Ch.4 Multiple Random Variables

- 4.1 Vector Random Variables
  - a function that assigns a vector of real numbers to each outcome ζ in S

- Events and Probabilities
  - $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ $\rightarrow$ n-dimensional r.v.

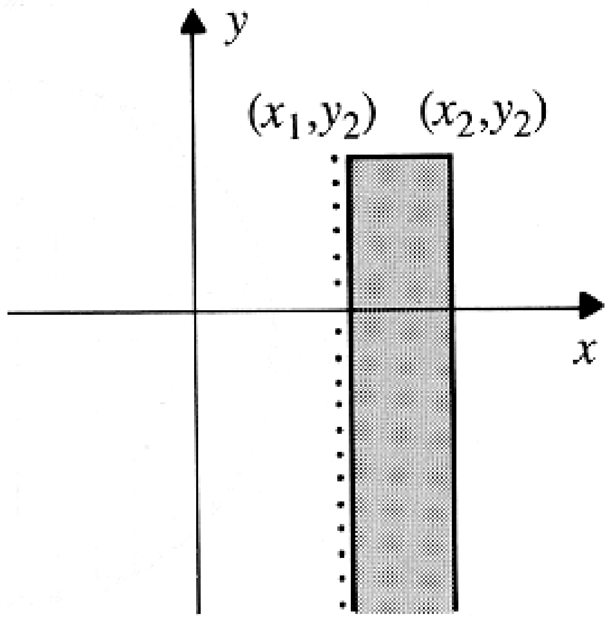
  $\rightarrow$ a corresponding region in an n-dimensional real space.

  - Events of the product form

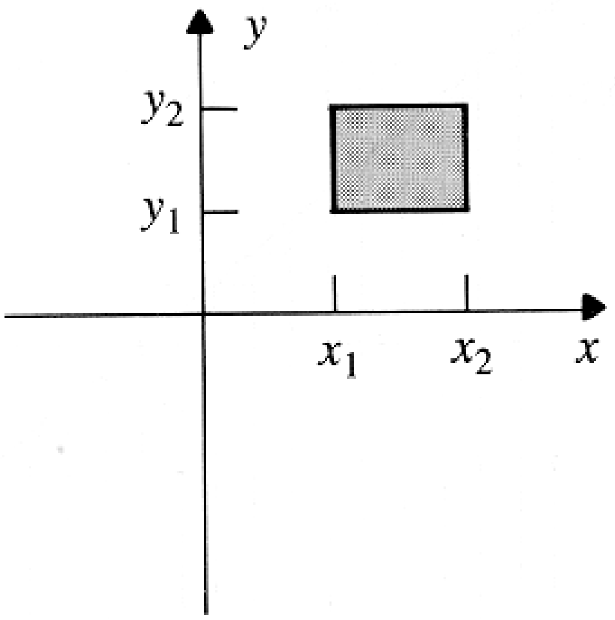
    $$A = \{X_1 \text{ in } A_1\} \cap \{X_2 \text{ in } A_2\} \cap \ldots \cap \{X_n \text{ in } A_n\}$$

    A_k : a one-dimensional event involving X_k only.

- E.g.



$\{x_1 < X \leqslant x_2\} \cap \{Y \leqslant y_2\}$



$\{x_1 \leqslant X \leqslant x_2\} \cap \{y_1 \leqslant Y \leqslant y_2\}$

❖ $$P[A] = P[\{X_1 \text{ in } A_1\} \cap \{X_2 \text{ in } A_2\} \cap \cdots \cap \{X_n \text{ in } A_n\}]$$

$$\equiv P[X_1 \text{ in } A_1, \cdots X_n \text{ in } A_n\}]$$

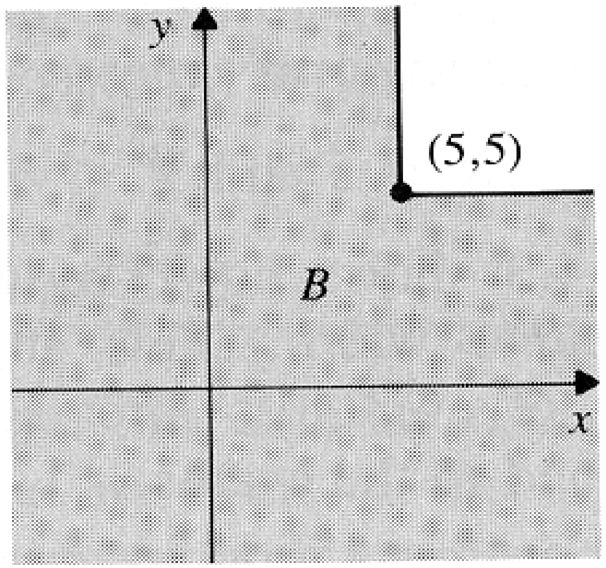
$$= P[\{\zeta \text{ in } S \text{ such that } \mathbf{X}(\zeta) \text{ in } A\}]$$

→ the probability of the equivalent event in the underlying sample space.

❖ Non-product-form events

$\rightarrow$ Approximated by the union of product-form events

E.g. $B = \{\min(X, Y) \le 5\}$



$\rightarrow \quad B = \{X \le 5 \ \text{ and } \ Y < \infty\} \cup \{\mathrm{X} > 5 \ \text{ and } \ \mathrm{Y} \le 5\}$

$$P[B] \cong P\left[\bigcup_k B_k\right] = \sum_k P[B_k]$$

:approximated by the union of disjoint product-form events.

❖ Independence

$$P[X_1 \text{ in } A_1, \ldots, X_n \text{ in } A_n] = P[X_1 \text{ in } A_1] \ldots P[X_n \text{ in } A_n]$$

where the A_k is an event involving X_k only

$\rightarrow X_1,\ X_2, ..., X_n$ are independent.

4.2 Pairs of Random Variables

❖ Pairs of Discrete Random Variables

Vector r.v. : $\mathbf{X} = (X, Y)$

Sample space : $S = \{(x_j, y_k),\ j = 1, 2, \ldots, k = 1, 2, \ldots\}$

➢ The joint probability mass function of $\mathbf{X}$

$$p_{X,Y}(x_j, y_k) = P[\{X = x_j\} \cap \{Y = y_k\}]$$

$$\equiv P[X = x_j, Y = y_k] \qquad j = 1,2,\ldots \quad k = 1,2,\ldots$$

: the probability of the occurrence of the pairs (x_j, y_k)

- The probability of any event A

$$P[\mathbf{X} \text{ in } A] = \sum_{(x_j, y_k)} \sum_{\text{in } A} p_{X,Y}(x_j, y_k)$$

- $$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p_{X,Y}(x_j, y_k) = 1$$

- The marginal probability mass function

$$
\begin{aligned}
p_X(x_j) &= P[X = x_j] \\
&= P[X = x_j, Y = \text{anything}] \\
&= \sum_{k=1}^{\infty} p_{X,Y}(x_j, y_k)
\end{aligned}
$$

$$
p_Y(y_k) = \sum_{j=1}^{\infty} p_{X,Y}(x_j, y_k)
$$

- cf) Knowledge of the marginal pmf's is insufficient to specify the joint pmf

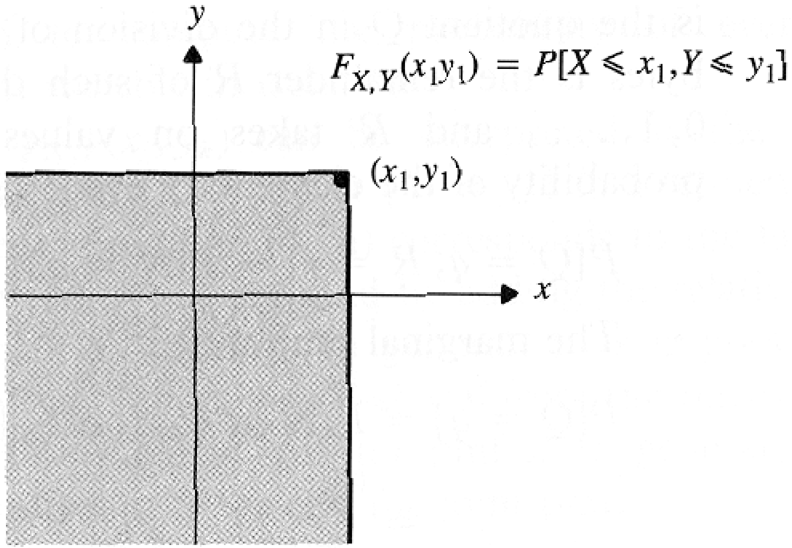
The Joint cdf of *X* and *Y*

- The probability of the product-form event

  $$\{X \le x_1\} \cap \{Y \le y_1\}$$

- $F_{X,Y}(x_1, y_1) = P[X \le x_1,\ Y \le y_1]$

  $\rightarrow$ the amounts of mass contained in the rectangular region.



Probabilities of the joint cdf

① $F_{X,Y}(x_1, y_1) \le F_{X,Y}(x_2, y_2) \quad \text{if } x_1 \le x_2,\ y_1 \le y_2$

cf) the semi-infinite rectangular defined by (x_1, y_1)
is contained in that defined by (x_2, y_2)

② $F_{X,Y}(-\infty, y_1) = F_{X,Y}(x_1, -\infty) = 0$

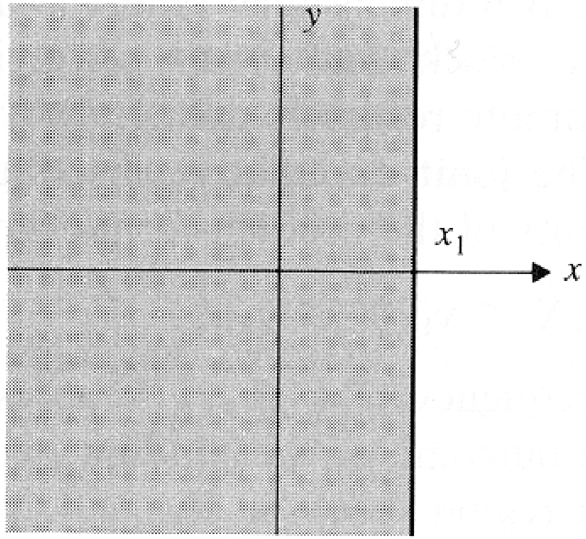
cf) It is impossible for either X or Y to assume a value
less than $-\infty$

③ $F_{X,Y}(\infty, \infty) = 1$

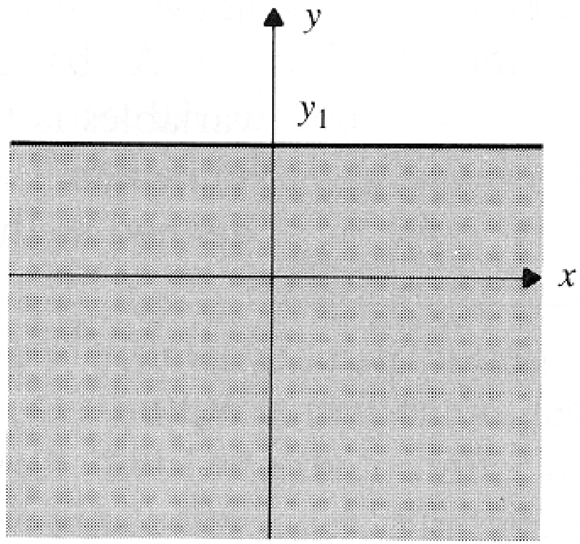
④ The marginal cumulative distribution functions

$$F_X(x) = F_{X,Y}(x,\infty) = P[X \le x, Y < \infty] = P[X \le x]$$

$$F_Y(y) = F_{X,Y}(\infty, y) = P[Y \le y]$$



$F_X(x_1) = P[X \leqslant x_1, Y < \infty]$



$F_Y(y_1) = P[X < \infty, Y \leqslant y_1]$

⑤ $\lim_{x \to a^+} F_{X,Y}(x, y) = F_{X,Y}(a, y)$

$\lim_{y \to b^+} F_{X,Y}(x, y) = F_{X,Y}(x, b)$

→ The joint cdf is continuous from the “east” and from the “north”

❖ Ex. 4.8

For $\mathbf{X} = (X, Y)$

$$F_{X,Y}(x, y) = \begin{cases} (1-e^{-\alpha x})(1-e^{-\beta y}) & x \geq 0,\ y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

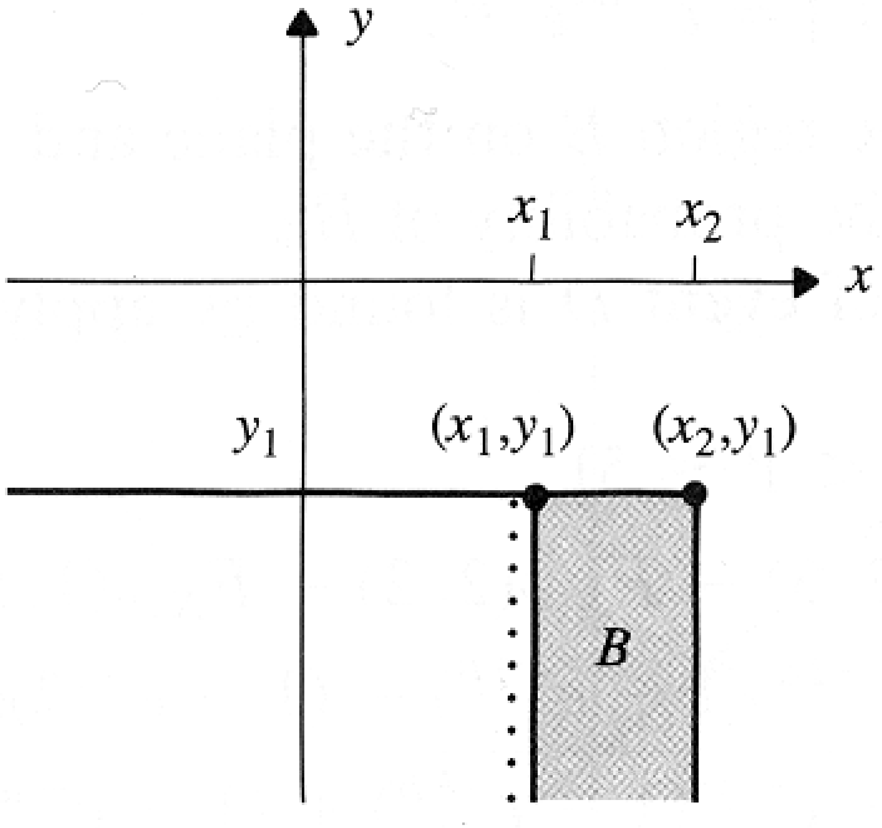
Find the marginal cdf

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y) = 1 - e^{-\alpha x} \qquad x \geq 0$$

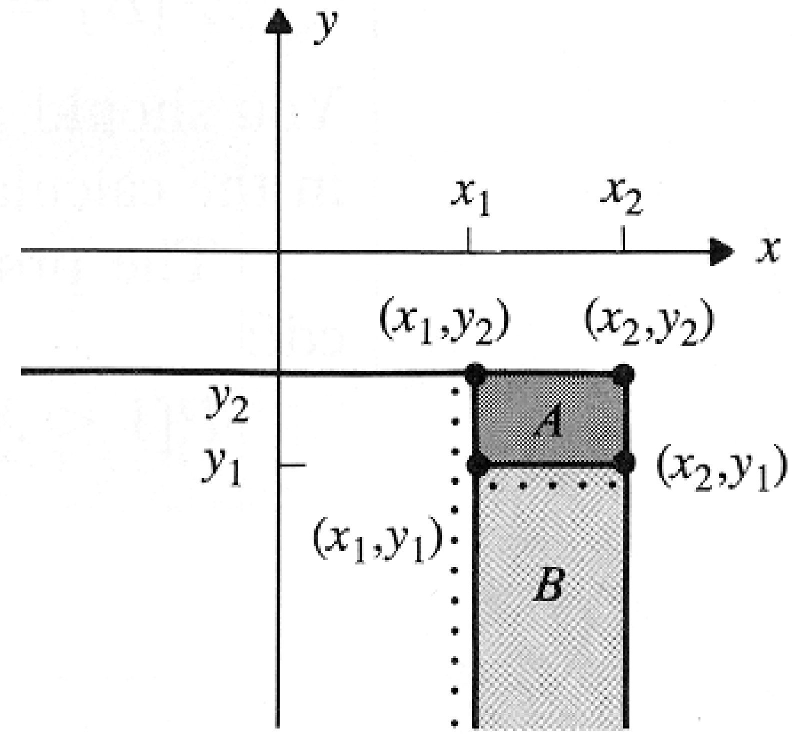
$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y) = 1 - e^{-\beta y} \qquad y \geq 0$$

$\rightarrow$ an exponential distribution

- The union and intersection of semi-infinite rectangles



$$P[x_1 < X \le x_2, Y \le y_1]$$
$$= F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_1)$$



⑥ $$P[x_1 < X \le x_2, y_1 < Y \le y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$

The Joint pdf of Two Jointly Continuous Random Variables

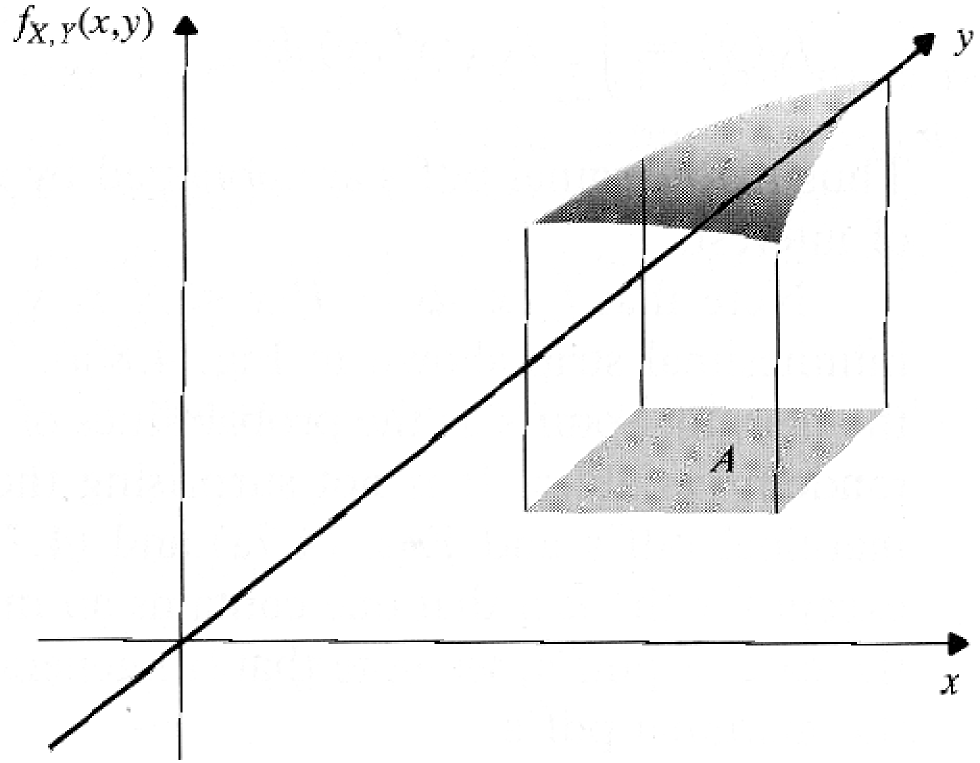
- Any reasonable shape (i.e., disk, polygon, or half-plane) can be approximated by the union of rectangles.

- For the sufficiently smooth cdf, as we increase the fineness of the rectangles, the sum approaches an integral over a probability density function.

- $\mathbf{X} = (X, Y)$

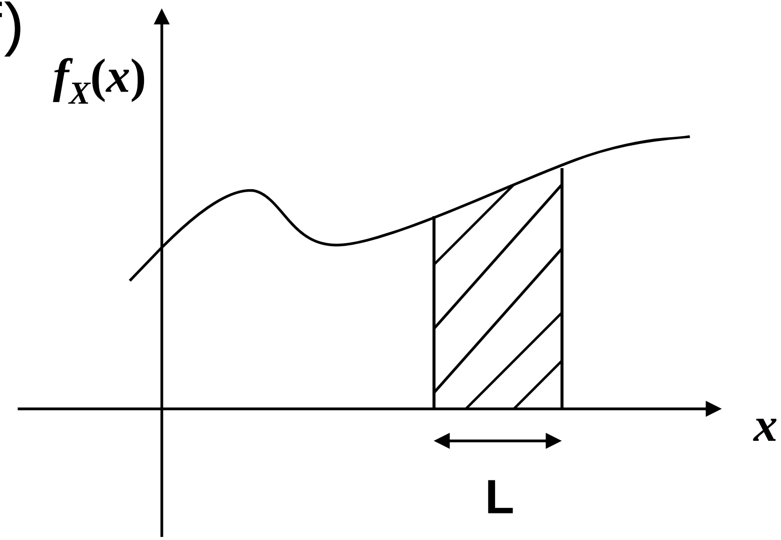
  $$P[\mathbf{X} \text{ in } A] = \int_A \int f_{X,Y}(x', y') dx' dy'$$

  where $f_{X,Y}(x', y')$ is the joint probability density function
  $\rightarrow$ the r.v.'s X and Y are jointly continuous.



$$P[\mathbf{X} \text{ in } A] \rightarrow \text{volume}$$

cf)



$P[X \text{ in L}] \rightarrow$ Area

- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x', y')dx'dy' = 1$

❖ The joint cdf

$$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x', y') dx' dy'$$

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}$$

For the jointly <u>continuous</u> r.v.'s X and Y.

- The probability of a rectangular region

$$P[a_1 < X \le b_1, a_2 < Y \le b_2]$$
$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} f_{X,Y}(x', y') dx' dy'$$

- The probability of an infinitesimal rectangle

$$P[x < X \le x + dx, y < Y \le y + dy]$$
$$= \int_{x}^{x+dx} \int_{y}^{y+dy} f_{X,Y}(x', y') dx' dy'$$
$$\cong f_{X,Y}(x, y) dx dy$$

❖ The marginal pdf's

$$f_X(x) = \frac{d}{dx}\int_{-\infty}^{x}\left\{\int_{-\infty}^{\infty} f_{X,Y}(x',y')dy'\right\}dx'$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(x,y')dy'$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x',y)dx'$$

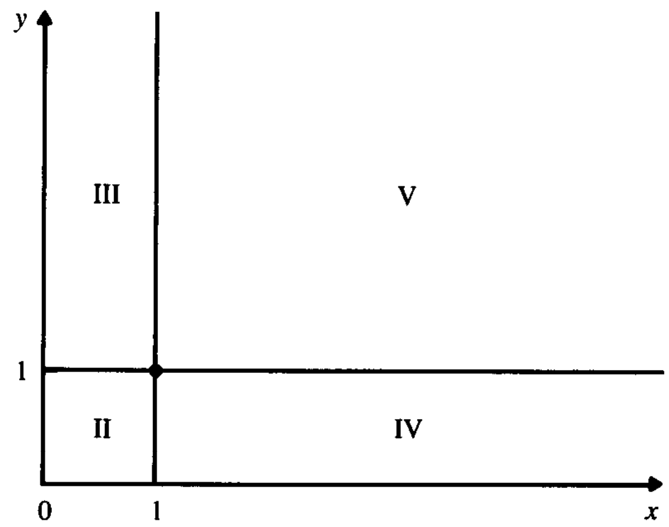
cf) The joint pdf cannot be obtained from the marginal pdf's

❖ Ex. 4.10

A randomly selected point (X, Y) in the unit square has the uniform joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the joint cdf



① $x < 0$ or $y < 0$

$F_{X,Y}(x, y) = 0$ ($\because$ the pdf is zero)

② $0 \le x \le 1$ and $0 \le y \le 1$

$$F_{X,Y}(x,y) = \int_0^x \int_0^y 1 dx'dy' = xy$$

③ $0 \le x \le 1, \; y > 1$

$$F_{X,Y}(x,y) = \int_0^x \int_0^1 1 dx'dy' = x$$

④ $x > 1, \quad 0 \le y \le 1$

$$F_{X,Y}(x,y) = \int_0^1 \int_0^y 1 dx'dy' = y$$

⑤ $x > 1, \quad y > 1$

$$F_{X,Y}(x,y) = \int_0^1 \int_0^1 1 dx'dy' = 1$$

Random Variables That Differ in Type

- Joint random variables of discrete and continuous r.v.'s
  : $P[X = k, Y \le y]$ or $P[X = k, y_1 \le Y \le y_2]$

- Ex. 4.14
  - X : input to the communication channel.
    = +1 or –1 with equal probability
  - Y : output = the input + a noise voltage N.
  - N : Uniformly distributed in the interval from –2 volts to the +2 volts.
  - $P[X = +1,\ Y \le 0] = ?$

❖ sol)

$$P[X=k,\ \ Y\le y]=P[Y\le y\mid X=k]\cdot P[X=k]$$

$$\therefore P[X=+1,\ Y\le y]=P[Y\le y\mid X=+1]P[X=+1]$$

For $X=1$, Y is uniformly distributed in the interval [-1,3]

$$P[Y\le y\mid X=+1]=\frac{y+1}{4} \qquad \text{for} \ \ -1\le y\le 3$$

$$\therefore\ P\left[X=+1, Y\le 0\right]=P[Y\le 0\mid X=+1]P[X=+1]$$

$$=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$$

4.3 Independence of Two r.v.'s

- "The discrete r.v.'s X and Y are independent if and only if the joint pmf is equal to the product of the marginal pmf's for all x_j, y_k"

cf) $p_{X,Y}(x_j, y_k) = P[X = x_j, Y = y_k]$

$= P[X = x_j]P[Y = y_k]$

$= p_X(x_j)p_Y(y_k)$ for all x_j and y_k

equivalent to the independence of the r.v.'s X and Y

Independence of Continuous r.v.'s

- The r.v.'s X and Y are independent
  if and only if their joint cdf is equal to the product of its marginal cdf's

$$F_{X,Y}(x, y) = F_X(x)\,F_Y(y) \quad \text{for all } x \text{ and } y$$

- If X and Y are jointly continuous, then X and Y are independent if and only if their joint pdf is equal to the product of the marginal pdf's

$$f_{X,Y}(x, y) = f_X(x)\,f_Y(y) \quad \text{for all } x \text{ and } y$$

❖ Ex.4.17

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y} = 2e^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

($\because$ $c = 2$ by normalization condition)

$$f_X(x) = 2e^{-x}(1-e^{-x}) \qquad 0 \le x < \infty$$

$$f_Y(y) = 2e^{-2y} \qquad 0 \le y < \infty$$

- $f_X(x)$ and $f_Y(y)$ are nonzero for all $x > 0$ and all $y > 0$
- $f_{X,Y}(x,y)$ is nonzero for $0 < y < x < \infty$

$\therefore$ $f_{X,Y}(x,y) \neq f_X(x)\, f_Y(y)$ for all $x > 0$ and $y > 0$

$\therefore$ X and Y are not independent.

- Ex. 4.13 and 4.18

  Jointly Gaussian r.v.'s X and Y

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)} \qquad -\infty < x, y < \infty$$

$$f_X(x) = \frac{e^{-x^2/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-(y^2-2\rho xy)/2(1-\rho^2)} dy$$

$$= \frac{e^{-x^2/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-(y^2-2\rho xy+\rho^2x^2-\rho^2x^2)/2(1-\rho^2)} dy$$

$$= \frac{e^{-x^2(1-\rho^2)/2(1-\rho^2)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-(y-\rho x)^2/2(1-\rho^2)}}{\sqrt{2\pi(1-\rho^2)}} dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

mean = ρx

variance $(1-\rho^2)$

$\longrightarrow$ **Gaussian**

$\longrightarrow$ zero mean, variance 1

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$f_X(x)\, f_Y(y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} \qquad -\infty < x, y < \infty$$

$\therefore\ f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ if and only if $\rho = 0$

cf) ρ is the correlation coefficient between X and Y.

❖ Note

If X and Y are independent r.v.'s

$g(X)$ and $h(Y)$ are independent.

- Proof) A : event involving only $g(X)$

  A' : equivalent event involving only X

  B : event involving only $h(Y)$

  B' : equivalent event involving only Y

$$
\begin{aligned}
P[g(X) \text{ in } A, h(Y) \text{ in } B] &= P[X \text{ in } A', Y \text{ in } B'] \\
&= P[X \text{ in } A']P[Y \text{ in } B'] \\
&= P[g(X) \text{ in } A]P[h(Y) \text{ in } B]
\end{aligned}
$$

4.4 Conditional Probability and Conditional Expectation

- The probability of events concerning the r.v. Y given that we know $X = x$
- The expected value of Y given $X = x$

❖ Conditional Probability

$$P[Y \text{ in } A \mid X = x] = \frac{P[Y \text{ in } A, X = x]}{P[X = x]}$$

- If X is discrete, then the conditional cdf of Y given $X = x_k$

$$F_Y(y \mid x_k) = \frac{P[Y \le y, X = x_k]}{P[X = x_k]}, \qquad \text{for } \; P[X = x_k] > 0$$

- The conditional pdf of Y given $X = x_k$

$$f_Y(y \mid x_k) = \frac{d}{dy} F_Y(y \mid x_k)$$

- The probability of an event A given $X = x_k$

$$P[Y \text{ in } A \mid X = x_k] = \int_{y \text{ in } A} f_Y(y \mid x_k) dy$$

- Note

  If X and Y are independent,

$$P[Y \le y, X = x_k] = P[Y \le y] P[X = x_k]$$

$$\therefore P[Y \text{ in } A \mid X = x_k] = P[Y \text{ in } A]$$

$$F_Y(y \mid x) = F_Y(y), \quad f_Y(y \mid x) = f_Y(y)$$

- If X and Y are discrete, the conditional pmf of Y given $X = x_k$

$$p_Y\left(y_j \mid x_k\right) = P[Y = y_j \mid X = x_k]$$

$$= \frac{P[X = x_k, Y = y_j]}{P[X = x_k]} \qquad \text{for } x_k \text{ such that } P[X = x_k] > 0$$

- Note

  We define

  $p_Y(y_j \mid x_k) = 0$ for x_k such that $P[X = x_k] = 0$

- The probability of any event A

$$P[Y \text{ in } A \mid X = x_k] = \sum_{y_j \text{ in } A} p_Y(y_j \mid x_k)$$

- Note

  If X and Y are independent

$$p_Y\left(y_j \mid x_k\right) = \frac{P[X = x_k]P[Y = y_j]}{P[X = x_k]} = P[Y = y_j]$$

$$= p_Y(y_j)$$

- If X is a continuous r.v., then $P[X = x] = 0$

$$\rightarrow \quad \frac{P[Y \text{ in } A, X = x]}{P[X = x]} \text{ is undefined}$$

- But, …

- Suppose that, for jointly continuous r.v.'s X and Y, with a joint pdf is continuous and nonzero over some region.

- The conditional cdf of Y given $X = x$,

$$F_Y(y \mid x) = \lim_{h \to 0} F_Y(y \mid x < X \leq x + h)$$

$$F_Y\left(y \mid x < X \leq x + h\right) = \frac{P[Y \leq y, x < X \leq x + h]}{P[x < X \leq x + h]}$$

$$= \frac{\int_{-\infty}^{y} \int_{x}^{x+h} f_{X,Y}(x', y')dx'dy'}{\int_{x}^{x+h} f_X(x')dx'}$$

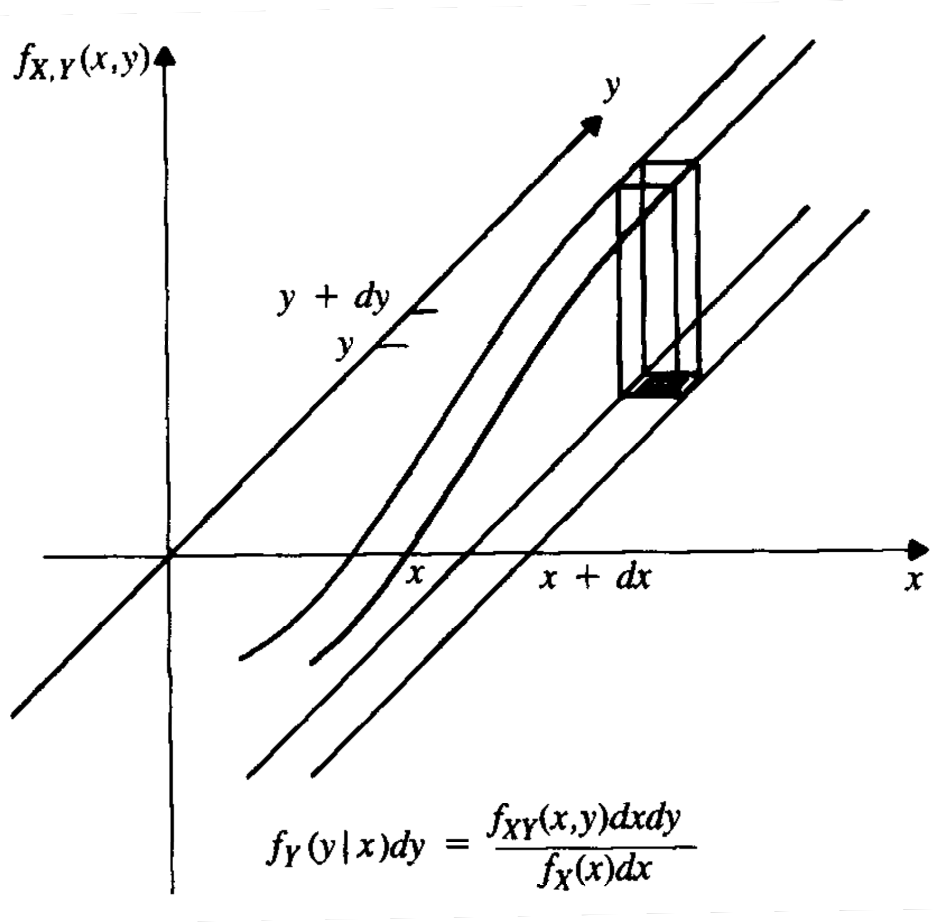
$$\cong \frac{\int_{-\infty}^{y} f_{X,Y}(x, y')dy'h}{f_X(x)h}$$

- Let $h \to 0$

$$F_Y(y \mid x) = \frac{\int_{-\infty}^{y} f_{X,Y}(x, y')dy'}{f_X(x)}$$

- The conditional pdf of Y given $X = x$

$$\boxed{f_Y(y \mid x) = \frac{d}{dy} F_Y(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)}}$$



$f_Y(y \mid x)dy \rightarrow$
the probability that Y is in the infinitesimal strip defined by $(y, y + dy)$ given that X is in the infinitesimal strip defined by $(x, x + dx)$

- Note

  a posteriori probability that Y is close to y given that X has been observed to be close to x.

- Note

  If X and Y are independent.

  $$f_{X,Y}(x, y) = f_X(x)\, f_Y(y)$$

  $$\rightarrow \quad f_Y(y \mid x) = f_Y(y) \quad \text{and} \quad F_Y(y \mid x) = F_Y(y)$$

- Note

$$\boxed{p_{X,Y}(x_k, y_j) = p_Y(y_j \mid x_k)\, p_X(x_k)}$$

$$\begin{aligned}
P[Y \text{ in } A] &= \sum_{\text{all } x_k,}\ \sum_{y_j \text{ in } A} p_{X,Y}(x_k, y_j) \\
&= \sum_{\text{all } x_k,}\ \sum_{y_j \text{ in } A} p_Y(y_j \mid x_k) p_X(x_k) \\
&= \sum_{\text{all } x_k} p_X(x_k) \sum_{y_j \text{ in } A} p_Y(y_j \mid x_k) \\
&= \sum_{\text{all } x_k} P[Y \text{ in } A \mid X = x_k] p_X(x_k)
\end{aligned}$$

First compute $P[Y \text{ in } A \mid X = x_k]$ and then average over x_k
$\rightarrow$ This is also valid when X is discrete and Y is continuous

- For continuous X and Y

$$P[Y \text{ in } A] = \int_{-\infty}^{\infty} P[Y \text{ in } A \mid X = x] f_X(x)\,dx$$

Ex. 4.22

- X : the total number of defects on a chip

  $\rightarrow$ Poisson r.v. with mean α
- p : a probability of falling in a specific region R.
- The location of defect : independent.

  Find the pmf of the number of defects Y that fall in the region R.

❖ sol) $P[Y=j]=\sum_{k=0}^{\infty} P[Y=j \mid X=k]P[X=k]$

$$P[Y=j \mid X=k]=\begin{cases} 0 & j>k \\ \binom{k}{j} p^{j}(1-p)^{k-j} & 0\le j\le k \end{cases}$$

$$P[Y=j]=\sum_{k=j}^{\infty}\frac{k!}{j!(k-j)!}p^{j}(1-p)^{j-k}\frac{\alpha^{k}}{k!}e^{-\alpha}$$

$$=\frac{(\alpha p)^{j}e^{-\alpha}}{j!}\sum_{k=j}^{\infty}\frac{\{(1-p)\alpha\}^{k-j}}{(k-j)!}$$

$$=\frac{(\alpha p)^{j}e^{-\alpha}}{j!}e^{(1-p)\alpha}=\frac{(\alpha p)^{j}}{j!}e^{-\alpha p}$$

Conditional Expectation

- The conditional expectation of Y given $X = x$

$$E[Y \mid x] = \int_{-\infty}^{\infty} y f_Y(y \mid x) dy$$

$$E[Y \mid x] = \sum_{y_j} y_j p_Y(y_j \mid x)$$ for discrete r.v.'s X and Y

- $E[Y \mid x] = g(x) \rightarrow$ a function of x
  then for a r.v. X,
  $g(X) = E[Y \mid X] \rightarrow$ a random variable

$\therefore\ E[g(X)] = E[E[Y \mid X]]$

→ Expectation w.r.t Y

→ Expectation w.r.t X

$$E\left[E[Y \mid X]\right] = \int_{-\infty}^{\infty} E[Y \mid x] f_X(x)\,dx \qquad : X \text{ continuous}$$

$$= \sum_{x_k} E[Y \mid x_k] p_X(x_k) \qquad : X \text{ discrete}$$

- Note $E[Y] = E[E[Y \mid X]]$

Proof) $$E\left[E[Y \mid X]\right] = \int_{-\infty}^{\infty} E[Y \mid x] f_X(x)dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_Y(y \mid x)dy f_X(x)dx$$
$$= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy$$
$$= \int_{-\infty}^{\infty} y f_Y(y)dy$$
$$= E[Y]$$

cf) $E[h(Y)] = E[E[h(Y) \mid X]]$

$E[Y^k] = E[E[Y^k \mid X]]$

- Ex. 4.25 Mean of Y in Ex. 4.22

$$P[Y=j]=\sum_{k=0}^{\infty}P[Y=j\mid X=k]P[X=k]$$

$$E[Y]=\sum_{k=0}^{\infty}E[Y\mid X=k]P[X=k]$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}jP[Y=j\mid X=k]\cdot P[X=k]$$

$$=\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{k}j\cdot\binom{k}{j}p^{j}(1-p)^{k-j}}_{\text{The first moment of Binomial r.v.}}P[X=k]$$

$$= \sum_{k=0}^{\infty} kpP[X = k] = p\underline{E[X]} = p\alpha$$

Mean of binomial r.v.

Mean of Poisson r.v. (given)

4.5 Multiple r.v.'s

❖ Joint Distributions

- The joint cdf $X_1, X_2, \ldots, X_n$

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P[X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n]$$

- Marginal cdf's

cf) the joint cdf for $X_1, \ldots, X_{n\text{-}1}$

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_{n-1}, \infty)$$

cf) the joint cdf for X_1 and X_2

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \infty, \ldots, \infty)$$

- Note : The prob. of all product-form events $\{X_1 \text{ in } A_1\} \cap \{X_2 \text{ in } A_2\} \ldots \cap \{X_n \text{ in } A_n\}$ can be expressed in terms of the joint cdf.

- The joint probability mass function : n discrete r.v.'s

$$p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n]$$

- The probability of any n-dimensional event A.

$$P[(X_1, \ldots, X_n) \text{ in } A] = \sum_{\mathbf{x} \text{ in } A} \cdots \sum p_{X_1, X_2, \ldots, X_n}(x_1, \ldots, x_n)$$

- Marginal pmf's

$$p_{X_j}(x_j) = P[X_j = x_j]$$
$$= \sum_{x_1} \cdots \sum_{x_{j-1}} \sum_{x_{j+1}} \cdots \sum_{x_n} p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$$

- $$p_{X_1, X_2, \ldots, X_{n-1}}(x_1, x_2, \ldots, x_{n-1}) = \sum_{x_n} p_{X_1, \ldots, X_n}(x_1, x_2, \ldots, x_n)$$

- The conditional pmf

$$p_{X_n}(x_n \mid x_1, \ldots, x_{n-1}) = \frac{p_{X_1,\ldots,X_n}(x_1, \ldots, x_n)}{p_{X_1,\ldots,X_{n-1}}(x_1, \ldots, x_{n-1})}$$

if $p_{X_1,\ldots,X_{n-1}}(x_1, \ldots, x_{n-1}) > 0$

- $$p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = p_{X_n}(x_n \mid x_1, \ldots, x_{n-1}) \times p_{X_{n-1}}(x_{n-1} \mid x_1, \ldots, x_{n-2}) \ldots p_{X_2}(x_2 \mid x_1) p_{X_1}(x_1)$$

- The joint probability density function for jointly continuous function

$$P[(X_1, X_2, \ldots, X_n) \text{ in } A] = \int \underset{\mathbf{x} \text{ in } A}{\cdots} \int f_{X_1,\ldots,X_n}(x_1', \ldots, x_n')dx_1' \ldots dx_n'$$

where $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$ is the joint pdf

- The joint cdf of $\mathbf{X}$

$$F_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(x_1', \ldots, x_n')dx_1' \ldots dx_n'$$

$$\therefore f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} F_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n)$$

- Marginal pdf's

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1, X_2, \ldots, X_n}(x_1, x_2', x_3', \ldots, x_n') dx_2' \ldots dx_n'$$

- Conditional pdf's

$$f_{X_n}(x_n \mid x_1, \ldots, x_{n-1}) = \frac{f_{X_1, \ldots, X_n}(x_1, \ldots, x_n)}{f_{X_1, \ldots, X_{n-1}}(x_1, \ldots, x_{n-1})} \qquad \text{for} \quad f_{X_1, \ldots, X_{n-1}}(x_1, \ldots, x_{n-1}) > 0$$

$$\begin{aligned} & f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) \\ & \quad = f_{X_n}(x_n \mid x_1, \ldots, x_{n-1}) f_{X_{n-1}}(x_{n-1} \mid x_1, \ldots, x_{n-2}) \ldots f_{X_2}(x_2 \mid x_1) f_{X_1}(x_1) \end{aligned}$$

Independence

- $X_1,\ldots,X_n$ are independent if and only if

$$F_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n)=F_{X_1}(x_1)\ldots F_{X_n}(x_n) \qquad \text{for all } x_1,\ldots,x_n$$

- For discrete r.v.'s

$$p(x_1,x_2,\ldots,x_n)=p_{X_1}(x_1)p_{X_2}(x_2)\ldots p_{X_n}(x_n) \qquad \text{for all } x_1,\ldots,x_n$$

- For jointly continuous r.v.'s

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)=f_{X_1}(x_1)\ldots f_{X_n}(x_n) \qquad \text{for all } x_1,\ldots,x_n$$