Electromagnetics

<Chap. 9> Transmission Lines

Section 9.3 ~ 9.4

(1st of **week 10**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 9 | Contents for 1st class of week 10

Review of last class

- Equivalent circuit model: Lossy transmission lines

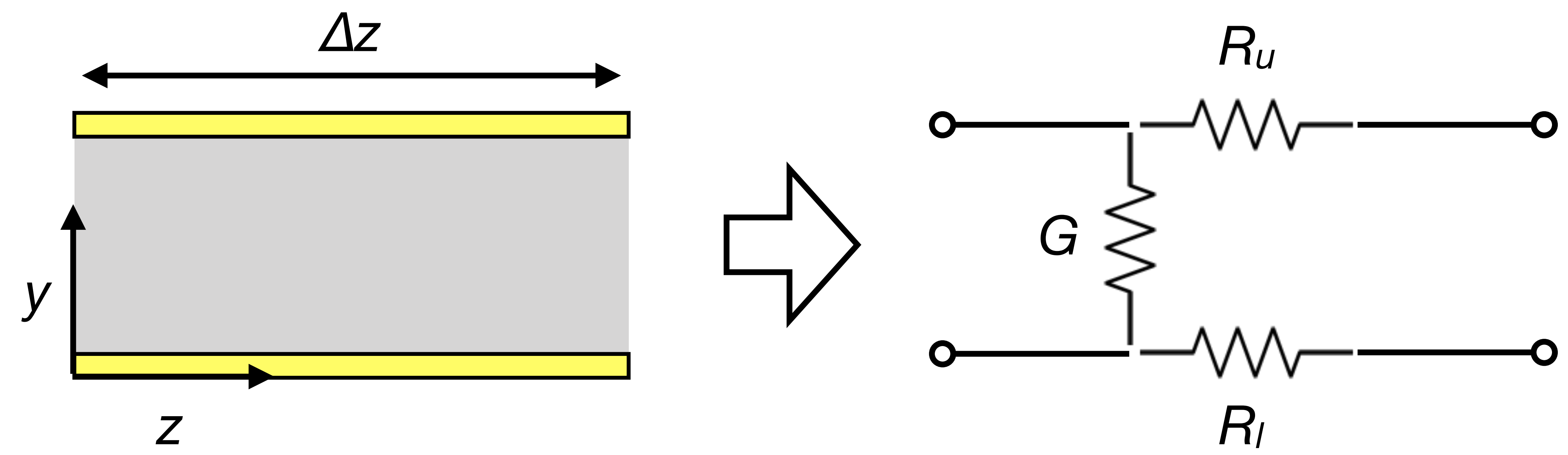
Sec 3. General Transmission-Line Equations

- General equations
- Special cases
- TR-line circuit parameters

Chap. 9 | Lossy TR lines: Equivalent circuit model (1/3)

- ***Attenuation in the parallel-plate transmission lines caused by…***
  - (1) Lossy dielectric ($\sigma \neq 0$)
  - (2) Imperfectly conducting walls ($\sigma_c \neq \infty$)

$$\alpha = \alpha_d + \alpha_c = \frac{\sigma}{2}\eta + \frac{1}{d}\sqrt{\frac{\pi f \varepsilon}{\sigma_c}}$$



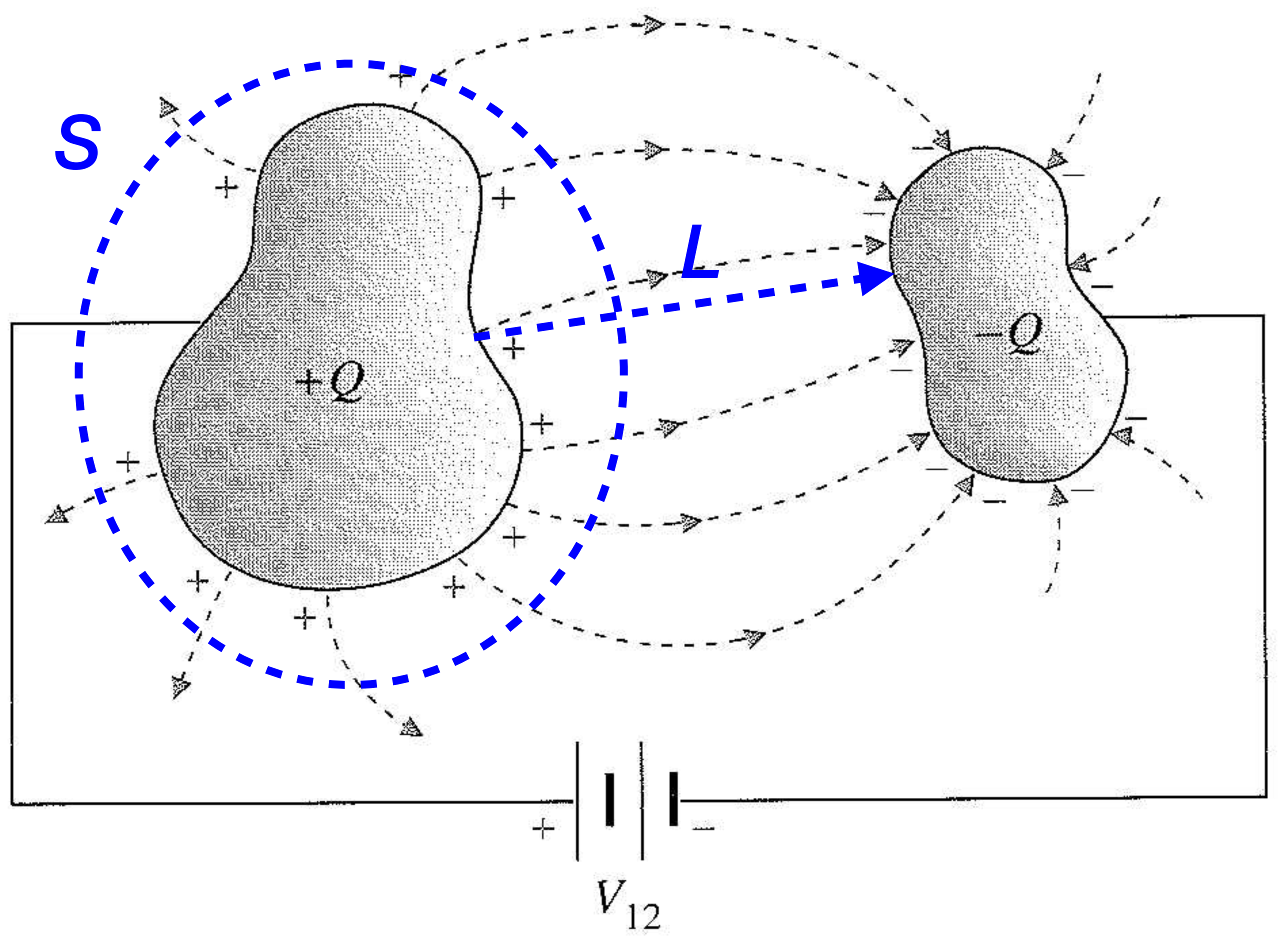
<Circuit representation>

- ***Conductance (G) between two conductors per unit length***

$$G = C\frac{\sigma}{\varepsilon} \quad \textit{(from right)}$$

$$= \varepsilon\frac{w}{d}\cdot\frac{\sigma}{\varepsilon} = \sigma\frac{w}{d} \quad \text{(S/m)}$$

where σ is the conductivity of the dielectric



$$C = \frac{Q}{V} = \frac{\oint_S \boldsymbol{D}\cdot d\boldsymbol{s}}{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}} = \frac{\oint_S \varepsilon\boldsymbol{E}\cdot d\boldsymbol{s}}{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}}$$

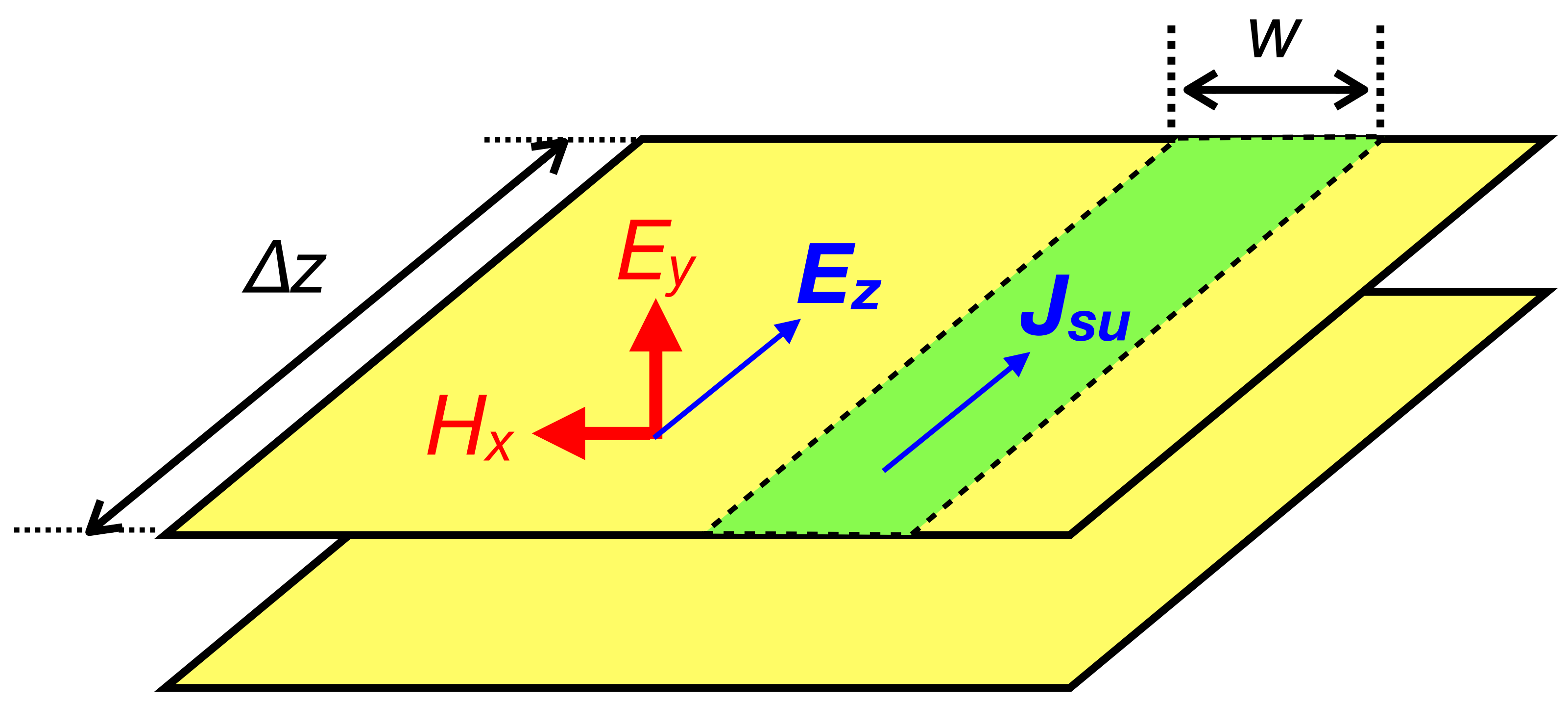
$$R = \frac{V}{I} = \frac{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}}{\oint_S \boldsymbol{J}\cdot d\boldsymbol{s}} = \frac{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}}{\oint_S \sigma\boldsymbol{E}\cdot d\boldsymbol{s}}$$

$$\rightarrow RC = \frac{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}}{\sigma\oint_S \boldsymbol{E}\cdot d\boldsymbol{s}}\cdot\frac{\varepsilon\oint_S \boldsymbol{E}\cdot d\boldsymbol{s}}{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}} = \frac{\varepsilon}{\sigma} = \frac{C}{G}$$

Chap. 9 | Lossy TR lines: Equivalent circuit model (2/3)

- ***Resistance** (R) along the conductors per unit length*
  - In actual cases, conductivity of the plate is ***finite*** ($\sigma_c \neq \infty$)

    $\therefore$ *small, yet non-vanishing longitudinal field (E_z) "induced!" ($H_x \rightarrow J_{su} \rightarrow E_z$!)*
  - *R* obtained by *<u>relationship between power loss at the surface vs. surface current</u>*



  - Time-average power dissipated on unit surface [W/m²] due to E_z

$$\boldsymbol{a_y} p_{\sigma_c}\ [\mathrm{W/m^2}] = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{a_z} E_z \times \boldsymbol{a_x} H_x^*\right) = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{a_y}|\boldsymbol{J_s}|^2 Z_s\right) \quad \cdots(1)$$

"Surface" impedance (Z_s) of the plate

$$Z_s = \frac{E_z}{H_x} = \frac{E_z}{J_{su}} = \eta_c \quad \cdots(2) \quad \text{(= Intrinsic impedance of the plate)}$$

$$\eta_c = R_s + jX_s = (1+j)\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \quad (\Omega) \quad \cdots(3) \quad \text{(Refer to lecture note 3-2)}$$

"Surface" current (J_s) *($\because f\uparrow \rightarrow$ skin depth$\downarrow$)*

$$\boldsymbol{J}_{su} = \boldsymbol{a_z} H_x = \boldsymbol{a_z}\sigma_c E_z \quad (\mathrm{A/m})$$

$$\boldsymbol{J}_{sl} = -\boldsymbol{J}_{su}$$

  - From **eqn. (2)**, we get

$$E_z = J_{su} Z_s \quad \text{and} \quad H_x = J_{su} \quad \cdots(4)$$

  - By plugging **eqn. (4)** and **(3)** into **eqn. (1)**, we get

$$p_{\sigma_c} = \frac{1}{2}\mathrm{Re}\left(|J_s|^2 Z_s\right) = \frac{1}{2}|J_s|^2 R_s \quad (\mathrm{W/m^2})$$

  - Power dissipated per unit length [W/m] through the plate of width *w*

$$P_{\sigma_c} = w p_{\sigma_c} = \frac{1}{2} w |J_s|^2 R_s \quad (\mathrm{W/m})$$

$$= \frac{1}{2}|wJ_s|^2\left(\frac{R_s}{w}\right) = \frac{1}{2} I^2 \left(\frac{R_s}{w}\right)$$

$$\therefore R = R_u + R_l = 2\left(\frac{R_s}{w}\right) = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \quad (\Omega / m)$$

Lossy TR lines: Equivalent circuit model (3/3)

- How significant is E_z?

$$\frac{|E_z|}{|E_y|} = \frac{|\eta_c H_x|}{|\eta H_x|} = \sqrt{\frac{\varepsilon}{\mu}}|\eta_c| = \sqrt{\frac{\varepsilon}{\mu}}\sqrt{2}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \sqrt{\frac{2\pi f \varepsilon}{\sigma_c}}$$

e.g.) For copper [σ_c = 5.8 x 10^7 (S/m)] and $\varepsilon = \varepsilon_0$ for dielectric at f = 3 (GHz),

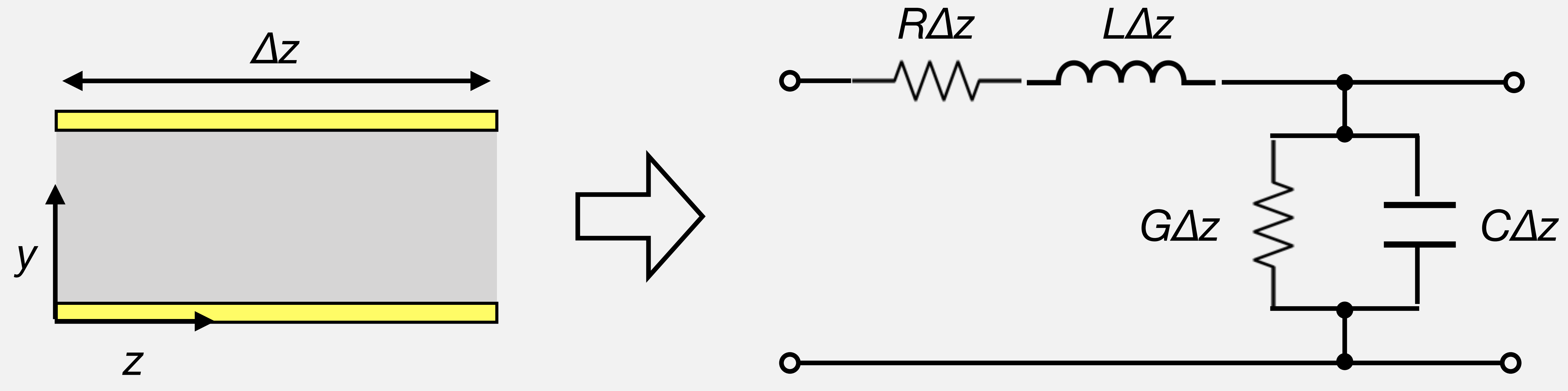
$$|E_z| \simeq 5.3\times 10^{-5}|E_y| \ll |E_y|$$

∴ E_z = a slight perturbation → TEM approximation holds!

→ ***"Quasi"-TEM mode*** in lossy transmission line!

- ***Distributed parameters of parallel-plate transmission line (width = w, separation = d)***

Parameter	Formula	Unit
R	$\frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}}$	(Ω / m)
L	$\mu \frac{d}{w}$	(H / m)
G	$\sigma \frac{w}{d}$	(S / m)
C	$\varepsilon \frac{w}{d}$	(F / m)

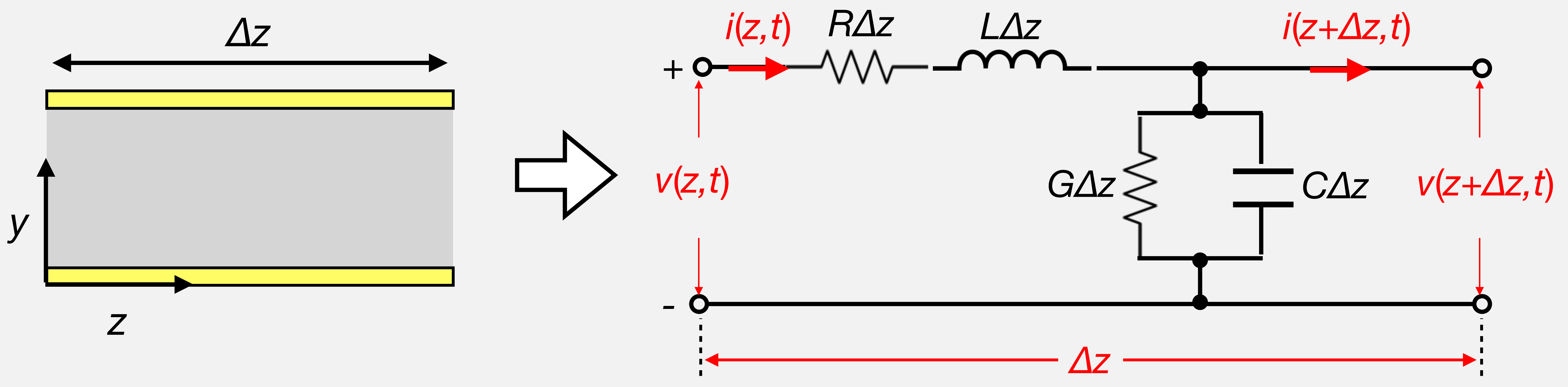


<Equivalent circuit of a two conductor transmission line>

Chap. 9 | General TR-line equations (1a/4)

- ***General TR-line equations***
  - Generally applied to ***parallel-plate, two-wire, coaxial*** TR-lines

  - ***"Disturbed-element" model***
    - TR-line = ***infinite*** series of an ***infinitesimally short segment (Δz) of the TR-line***
    - Short segment represented with ***circuit-elements (R, L, C, G) "distributed" uniformly*** throughout entire TR-line (R, L, C, G given per unit length)

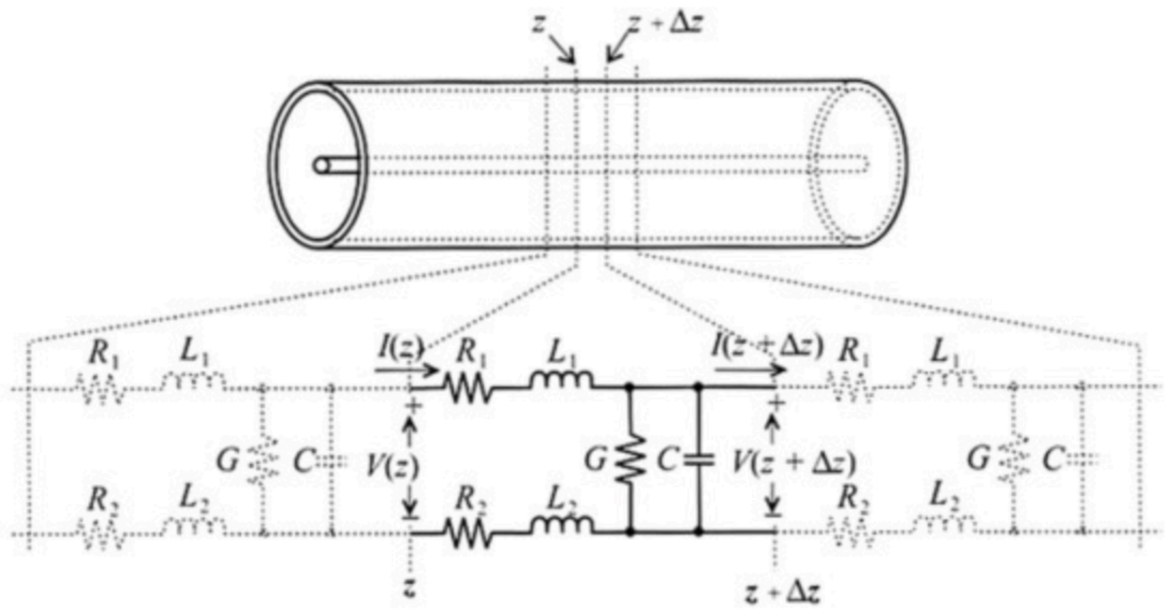


Series components
- R (Ω/m): finite conductivity of the plates (σ_c)
- L (H/m): H-field in the wire and self-inductance

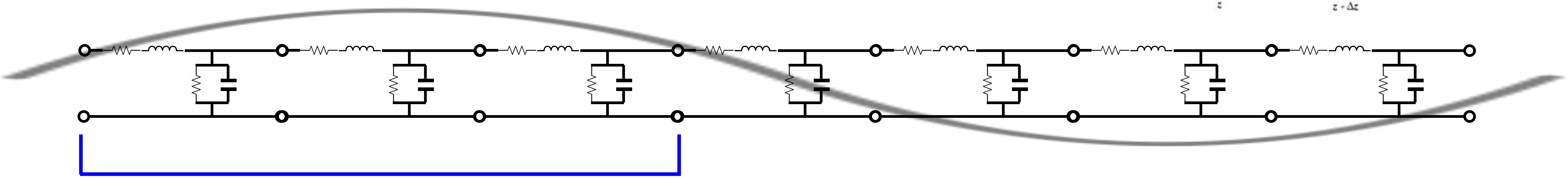
Shunt components
- C (F/m): Two conductors + dielectric
- G (S/m): non-zero conductivity of dielectric (σ)

  - When used?
    - ***Very, very long TR-line (>240 km)***: AC voltage and current at one location different from those at other (∵signal speed ≠ ∞)
    - ***At very high frequency***: physical dimension of circuit ~ wavelength of electrical signal
      - For these cases, wires or lines are not perfect conductors and their impedance matters (represented by R, L, C, G)
      - *c.f) Lumped-element model (R, L, C, G NOT depending on length and concentrated at singular points) e.g. regular circuit we use*

Chap. 9 | General TR-line equations (1b/4)



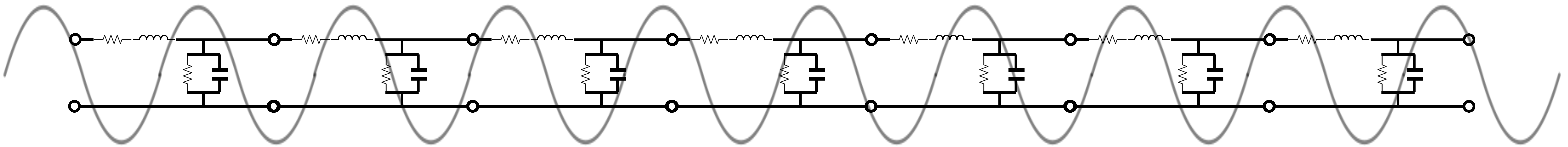
- *Lumped-element model (low frequency)*



Can be lumped into single *R, G, C, G*

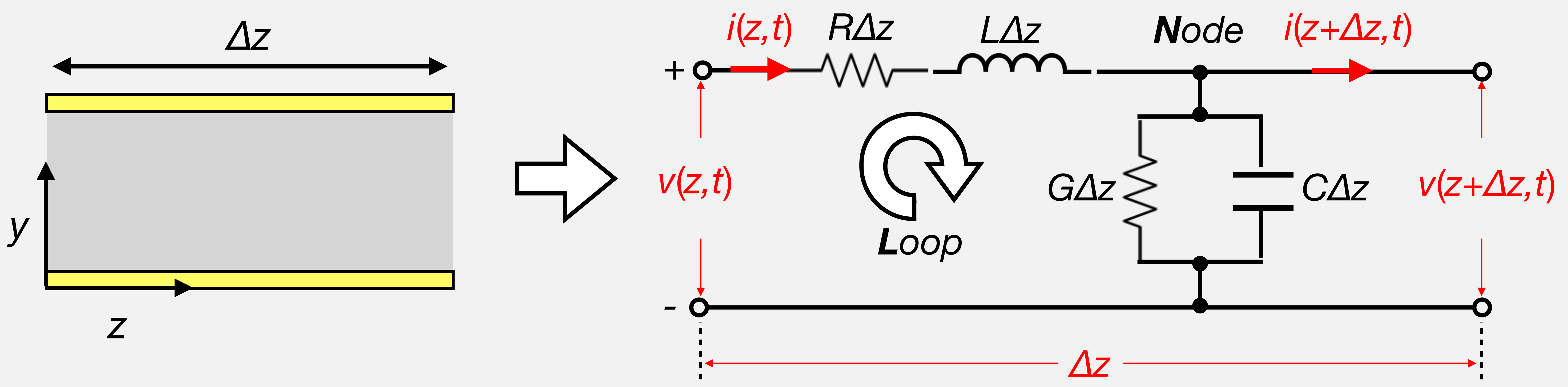
(∴not dependent on *z*)

- *Disturbed-element model (high frequency)*



Chap. 9 | General TR-line equations (2/4)

• *General TR-line equations*



<Infinitesimal TR line element> <Equivalent circuit>

Series components

- R (Ω/m): finite conductivity of the plates (σ_c)
- L (H/m): H-field in the wire and self-inductance

Shunt components

- C (F/m): Two conductors + dielectric
- G (S/m): non-zero conductivity of dielectric (σ)

- Kirchoff's voltage law (around loop ***L***)

$$-v(z,t)+R\Delta z\cdot i(z,t)+L\Delta z\frac{\partial i(z,t)}{\partial t}+v(z+\Delta z,t)=0 \quad\rightarrow\quad -\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z}\overset{\Delta z\rightarrow 0}{\triangleq}\boxed{-\frac{\partial v(z,t)}{\partial z}=Ri(z,t)+L\frac{\partial i(z,t)}{\partial t}}$$

- Kirchoff's current law (to node ***N***)

$$i(z,t)-G\Delta z\cdot v(z+\Delta z,t)-C\Delta z\frac{\partial v(z+\Delta z,t)}{\partial t}-i(z+\Delta z,t)=0 \quad\rightarrow\quad -\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z}\triangleq\boxed{-\frac{\partial i(z,t)}{\partial z}=Gv(z,t)+C\frac{\partial v(z,t)}{\partial t}}$$

General TR-line equations (3/4)

- ***General transmission-line equations***

: A pair of 1st-order PDEs in $v(z,t)$ and $i(z,t)$

$$\begin{cases} -\dfrac{\partial v(z,t)}{\partial z} = Ri(z,t) + L\dfrac{\partial i(z,t)}{\partial t} \\ -\dfrac{\partial i(z,t)}{\partial z} = Gv(z,t) + C\dfrac{\partial v(z,t)}{\partial t} \end{cases}$$

Time-harmonic TR-line equation

since $\begin{cases} v(z,t) = \mathrm{Re}\left[V(z)e^{j\omega t}\right] \\ i(z,t) = \mathrm{Re}\left[I(z)e^{j\omega t}\right] \end{cases}$,

$$\begin{cases} -\dfrac{dV(z)}{dz} = (R + j\omega L)I(z) \\ -\dfrac{dI(z)}{dz} = (G + j\omega C)V(z) \end{cases}$$

c.f.) Under lossless condition (R = 0, G = 0)
(where $\sigma_c \to \infty$, $\sigma \to 0$)

$$\begin{cases} -\dfrac{dV(z)}{dz} = j\omega LI(z) \\ -\dfrac{dI(z)}{dz} = j\omega CV(z) \end{cases}$$

Consistent with ideal case!
(in previous class)

- ***Wave characteristics on an infinite TR line***

- To solve for $V(z)$ and $I(z)$, coupled time-harmonic TR equations combined as

(Double derivative)

$$\begin{cases} -\dfrac{d^2V(z)}{dz^2} = (R + j\omega L)\dfrac{dI(z)}{dz} = -(R + j\omega L)(G + j\omega C)V(z) \\ -\dfrac{d^2I(z)}{dz^2} = (G + j\omega C)\dfrac{dV(z)}{dz} = -(G + j\omega C)(R + j\omega L)I(z) \end{cases}$$

$\Rightarrow$

$$\begin{cases} \dfrac{d^2V(z)}{dz^2} = \gamma^2 V(z) \\ \dfrac{d^2I(z)}{dz^2} = \gamma^2 I(z) \end{cases}$$

where $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$

 General TR-line equations (4/4)

- *Solution for an infinite TR-line*

$$\begin{cases} V(z) = V_0 e^{-\gamma z} + \cancel{V_1} e^{\gamma z} \\ I(z) = I_0 e^{-\gamma z} + \cancel{I_1} e^{\gamma z} \end{cases} \cdots(1)$$ *(∵No reflection!)*

- If we plug (1) into (2),

$$\left[-\frac{d}{dz}\left(V_0 e^{-\gamma z}\right) = \gamma V_0 e^{-\gamma z}\right] = (R + j\omega L) I_0 e^{-\gamma z}$$

$$\rightarrow \frac{V_0}{I_0} \triangleq Z_0 = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

First order ODEs

$$\begin{cases} -\dfrac{dV(z)}{dz} = (R + j\omega L) I(z) \\ -\dfrac{dI(z)}{dz} = (G + j\omega C) V(z) \end{cases} \cdots(2)$$

Second order ODEs

$$\begin{cases} \dfrac{d^2V(z)}{dz^2} = \gamma^2 V(z) \\ \dfrac{d^2I(z)}{dz^2} = \gamma^2 I(z) \end{cases}$$ where $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$

Characteristic Impedance (Z_0) & propagation constant (γ)

- Z_0 is uniform across the TR-line (∵uniform cross-section)
- Both NOT depend on *z* (particular position TR-line)
- Both ONLY depend on distributed parameters (*R*, *L*, *G*, *C*) and *ω*

Chap. 9 | Special cases for TR-lines (1/3)

- ***1) Lossless line*** **(R = 0, G = 0)**
  - ***Ideal*** case → infinite conductor conductivity ($\sigma_c = \infty$), zero dielectric conductivity ($\sigma = 0$)
  - ***Propagation constant:***

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = j\omega\sqrt{LC}$$

$$\alpha = 0 \quad \textit{(No attenuation)}$$

$$\beta = \omega\sqrt{LC} \quad \text{(a “linear” function of } \omega\text{)}$$

  - ***Phase velocity:***

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \quad \text{(constant)}$$

Non-dispersive!
(i.e. frequency-independent)
→ No signal distortion!

  - ***Characteristic impedance:***

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{L}{C}}$$

$$R_0 = \sqrt{\frac{L}{C}} \quad \text{(constant)}$$

$$X_0 = 0$$

- ***2) Low-Loss line*** **($R << \omega L$, $G << \omega C$)**
  - Realistic case, easily satisfied at ***very high frequencies***
  - ***Propagation constant:***

Binomial approx.

$$\boxed{\lim_{x \to 0}(1 + x)^n \simeq (1 + nx)}$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{j\omega L\left(1 + \frac{R}{j\omega L}\right) j\omega C\left(1 + \frac{G}{j\omega C}\right)} \simeq j\omega\sqrt{LC}\left(1 + \frac{R}{2j\omega L}\right)\left(1 + \frac{G}{2j\omega C}\right)$$

$$\simeq j\omega\sqrt{LC}\left(1 + \frac{1}{2j\omega}\left(\frac{R}{L} + \frac{G}{C}\right) - \cancel{\frac{1}{4\omega^2}\frac{RG}{LC}}\right)$$

(Neglected at high frequency!)

$$\therefore \alpha \simeq \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right),$$

$$\beta \simeq \omega\sqrt{LC} \quad \text{(“nearly” a linear function of } \omega\text{)}$$

Special cases for TR-lines (2/3)

• *2) Low-Loss line* ($R \ll \omega L$, $G \ll \omega C$)

- Phase velocity:

$$u_p = \frac{\omega}{\beta} \simeq \frac{1}{\sqrt{LC}}$$ *(Approximately constant)*

- Characteristic impedance:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{j\omega L}{j\omega C}}\left(1 + \frac{R}{j\omega L}\right)^{1/2}\left(1 + \frac{G}{j\omega C}\right)^{-1/2} \simeq \sqrt{\frac{L}{C}}\left(1 + \frac{R}{2j\omega L}\right)\left(1 - \frac{G}{2j\omega C}\right)$$

$$\simeq \sqrt{\frac{L}{C}}\left[1 + \frac{1}{2j\omega}\left(\frac{R}{L} - \frac{G}{C}\right) - \cancel{\frac{1}{4\omega^2}\frac{RG}{LC}}\right]$$

(Neglected at high frequency!)

$$\therefore R_0 \simeq \sqrt{\frac{L}{C}},$$

$$X_0 \simeq -\sqrt{\frac{L}{C}}\,\frac{1}{2\omega}\left(\frac{R}{L} - \frac{G}{C}\right) \simeq 0$$ *(minor phase shift between E and H)*

Lossy transmission line

- At ***high*** frequency → *"nearly"* non-dispersive system (Good!)
- At ***low*** frequency → dispersive system (Bad, ***signal distortion!***) →

(Signal = a band of ***multiple, continuous frequencies***)

i.e.) α, β: functions of frequency (ω)
→ attenuates differently vs. ω
→ travels differently vs. ω

Chap. 9 | Special cases for TR-lines (3/3)

- *3) "Distortionless" line*
  - If lossy TR-line satisfies the condition as $\dfrac{R}{L} = \dfrac{G}{C}$ or $G = \dfrac{RC}{L}$,

- Propagation constant:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(R + j\omega L)\left(\frac{RC}{L} + j\omega C\right)} = \sqrt{(R + j\omega L)(R + j\omega L)\frac{C}{L}} = \sqrt{\frac{C}{L}}(R + j\omega L)$$

$$\alpha = R\sqrt{\frac{C}{L}}, \quad \beta = \omega\sqrt{LC} \quad \textit{(A linear function of } \omega\textit{)} \quad \rightarrow \quad u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \quad \textit{(Constant)}$$

- Characteristic impedance:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{R + j\omega L}{RC/L + j\omega C}} = \sqrt{\frac{L}{C}} \quad \textit{(Constant)}$$

∴ ***Distortionless TR line can be designed such that*** $\dfrac{R}{L} = \dfrac{G}{C}$

Q: How would you minimize *α* (i.e. attenuation) though?

Series components
- *R* (Ω/m): finite conductivity of the plates (σ_c)
- *L* (H/m): *H*-field in the wire and self-inductance

Shunt components
- *C* (F/m): Two conductors + dielectric
- *G* (S/m): non-zero conductivity of dielectric (σ)

Chap. 9 | TR-line parameters (1/3)

- ***TR-line parameters***
  - Electrical properties of TR-line ***COMPLETELY*** explained by ***distributed parameters (R, L, C, G)*** at given ω
  - For simplicity: $\sigma_c \to \infty$ (i.e. very good conductors) → $R = 0$ so that waves ***nearly TEM!***

$$\gamma = \sqrt{(\cancel{R} + j\omega L)(G + j\omega C)} = j\omega\sqrt{LC}\left(1 + \frac{G}{j\omega C}\right)^{1/2}$$

$$= j\omega\sqrt{LC}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2} \quad \cdots(1) \qquad \left(\because \frac{G}{C} = \frac{\sigma}{\varepsilon} \quad \cdots(2)\right)$$

(From p.3 of this slide)

  - By comparing (2) and (1), we know that $\boxed{LC = \mu\varepsilon \quad \cdots(4)}$

Propagation constant in lossy medium

$$\gamma = jk_c = j\omega\sqrt{\mu\varepsilon_c} \quad \text{where} \quad \varepsilon_c = \varepsilon + \frac{\sigma}{j\omega}$$

$$= j\omega\sqrt{\mu\left(\varepsilon + \frac{\sigma}{j\omega}\right)}$$

$$= j\omega\sqrt{\mu\varepsilon}\sqrt{1 + \frac{\sigma}{j\omega\varepsilon}} \quad \cdots(3)$$

- ***Procedures to obtain R, L, C, G***
  - If ***L*** is known, we know ***C*** from **equation (4)** and vice versa
  - Once ***C*** is determined, we know ***G*** from **equation (2)**
  - ***R*** can be obtained by introducing a small E_z as perturbation & by finding ohmic power dissipated in a unit length of the TR line (see right)

Intrinsic impedance of a good conductor

$$Z_s = R_s + jX_s = (1 + j)\sqrt{\frac{\pi f \mu_c}{\sigma_c}}$$

Ohmic power dissipated per unit area

$$p_\sigma = \mathrm{Re}\left(\frac{1}{2}|J_s|^2 Z_s\right) = \frac{1}{2}|J_s|^2 R_s \quad (\mathrm{W/m^2})$$

where J_s: surface current (A/m)

Chap. 9 | TR-line parameters (2/3)

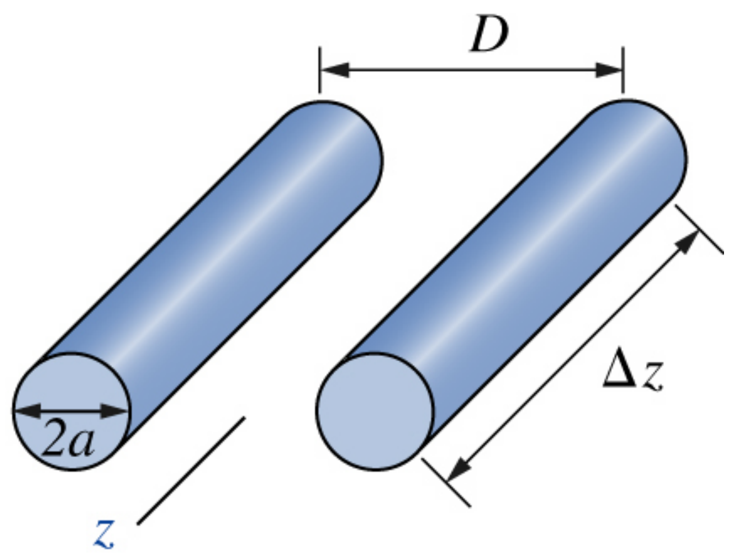
• *Two-wire TR line*

- ***Capacitance*** per unit length of a two-wire TR line

$$C = \frac{\pi\varepsilon}{\cosh^{-1}(D/2a)}$$ *(Chap. 4-4; Image method)*

- ***Inductance*** per unit length

$$L = \frac{\mu\varepsilon}{C} = \frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right) \quad (\because LC = \mu\varepsilon)$$



- ***Conductance*** per unit length

$$G = C\frac{\sigma}{\varepsilon} = \frac{\pi\sigma}{\cosh^{-1}(D/2a)} \quad \left(\because \frac{G}{C} = \frac{\sigma}{\varepsilon}\right)$$

- ***Resistance*** per unit length of "single" conductor

$$P_\sigma\ (\mathrm{W/m}) = 2\pi a \cdot p_\sigma = 2\pi a \frac{1}{2}|J_s|^2 R_s$$

$$= \frac{1}{2}\underline{|2\pi a J_s|}^2\left(\frac{R_s}{2\pi a}\right) = \frac{1}{2}I^2\left(\frac{R_s}{2\pi a}\right)$$

$$R = 2\left(\frac{R_s}{2\pi a}\right) = \frac{1}{\pi a}\sqrt{\frac{\pi f \mu_c}{\sigma_c}}$$

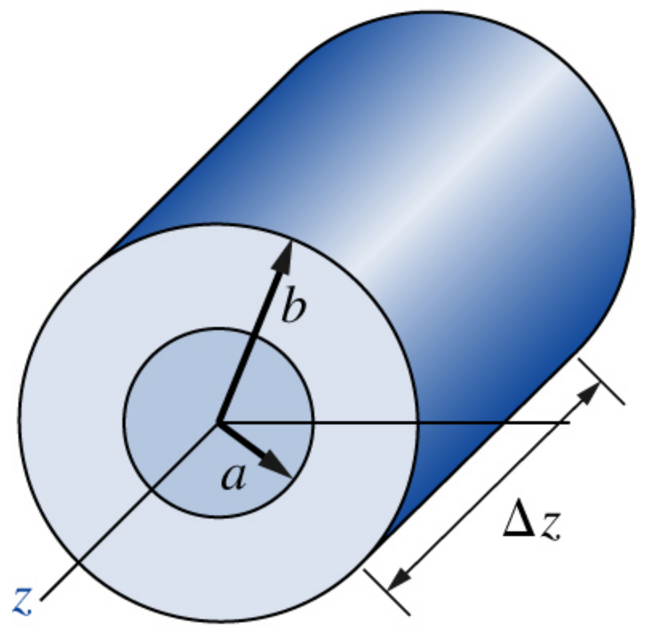
• *Coaxial TR line*

- ***Inductance*** per unit length of a coaxial TR line

$$L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$$ *(Chap. 6-11; Mutual inductance)*

- ***Capacitance*** per unit length

$$C = \frac{\mu\varepsilon}{L} = \frac{2\pi\varepsilon}{\ln(b/a)}$$



- ***Conductance*** per unit length

$$G = C\frac{\sigma}{\varepsilon} = \frac{2\pi\sigma}{\ln(b/a)}$$

- ***Resistance*** per unit length of inner & outer conductors

$$P_\sigma\ (\mathrm{W/m}) = 2\pi a \cdot p_{\sigma i} + 2\pi b \cdot p_{\sigma o} = 2\pi a \frac{1}{2}|J_{si}|^2 R_s + 2\pi b \frac{1}{2}|J_{so}|^2 R_s$$

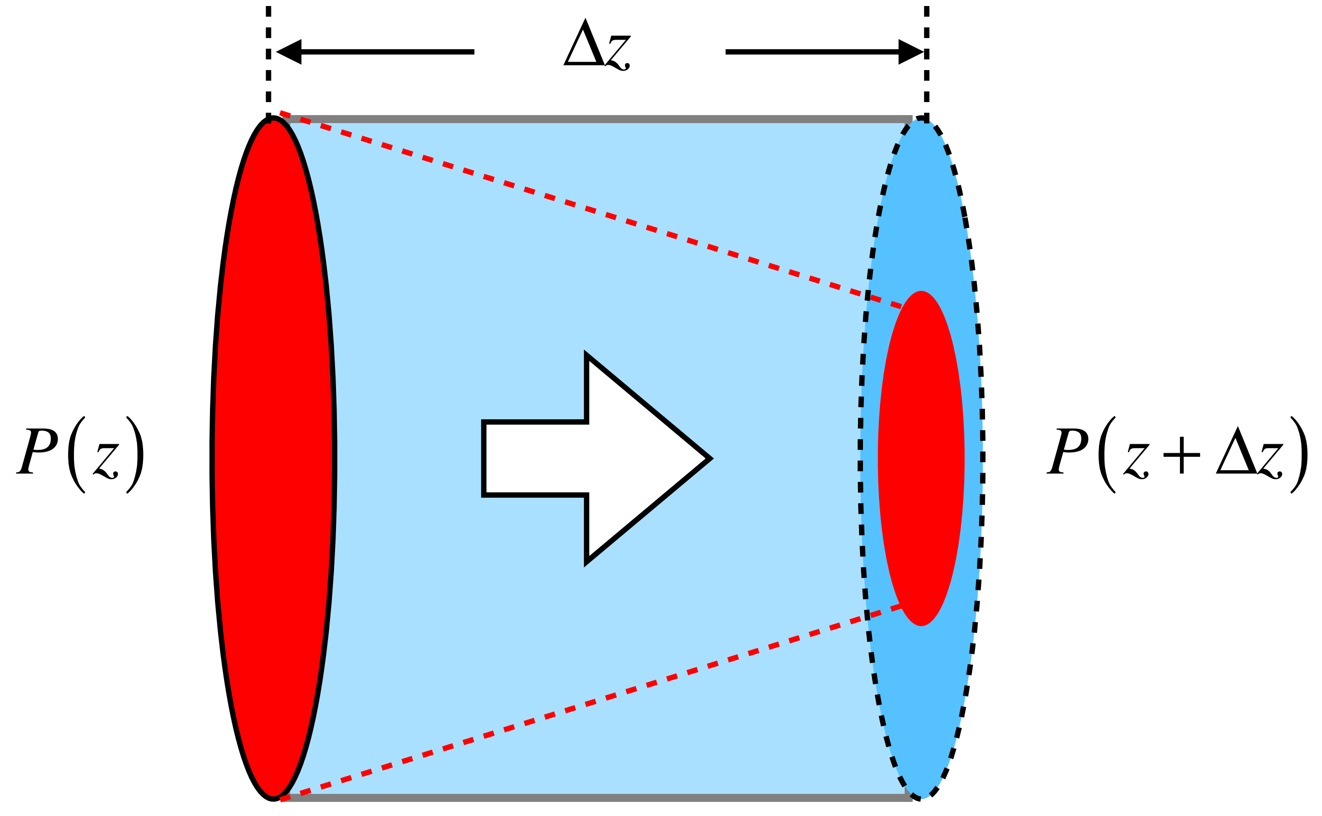
$$= \frac{1}{2}(2\pi a|J_{si}|)^2\left(\frac{R_s}{2\pi a}\right) + \frac{1}{2}(2\pi b|J_{so}|)^2\left(\frac{R_s}{2\pi b}\right) = \frac{1}{2}I^2\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$R = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$$

Chap. 9 | TR-line parameters (3/3)

- *Attenuation constant from power relations*

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \rightarrow \alpha = \mathrm{Re}\left[\sqrt{(R + j\omega L)(G + j\omega C)}\right]$$ *(One method!)*



- Time-average power propagated along the TR line **at any z**

$$P(z) = \frac{1}{2}\mathrm{Re}\left[V(z)I^*(z)\right] = \frac{V_0^2}{2|Z_0|^2}R_0 e^{-2\alpha z} \quad \text{where} \quad \begin{cases} V(z) = V_0 e^{-\gamma z} = V_0 e^{-(\alpha + j\beta)z} \\ I(z) = \dfrac{V_0}{Z_0} e^{-(\alpha + j\beta)z} \end{cases}$$

$$Z_0 = \frac{V_0}{I_0} = R_0 + jX_0 = \frac{R + j\omega L}{\gamma}$$

- Rate of power decrease along the line of length Δz

$$-\lim_{\Delta z \to 0}\frac{P(z + \Delta z) - P(z)}{\Delta z} \rightarrow -\frac{\partial P(z)}{\partial z} = 2\alpha P(z) = \frac{\alpha V_0^2}{|Z_0|^2}R_0 e^{-2\alpha z} \quad \cdots(1)$$

- Rate of energy dissipation P_L per Δz

$$P_L(z) = \frac{1}{2}\left[|I(z)|^2 R + |V(z)|^2 G\right] = \frac{V_0^2}{2|Z_0|^2}\left(R + G|Z_0|^2\right)e^{-2\alpha z} \quad \cdots(2)$$

- Since (1) = (2),

$$2\alpha P(z) = P_L(z) \rightarrow \boxed{\therefore \alpha = \frac{P_L(z)}{2P(z)} = \frac{1}{2R_0}\left(R + G|Z_0|^2\right)}$$

Consistent results as in pp. 9~11

- For a low-loss line $\left(Z_0 \cong R_0 = \sqrt{\dfrac{L}{C}}\right)$

$$\alpha = \frac{1}{2R_0}\left(R + G|Z_0|^2\right) = \frac{1}{2}\left(\frac{R}{R_0} + GR_0\right) = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$$

- For a distortion-less line $\left(Z_0 = R_0 = \sqrt{\dfrac{L}{C}} \text{ and } \dfrac{R}{L} = \dfrac{G}{C}\right)$

$$\alpha = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) = \frac{1}{2}R\sqrt{\frac{C}{L}}\left(1 + \frac{G}{R}\cdot\frac{L}{C}\right) = R\sqrt{\frac{C}{L}}$$

Electromagnetics

<Chap. 9> Transmission Lines

Section 9.3 ~ 9.4

(2nd of **week 10**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 9 | Contents for 2nd class of week 10

Sec 4. Wave characteristics on Finite Transmission Lines

- Equivalent circuit model
- Impedance matching: load impedance & TR-line length
- Lossy finite-length TR-lines

Chap. 9 | Circuit model for "finite" TR-line (1/3)

- ***General solution for finite-length transmission lines***

$$\begin{cases} \dfrac{d^2V(z)}{dz^2} = \gamma^2 V(z) \\ \dfrac{d^2I(z)}{dz^2} = \gamma^2 I(z) \end{cases} \Rightarrow \begin{cases} V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \\ I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \end{cases}$$

where $\dfrac{V_0^+}{I_0^+} = -\dfrac{V_0^-}{I_0^-} = Z_0$ ***(Proof; HW)***

(Time-harmonic Helmholtz's equations)

$I_0^- e^{\gamma z}$: ***Reflected waves*** *(When TR-line is finite in length, and not impedance-matched)*

$I_0^+ e^{-\gamma z}$: Forward waves

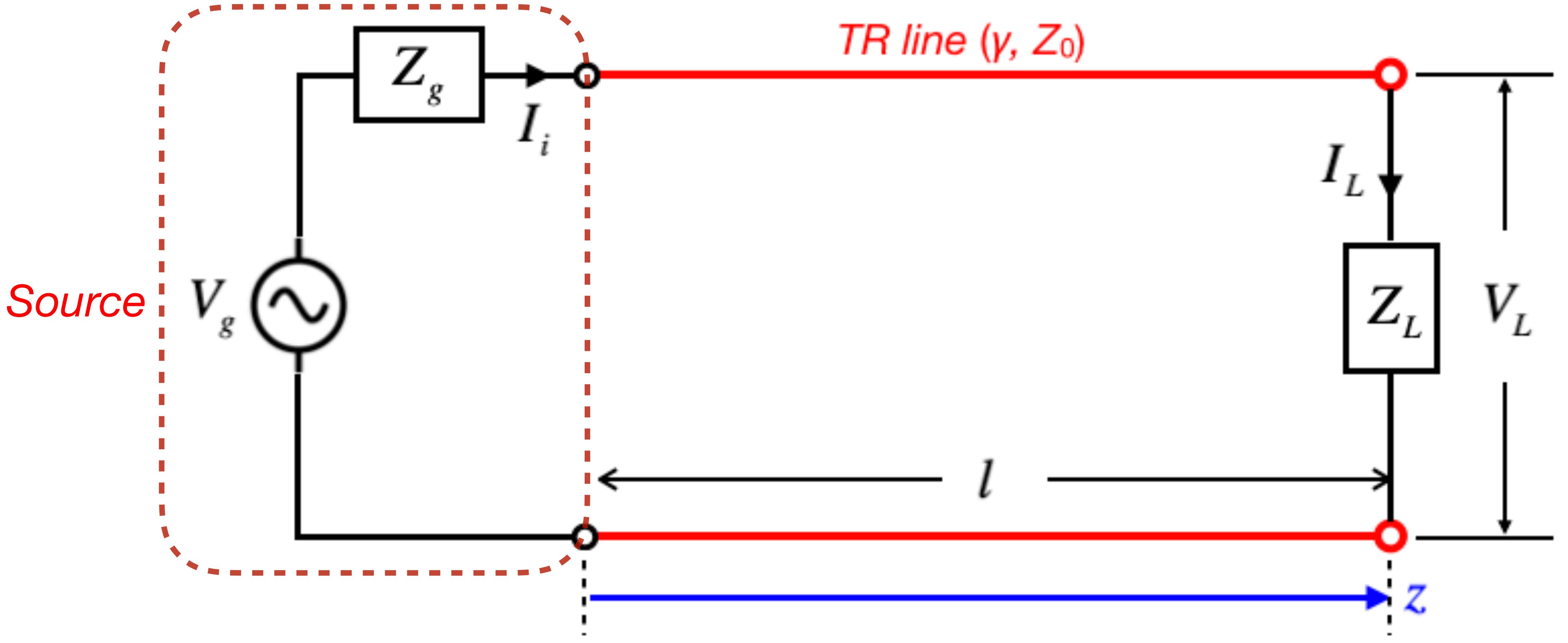
"Reflection-less" matching condition

$Z_L = Z_0$

Z_L: load impedance

Z_0: characteristic impedance of TR line

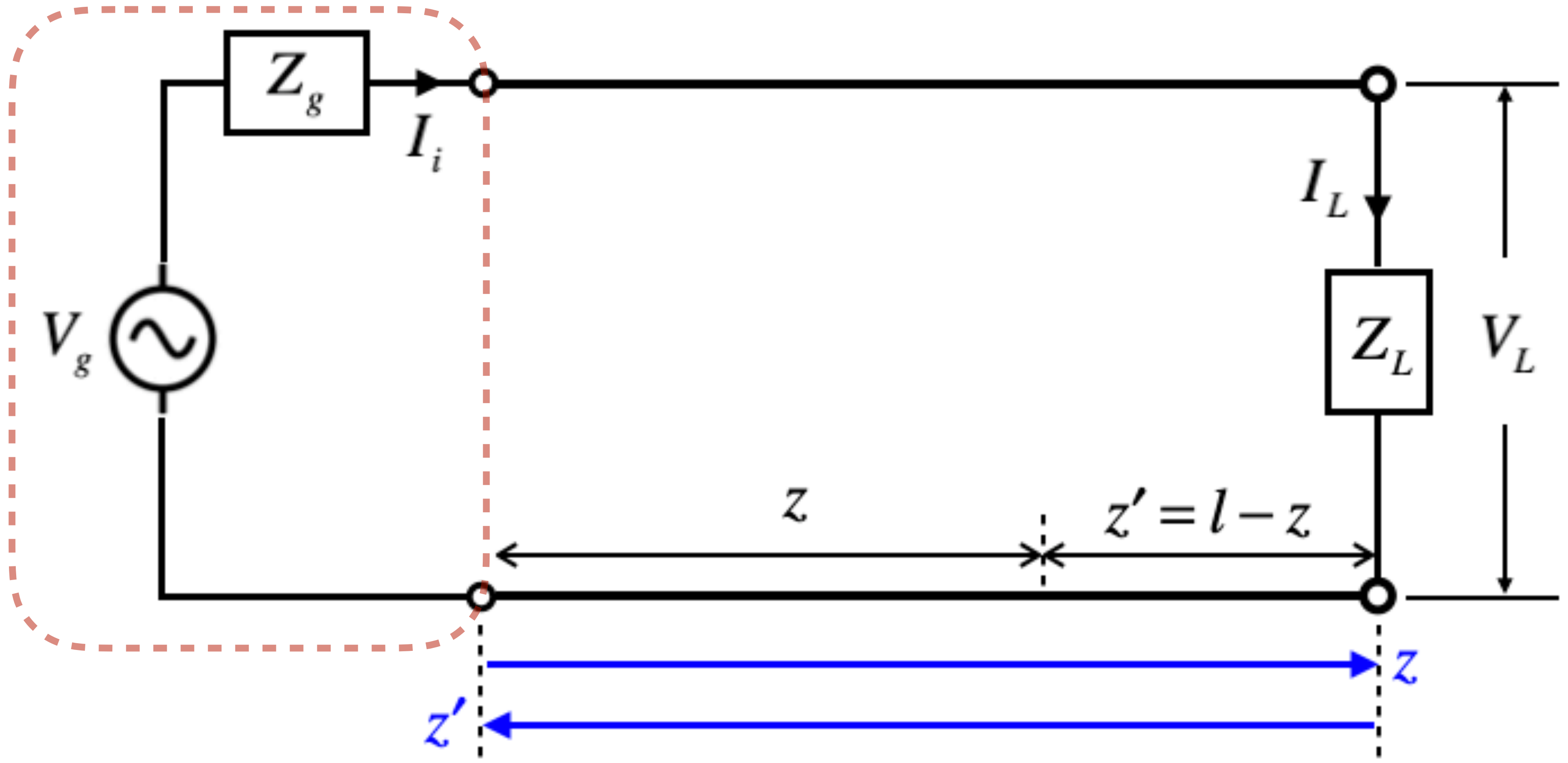
- ***Equivalent circuit model***
  - Sinusoidal voltage V_g with an internal impedance Z_g
  - TR-line characterized by γ (propagation constant) and Z_0 (characteristic impedance)
  - Line of length l terminated in an arbitrary impedance Z_L



<Equivalent circuit for finite transmission line>

- ***What to do to know V(z) and I(z)?***
  - Solving for 4 unknowns (V_0^+, V_0^-, I_0^+, I_0^-)
  - They are partly dependent on each other

Circuit model for "finite" TR-line (2/3)



- ***General solution for finite-length transmission lines***
  - *4 unknowns (V_0^+, V_0^-, I_0^+, I_0^-) should be identified*

$$\begin{cases} V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \\ I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \end{cases} \quad \cdots(1)$$

- Express "*unknowns*" *vs.* V and I at the load end ($z = l$)
- At $z = l$, we have

$$\begin{cases} V(l) = V_L = V_0^+ e^{-\gamma l} + V_0^- e^{\gamma l} \\ I(z) = I_L = \dfrac{V_0^+}{Z_0} e^{-\gamma l} - \dfrac{V_0^-}{Z_0} e^{\gamma l} \quad \text{where} \quad \dfrac{V_0^+}{I_0^+} = -\dfrac{V_0^-}{I_0^-} = Z_0 \end{cases}$$

- Express ***unknown V_0^+ and V_0^-*** vs. V_L and I_L

$$\begin{cases} V_0^+ = \dfrac{1}{2}(V_L + I_L Z_0) e^{\gamma l} = \dfrac{1}{2} I_L (Z_L + Z_0) e^{\gamma l} \\ V_0^- = \dfrac{1}{2}(V_L - I_L Z_0) e^{-\gamma l} = \dfrac{1}{2} I_L (Z_L - Z_0) e^{-\gamma l} \end{cases} \quad \cdots(2)$$

$$\boxed{V_L = I_L Z_L}$$

- If we substitute ***eqn. (2)*** into ***(1)***, we get

$$\begin{cases} V(z) = \dfrac{I_L}{2}\left[(Z_L + Z_0) e^{\gamma(l-z)} + (Z_L - Z_0) e^{-\gamma(l-z)}\right] \\ I(z) = \dfrac{I_L}{2Z_0}\left[(Z_L + Z_0) e^{\gamma(l-z)} - (Z_L - Z_0) e^{-\gamma(l-z)}\right] \end{cases}$$

  - Voltage and current at ***z (distance from source)*** are expressed in terms of Z_0 and Z_L and l

- Equivalent expression with ***z' = l - z (distance from load):***

$$\boxed{\begin{cases} V(z') = \dfrac{I_L}{2}\left[(Z_L + Z_0) e^{\gamma z'} + (Z_L - Z_0) e^{-\gamma z'}\right] \\ I(z') = \dfrac{I_L}{2Z_0}\left[(Z_L + Z_0) e^{\gamma z'} - (Z_L - Z_0) e^{-\gamma z'}\right] \end{cases}}$$

(Note: z and z' dependence on V and I are different!)

Chap. 9 | Circuit model for "finite" TR-line (3/3)

- ***General solution for finite-length transmission lines***
  - Simplified form by using hyperbolic functions

$$\begin{cases} V(z') = \dfrac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}\right] \\ I(z') = \dfrac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma z'} - (Z_L - Z_0)e^{-\gamma z'}\right] \end{cases} \Rightarrow$$

$$\left(\because e^{\gamma z'} + e^{-\gamma z'} = 2\cosh\gamma z' \text{ and } e^{\gamma z'} - e^{-\gamma z'} = 2\sinh\gamma z'\right)$$

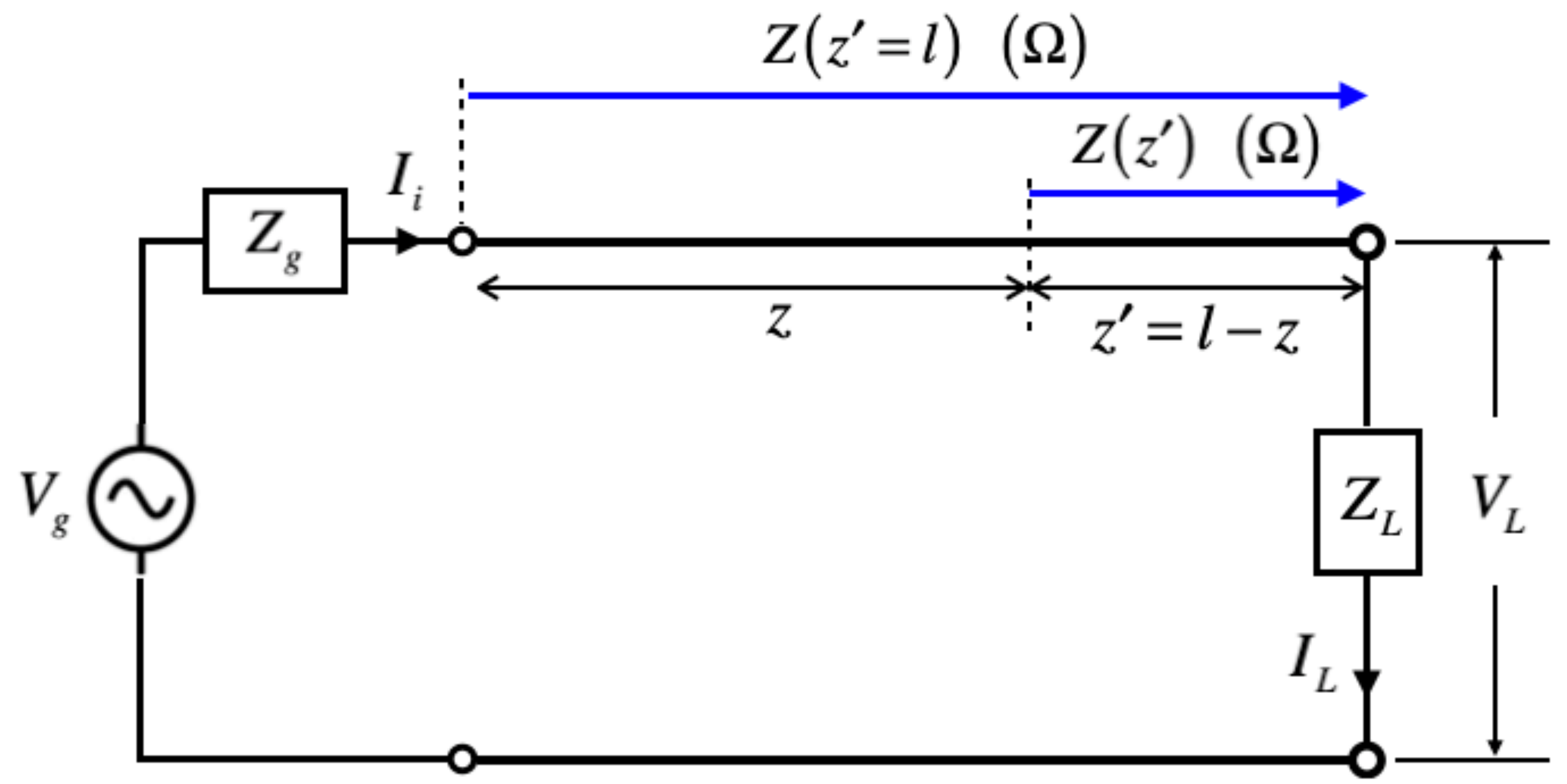
$$\boxed{\begin{aligned} V(z') &= I_L\left[Z_L \cosh\gamma z' + Z_0 \sinh\gamma z'\right] \\ I(z') &= \frac{I_L}{Z_0}\left[Z_L \sinh\gamma z' + Z_0 \cosh\gamma z'\right] \end{aligned}}$$

Can find voltage and current at any z' using I_L, Z_L, γ, Z_0!

- ***Input impedance***
  - *Z(z')=V(z')/I(z')*: Impedance ***looking "toward"*** the load end from *z'*

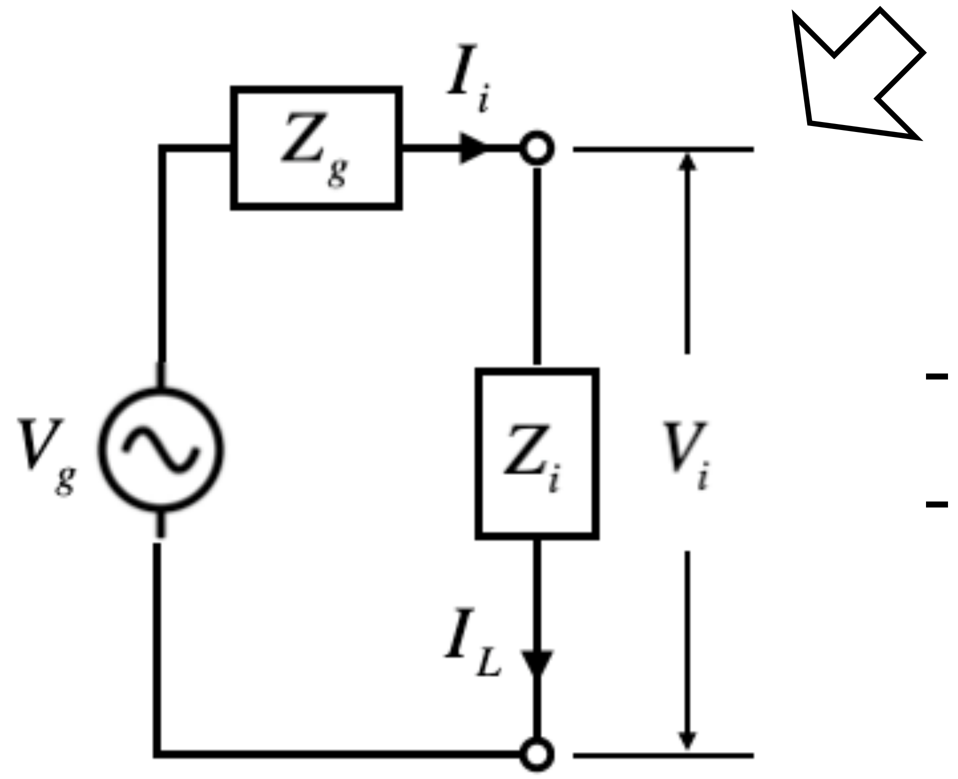
$$Z(z') = \frac{V(z')}{I(z')} = Z_0 \frac{Z_L \cosh\gamma z' + Z_0 \sinh\gamma z'}{Z_L \sinh\gamma z' + Z_0 \cosh\gamma z'}$$

$$= Z_0 \frac{Z_L + Z_0 \tanh\gamma z'}{Z_0 + Z_L \tanh\gamma z'} \quad (\Omega)$$



  - At the source end (*z'* = *l*), the source sees an input impedance

$$\boxed{\left[(Z)_{z'=l} = (Z)_{z=0}\right] \triangleq Z_i = Z_0 \frac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l} \quad (\Omega)}$$



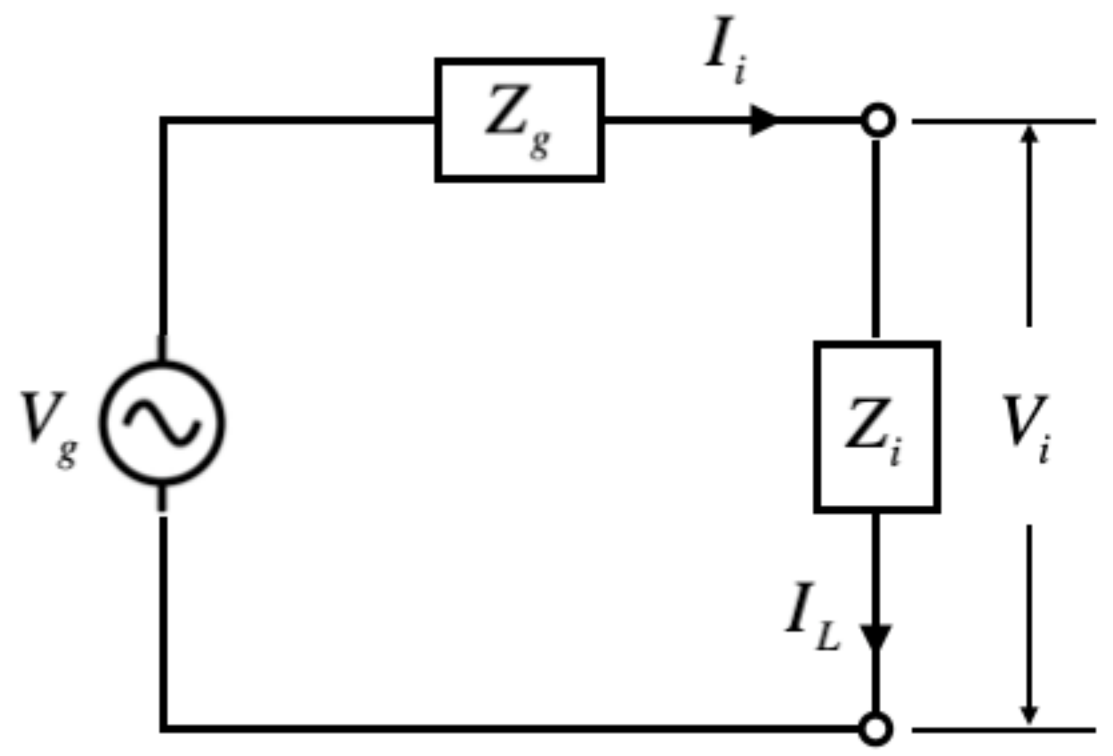
- Z_i = Finite TR-line + load *L*
- V_i and I_i easily obtainable!

Impedance matching condition 1: Reflection-less

- ***"Reflection-less" impedance-matching condition***
  - load impedance (Z_L) = characteristic impedance of the line (Z_0)

  → input impedance Z_i becomes Z_0 as below

$$Z_i = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} = Z_0 \quad (\Omega)$$



  - Under such condition where $Z_L = Z_0$, Voltage and current expressions become

$$\begin{cases} V(z) = \dfrac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma(l-z)} + (Z_L - Z_0)e^{-\gamma(l-z)}\right] = \left(I_L Z_0 e^{\gamma l}\right)e^{-\gamma z} = V_i e^{-\gamma z} \\ I(z) = \dfrac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma(l-z)} - (Z_L - Z_0)e^{-\gamma(l-z)}\right] = \left(I_L e^{\gamma l}\right)e^{-\gamma z} = I_i e^{-\gamma z} \end{cases}$$

→ *Wave only traveling in +z direction, **no reflected wave! (in -z direction)***

∴ *When a finite TR-line terminated with its characteristic impedance ($Z_L = Z_0$), its V(z) and Z(z) are the same as if the line is extended to infinity*

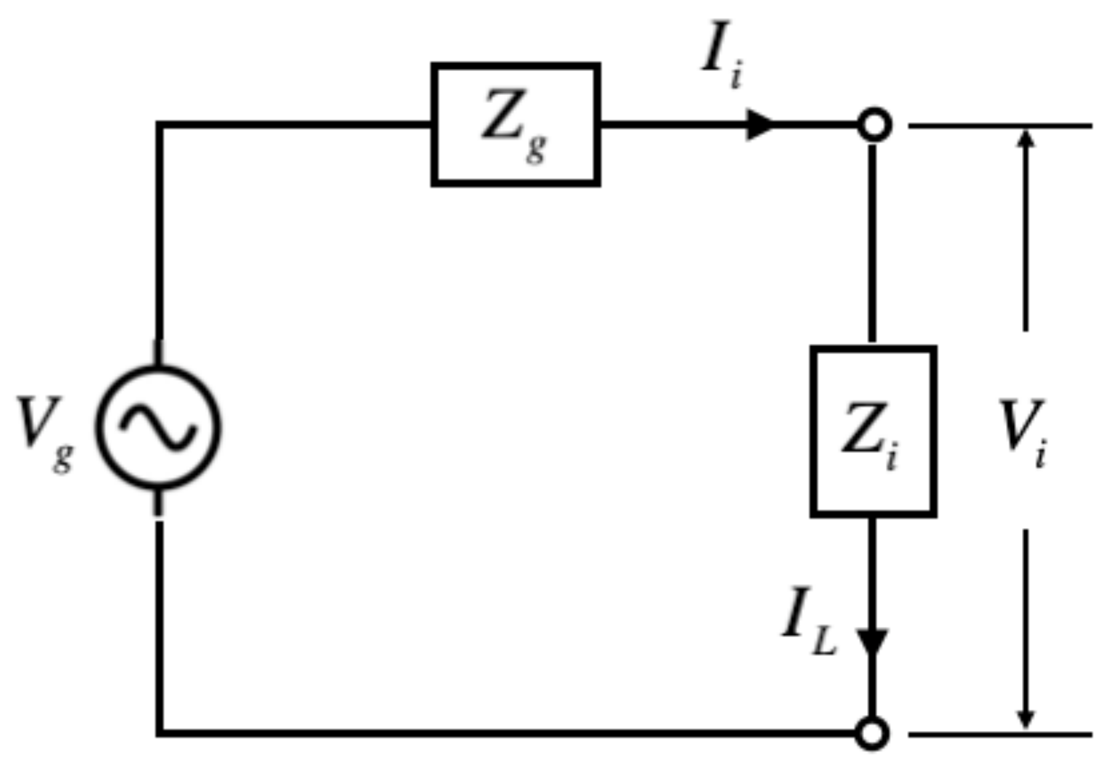
Reflection-less condition

$$\boxed{Z_L = Z_0 \quad (\Omega)}$$

Chap. 9 | Impedance matching condition 2: Maximum power transfer

- *Impedance matching for maximum power transfer*
  - Input impedance (Z_i) = Complex conjugate of internal impedance of the source (Z_g^*)

    → ***Maximum power transfer!*** *(Proof as below)*



$$\begin{cases} V_i = \dfrac{Z_i}{Z_g + Z_i} V_g \\ I_i = \dfrac{V_g}{Z_g + Z_i} \end{cases}$$

$$(P_{av})_L = (P_{av})_i = \frac{1}{2}\mathrm{Re}\left[V_i I_i^*\right]$$

Output power = Input power delivered to input terminal (assuming "lossless" TR-line)

(see right figure→) $$= \frac{1}{2}\mathrm{Re}\left[\frac{V_g V_g^* Z_i}{(Z_g + Z_i)(Z_g + Z_i)^*}\right] = \frac{1}{2}\frac{|V_g|^2 R_i}{(R_g + R_i)^2 + (X_g + X_i)^2}$$

where $Z_g = R_g + jX_g$ and $Z_i = R_i + jX_i$

To maximize output power: [i.e. max$(P_{av})_L$]

$$\begin{cases} \dfrac{\partial (P_{av})_L}{\partial R_i} = 0 \quad \rightarrow \quad R_i = R_g \\ \dfrac{\partial (P_{av})_L}{\partial X_i} = 0 \quad \rightarrow \quad X_i = -X_g \end{cases}$$

Power dissipation: $$(P_{av})_L = (P_{av})_G = \frac{|V_g|^2}{8R_g}$$ *(50 % of the generated power!)*

Maximum power-transfer matching

$$\boxed{\begin{aligned} & Z_i = R_i + jX_i = R_g - jX_g \triangleq Z_g^* \\ & \therefore Z_i = Z_g^* \end{aligned}}$$

Input impedance vs. load impedance (Open-circuit)

- ***Open-circuit = infinite load impedance ($Z_L \to \infty$)***

  - For simplicity, let's assume "lossless" TR-line represented by

$$\begin{cases} \gamma = j\beta \\ Z_0 = R_0 \end{cases} \Rightarrow$$

Then, input impedance given by

$$\boxed{Z_i = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} = R_0 \frac{Z_L + jR_0 \tan \beta l}{R_0 + jZ_L \tan \beta l} \quad (\Omega)}$$

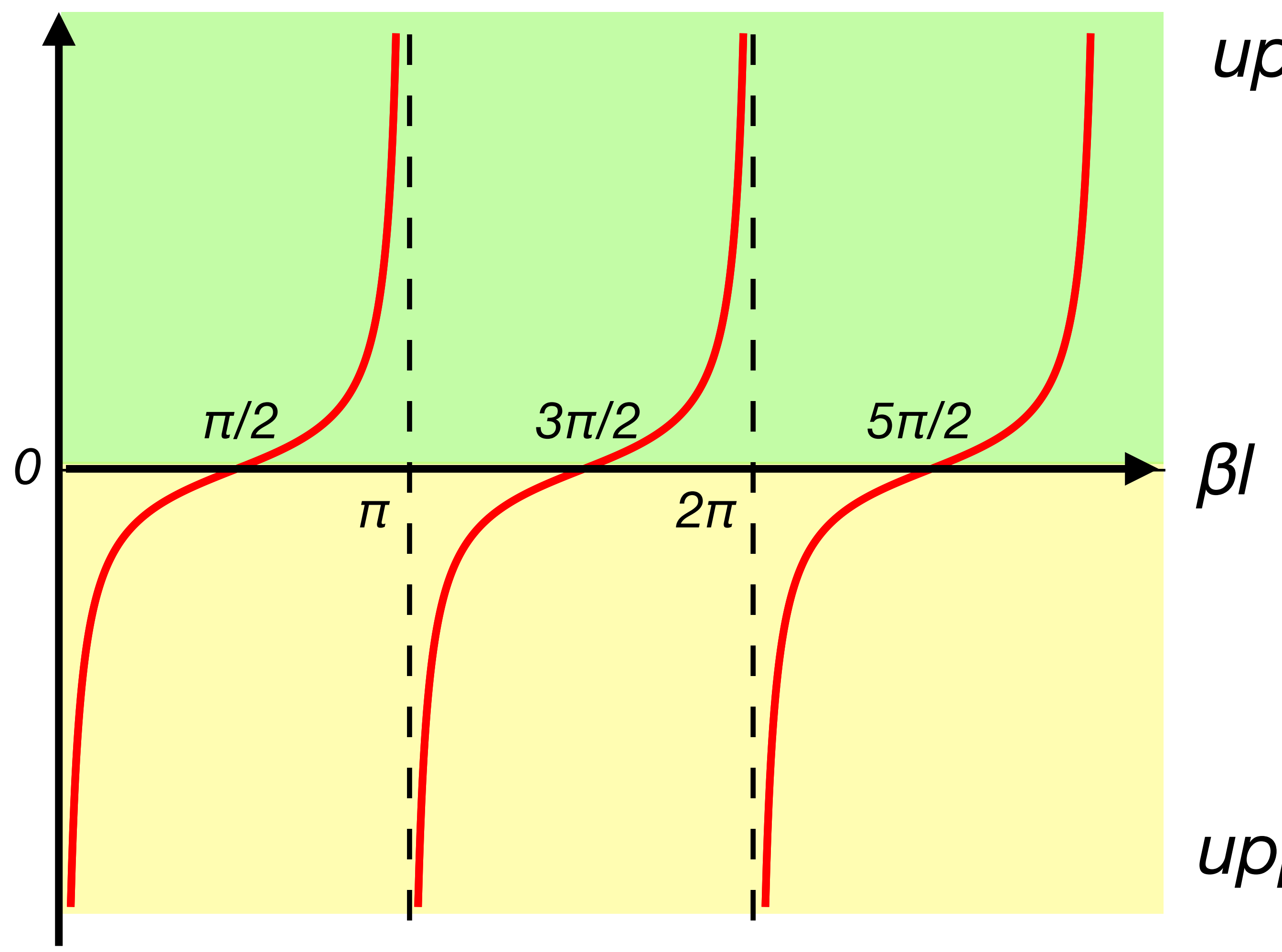
$$\left(\because \tanh(\gamma l) = \tanh(j\beta l) = j\tan(\beta l)\right)$$

  - Input impedance with $Z_L \to \infty$

$$\lim_{Z_L\to\infty} Z_i = \lim_{Z_L\to\infty} R_0 \frac{Z_L + jR_0 \tan \beta l}{R_0 + jZ_L \tan \beta l} = \frac{R_0}{j\tan \beta l} = \boxed{-jR_0 \cot \beta l \quad (\Omega)}$$

→ Purely imaginary [reactive] (either ***capacitive*** or ***inductive***)

$$Z_i = jX_i = -jR_0 \cot \beta l$$



- Depending on choice of length *l*, input impedance can be modified
- For ***O.C. TR-line with very short length (l) comparable to wavelength***,

$$\beta l = \frac{2\pi l}{\lambda} \ll 1 \quad \rightarrow \quad \lim_{Z_L\to\infty} Z_i = \frac{R_0}{j\tan \beta l} \cong \frac{R_0}{j\beta l} = -j\frac{\sqrt{L/C}}{\omega\sqrt{LC}\, l} = \boxed{-j\frac{1}{\omega C l}}$$

$$\boxed{\because \beta = \omega\sqrt{LC},\ R_0 = \sqrt{\frac{L}{C}}}$$

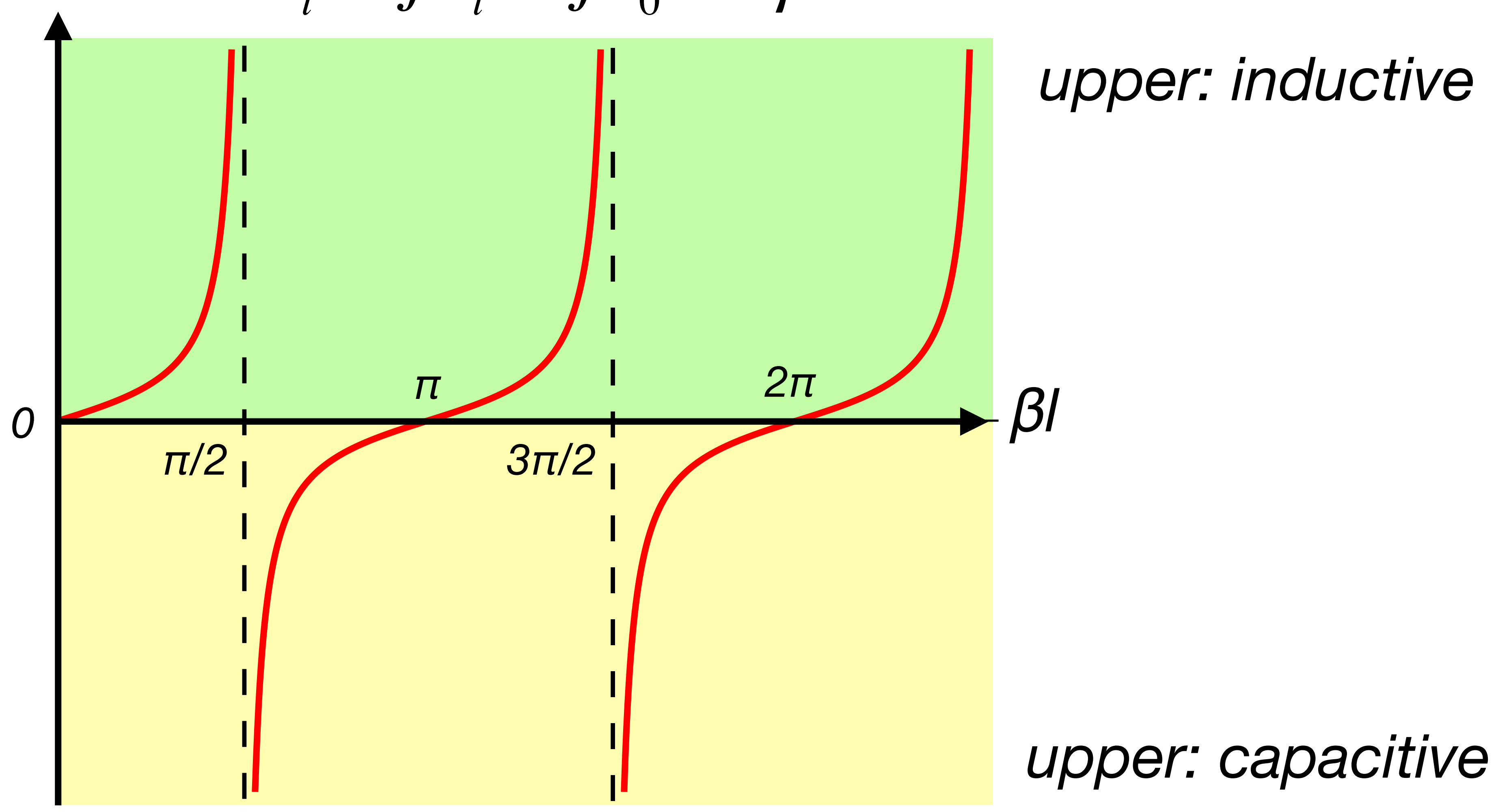
Purely Capacitive!

Input impedance vs. load impedance (short-circuit)

- ***Short circuit = zero load impedance (Z_L = 0)***
  - Input impedance with $Z_L = 0$

$$\lim_{Z_L \to 0} Z_i = \lim_{Z_L \to 0} R_0 \frac{Z_L + jR_0 \tan\beta l}{R_0 + jZ_L \tan\beta l} = jR_0 \tan\beta l \quad (\Omega)$$

$$Z_i = jX_i = jR_0 \tan\beta l$$



- Depending on choice of length l, input impedance can be modified
- For ***S.C. TR-line with very short length (l) comparable to wavelength***,

$$\beta l = \frac{2\pi l}{\lambda} \ll 1 \quad \rightarrow \quad \lim_{Z_L \to \infty} Z_i = jR_0 \tan\beta l \cong jR_0 \beta l = j\sqrt{\frac{L}{C}}\omega\sqrt{LC}\, l = \boxed{j\omega L l}$$

$$\because \beta = \omega\sqrt{LC}, \; R_0 = \sqrt{\frac{L}{C}}$$

Purely inductive!

Input impedance vs. TR-line length

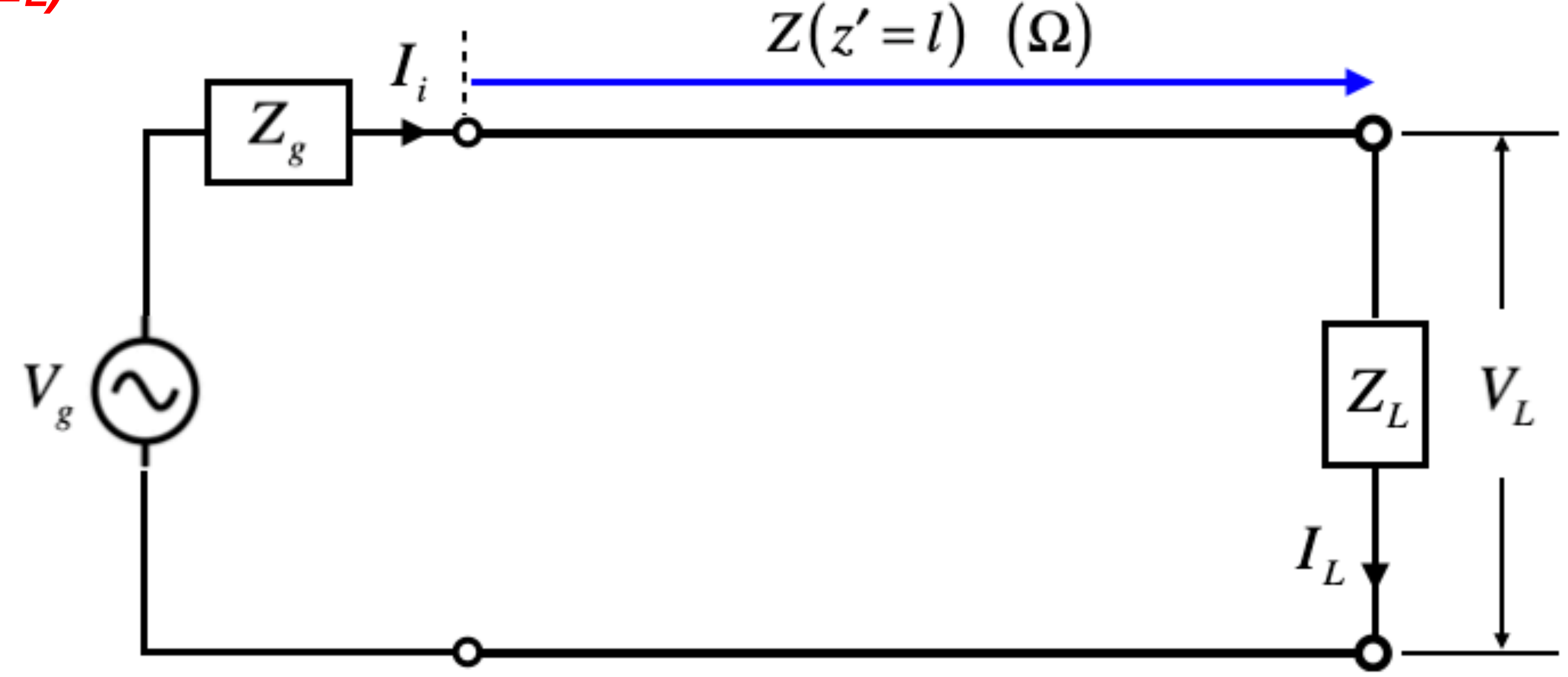
- ***"Quarter-wave" TR-line***

  - Length of the line, l = odd multiple of $\lambda/4$

$$l=(2n-1)\frac{\lambda}{4},\quad (n=1,2,3,\cdots)\qquad \text{Then, } \beta l=\frac{2\pi}{\lambda}\cdot(2n-1)\frac{\lambda}{4}=(2n-1)\frac{\pi}{2}\ \rightarrow\ \tan\beta l=\tan\left[(2n-1)\frac{\pi}{2}\right]\rightarrow\pm\infty$$

$$\boxed{\therefore \lim_{\tan(\beta l)\to\pm\infty} Z_i=\lim_{\tan(\beta l)\to\pm\infty} R_0\frac{Z_L+jR_0\tan\beta l}{R_0+jZ_L\tan\beta l}=\frac{R_0^2}{Z_L}\quad(\Omega)}$$

Quarter-wave lines
= Impedance inverter ($1/Z_L$)

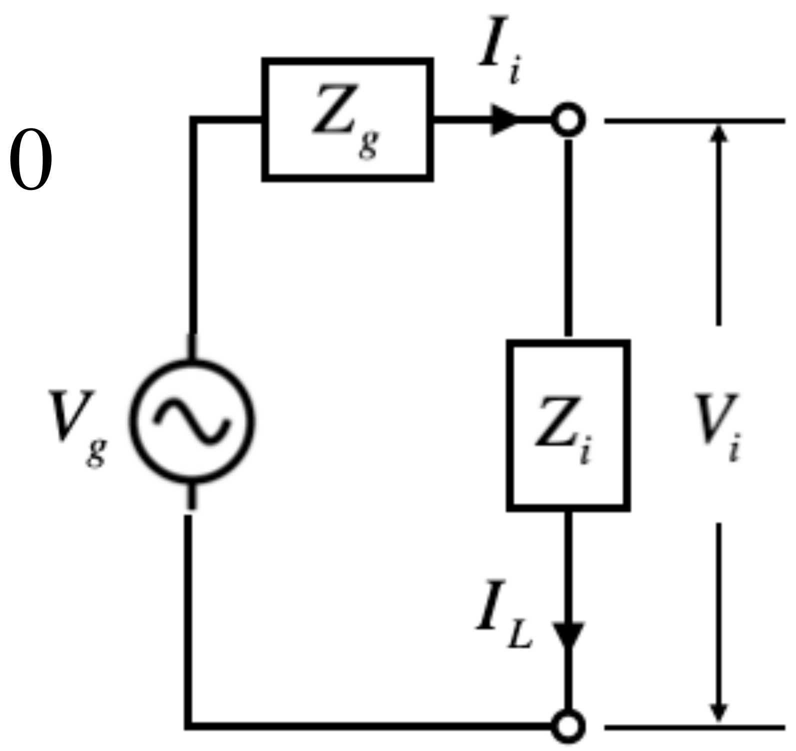


  - If load impedance Z_L → ∞ (Open), Z_i → 0 (Short)
  - If load impedance Z_L → 0 (short), Z_i → ∞ (Open)

- ***"Half-wave" TR-line***

  - Length of the line, l = integer multiple of $\lambda/2$

$$l=\frac{n}{2}\lambda,\quad (n=1,2,3,\cdots)\qquad \text{Then, } \beta l=\frac{2\pi}{\lambda}\cdot\left(\frac{n\lambda}{2}\right)=n\pi\ \rightarrow\ \tan\beta l=0$$



$$\boxed{\therefore \lim_{\tan(\beta l)\to 0} Z_i=\lim_{\tan(\beta l)\to 0} R_0\frac{Z_L+jR_0\tan\beta l}{R_0+jZ_L\tan\beta l}=Z_L\quad(\Omega)}$$

half-wave lines

Chap. 9 | TR-line characteristics vs. input impedance

- ***Determination of TR-line characteristics***
  - TR-line represented by Z_0 (characteristic impedance) and γ (propagation constant)
  - Can be obtained by measuring input impedance Z_i under open & short-circuit condition

Open-circuit: $Z_L \to \infty \Rightarrow Z_{io} = Z_0 \dfrac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l} = Z_0 \coth\gamma l \quad (\Omega)$ $\quad Z_{io} \cdot Z_{is} = Z_0^2 \tanh\gamma l \cdot \coth\gamma l = Z_0^2 \rightarrow \boxed{\therefore Z_0 = \sqrt{Z_{io} \cdot Z_{is}}}$

Short-circuit: $Z_L = 0 \Rightarrow Z_{is} = Z_0 \dfrac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l} = Z_0 \tanh\gamma l \quad (\Omega)$ $\quad \dfrac{Z_{is}}{Z_{io}} = \tanh^2 \gamma l \rightarrow \boxed{\therefore \gamma = \dfrac{1}{l}\tanh^{-1}\sqrt{\dfrac{Z_{is}}{Z_{io}}}}$

These apply to lossy TR-line as well!

- ***Input impedance for a "lossy" and "short-circuited" TR-line***
  - Lossy → $\gamma = \alpha + j\beta$
  - Short-circuit → $Z_L = 0$

$$
\begin{aligned}
Z_{is} = Z_0 \frac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l} &= Z_0 \tanh\gamma l \\
&= Z_0 \frac{\sinh(\alpha + j\beta)l}{\cosh(\alpha + j\beta)l} \\
&= Z_0 \frac{\sinh(\alpha l)\cos(\beta l) + j\cosh(\alpha l)\sin(\beta l)}{\cosh(\alpha l)\cos(\beta l) + j\sinh(\alpha l)\sin(\beta l)}
\end{aligned}
$$

Simple hyperbolic math!

$$
\begin{cases}
\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b) \\
\sinh(a+b) = \sinh(a)\cosh(b) + \cosh(a)\sinh(b)
\end{cases}
$$

$$
\begin{cases}
\cosh(ja) = \cos a \\
\sinh(ja) = j\sin a
\end{cases}
$$

Chap. 9 | Lossy TR-line

$$Z_{is} = Z_0 \frac{\sinh(\alpha l)\cos(\beta l) + j\cosh(\alpha l)\sin(\beta l)}{\cosh(\alpha l)\cos(\beta l) + j\sinh(\alpha l)\sin(\beta l)}$$

- *"lossy" and "short-circuited" TR-line (realistic case)*
  - For a half-wave line

$$l = \frac{n}{2}\lambda, \quad (n = 1, 2, 3, \cdots) \qquad \beta l = \frac{2\pi}{\lambda}\cdot\left(\frac{n\lambda}{2}\right) = n\pi \;\rightarrow\; \sin\beta l = 0 \;\rightarrow\; Z_{is} = Z_0\tanh(\alpha l) \cong \boxed{Z_0 \alpha l}$$

Small, but non-zero

  - For a quarter-wave line

$$l = (2n-1)\frac{\lambda}{4}, \quad (n = 1, 2, 3, \cdots) \qquad \beta l = \frac{2\pi}{\lambda}\cdot(2n-1)\frac{\lambda}{4} = (2n-1)\frac{\pi}{2} \;\rightarrow\; \cos\beta l = 0 \;\rightarrow\; Z_{is} = \frac{Z_0}{\tanh(\alpha l)} \cong \boxed{\frac{Z_0}{\alpha l}}$$

Large, but finite

  - Both assumed low loss condition, $\alpha l << 1$.

- *"lossless" and "short-circuited" TR-line (Ideal case)*
  - For a half-wave line

$$\lim_{\tan(\beta l)\to 0} Z_i = \lim_{\tan(\beta l)\to 0} R_0 \frac{Z_L + jR_0\tan\beta l}{R_0 + jZ_L\tan\beta l} = Z_L \;\rightarrow\; \boxed{0}$$

Zero

  - For a quarter-wave line

$$\lim_{\tan(\beta l)\to \pm\infty} Z_i = \lim_{\tan(\beta l)\to \pm\infty} R_0 \frac{Z_L + jR_0\tan\beta l}{R_0 + jZ_L\tan\beta l} = \frac{R_0^2}{Z_L} \;\rightarrow\; \boxed{\infty}$$

Infinity

	Half-wave line	**Quarter-wave line**
Impedance at resonance	Minimum (0 for lossless)	Maximum (∞ for lossless)
Frequency dependent	Band-pass	Band-stop
Similarity	Series-RLC resonant circuit	Parallel-RLC resonant circuit