Outline

- Autocorrelation - F.T - power spectrum
  - Discrete and continuous
- Power spectrum density
  - realization by time average

- Review
  - Even function property of autocorrelation

7.1 Power Spectral Density

- The spectrum of the time function
  : The weighting function of the Fourier series or transform
- A sample function of a random process
  : selected from an ensemble of allowable time functions
- The weighting function or spectrum for a random process
  - The average rate of change of the ensemble of allowable time functions
  - The autocorrelation function $R_X(\tau)$ is an appropriate measure for the avg. rate of change of a random process
- Einstein-Wiener-Khinchin theorem
  : the power spectral density of a wide-sense stationary random process is given by the Fourier transform of the autocorrelation function

Continuous-Time Random Processes

- $X(t)$
  - a continuous-time WSS random process
  - mean = m_X
  - autocorrelation function = $R_X(\tau)$
  - power spectral density of $X(t)$

$$S_X(f) = \mathcal{F}\{R_X(\tau)\}$$

$$= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

- $R_X(\tau) = R_X(-\tau)$: an even function of τ with assumption of a real valued random process

$$\therefore\ S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)(\cos 2\pi f\tau - j\sin 2\pi f\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} R_X(\tau)\cos 2\pi f\tau\,d\tau$$

- $S_X(f)$
  - real-valued
  - an even function of f
  - $S_X(f) \geq 0$ for all f

- The inverse Fourier transform of the power spectral density

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\}$$

$$= \int_{-\infty}^{\infty} S_X(f)e^{j2\pi f\tau}\,df$$

- The average power of $X(t)$

$$E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\,df$$

- $R_X(\tau) = C_X(\tau) + m_X^2$

$$S_X(f) = \mathcal{F}\{C_X(\tau) + m_X^2\}$$

$$= \mathcal{F}\{C_X(\tau)\} + m_X^2 \delta(f)$$

$\Rightarrow m_X$: the “dc” component of $X(t)$

- Cross power spectral density $S_{X,Y}(f)$

  : two jointly wide-sense stationary processes

$$S_{X,Y}(f) = \mathcal{F}\{R_{X,Y}(\tau)\}$$

where $R_{X,Y}(\tau) = E[X(t+\tau)Y(t)]$

$S_{X,Y}(f)$: a complex function of f even if $X(t)$ and $Y(t)$ are both real-valued.

❖ Ex. 7.1

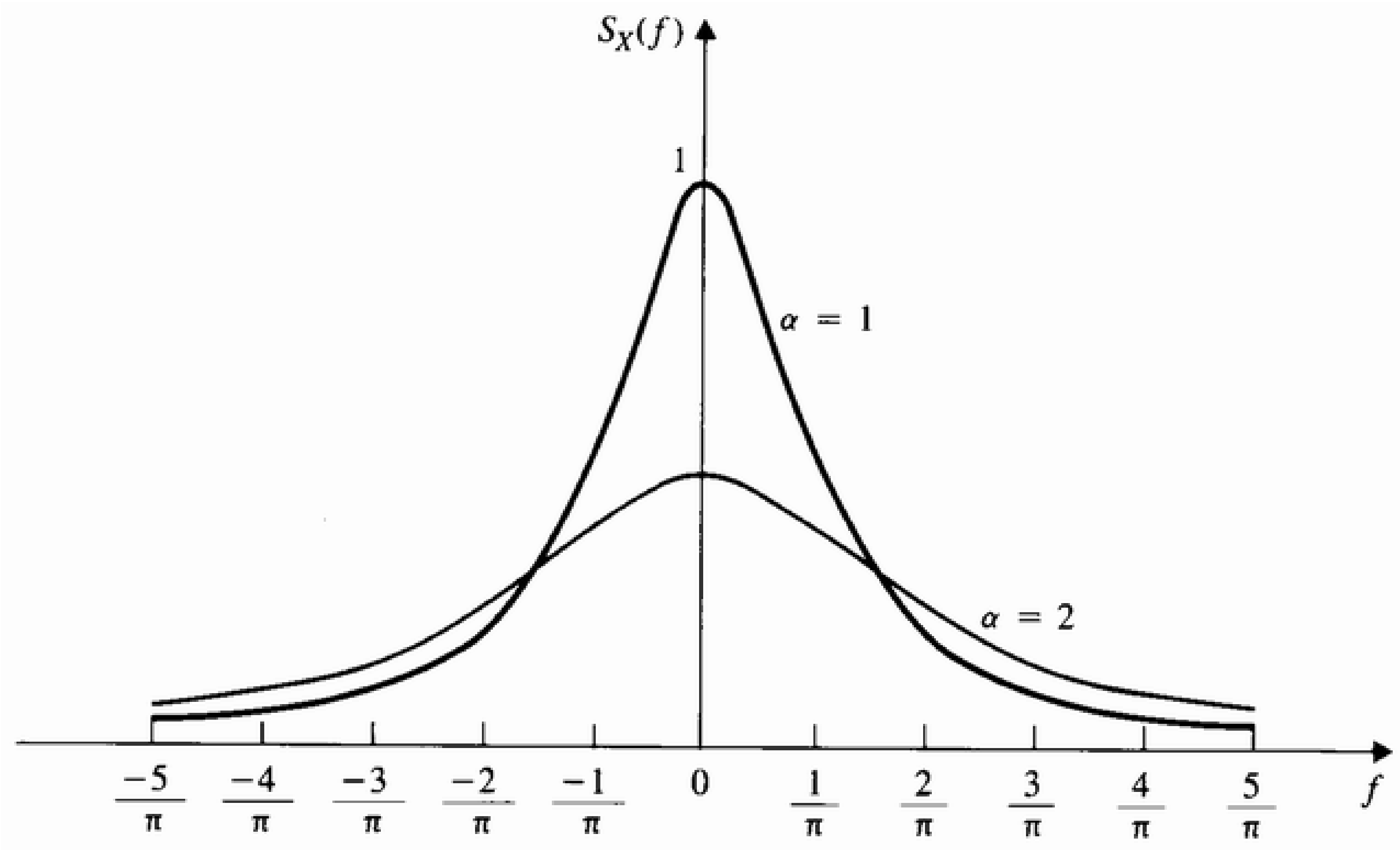
Find the power spectral density of the random telegraph signal

sol) $R_X(\alpha) = e^{-2\alpha|\tau|}$,

α : the average transition rate of the signal

$$S_X(f) = \int_{-\infty}^{0} e^{2\alpha\tau} e^{-j2\pi f\tau} d\tau + \int_{0}^{\infty} e^{-2\alpha\tau} e^{-j2\pi f\tau} d\tau$$

$$= \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}$$



$S_X(f)$
1
$\alpha = 1$
$\alpha = 2$
$\frac{-5}{\pi}$ $\frac{-4}{\pi}$ $\frac{-3}{\pi}$ $\frac{-2}{\pi}$ $\frac{-1}{\pi}$ 0 $\frac{1}{\pi}$ $\frac{2}{\pi}$ $\frac{3}{\pi}$ $\frac{4}{\pi}$ $\frac{5}{\pi}$
f

❖ Ex. 7.2

Let $X(t) = a\cos(2\pi f_0 t + \Theta)$, where Θ is uniformly distributed in the interval $(0, 2\pi)$. Find $S_X(f)$.

sol) $R_X(\tau) = \dfrac{a^2}{2}\cos 2\pi f_0 \tau$

$$\therefore\ S_X(f) = \frac{a^2}{2}\mathcal{F}\{\cos 2\pi f_0 \tau\}$$

$$= \frac{a^2}{4}\delta(f - f_0) + \frac{a^2}{4}\delta(f + f_0)$$

- Note
  - The average power of the signal

$$R_X(0) = \frac{a^2}{2}$$

  - All of this power is concentrated at the frequencies $\pm f_0$

Discrete-Time Random Processes

- X_n : a discrete-time WSS random process with mean m_X and autocorrelation function $R_X(k)$

- Power spectral density of X_n

$$S_X(f) = \mathcal{F}\{R_X(k)\}$$

$$= \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi fk}$$

❖ <u>Note</u>

- Only consider frequencies in the range $-\frac{1}{2} \le f \le \frac{1}{2}$.

$\because$ $S_X(f)$ is periodic in f with period 1.

- a real valued, nonnegative, even function of f.

❖ The inverse Fourier transform of $S_X(f)$

$$R_X(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) e^{j2\pi fk} df$$

<u>Note</u>

$R_X(k)$: the coefficients of the Fourier series of the periodic functions $S_X(f)$

- The cross-power spectral density $S_{X,Y}(f)$ of two jointly WSS discrete-time processes X_n and Y_n

$$S_{X,Y}(f) = \mathcal{F}\{R_{X,Y}(k)\}$$

where $R_{X,Y}(k) = E[X_{n+k}Y_n]$

- Ex. 7.7

  Let the process Y_n be defined by $Y_n = X_n + \alpha X_{n-1}$, where X_n is the white noise process

  Find $S_Y(f)$.

sol)

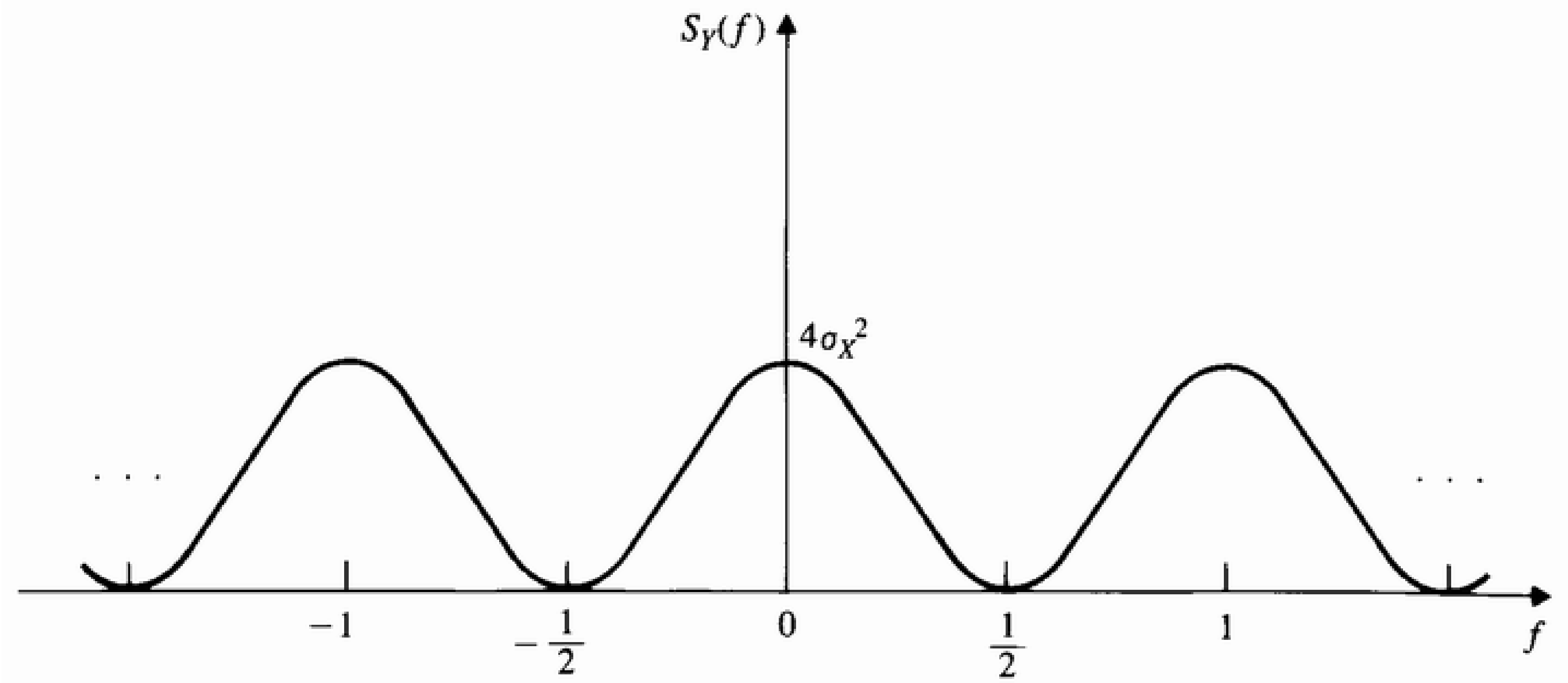
$$E[Y_n] = 0$$

$$R(k) = E[Y_n Y_{n+k}] = \begin{cases} (1+\alpha^2)\sigma_X^2 & k = 0 \\ \alpha\sigma_X^2 & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad S_Y(f) = (1+\alpha^2)\sigma_X^2 + \alpha\sigma_X^2 \{e^{j2\pi f} + e^{-j2\pi f}\}$$

$$= \sigma_X^2 \{(1+\alpha^2) + 2\alpha \cos 2\pi f\}$$

- For $\alpha = 1$



Power Spectral Density as a Time Average

- Let $X_0,\ldots,X_{k-1}$ be k (time) observations from the discrete-time WSS process

- Let $\tilde{x}_k(f)$: the discrete Fourier transform of this sequence

$$\tilde{x}_k(f) = \sum_{m=0}^{k-1} X_m e^{-j2\pi fm}$$

<u>Note</u>

$\tilde{x}_k(f)$: a complex-valued random variable

measure of the energy at f

- The magnitude squared of $\tilde{x}_k(f)$

  : a measure of the energy at the frequency f

- The "power" at the frequency f
  - ✓ $$\tilde{p}_k(f) = \frac{1}{k}\left|\tilde{x}_k(f)\right|^2$$ (time average)

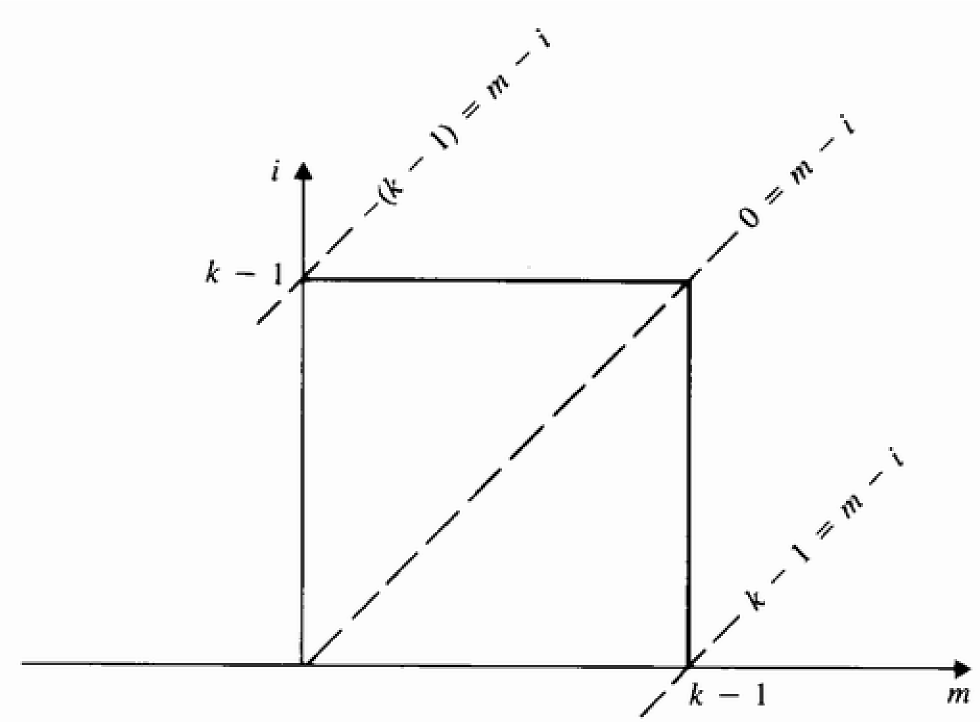
  : the periodogram estimate for the power spectral density

> <u>Note</u>
>
> Divide the energy by the total "time" k.

- The expected value of the periodogram estimate

$$
\begin{aligned}
E[\widetilde{p}_k(f)] &= \frac{1}{k} E[\widetilde{x}_k(f)\widetilde{x}_k^*(f)] \\
&= \frac{1}{k} E\left[\sum_{m=0}^{k-1} X_m e^{-j2\pi fm} \sum_{i=0}^{k-1} X_i e^{-j2\pi fi}\right] \\
&= \frac{1}{k} \sum_{m=0}^{k-1} \sum_{i=0}^{k-1} E[X_m X_i] e^{-j2\pi f(m-i)} \\
&= \frac{1}{k} \sum_{m=0}^{k-1} \sum_{i=0}^{k-1} R_X(m-i) e^{-j2\pi f(m-i)}
\end{aligned}
$$

cf)



- $R_X(m-i)$ is constant along the diagonal $m' = m-i$.
- m' ranges from $-(k-1)$ to $k-1$
- $k-|m'|$ terms along the diagonal $m' = m-i$

$$\therefore E[\tilde{p}_k(f)] = \frac{1}{k} \sum_{m'=-(k-1)}^{k-1} \{k - |m'|\} R_X(m') e^{-j2\pi f m'}$$

$$= \sum_{m'=-(k-1)}^{k-1} \left\{1 - \frac{|m'|}{k}\right\} R_X(m') e^{-j2\pi f m'}$$

❖ <u>Note</u>

$$E[\tilde{p}_k(f)] \neq S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi f k}$$

Differences (replace $2T$ with k in Chap. 6.7)

1. $\left\{1 - \frac{|m'|}{k}\right\}$ term

2. limits of $\sum$

- Note
  - $\widetilde{p}_k(f)$ is a "biased" estimator for $S_X(f)$
  - As k goes infinite, $\left\{1-\frac{|m'|}{k}\right\}$ approaches 1

    and limits of summation approaches $\pm\infty$.
  - $E[\widetilde{p}_k(f)] \rightarrow S_X(f)$ as $k \rightarrow \infty$
  - $S_X(f)$ is nonnegative for all f

    $\because\ \widetilde{p}_k(f) = \frac{1}{k}\left|\widetilde{x}_k(f)\right|^2$ is nonnegative for all f
  - The variance of the periodogram estimate should also approaches zero.

Homework

- Chapter 7
- 4,8,12,14,16