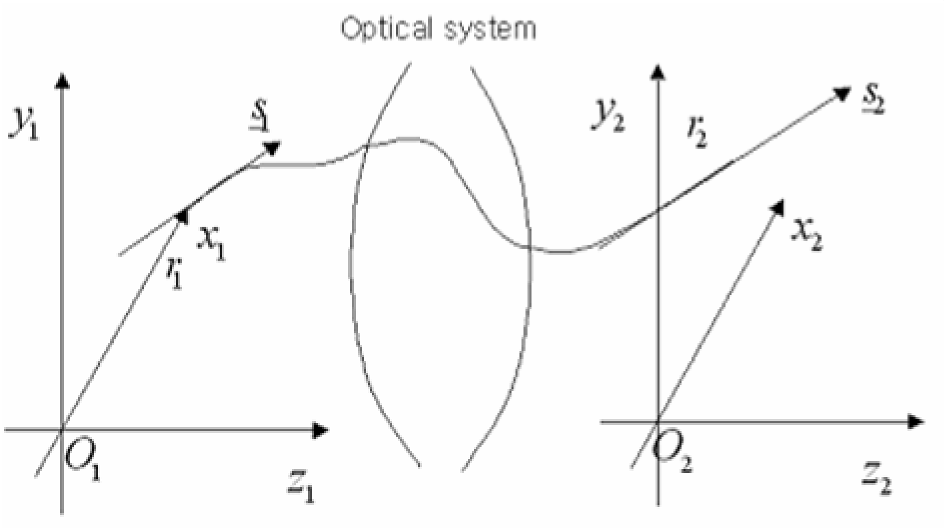


A typical resonator with two curved mirrors. The minimum beam waist will fall somewhere in the middle of the cavity.



A lens equivalent to the resonator with two curved mirrors.

Wave propagation through the first order optical systems



Hamilton's point characteristic function

$$V\left(x_1, y_1, z_1, x_2, y_2, z_2\right)$$

Eikonal function

$$S\left(x_1, y_1, z_1\right)$$

ray-direction vector

$$\underline{s}_i = \left(p_i, q_i, m_i\right)$$

Relationship between V-function and Eikonal function

$$V\left(x_1, y_1, z_1, x_2, y_2, z_2\right)$$

$$= \int_{r_1}^{r_2} n ds = S\left(x_2, y_2, z_2\right) - S\left(x_1, y_1, z_1\right)$$

Functional relationship between ray-direction and V-function

$$p_1 = -\frac{\partial V}{\partial x_1} \qquad q_1 = -\frac{\partial V}{\partial y_1} \qquad m_1 = -\frac{\partial V}{\partial z_1}$$

$$p_2 = \frac{\partial V}{\partial x_2} \qquad q_2 = \frac{\partial V}{\partial y_2} \qquad m_2 = \frac{\partial V}{\partial z_2}$$

$$p_1^2 + q_1^2 + m_1^2 = n_1^2$$

$$p_2^2 + q_2^2 + m_2^2 = n_2^2$$

Wave propagation through the first order optical systems

When the first order approximation is validated in calculating the V-function, the optical system is called the first order optical system. It is equivalent to the paraxial optical system. Then the transform of a ray by the first order optical system can be described by the 4×4 matrix equation.

$$\begin{pmatrix} x_2 \\ y_2 \\ p_2 \\ q_2 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \\ p_1 \\ q_1 \end{pmatrix}$$

position

direction

Differential equation of the point characteristic function

$$\begin{pmatrix} x_2 \\ y_2 \\ p_2 \\ q_2 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \\ p_1 \\ q_1 \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^{-1} \left\{ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \left\{ \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} - \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^{-1} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right\} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$+ \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^{-1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$p_1 = -\frac{\partial V}{\partial x_1} \quad q_1 = -\frac{\partial V}{\partial y_1} \quad m_1 = -\frac{\partial V}{\partial z_1}$$

$$p_2 = \frac{\partial V}{\partial x_2} \quad q_2 = \frac{\partial V}{\partial y_2} \quad m_2 = \frac{\partial V}{\partial z_2}$$

The first order approximation

$$\begin{pmatrix} \dfrac{\partial V}{\partial x_1} \\ \dfrac{\partial V}{\partial y_1} \end{pmatrix} = B^{-1} A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - B^{-1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{\partial V}{\partial x_2} \\ \dfrac{\partial V}{\partial y_2} \end{pmatrix} = \left(C - DB^{-1}A \right) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + DB^{-1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Integral transformation of the first order optical system

$$\begin{pmatrix} \frac{dV}{dx_2} \\ \frac{dV}{dy_2} \\ \frac{dV}{dx_1} \\ \frac{dV}{dy_1} \end{pmatrix} = \begin{bmatrix} DB^{-1} & C - DB^{-1}A \\ -B^{-1} & B^{-1}A \end{bmatrix} \begin{pmatrix} x_2 \\ y_2 \\ x_1 \\ y_1 \end{pmatrix} \quad \Rightarrow \quad V = \frac{1}{2} \begin{pmatrix} x_2 \\ y_2 \\ x_1 \\ y_1 \end{pmatrix}^T \begin{bmatrix} DB^{-1} & C - DB^{-1}A \\ -B^{-1} & B^{-1}A \end{bmatrix} \begin{pmatrix} x_2 \\ y_2 \\ x_1 \\ y_1 \end{pmatrix} + const.$$

The integral transformation of the first order optical system

$$g(x_2, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x_2, y_2, x_1, y_1) f(x_1, y_1) dx_1 dy_1$$

Output field: $g(x_2, y_2)$

Input field: $f(x_1, y_1)$

$$h(x_2, y_2, x_1, y_1) = \left| h(x_2, y_2, x_1, y_1) \right| e^{jk_0 V}$$

Integral transformation of the separable (orthogonal) optical system

Separable optical system

$$h(x_2,y_2,x_1,y_1) = -j\sqrt{\frac{1}{\det|B|}}\exp\left[j\pi(r_2,r_1)\begin{bmatrix} D_sB_s^{-1} & -B_s^{-1} \\ -B_s^{-1} & B_s^{-1}A_s \end{bmatrix}\begin{pmatrix} r_2 \\ r_1 \end{pmatrix}\right] = -j\sqrt{\frac{1}{B_xB_y}}\exp\left[j\pi(x_2,y_2,x_1,y_1)\begin{bmatrix} \frac{D_x}{B_x} & 0 & \frac{-1}{B_x} & 0 \\ 0 & \frac{D_y}{B_y} & 0 & \frac{-1}{B_x} \\ \frac{-1}{B_x} & 0 & \frac{A_x}{B_x} & 0 \\ 0 & \frac{-1}{B_x} & 0 & \frac{A_x}{B_x} \end{bmatrix}\begin{pmatrix} x_2 \\ y_2 \\ x_1 \\ y_1 \end{pmatrix}\right]$$

Free space of length *d*

$$D(d) = \begin{bmatrix} 1 & 0 & \lambda d & 0 \\ 0 & 1 & 0 & \lambda d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Cylindrical lens with *x*-focal length

$$L(\infty, f_y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{\lambda f_y} & 0 & 1 \end{bmatrix}$$

Cylindrical lens with *y*-focal length

$$L(f_x, \infty) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\lambda f_x} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

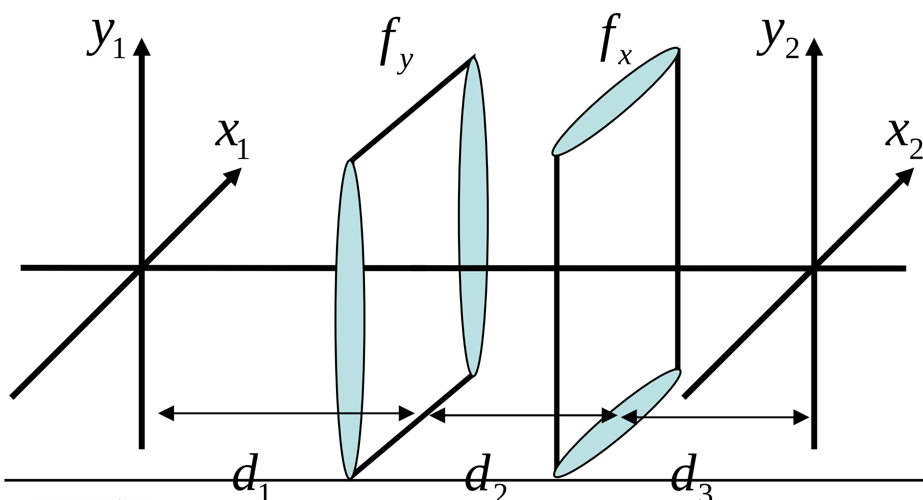
Total transfer matrix

$$T = D(d_3)L(f_x,\infty)D(d_2)L(\infty,f_y)D(d_1) = \begin{bmatrix} 1-d_3/f_x & 0 & \lambda(d_1+d_2+d_3)-\lambda(d_1+d_2)d_3/f_x & 0 \\ 0 & 1-(d_2+d_3)/f_y & 0 & \lambda(d_1+d_2+d_3)-\lambda(d_2+d_3)d_1/f_y \\ -1/(\lambda f_x) & 0 & 1-(d_1+d_2)/f_x & 0 \\ 0 & -1/(\lambda f_y) & 0 & 1-d_1/f_y \end{bmatrix}$$

Integral transform kernel

$$h(x_2,y_2,x_1,y_1) = -j\frac{1}{\lambda\sqrt{\left[(d_1+d_2+d_3)-(d_1+d_2)d_3/f_x\right]\left[(d_1+d_2+d_3)-(d_2+d_3)d_1/f_y\right]}} \times \exp\left[j\frac{\pi}{\left[\lambda(d_1+d_2+d_3)-\lambda(d_1+d_2)d_3/f_x\right]}\left[\left(1-(d_1+d_2)/f_x\right)x_2^2-2x_2x_1+\left(1-d_3/f_x\right)x_1^2\right]\right]$$

$$\times\exp\left[j\frac{\pi}{\left[\lambda(d_1+d_2+d_3)-\lambda(d_2+d_3)d_1/f_y\right]}\left[\left(1-d_1/f_y\right)y_2^2-2y_2y_1+\left(1-(d_2+d_3)/f_y\right)y_1^2\right]\right]$$



When f_x and f_y are infinite, the integral transform kernel becomes that of the Fresnel diffraction integral.

H. Kim and B. Lee, *Optics Communications*, vol. 260, no. 2, pp. 383-397, 2006.

참고문헌

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