확률변수 및 확률과정의 기초

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Ch.1 Probability Models

- Probability models: (cf. Deterministic models)
  - Systems work in a chaotic environment
  - Probability models
    - Make sense out of the chaos
    - Build efficient, reliable, and cost-effective systems
  - Introduction to
    - The theory underlying probability models
    - The basic techniques used in the development of such models

Mathematical Models

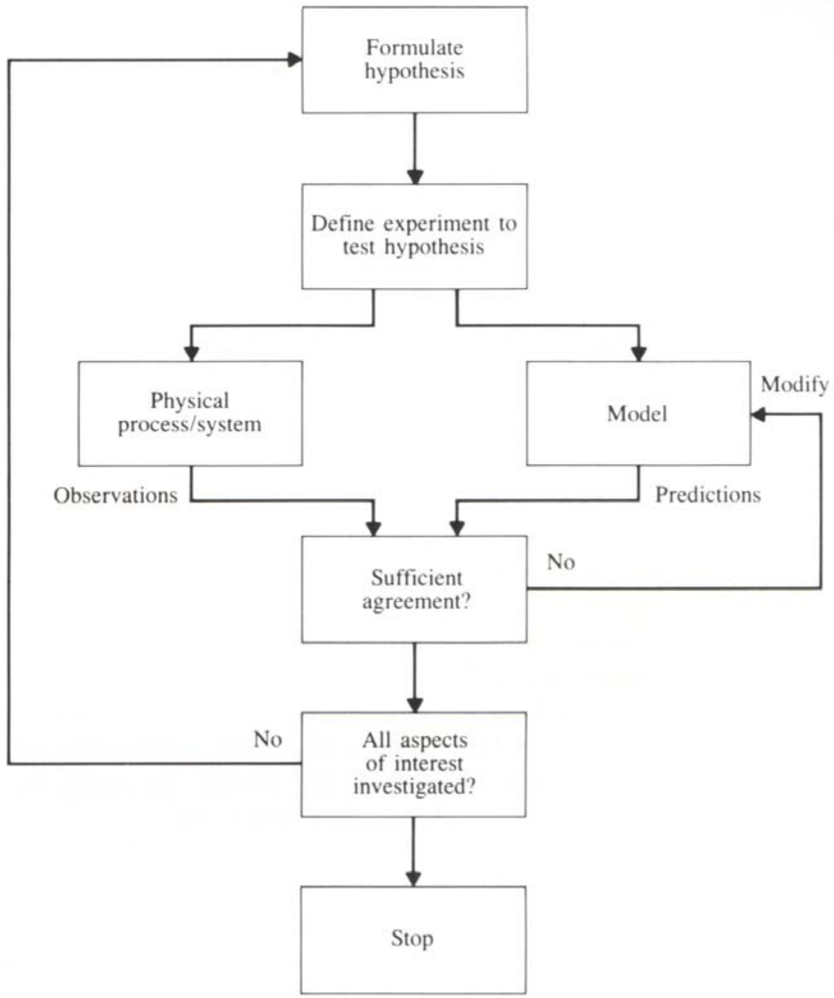
- Design or modification of any complex system: Choices from various feasible alternatives
  - Criteria for choices: cost, reliability and performance
  - The quantitative evaluation of these criteria
    - No actual implementation and experimental evaluation of the alternatives
    - Based on estimates obtained using models of the alternatives
  - Model: an appropriate representation of a physical situation
    - Explain observed behavior using a set of simple and understandable rules
    - Used to predict the outcome of experiments involving the given physical situation
    - Explains all relevant aspects of a given situation
    - Avoid the cost of experiments (labor, equipment, and time)

Mathematical Models

- When the observational phenomenon has measurable properties
- Consist of a set of assumptions how a system in physical process works
  - In the form of mathematical relations involving the important parameters and variables of the system
- The conditions of the experiment: Given in the mathematical relations
- The Solution of these relations: The prediction of the measured results of the experiment
- Usefulness of mathematical models
  - Intuition and rules of thumb: Unreliable
  - Experimentation is not possible during the initial phases of a system design
  - The cost of extensive experimentation in existing systems

The modeling process

FIGURE 1.1
The modeling process.



Deterministic Models

- Deterministic models
  - The solution of a set of mathematical equations
  - The exact outcomes of the experiment

Ex) Circuit theory, electromagnetic theory

Probability Models

- **Random experiment**: an experiment in which **the outcome varies in an unpredictable fashion** for repeated experiments **under the same conditions**
  - Deterministic models: Not appropriate for random experiments
  - Probability models: Intended for random experiments
  - Sample space: the set S of all possible outcomes

Statistical Regularity

- Statistical regularity: Averages obtained in long sequences of trials of random experiments consistently yield approximately the same value
- The relative frequency: $$f_k(n) = \frac{N_k(n)}{n}$$

  k: outcome, n: total no. of trials

- The probability of the outcome k: $$\lim_{n \to \infty} f_k(n) = p_k$$
- The conditions under which a random experiment is performed determine the probabilities of the outcomes of an experiment

Properties of Relative Frequency

- Random experiment having K possible outcomes:
  - Sample space $S=\{1,2, \ldots, K\}$
  - The number of outcomes: $0 \le N_k(n) \le n \quad \text{for } k = 1, 2, \ldots, K$
  - The relative frequencies: $0 \le f_k(n) \le 1 \quad \text{for } k = 1, 2, \ldots, K$
  - The sum of the number of occurrences: $\sum_{k=1}^{K} N_k(n) = n$
  - The sum of the relative frequencies: $\sum_{k=1}^{K} f_k(n) = 1$
  - Events: The association of outcomes of an experiment
    - ✓ The relative frequency of an event is the sum of the relative frequencies of the associated outcomes

Ex) Let C be the event A or B occurs where A and B cannot occur simultaneously, then $f_C(n) = f_A(n) + f_B(n)$

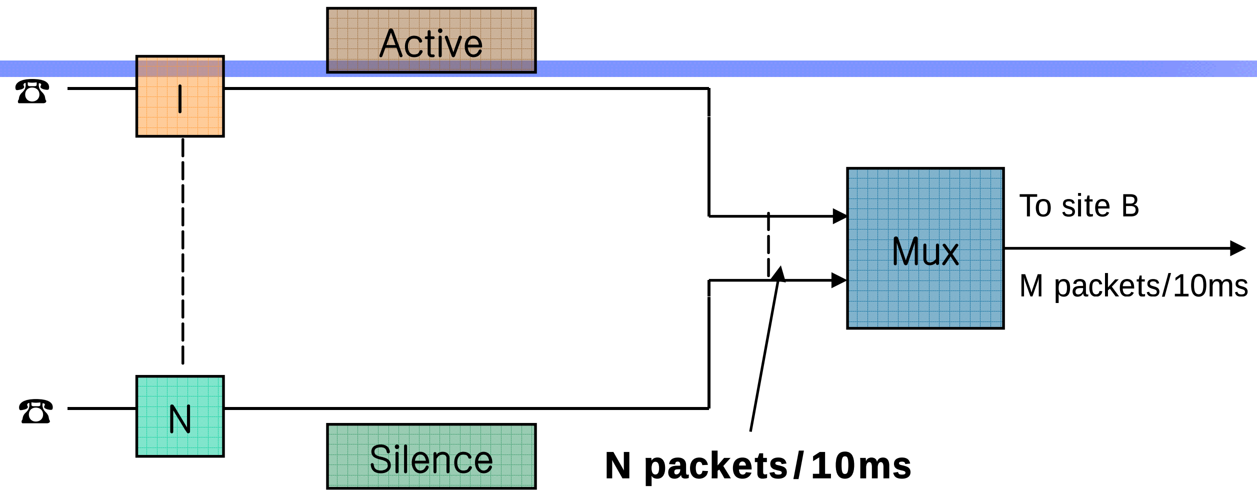
The Axiomatic Approach to a Theory of Probability

- Axioms: Specify that probability assignments must satisfy the properties of the relative frequencies
  - A random experiment has been defined and a set S of all possible outcomes has been identified
  - A class of subsets of S called events has been specified
  - Each event A has been assigned a number $P[A]$ to satisfy the following axioms
    - $0 \leq P[A] \leq 1$
    - $P[S] = 1$
    - If A and B are events that cannot occur simultaneously then $P[A \text{ or } B]=P[A] + P[B]$

Building a Probability Model

- How to build a probability model from a real-world problem
- We identify the elements in the axioms:
  - Defining the random experiment inherent in the application
  - Specifying the set S of all possible outcomes and the events of interests
  - Specifying a probability assignment from which the probabilities of all events of interest can be computed
- The challenge: To develop the simplest model

Ex) A Packet Voice Transmission System



- System requirement : 48 simultaneous conversations from city A to city B. using “Packets” of voice information

- Speech signal → digitized and bundled into packets for 10 ms segments of speech
  → address is appended to each voice packet and transmitted
  cf. circuit switching and packet switching

- Simplest design : 48 packets every 10 ms in each direction

- Voice activity : On the average, 2/3 of all packets contain silence, and hence no speech information.
  → On the average 48/3 = 16 packets/10ms are active.

- Random experiment : the number of active packets:
  outcome=A, Transmit rate= M packets/10ms < 48 packets/10ms

$A \le M \Rightarrow$ All packets transmitted

$A > M \Rightarrow (A\text{-}M)$ packets are discarded

- n trials for A
  - $A(j)$ = the outcome in the j th trial
  - $N_k(n)$ = the number of trials having k active packets in the first n trials

The relative frequency of the outcome k in the first n trials

$$\Rightarrow f_k(n) = \frac{N_k(n)}{n}$$

$$\lim_{n\to\infty} f_k(n) = p_k \qquad \text{for} \quad 0 \le k \le 48$$

- Rate of active packets

⇒ The average number of active packets per 10-ms = **Sample mean**

$$\langle A \rangle_n = \frac{1}{n}\sum_{j=1}^{n} A(j)$$

$$= \frac{1}{n}\sum_{k=0}^{48} kN_k(n)$$

$$\langle A \rangle_n \rightarrow \sum_{k=0}^{48} k p_k \equiv E[A]$$: The expected value of A

- The fraction of active packets discarded by the system in n trials

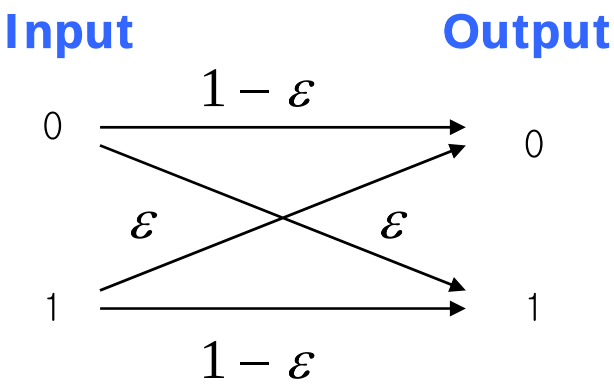
$$\frac{\text{no. of active packets discarded}}{\text{no. of active packets produced}} = \frac{\sum_{k=M+1}^{48} (k-M) N_k(n)}{\sum_{k=0}^{48} k N_k(n)} = \frac{\sum_{k=M+1}^{48} (k-M) N_k(n)/n}{\sum_{k=0}^{48} k N_k(n)/n}$$

$$\Rightarrow \frac{\sum_{k=M+1}^{48} (k-M) p_k}{\sum_{k=0}^{48} k p_k}$$ $\Rightarrow$ The long-term fraction of active packets discarded

- $p_k = ?$

→ Given by the binomial distribution

Ex) Communication over Unreliable Channels



ε **: error rate**

: long-term proportion of bits delivered in error

Error correcting code (Channel code)

$0 \rightarrow 000$
$1 \rightarrow 111$ $\Rightarrow$ The wrong decision only if two or three of the bits be in error

$\therefore$ New error rate = $3\varepsilon^2 - 2\varepsilon^3$

e.g. $\varepsilon = 10^{-3} \Rightarrow 3\varepsilon^2 - 2\varepsilon^3 = 3\times10^{-6}$

Processing of Random Signals

- Outcome of a random experiment
  - A single number (Discrete)
  - An entire function of time (Continuous)
- Observed voltage wave form

$$Y(t) = S(t) + N(t) \quad \Rightarrow \quad \text{SNR(Signal to Noise Ratio)}$$

  - Filtering may improve SNR: A kind of processing

Resource Sharing System

- Resource demand: random and unsteady
  - Dedicated sufficient resources: Waste of resource
  - Ex) Computer sharing by providing Queue
    - Performance measure of interest:
      the average response time and throughput

- Reliability of systems
  - Measures of reliability: (1) the average time to failure, (2) the probability that a component is still functioning after a given time

- H.W.: 3, 6, 7, 9, 11